



Fujita exponent and asymptotics of solutions to nonlinear equation with fractional laplacian

Ahmad Fino, Grzegorz Karch

► To cite this version:

Ahmad Fino, Grzegorz Karch. Fujita exponent and asymptotics of solutions to nonlinear equation with fractional laplacian. 2008. hal-00333433v1

HAL Id: hal-00333433

<https://hal.science/hal-00333433v1>

Preprint submitted on 8 Nov 2008 (v1), last revised 10 Jan 2009 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

FUJITA EXPONENT AND ASYMPTOTICS OF SOLUTIONS TO A NONLINEAR EQUATION WITH FRACTIONAL LAPLACIAN

AHMAD FINO AND GRZEGORZ KARCH

ABSTRACT. The large time behavior of nonnegative solutions to the reaction-diffusion equation $\partial_t u = -(-\Delta)^{\alpha/2} u + \lambda u^p$, ($\alpha \in (0, 2]$, $p \geq 1$, $\lambda \in \{-1, 1\}$) posed on \mathbb{R}^N and supplemented with an integrable initial condition is studied. The solutions for the equation with absorbing term ($\lambda = -1$) are global-in-time. In this case, we show that the anomalous diffusion term determines the large time asymptotics for $p > 1 + \alpha/N$, while nonlinear effects win if $p \leq 1 + \alpha/N$.

It is known that some solutions to the equation with $\lambda = 1$ blow up in a finite time. Using the method from this article, we show the blow-up of all positive solutions in the critical case $p = 1 + \alpha/N$.

1. INTRODUCTION

We study the behavior, as $t \rightarrow \infty$, of solutions to the following initial value problem for the reaction-diffusion equation with the anomalous diffusion

$$(1) \quad \partial_t u = -\Lambda^\alpha u + \lambda u^p, \quad x \in \mathbb{R}^N, t > 0,$$

$$(2) \quad u(x, 0) = u_0(x),$$

where the pseudodifferential operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ with $0 < \alpha \leq 2$ is defined by the Fourier transformation: $\widehat{\Lambda^\alpha u}(\xi) = |\xi|^\alpha \widehat{u}(\xi)$. Moreover, we assume that $\lambda \in \{-1, 1\}$ and $p > 1$.

Nonlinear evolution problems involving the fractional Laplacian describing *the anomalous diffusion* (or the α -stable Lévy diffusion) have been extensively studied in the mathematical and physical literature (see [2, 11, 5] for references). One of possible ways to understand the interaction between

Date: October 20, 2008.

2000 Mathematics Subject Classification. Primary 35K55; Secondary 35B40, 60H99.

Key words and phrases. Large time behavior of solutions; fractional Laplacian; blow-up of solutions; critical exponent.

The authors express their gratitude to Piotr Biler for careful reading of the manuscript and helpful remarks. The first author gratefully thanks the Mathematical Institute of University of Wrocław for the warm hospitality. The preparation of this paper was supported in part by the European Commission Marie Curie Host Fellowship for the Transfer of Knowledge “Harmonic Analysis, Nonlinear Analysis and Probability” MTKD-CT-2004-013389 and by the MNiSW grant N201 022 32 / 09 02.

the anomalous diffusion operator (given by Λ^α or, more generally, by the Lévy diffusion operator) and the nonlinearity in the equation under consideration is the study of the large time asymptotics of solutions to such equations. Our goal is to contribute to this theory and our results can be summarized as follows. For $\lambda = -1$ in equation (1), nonnegative solutions to the Cauchy problem exist globally in time. Hence, we study the decay properties of mass $M(t) = \int_{\mathbb{R}^N} u(x, t) dx$ of the solutions $u = u(x, t)$ to problem (1)-(2). We prove that $\lim_{t \rightarrow \infty} M(t) = M_\infty > 0$ for $p > 1 + \alpha/N$ (cf. Theorem 1, below), while $M(t)$ tends to zero as $t \rightarrow \infty$ if $1 < p \leq 1 + \alpha/N$ (cf. Theorem 2). As a by-product of our analysis, we show the blow-up of all nonnegative solutions to (1)-(2) with $\lambda = 1$ in the case of the critical nonlinearity exponent $p = 1 + \alpha/N$ (see Theorem 3, below).

