



HAL
open science

Clifford algebras bundles to multidimensional image segmentation

Thomas Batard, Michel Berthier

► **To cite this version:**

Thomas Batard, Michel Berthier. Clifford algebras bundles to multidimensional image segmentation. 2008. hal-00333016

HAL Id: hal-00333016

<https://hal.science/hal-00333016>

Preprint submitted on 22 Oct 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Clifford algebras bundles to multidimensional image segmentation

Thomas Batard and Michel Berthier

Abstract. We present a new theoretical framework for multidimensional image processing using Clifford algebras. The aim of the paper is to detect edges by computing the first fundamental form of a surface associated to an image. We propose to construct this metric in the Clifford bundles setting. A n D image, i.e. an image of dimension n , is considered as a section of a trivial Clifford bundle $(CT(D), \tilde{\pi}, D)$ over the domain D of the image and with fiber $Cl(\mathbb{R}^n, \|\cdot\|_2)$. Due to the triviality, any connection ∇_1 on the given bundle is the sum of the trivial connection $\tilde{\nabla}_0$ with ω , a 1-form on D with values in $End(CT(D))$. We show that varying ω and derivating well-chosen sections with respect to ∇_1 provides all the information needed to perform various kind of segmentation. We present several illustrations of our results, dealing with color ($n=3$) and color/infrared ($n=4$) images. As an example, let us mention the problem of detecting regions of a given color with constraints on temperature; the segmentation results from the computation of $\nabla_1(I) = \tilde{\nabla}_0(I) + (dx + dy) \otimes \mu I$, where I is the image section and μ is a vector section coding the given color.

Mathematics Subject Classification (2000). 68U10, 55R10, 15A66, 53C05.

Keywords. Image processing, Clifford bundles, Covariant derivatives, Geometric algebra.

1. Introduction

Clifford (Kähler-Atiyah [6]) algebras are widely used in computer sciences for several reasons (see [13] for examples of applications). In this paper, we will take advantage of the following two facts:

1. As it is well known, Clifford algebras provide a particularly efficient framework to make computations without coordinates [7]. We use in the sequel inner, wedge and geometric products as well as the related geometric interpretations of these

operations. More precisely, the acquisition space of the image, i.e. the space where the image gets its values, is embedded into a Clifford algebra. We measure the local variations of the values of the pixels, seen as integer coordinates points of the domain D of the image, using geometric information and transformations. We get in this way the metric data of the image surface needed for the edge detection. It appears to be an asset that we can take into account variations of the metric by introducing Clifford bundles.

2. A Clifford algebra contains elements of different degrees (scalar, vectors, bivectors, ..., pseudoscalar). This allows to combine information of different nature in a single multivector to have a global and concise treatment of the used data. For example, when computing the covariant derivative $\nabla_1(S)$, for S a section of a Clifford bundle over D , we obtain a 1-form on D that may be decomposed as a sum of 1-forms with values in the parts of degree k , for $k = 1, \dots, n$, of the total space of the bundle, each part coding a particular information.

The aim of the paper is to generalize a method of edge detection based on the definition of an image as 2-dimensional surface, in the Clifford bundles framework. As we will see Section 2.1, edges detected by such a method depend completely on the first fundamental form of the surface, i.e the metric induced by the metric of the ambient space. Let us cite some works where images are defined as surfaces with the metric of the ambient space varying with the point. [8] and [1] both propose application to color edge detection. The first deals with hyperbolic coordinates and a corresponding metric, whereas we deal in the second with luminance, saturation and hue components (see Section 2.2). In [8], edge detection arises from a diffusion process whereas in [1], edge detection arises from a local definition of edges [11]. Note that in [1], we already proposed to detect only specific edges, and to construct first fundamental forms using Clifford bundles formalism with applications to color and color/infrared images. We also mentioned extensions to nD images, but did not propose there explicit methods.

All the applications we present in this paper follow the same schema: First, we consider appropriate connection ∇_1 and section S relative to a trivial Clifford bundle over D . A metric generating a new Clifford bundle over D is constructed using the parts of degree $\neq 1$ of $\nabla_1(S)$. Roughly speaking, this metric corresponds to the metric of the ambient space of the surface. Then, a 1-form with values in this new bundle is built from the part of degree 1 of $\nabla_1(S)$. At last, by a tensor product over a well-chosen ring, we associate to this 1-form a symmetric tensor of rank 2 with values in the scalar part of the bundle. It measures variations of S with respect to the metric of the new bundle. Identifying the ring \mathbb{R} and its injection into Clifford algebras, this tensor can be assimilated as the first fundamental form of a surface.