The idea which allows us to express the competition between diffusive and nonlinear terms in an evolution equation by studying the large time behavior of the space integral of a solution was already introduced by Ben-Artzi & Koch [1] who considered the viscous Hamilton-Jacobi equation $u_t = \Delta u - |\nabla u|^p$ (see also Pinsky [16]). An analogous result for the equation $u_t = \Delta u + |\nabla u|^p$ (with growing-in-time mass of solutions) was proved by Laurençot & Souplet [13]. Such questions concerning the asymptotic behavior of solutions to the Hamilton-Jacobi equation with the Lévy diffusion operator have been answered in [11].

In the case of the classical reaction-diffusion equation (*i.e.* equation (1) with $\alpha = 2$), for $p < 1 + 2/N$, Fujita [6] proved the nonexistence of nonnegative global-in-time solution for any nontrivial initial condition. On the other hand, if $p > 1 + 2/N$, global solutions do exist for any sufficiently small nonnegative initial data. The proof of the blow-up of all nonnegative solutions in the critical case $p = 1 + 2/N$ was completed in [9, 17, 12]. Analogous blow-up results for problem (1)-(2) with the fractional Laplacian and the critical exponent $p = 1 + \alpha/N$ for the existence/nonexistence of solutions are contained *e.g.* in [17, 7, 8, 3].

2. STATEMENT OF RESULTS

In all theorems below, we always assume that $u = u(x, t)$ is a nonnegative (possibly weak) solution of problem (1)-(2) corresponding to the nonnegative initial datum $u_0 \in L^1(\mathbb{R}^N)$. Let $u_0 \not\equiv 0$, for simplicity of the exposition. We refer the reader to [5] for several results on the existence, the uniqueness and the regularity of solutions to (1)-(2) as well as for the proof of the

maximum principle (which assures that the solution is nonnegative if the corresponding initial datum is so).

First, we deal with the equation (1) containing the absorbing nonlinearity ($\lambda = -1$) and we study the decay of “mass”

$$(3) \quad M(t) \equiv \int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx - \int_0^t \int_{\mathbb{R}^N} u^p(x, s) dx ds.$$

Remark. In order to obtain equality (3), it suffices to integrate equation (1) with respect to x and t . Another method which leads to (3) and which requires weaker regularity assumptions on a solution consists in integrating with respect to x the integral formulation of problem (1)-(2) (see (12), below) and using the Fubini theorem. \square

Since we limit ourselves to nonnegative solutions, the function $M(t)$ defined in (3) is nonnegative and nonincreasing. Hence, the limit $M_\infty = \lim_{t \rightarrow \infty} M(t)$ exists and we answer the question whether it is equal to zero or not.

In our first theorem, the diffusion phenomena determine the large time asymptotics of solutions to (1)-(2).

Theorem 1. *Assume that $u = u(x, t)$ is a nonnegative nontrivial solution of (1)-(2) with $\lambda = -1$ and $p > 1 + \alpha/N$. Then $\lim_{t \rightarrow \infty} M(t) = M_\infty > 0$.*

Moreover, for all $q \in [1, \infty)$

$$(4) \quad t^{\frac{N}{\alpha}(1-\frac{1}{q})} \|u(t) - M_\infty P_\alpha(t)\|_q \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where the function $P_\alpha(x, t)$ denotes the fundamental solution of the linear equation $u_t + \Lambda^\alpha u = 0$, cf. (7) below.

In the remaining range of p , mass $M(t)$ converges to zero and this phenomenon can be interpreted as the domination of nonlinear effects in the large time asymptotics of solutions to (1)-(2). Note here that mass $M(t) = \int_{\mathbb{R}^N} u(x, t) dx$ of every solution to the linear equation $u_t + \Lambda^\alpha u = 0$ is constant in time.