The paper is organized as follows. Section 2 is devoted to first, remind the method of edge detection on color images we propose to extend, then to give some basic

results on color and color/infrared spaces when embedded into Clifford algebras. In Section 3, we detail the global framework we propose for nD image edge detection. It contains the construction of the trivial Clifford bundle wherein a multidimensional image is represented as a section and the construction of a connection compatible with the Clifford product. Moreover, we develop a general method to construct first fundamental forms of 2-dimensional surfaces from the information given by $\nabla_1(S)$. In Section 4, we apply the method of Section 3 to color and color/infrared images. We present several illustrations due to particular choices of the connection ∇_1 and the section S .

2. Preliminaries

2.1. Multidimensional image edge detection using metrics of surfaces

A usual method of edge detection on color images is to consider a color image, of components (r, g, b) (see Section 2.2) defined on a subset D of \mathbb{R}^2 as a 2-dimensional surface S parametrized by

$$\varphi : (x, y) \mapsto (x, y, r(x, y), g(x, y), b(x, y))$$

embedded into $(\mathbb{R}^5, \|\cdot\|_2)$. The euclidean metric of the embedding space \mathbb{R}^5 induces a metric on S called the first fundamental form of S , which takes the following form

$$dS^2 := dx^2 + dy^2 + dr^2 + dg^2 + db^2 \quad (1)$$

Then, color variations on the image are assimilated to tangent vectors of S and a measure of these variations is given by dS^2 . The rest of the method is devoted to select the strongest local variations, called edges.

More precisely, let $I(q)$ be the matrix representation of the metric dS^2 at $q = \varphi(p)$ in the coordinates system given by $(d_p\varphi(1, 0), d_p\varphi(0, 1))$. Let $\theta_+(q)$ and $\theta_-(q)$, $\theta_+(q) \geq \theta_-(q)$, be the two eigenvalues of $I(q)$ and $\Theta_+(q)$, $\Theta_-(q)$ the corresponding eigenvectors. The edge measure is then given by

$$\varpi(q) = \sqrt{\theta_+(q) - \theta_-(q)} \quad (2)$$

and we say that $p \in D$ is an edge point if the function ϖ has a local maximum at $\varphi(p)$ in the direction given by $\Theta_+(\varphi(p))$.

We propose to generalize this method in two-folds. First, by considering images of higher dimension, that are represented in this context by 2-dimensional surfaces embedded in higher dimensional spaces. For example, a color/infrared image is represented by a 2-dimensional surface embedded into \mathbb{R}^6 . Secondly, by considering embedding spaces equipped with metrics varying with the point, so that to detect only specific edges on the image.

2.2. Color and color/infrared spaces as subsets of Clifford algebras

There are many ways to represent the set of colors i.e. there are many color spaces. The standard color space, based on the physical properties of the human vision is *RGB* (red, green, blue). Each color is identified by its red, green and blue levels, which are generally encoded by integers from 0 to 255. The *RGB* space is geometrically represented by a cube.

In the seminal papers [9] and [10], Sangwine & al. propose to embed *RGB* into the space of imaginary quaternions \mathbb{H}_0 by

$$(r, g, b) \mapsto ri + gj + bk$$

so that to compute geometric transformations on colors, as rotations, using only the addition and product laws of the algebra of quaternions \mathbb{H} . As a consequence, applying such transformations on some pixels of a color image, seen as a \mathbb{H} -valued function

$$f : \begin{array}{ccc} D & \longrightarrow & \mathbb{H} \\ (x, y) & \longmapsto & r(x, y)i + g(x, y)j + b(x, y)k \end{array}$$

the authors define new kinds of color image segmentation, as chrominance (see later) edge detection.

Such results may be recovered by embedding *RGB* into the vector part of the Clifford algebra $Cl(\mathbb{R}^3, Q)$ associated to \mathbb{R}^3 equipped with an euclidean quadratic form Q .

$$(r, g, b) \mapsto re_1 + ge_2 + be_3$$

Using this last framework presents two assets comparing to quaternions. First, as $Cl(\mathbb{R}^3, Q)$ is of dimension 8 and \mathbb{H} is of dimension 4, the former may carry more information about colors. Moreover, as one can associate a Clifford algebra to a vector space of any dimension, it is possible to deal with nD images: The acquisition space of a nD image is embedded, as before, into the vector part of $Cl(\mathbb{R}^n, Q)$ where Q is an euclidean quadratic form. In particular, for color/infrared images, we embed *RGBT* into $Cl(\mathbb{R}^4, Q)$, where T stands for temperature, by the following map

$$(r, g, b, t) \mapsto re_1 + ge_2 + be_3 + te_4$$

There exist other color spaces based on the perceptual properties of the human vision and parametrized by luminance (or intensity), saturation and hue. Let us mention the *HSL* color spaces represented by a double cone. Such color spaces are called perceptual since effects of variations of each one of the components can be interpreted. In particular, the hue corresponds to the usual notion of color, i.e. we distinguish red, yellow, green, blue, purple...hues. We refer to [3] for more details.

Beside *RGB* color space we consider the *HSL* color space defined as follows.

We set first

$$\begin{pmatrix} Y \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix}$$

Then the luminance l , the saturation s and the hue h are respectively given by

$$\begin{aligned} l &= Y \\ s &= \sqrt{C_1^2 + C_2^2} \\ h &= \begin{cases} \arccos(C_2/s) & \text{if } C_2 > 0 \\ 2\pi - \arccos(C_2/s) & \text{otherwise} \end{cases} \end{aligned}$$

Note that the hue is an angular information. At the end of this part, we will see another representation of hues.

We can compute h from r, g and b in the setting of the Clifford algebra $\mathbb{R}_{3,0}$ as follows. Let $a = re_1 + ge_2 + be_3 \in \mathbb{R}_{3,0}$ be a color, and let $h(a)$ denote its hue. The vector a may be decomposed into its projection on the achromatic axis, generated by the unit vector $(e_1 + e_2 + e_3)/3$ and its rejection $v(a)$, which corresponds obviously to the projection on the plane generated by the dual of $(e_1 + e_2 + e_3)/3$, called the chrominance plane. The vector $v(a)$ is called the chrominance vector of the color a . Simple computations show that

$$h(a) = 2\pi + \text{sign}(g(a) - b(a)) \arccos\left(\frac{v(a)}{\|v(a)\|} \cdot \rho(a)\right)$$

with $\rho(a)$ the unit chrominance vector corresponding to colors colinear to e_1 and $h(a)$ defined modulo 2π . In other words, $h(a)$ is the oriented angle from $\rho(a)$ to $v(a)$. To conclude this part, let us explain how to define the hue using bivectors.

Proposition Let T be the set of bivectors

$$T = \{(e_1 + e_2 + e_3) \wedge \alpha, \alpha \in RGB\}$$

with the following equivalence relation:

$$B \simeq C \iff B = \lambda C \quad \text{for } \lambda > 0$$

Then, there is a bijection between T/\simeq and the set of hues.

Proof. We have

$$(e_1 + e_2 + e_3) \wedge \alpha = (e_1 + e_2 + e_3)v_\alpha$$

where v_α is the projection of α on the chrominance. Then, there is a bijection between T/\simeq and the set $(e_1 + e_2 + e_3)v$ for v a unit vector in the chrominance plane. This latter being in bijection with the set of different hues, we conclude that there exists a bijection between T/\simeq and the set of hues. \square

As complementary hues are given by opposite chrominance axis, they generate opposite bivectors.

Figure 1.¹ shows the representation of the set of colors having the same hue in *RGB* color space, for $h = 330^\circ, 0^\circ, 30^\circ$.