Theorem 2. *Assume that $u = u(x, t)$ is a nonnegative solutions of problem (1)-(2) with $\lambda = -1$ and $1 < p \leq 1 + \alpha/N$. Then $\lim_{t \rightarrow \infty} M(t) = 0$.*

Let us emphasize that the proof of Theorem 2 is based on the, so-called, rescaled test function method which was used by Mitidieri & Pokhozhaev (cf. e.g. [14, 15] and the references therein) to prove the nonexistence of solution to nonlinear elliptic and parabolic equations.

As a by-product of our analysis, we can also contribute to the theory of the blow-up of solutions to (1)-(2) with $\lambda = +1$. Recall that the method

of the rescaled test function (which we also apply here) has been used in [7, 8] to show the blow-up of all positive solutions to (1)-(2) with $\lambda = 1$ and $p < 1 + \alpha/N$. Here, we complete that result by a simple proof of the blow-up in the critical case $p = 1 + \alpha/N$.

Theorem 3. *If $\lambda = 1$, $\alpha \in (0, 2]$ and $p = 1 + \alpha/N$, then any nonnegative nonzero solution of (1)-(2) blows up in a finite time.*

3. PROOFS OF THEOREMS 1, 2, AND 3

First note that any sufficiently regular nonnegative solution to (1)-(2) satisfies

$$(5) \quad 0 \leq \int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx + \lambda \int_0^t \int_{\mathbb{R}^N} u^p(x, s) dx ds.$$

Hence, for $\lambda = -1$ and $u_0 \in L^1(\mathbb{R}^N)$, we immediately obtain

$$(6) \quad u \in L^\infty([0, \infty), L^1(\mathbb{R}^N)) \cap L^p(\mathbb{R}^N \times (0, \infty)).$$

Proof of Theorem 1. First, we recall that the fundamental solution $P_\alpha = P_\alpha(x, t)$ of the linear equation $\partial_t u + \Lambda^\alpha u = 0$ can be written *via* the Fourier transform as follows

$$(7) \quad P_\alpha(x, t) = t^{-N/\alpha} P_\alpha(xt^{-1/\alpha}, 1) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix \cdot \xi - t|\xi|^\alpha} d\xi.$$

It is well known that for each $\alpha \in (0, 2]$, this function satisfies

$$(8) \quad P_\alpha(\cdot, 1) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad P_\alpha(x, t) \geq 0, \quad \int_{\mathbb{R}^N} P_\alpha(x, t) dx = 1,$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Hence, using the Young inequality for the convolution and the self-similar form of P_α , we have

$$(9) \quad \|P_\alpha(t) * u_0\|_p \leq Ct^{-N(1-1/p)/\alpha} \|u_0\|_1,$$

$$(10) \quad \|\nabla P_\alpha(t)\|_p = Ct^{-N(1-1/p)/\alpha - 1/\alpha},$$

$$(11) \quad \|P_\alpha(t) * u_0\|_p \leq \|u_0\|_p,$$

for all $p \in [1, \infty]$ and $t > 0$.

In the next step, using the following well known integral representation of solutions to (1)-(2)

$$(12) \quad u(t) = P_\alpha(t) * u_0 - \int_0^t P_\alpha(t-s) * u^p(s) ds,$$

we immediately obtain the estimate $0 \leq u(x, t) \leq P_\alpha(x, t) * u_0(x)$. Hence, by (9) and (11) we get

$$(13) \quad \begin{aligned} \|u(t)\|_p^p &\leq \|P_\alpha(t) * u_0\|_p^p \\ &\leq \min \{Ct^{-N(p-1)/\alpha} \|u_0\|_1^p; \|u_0\|_p^p\} \equiv H(t, p, \alpha, u_0). \end{aligned}$$

Now, for fixed $\varepsilon \in (0, 1]$, we consider the solution $u^\varepsilon = u^\varepsilon(x, t)$ of (1)-(2) with the initial condition $\varepsilon u_0(x)$. The comparison principle implies that $0 \leq u^\varepsilon(x, t) \leq u(x, t)$ for every $x \in \mathbb{R}^N$ and $t > 0$. Hence, it suffices to show that for small $\varepsilon > 0$, which will be determinate later, we have

$$M_\infty^\varepsilon \equiv \lim_{t \rightarrow \infty} \int_{\mathbb{R}^N} u^\varepsilon(x, t) dx > 0.$$