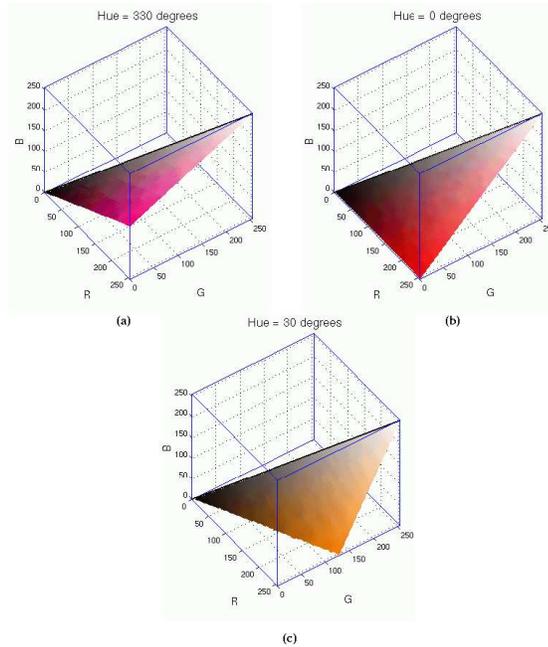


FIGURE 1. (a) colors of hue 330° - (b) colors of hue 0° - (c) colors of hue 30°

3. A general method to multidimensional image edge detection

3.1. A family $\mathcal{G} = \{\mathcal{G}_n, n \in \mathbb{N}\}$ of Clifford bundles

The aim of this subsection is to describe the family of Clifford bundles we consider to perform the edge detection. As before D denotes the domain of the image. To each point $p \in D$, we associate a vector space E_p of dimension n endowed with a definite positive quadratic form $g(p)$ such that in the basis $(e_1(p), e_2(p), \dots, e_n(p))$

¹This picture was found at the address <http://gug.sunsite.dk>.

of E_p , $g(p)$ takes the following form

$$g(p) = \begin{pmatrix} g_1(p) & 0 & 0 & \cdots & 0 \\ 0 & g_2(p) & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & g_n(p) \end{pmatrix}$$

Let E be the disjoint union of E_p for p in D , and $\pi : E \rightarrow D$ that maps $v \in E_p$ to p . We construct a trivial Riemannian vector bundle of rank n (E, π, D) as follows. Let φ be the bijection

$$D \times (\mathbb{R}^n, \|\cdot\|_2) \longrightarrow E$$

$$\left(p, (u_1, u_2, \dots, u_n) \right) \longmapsto u_1 \frac{e_1(p)}{\sqrt{g_1(p)}} + u_2 \frac{e_2(p)}{\sqrt{g_2(p)}} + \dots + u_n \frac{e_n(p)}{\sqrt{g_n(p)}}$$

where for a fixed p , $\varphi : (\mathbb{R}^n, \|\cdot\|_2) \rightarrow E_p$ is an isometry from the euclidean space \mathbb{R}^n to E_p . We define a topology on E by stating that $U \in E$ is open iff $\varphi^{-1}(U)$ is open in $D \times (\mathbb{R}^n, \|\cdot\|_2)$. This topology makes φ an homeomorphism and E a manifold with φ^{-1} as global chart. Consequently, φ is a diffeomorphism and therefore (E, π, D) is a trivial Riemannian vector bundle of rank n having φ as global trivialization. We denote \mathcal{F}_n the family of bundles (E, π, D) of rank n obtained in this way.

In the trivialization φ , any section S can be written:

$$S(p) = s_1(p) \frac{e_1(p)}{\sqrt{g_1(p)}} + s_2(p) \frac{e_2(p)}{\sqrt{g_2(p)}} + \dots + s_n(p) \frac{e_n(p)}{\sqrt{g_n(p)}}$$

Each fiber E_p endowed with the quadratic form $g(p)$ generates a Clifford algebra $Cl(E_p, g(p))$. In particular, the basis $(e_1(p), e_2(p), \dots, e_n(p))$ of E_p generates a basis of $Cl(E_p, g(p))$:

$$\left\{ 1(p), e_1(p), e_2(p), \dots, e_n(p), e_1(p)e_2(p), \dots, e_1(p)e_2(p) \dots e_n(p) \right\}$$

Let us denote $CT(D)$ the disjoint union of $Cl(E_p, g(p))$ for $p \in D$, and $\tilde{\pi}$ that maps $v \in Cl(E_p, g(p))$ to p . Then, from (E, π, D) , we construct a trivial Clifford bundle $(CT(D), \tilde{\pi}, D)$ as follows. Let $\tilde{\varphi}$ be the bijection

$$D \times Cl(\mathbb{R}^n, \|\cdot\|_2) \longrightarrow CT(D)$$

$$\left(p, (u_0, u_1, \dots, u_{2^n-1}) \right) \longmapsto u_0 1(p) + u_1 \frac{e_1(p)}{\sqrt{g_1(p)}} + \dots + u_{2^n-1} \frac{e_1(p) \dots e_n(p)}{\sqrt{g_1(p) \dots g_n(p)}}$$

where for a fixed p , $\tilde{\varphi} : Cl(\mathbb{R}^n, \|\cdot\|_2) \rightarrow Cl(E_p, g_p)$ is an algebra isomorphism. As previously done, we endow $CT(D)$ with a topology by stating that $U \in CT(D)$ is

open iff $\tilde{\varphi}^{-1}(U)$ is open in $D \times Cl(\mathbb{R}^n, \|\cdot\|_2)$. Then, the bijection $\tilde{\varphi}$ is an homeomorphism, and $CT(D)$ is a manifold having $\tilde{\varphi}^{-1}$ as global chart. Consequently, $\tilde{\varphi}$ is a diffeomorphism. We conclude that $(CT(D), \tilde{\pi}, D)$ is a trivial Clifford bundle having $\tilde{\varphi}$ as global trivialization. We denote \mathcal{G}_n the family of those bundles $(CT(D), \tilde{\pi}, D)$.

In the trivialization $\tilde{\varphi}$, any section S may be written:

$$S(p) = s_0(p) 1(p) + s_1(p) \frac{e_1(p)}{\sqrt{g_1(p)}} + \cdots + s_{2^n-1}(p) \frac{e_1(p)e_2(p)\cdots e_n(p)}{\sqrt{g_1(p)g_2(p)\cdots g_n(p)}}$$

Let us mention that $(CT(D), \tilde{\pi}, D)$ is, in particular, a trivial vector bundle.

3.2. Algebra connection on $(CT(D), \tilde{\pi}, D)$

In what follows, $\Gamma(H)$ denotes the space of C^k sections, $k \geq 1$, of a vector bundle of total space H . Let us recall that any connection ∇ on a trivial vector bundle (H, π_1, M) takes the form

$$\nabla = d + \omega \tag{3}$$

for some $\omega \in \Gamma(T^*M \otimes \text{End}(H))$ and d being the usual differentiation of functions. More precisely, let f_1, f_2, \dots, f_n be a basis of the fiber of (H, π_1, M) , and Φ be a global trivialization. Then $(\Phi(\cdot, f_1), \Phi(\cdot, f_2), \dots, \Phi(\cdot, f_n))$ defines a global frame, i.e. for each $p \in M$, $(\Phi(p, f_1), \Phi(p, f_2), \dots, \Phi(p, f_n))$ is a basis of H_p .

Then, any section $u \in \Gamma(H)$ may be written

$$u = \sum_{i=1}^n u_i \Phi(\cdot, f_i)$$

and

$$\nabla u = \sum_{i=1}^n du_i \Phi(\cdot, f_i) + u_i \omega(\Phi(\cdot, f_i))$$

On a vector bundle $(E, \pi, D) \in \mathcal{F}_n$ of metric g , we denote ∇_0 the connection where $\omega \equiv 0$ in the trivialization φ . For $s \in \Gamma(E)$, we have therefore

$$\nabla_0(s) = \sum_{i=1}^n ds_i \frac{e_i}{\sqrt{g_i}}$$

Proposition

Let $(CT(D), \tilde{\pi}, D) \in \mathcal{G}_n$ generated by $(E, \pi, D) \in \mathcal{F}_n$. The connection ∇_0 induces an algebra connection on $(CT(D), \tilde{\pi}, D)$, i.e. a connection $\tilde{\nabla}_0$ satisfying

$$\tilde{\nabla}_0(MN) = \tilde{\nabla}_0(M)N + M\tilde{\nabla}_0(N)$$

for all M and N in $\Gamma(CT(D))$.

Proof. First, let us remark that ∇_0 is compatible with the fiber metric, i.e

$$d_x g(s, s') = g(\nabla_{0_x} s, s') + g(s, \nabla_{0_x} s')$$

for $s, s' \in \Gamma(E)$ and $X \in \Gamma(TD)$.