Note that using equality (5) in the case of the solution u^ε , we obtain

$$(14) \quad M_\infty^\varepsilon = \varepsilon \left\{ \int_{\mathbb{R}^N} u_0(x) dx - \frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^N} (u^\varepsilon(x, t))^p dx dt \right\}.$$

Now, we apply (13) with u replaced by u^ε . Observe that the function H defined in (13) satisfies $H(t, p, \alpha, \varepsilon u_0) = \varepsilon^p H(t, p, \alpha, u_0)$. Hence

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^N} (u^\varepsilon(x, t))^p dx dt &\leq \frac{1}{\varepsilon} \int_0^\infty H(t, p, \alpha, \varepsilon u_0) dt \\ &= \varepsilon^{p-1} \int_0^\infty H(t, p, \alpha, u_0) dt. \end{aligned}$$

It follows immediately from the definition of the function H that the integral on the right hand side is convergent for $p > 1 + \alpha/N$. Consequently,

$$\frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^N} (u^\varepsilon(x, t))^p dx dt \rightarrow 0 \quad \text{as } \varepsilon \searrow 0,$$

and the constant M_∞^ε given by (14) is positive for sufficiently small $\varepsilon > 0$.

From now on, the proof of the asymptotic relation (4) is standard, hence, we shall be brief in details. First, we recall that for every $u_0 \in L^1(\mathbb{R}^N)$ we have

$$(15) \quad \lim_{t \rightarrow \infty} \|P_\alpha(t) * u_0 - MP_\alpha(t)\|_1 = 0,$$

where $M = \int_{\mathbb{R}^N} u_0(x) dx$. This is an immediate consequence of the Taylor argument combined with an approximation argument. A detailed reasoning can be found in [2, Lemma 3.3].

Now, to complete the proof of Theorem 1, we adopt the reasoning from [13]. It follows from the integral equation (12) and inequality (11) with $p = 1$ that

$$\|u(t) - P_\alpha(t - t_0) * u(t_0)\|_1 \leq \int_{t_0}^t \|u(s)\|_p^p ds \quad \text{for all } t \geq t_0 \geq 0.$$

Hence, using the triangular inequality we infer

$$\begin{aligned}
 \|u(t) - M_\infty P_\alpha(t)\|_1 &\leq \int_{t_0}^t \|u(s)\|_p^p ds \\
 &+ \|P_\alpha(t - t_0) * u(t_0) - M(t_0)P_\alpha(t)\|_1 \\
 (16) \quad &+ \|P_\alpha(t)\|_1 |M(t_0) - M_\infty|.
 \end{aligned}$$

Applying first (8) and (15) with $u_0 = u(t_0)$, and next passing to limit $t \rightarrow \infty$ on the right hand side of (16), we obtain

$$\limsup_{t \rightarrow \infty} \|u(t) - M_\infty P_\alpha(t)\|_1 \leq \int_{t_0}^{\infty} \|u(s)\|_p^p ds + |M(t_0) - M_\infty|.$$

By letting t_0 go to $+\infty$ and using (6) we conclude

$$(17) \quad \|u(t) - M_\infty P_\alpha(t)\|_1 \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In order to obtain the asymptotic term for $p > 1$, observe that by the integral equation (12) and estimate (9), for each $m \in [1, \infty]$, we have

$$(18) \quad \|u(t)\|_m \leq \|P_\alpha(t) * u_0\|_m \leq Ct^{-N(1-1/m)/\alpha} \|u_0\|_1.$$

Hence, for every $q \in [1, m)$, using the Hölder inequality, we obtain

$$\begin{aligned}
 \|u(t) - M_\infty P_\alpha(t)\|_q &\leq \|u(t) - M_\infty P_\alpha(t)\|_1^{1-\delta} (\|u(t)\|_m^\delta + \|M_\infty P_\alpha(t)\|_m^\delta) \\
 &\leq Ct^{-N(1-1/q)/\alpha} \|u(t) - M_\infty P_\alpha(t)\|_1^{1-\delta},
 \end{aligned}$$

with $\delta = (1 - 1/q)/(1 - 1/m)$. Finally, applying (17) we complete the proof of Theorem 1. \square