Indeed, in the global frame $\left(\frac{e_1}{\sqrt{g_1}}, \frac{e_2}{\sqrt{g_2}}, \dots, \frac{e_n}{\sqrt{g_n}}\right)$ induced by φ , the metric g takes the form

$$g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Then, if $s = \sum_{i=1}^n s_i \frac{e_i}{\sqrt{g_i}}$ and $s' = \sum_{i=1}^n s'_i \frac{e_i}{\sqrt{g_i}}$, we have

$$d_x g(s, s') = \sum_{i=1}^n d_x s_i s'_i + s_i d_x s'_i$$

and

$$g(\nabla_{0_x} s, s') = \sum_{i=1}^n d_x s_i s'_i \quad g(s, \nabla_{0_x} s') = \sum_{i=1}^n s_i d_x s'_i$$

Secondly, the condition for a connection ∇ on a vector bundle equipped with a metric to be compatible with the fiber metric is in fact a sufficient condition to construct an algebra connection on the induced Clifford bundle. Indeed, ∇ can be extended in a unique way to the corresponding tensor algebra bundle by linearity and Leibniz's rule, and stating that $\nabla(f) = df$ for $f \in C^k(M)$ where M is the base manifold. Then, the compatibility with the fiber metric ensures that ∇ preserves the ideal generated by $(x \otimes x - g(x, x))$ [2]. Consequently, ∇ defines a connection on the quotient that preserves the product in the quotient algebra bundle which is a Clifford bundle. In our case, this means that ∇_0 induces a connection $\tilde{\nabla}_0$ on $(CT(D), \tilde{\pi}, D)$ that satisfies

$$\tilde{\nabla}_{0_x}(MN) = \tilde{\nabla}_{0_x}(M)N + M\tilde{\nabla}_{0_x}(N)$$

for all M, N in $\Gamma(CT(D))$ and $X \in \Gamma(TD)$.

At last, from the fact that $\Gamma(T^*D \otimes CT(D))$ may be endowed with a $\Gamma(CT(D))$ -bimodule structure by the two following maps

$$\begin{aligned} \Gamma(CT_2(D)) \times \Gamma(T^*D \otimes CT_2(D)) &\longrightarrow \Gamma(T^*D \otimes CT_2(D)) \\ (b, \xi \otimes c) &\longmapsto \xi \otimes bc \end{aligned} \tag{4}$$

$$\begin{aligned} \Gamma(T^*D \otimes CT_2(D)) \times \Gamma(CT_2(D)) &\longrightarrow \Gamma(T^*D \otimes CT_2(D)) \\ (\xi \otimes c, b) &\longmapsto \xi \otimes cb \end{aligned} \quad (5)$$

we have

$$\tilde{\nabla}_0(MN) = \tilde{\nabla}_0(M)N + M\tilde{\nabla}_0(N) \quad \square$$

Let us verify that the algebra connection $\tilde{\nabla}_0$ we have defined on $(CT(D), \tilde{\pi}, D)$ corresponds to the connection on $(CT(D), \tilde{\pi}, D)$ with $\omega \equiv 0$. For this, we just need to verify that

$$\tilde{\nabla}_0 \left(s_{i_1 i_2 \dots i_k} \frac{e_{i_1}}{\sqrt{g_{i_1}}} \frac{e_{i_2}}{\sqrt{g_{i_2}}} \dots \frac{e_{i_k}}{\sqrt{g_{i_k}}} \right) = d(s_{i_1 i_2 \dots i_k}) \left(\frac{e_{i_1}}{\sqrt{g_{i_1}}} \frac{e_{i_2}}{\sqrt{g_{i_2}}} \dots \frac{e_{i_k}}{\sqrt{g_{i_k}}} \right)$$