Proof of Theorem 2. Let us define the function $\varphi(x, t) = (\varphi_1(x))^\ell (\varphi_2(t))^\ell$ where

$$\ell = \frac{2p-1}{p-1}, \quad \varphi_1(x) = \psi\left(\frac{|x|}{BR}\right), \quad \varphi_2(t) = \psi\left(\frac{t}{R^\alpha}\right), \quad R > 0,$$

and ψ is a smooth nonincreasing function on $[0, \infty)$ such that

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

The constant $B > 0$ in the definition of φ_1 is fixed and will be chosen later. In fact, it plays some role in the critical case $p = 1 + \alpha/N$ only, while in the subcritical case $p < 1 + \alpha/N$ we simply put $B = 1$. In the following, we denote by Ω_1 and Ω_2 the supports of φ_1 and φ_2 , respectively:

$$\Omega_1 = \{x \in \mathbb{R}^N : |x| \leq 2BR\}, \quad \Omega_2 = \{t \in [0, \infty) : t \leq 2R^\alpha\}.$$

Now, we multiply equation (1) by $\varphi(x, t)$ and integrate with respect to x and t to obtain

$$\begin{aligned}
 (19) \quad & \int_{\Omega_1} u_0(x) \varphi(x, 0) dx - \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) \varphi(x, t) dx dt \\
 &= \int_{\Omega_2} \int_{\mathbb{R}^N} u(x, t) \varphi_2(t)^\ell \Lambda^\alpha(\varphi_1(x))^\ell dx dt \\
 &\quad - \int_{\Omega_2} \int_{\Omega_1} u(x, t) \varphi_1(x)^\ell \partial_t \varphi_2(t)^\ell dx dt \\
 &\leq \ell \int_{\Omega_2} \int_{\Omega_1} u(x, t) \varphi_2(t)^\ell \varphi_1(x)^{\ell-1} \Lambda^\alpha \varphi_1(x) dx dt \\
 &\quad - \ell \int_{\Omega_2} \int_{\Omega_1} u(x, t) \varphi_1(x)^\ell \varphi_2(t)^{\ell-1} \partial_t \varphi_2(t) dx dt.
 \end{aligned}$$

In (19), we have used the following inequality $\Lambda^\alpha \varphi_1^\ell \leq \ell \varphi_1^{\ell-1} \Lambda^\alpha \varphi_1$, see [4, Prop. 2.3] and [10, Prop. 3.3] for its proof, which is valid for all $\alpha \in (0, 2]$, $\ell \geq 1$, and any sufficiently regular, nonnegative, decaying at infinity function φ_1 .

Hence, by the ε -Young inequality $ab \leq \varepsilon a^p + C(\varepsilon) b^{\ell-1}$ (note that $1/p + 1/(\ell-1) = 1$) with $\varepsilon > 0$, we deduce from (19)

$$\begin{aligned}
 (20) \quad & \int_{\Omega_1} u_0(x) \varphi(x, 0) dx - (1 + 2\ell\varepsilon) \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) \varphi(x, t) dx dt \\
 &\leq C(\varepsilon) \ell \left\{ \int_{\Omega_2} \int_{\Omega_1} \varphi_1 \varphi_2^\ell |\Lambda^\alpha \varphi_1|^{\ell-1} dx dt + \int_{\Omega_2} \int_{\Omega_1} \varphi_1^\ell \varphi_2 |\partial_t \varphi_2|^{\ell-1} dx dt \right\}.
 \end{aligned}$$

Recall now that the functions φ_1 and φ_2 depend on $R > 0$. Hence changing the variables $\xi = R^{-1}x$ and $\tau = R^{-\alpha}t$, we easily obtain from (20) the following estimate

$$(21) \quad \int_{\Omega_1} u_0(x) \varphi(x, 0) dx - (1 + 2\ell\varepsilon) \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) \varphi(x, t) dx dt \leq C R^{N+\alpha-\alpha(\ell-1)},$$

where the constant C on the right hand side of (21) is independent of R . Note that $N + \alpha - \alpha(\ell-1) \leq 0$ if and only if $p \leq 1 + \alpha/N$. Now, we consider two cases.