By compatibility of $\tilde{\nabla}_0$ with the Clifford product, the expression

$$\tilde{\nabla}_0 \left(s_{i_1 i_2 \dots i_k} \frac{e_{i_1}}{\sqrt{g_{i_1}}} \frac{e_{i_2}}{\sqrt{g_{i_2}}} \dots \frac{e_{i_k}}{\sqrt{g_{i_k}}} \right)$$

can be decomposed into

$$\begin{aligned} &\tilde{\nabla}_0 \left(s_{i_1 i_2 \dots i_k} \frac{e_{i_1}}{\sqrt{g_{i_1}}} \right) \left(\frac{e_{i_2}}{\sqrt{g_{i_2}}} \dots \frac{e_{i_k}}{\sqrt{g_{i_k}}} \right) + \left(s_{i_1 i_2 \dots i_k} \frac{e_{i_1}}{\sqrt{g_{i_1}}} \right) \tilde{\nabla}_0 \left(\frac{e_{i_2}}{\sqrt{g_{i_2}}} \dots \frac{e_{i_k}}{\sqrt{g_{i_k}}} \right) \\ &= \left(d(s_{i_1 i_2 \dots i_k}) \frac{e_{i_1}}{\sqrt{g_{i_1}}} + s_{i_1 i_2 \dots i_k} \tilde{\nabla}_0 \left(\frac{e_{i_1}}{\sqrt{g_{i_1}}} \right) \right) \left(\frac{e_{i_2}}{\sqrt{g_{i_2}}} \dots \frac{e_{i_k}}{\sqrt{g_{i_k}}} \right) \\ &= \left(d(s_{i_1 i_2 \dots i_k}) \frac{e_{i_1}}{\sqrt{g_{i_1}}} \right) \left(\frac{e_{i_2}}{\sqrt{g_{i_2}}} \dots \frac{e_{i_k}}{\sqrt{g_{i_k}}} \right) \\ &= d(s_{i_1 i_2 \dots i_k}) \left(\frac{e_{i_1}}{\sqrt{g_{i_1}}} \frac{e_{i_2}}{\sqrt{g_{i_2}}} \dots \frac{e_{i_k}}{\sqrt{g_{i_k}}} \right) \end{aligned}$$

Therefore, any connection ∇_1 on $(CT(D), \tilde{\pi}, D)$ may be written

$$\nabla_1 = \tilde{\nabla}_0 + \omega \quad (6)$$

for some $\omega \in \Gamma(T^*D \otimes \text{End}(CT(D)))$.

As it is written in the Introduction, an image I is considered as a section of a bundle $(CT(D), \tilde{\pi}, D) \in \mathcal{G}_n$, where $n = 3$ for color images, and $n = 4$ for color/infrared images.

Let I be a color/infrared image. Then, in the global frame induced by the global trivialization φ , we have

$$I(p) = r(p) \frac{e_1(p)}{\sqrt{g_1(p)}} + g(p) \frac{e_2(p)}{\sqrt{g_2(p)}} + b(p) \frac{e_3(p)}{\sqrt{g_3(p)}} + t(p) \frac{e_4(p)}{\sqrt{g_4(p)}}$$

where r, g, b, t are respectively red, green, blue and temperature components in the *RGBT* color/infrared space.

3.3. Computing first fundamental forms from connections on Clifford bundles

This is the main part of the paper. We propose a general method to construct first fundamental forms of 2-dimensional surfaces representing nD images in the setting of Clifford algebras bundles. This is done in such a way that all the applications we present in this paper appear as particular cases, the embedding spaces being \mathbb{R}^6 for color/infrared images and \mathbb{R}^5 for color images.

Let $(E_1, \pi_1, D) \in \mathcal{F}_n$ of metric g , and (F_1, π_1, D) a subbundle² of rank m with global frame $(e_{j_1}, e_{j_2}, \dots, e_{j_m})$. Following the construction of the previous part, (E_1, π_1, D) generates a Clifford bundle $(CT_1(D), \tilde{\pi}_1, D) \in \mathcal{G}_n$. Let ∇_1 and S be respectively a connection and a section of $(CT_1(D), \tilde{\pi}_1, D)$. The 1-form $\nabla_1(S)$ defined on D with values in $CT_1(D)$ may be written

$$\nabla_1(S) = \langle \nabla_1(S) \rangle_0 + \langle \nabla_1(S) \rangle_1 + \langle \nabla_1(S) \rangle_2 + \dots + \langle \nabla_1(S) \rangle_{2^n - 1}$$

where $\langle \nabla_1(S) \rangle_k$ denotes the part of $\nabla_1(S)$ of degree k . Strictly speaking, $\langle \nabla_1(S) \rangle_k$ is a 1-form on D with values in the part of $CT_1(D)$ of degree k .