For $p < 1 + \alpha/N$, we have $N + \alpha - \alpha(\ell-1) < 0$. Hence, computing the limit $R \rightarrow \infty$ in (21) and using the Lebesgue dominated convergence theorem, we obtain

$$M_\infty = \int_{\mathbb{R}^N} u_0(x) dx - \int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt \leq 2\ell\varepsilon \int_0^\infty \int_{\mathbb{R}^N} u^p dx dt.$$

Since $u \in L^p(\mathbb{R}^N \times (0, \infty))$ (cf. (6)) and since $\varepsilon > 0$ can be chosen arbitrarily small, we immediately obtain that $M_\infty = 0$.

In the critical case $p = 1 + \alpha/N$, we estimate the first term on the right hand side of inequality (19) using again the ε -Young inequality, and the second term by the Hölder inequality (with $\bar{p} = p/(p-1) = \ell-1$) as follows

$$\begin{aligned}
& \int_{\Omega_1} u_0(x) \varphi(x, 0) dx - \int_{\Omega_2} \int_{\Omega_1} u^p \varphi(x, t) dx dt \\
& \leq \ell \varepsilon \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) dx dt \\
(22) \quad & + C(\varepsilon) \int_{\Omega_2} \int_{\Omega_1} \varphi_2^{\ell \bar{p}}(t) \varphi_1^{(\ell-1)\bar{p}}(x) |\Lambda^\alpha \varphi_1(x)|^{\bar{p}} dx dt \\
& + \ell \left(\int_{\Omega_3} \int_{\Omega_1} u^p(x, t) dx dt \right)^{1/p} \\
& \times \left(\int_{\Omega_2} \int_{\Omega_1} \varphi_1^{\ell \bar{p}}(x) \varphi_2^{(\ell-1)\bar{p}}(t) |\partial_t \varphi_2(t)|^{\bar{p}} dx dt \right)^{1/\bar{p}}.
\end{aligned}$$

Here, $\Omega_3 = \{t \in [0, \infty) : R^\alpha \leq t \leq 2R^\alpha\}$ is the support of $\partial_t \varphi_2$. Note that

$$\int_{\Omega_3} \int_{\Omega_1} u^p(x, t) dx dt \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

because $u \in L^p(\mathbb{R}^N \times [0, \infty))$ (cf. (6)).

Now, introducing the new variables $\xi = (BR)^{-1}x$, $\tau = R^{-\alpha}t$ and recalling that $p = 1 + \alpha/N$, we rewrite (22) as follows

$$\begin{aligned}
(23) \quad & \int_{\Omega_1} u_0(x) \varphi(x, 0) dx - \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) \varphi(x, t) dx dt - \varepsilon \ell \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) dx dt \\
& \leq C_1 B^{N/\bar{p}} \left(\int_{\Omega_3} \int_{\Omega_1} u^p(x, t) dx dt \right)^{1/p} + C_2 C(\varepsilon) B^{-\alpha},
\end{aligned}$$

where the constants C_1, C_2 are independent of R, B , and of ε . Passing to the limit $R \rightarrow +\infty$ in (23) and using the Lebesgue dominated convergence theorem we get

$$\begin{aligned}
(24) \quad & \int_{\mathbb{R}^N} u_0(x) dx - \int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt - \varepsilon \ell \int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt \\
& \leq C_2 C(\varepsilon) B^{-\alpha}.
\end{aligned}$$

Finally, computing the limit $B \rightarrow \infty$ in (24) we infer that $M_\infty = 0$ because $\varepsilon > 0$ can be arbitrarily small. This complete the proof of Theorem 2. \square

Proof of Theorem 3. The proof proceeds by contradiction. Let u be a nonnegative nontrivial solution of (1)-(2) with $\lambda = 1$. Take the same test function φ as in the proof of Theorem 2. Repeating the estimations which

lead to (23), we obtain

$$\begin{aligned} & \int_{\Omega_1} u_0(x) \varphi(x, 0) dx + \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) \varphi(x, t) dx dt \\ & \quad - \varepsilon \ell \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) dx dt \\ & \leq C_1 B^{N/\bar{p}} \left(\int_{\Omega_3} \int_{\Omega_1} u^p(x, t) dx dt \right)^{1/p} + C_2 C(\varepsilon) B^{-\alpha}. \end{aligned}$$

Now, we choose $\varepsilon = 1/(2\ell)$ in (25) and we pass to the following limits: first $R \rightarrow \infty$, next $B \rightarrow \infty$. Using the Lebesgue dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^N} u_0(x) dx + \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt \leq 0.$$

Hence, $u(x, t) = 0$ which contradicts our assumption imposed on u . \square

REFERENCES

- [1] M. BEN-ARTZI & H. KOCH, *Decay of mass for a semilinear parabolic equation*, Comm. Partial Differential Equations **24** (1999), 869–881.
- [2] P. BILER, G. KARCH & W. A. WOYCZYŃSKI, *Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws*, Ann. Inst. H. Poincaré-Analyse non linéaire, **18** (2001), 613–637.
- [3] M. BIRKNER, J.A. LOPEZ-MIMBELA, A. WAKOLBINGER, *Blow-up of semilinear PDE's at the critical dimension. A probabilistic approach*. Proc. Amer. Math. Soc. **130** (2002), 2431–2442.
- [4] A. CÓRDOBA & D. CÓRDOBA, *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys. **249** (2004), 511–528.
- [5] J. DRONIOU & C. IMBERT, *Fractal first-order partial differential equations*, Arch. Rational Mech. Anal. **182** (2006), 299–331.
- [6] H. FUJITA, *On the blowing up of solutions of the problem for $u_t = \Delta u + u^{1+\alpha}$* , J. Fac. Sci. Univ. Tokyo **13** (1966), 109–124.
- [7] M. GUEDDA & M. KIRANE, *Criticality for some evolution equations*, Differ. Uravn. **37** (2001), 511–520, 574–575.
- [8] M. GUEDDA & M. KIRANE, *A note on nonexistence of global solutions to a nonlinear integral equation*, Bull. Belgian Math. Soc. Simon Stevin **6** (1999), 491–497.
- [9] K. HAYAKAWA, *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan Acad. **49** (1973), 503–505.
- [10] N. JU, *The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations*, Comm. Math. Phys. **225** (2005), 161–181.
- [11] G. KARCH & W.A. WOYCZYŃSKI, *Fractal Hamilton-Jacobi-KPZ equations*, Trans. Amer. Math. Soc. **360** (2008), 2423–2442.
- [12] K. KOBAYASHI, T. SIRAO & H. TANAKA, *On the growing up problem for semilinear heat equations*, J. Math. Soc. Japan **29** (1977), 407–424.
- [13] PH. LAURENÇOT & PH. SOUPLET, *On the growth of mass for a viscous Hamilton-Jacobi equation*, J. Analyse Math. **89** (2003), 367–383.
- [14] È. MITIDIERI & S. I. POKHOZHAEV, *Apriori estimates and the absence of solutions of nonlinear partial differential equations and inequalities*, Tr. Mat. Inst. Steklova **234** (2001), 1–384; translation in Proc. Steklov Inst. Math. **234** (2001), no. 3, 1–362.
- [15] E. MITIDIERI & S. I. POKHOZHAEV, *Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on \mathbb{R}^N* , J. Evol. Equ. **1** (2001), 189–220.

- [16] R. G. PINSKY, *Decay of mass for the equation $u_t = \Delta u - a(x)u^p|\nabla u|^q$* , J. Differential Equations **165** (2000), 1–23.
- [17] S. SUGITANI, *On nonexistence of global solutions for some nonlinear integral equations*, Osaka J. Math. **12** (1975), 45–51.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE LA ROCHELLE, LA ROCHELLE
17042, FRANCE

E-mail address: `afino01@univ-lr.fr`

INSTYTUT MATEMATYCZNY, UNIwersYTET WROCLAWSKI, PL. GRUNWALDZKI
2/4, 50-384 WROCLAW, POLAND

E-mail address: `karch@math.uni.wroc.pl`

URL: `http://www.math.uni.wroc.pl/~karch`