Next we construct a vector bundle $(E_2, \pi_2, D) \in \mathcal{F}_m$ (m is the rank of the subbundle F_1) whose metric h arises from the information given by all the parts of $\nabla_1(S)$ except the one of degree 1. Roughly speaking, this metric corresponds to the metric of the ambient space of the surface. Then using a vector bundle morphism and a tensor product over a well-chosen ring, the vector part of $\nabla_1(S)$ provides a way to measure variations of S with respect to the metric h , which corresponds to the metric of the surface induced by the metric of its ambient space, i.e. its first fundamental form.

We now detail the last part of the construction. Let us denote $(CT_2(D), \tilde{\pi}_2, D)$ the Clifford bundle of \mathcal{G}_m generated by (E_2, π_2, D) . We define a vector bundle morphism ψ from (E_1, π_1, D) to (E_2, π_2, D) by

$$\psi(p) \left(\frac{e_i(p)}{\sqrt{g_i(p)}} \right) = \begin{cases} \frac{e_i(p)}{\sqrt{h_i(p)}} & \text{if } e_i(p) = e_{j_i}(p) \in F_1(p) \\ 0 & \text{otherwise} \end{cases}$$

This morphism induces a vector bundle morphism $\tilde{\psi}$ from $(CT_1(D), \tilde{\pi}_1, D)$ to $(CT_2(D), \tilde{\pi}_2, D)$ defined by

$$\tilde{\psi}(p) \left(\frac{e_{i_1}(p)e_{i_2}(p) \cdots e_{i_k}(p)}{\sqrt{g_{i_1}(p)g_{i_2}(p) \cdots g_{i_k}(p)}} \right) = \begin{cases} \frac{e_{\alpha_1}(p)e_{\alpha_2}(p) \cdots e_{\alpha_k}(p)}{\sqrt{h_{\alpha_1}(p)h_{\alpha_2}(p) \cdots h_{\alpha_k}(p)}} & \text{if } \begin{cases} e_{i_1}(p) = e_{j_{\alpha_1}}(p) \\ e_{i_2}(p) = e_{j_{\alpha_2}}(p) \\ \vdots \\ e_{i_k}(p) = e_{j_{\alpha_k}}(p) \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

²The reason why to consider such subbundles will be clarify in Sec. 4 devoted to applications

From $\tilde{\psi}$, we construct a vector bundle morphism Ψ from $(T^*D \otimes CT_1(D), p_1, D)$ to $(T^*D \otimes CT_2(D), p_2, D)$ that maps

$$\eta(p) \otimes \frac{e_{i_1}(p)e_{i_2}(p) \cdots e_{i_k}(p)}{\sqrt{g_{i_1}(p)g_{i_2}(p) \cdots g_{i_k}(p)}}$$

to

$$\left\{ \begin{array}{ll} \eta(p) \otimes \frac{e_{\alpha_1}(p)e_{\alpha_2}(p) \cdots e_{\alpha_k}(p)}{\sqrt{h_{\alpha_1}(p)h_{\alpha_2}(p) \cdots h_{\alpha_k}(p)}} & \text{if } \begin{cases} e_{i_1}(p) = e_{j_{\alpha_1}}(p) \\ e_{i_2}(p) = e_{j_{\alpha_2}}(p) \\ \vdots \\ e_{i_k}(p) = e_{j_{\alpha_k}}(p) \end{cases} \\ 0 & \text{otherwise} \end{array} \right. \quad (7)$$

Let η_1 be the vector part of $\nabla_1(S)$. It is important to notice that for all the applications to be treated in Sec. 4, we may suppose that

$$\eta_1 = \sum_{i=1}^n \eta_{1i} \otimes \frac{e_i}{\sqrt{g_i}}$$

where for each i , the 1-form η_{1i} is exact, i.e. may be written as the differential df_i of $f_i \in C^k(D)$. We have therefore

$$\begin{aligned} \Psi(p)(\eta_1(p)) &= \sum_{i=1}^n df_i(p) \otimes \psi(p) \left(\frac{e_i(p)}{\sqrt{g_i(p)}} \right) \\ &= \sum_{l=1}^m df_{j_l}(p) \otimes \left(\frac{e_l(p)}{\sqrt{h_l(p)}} \right) \end{aligned}$$

Let η_2 be the section of $(T^*D \otimes CT_2(D), p_2, D)$ given by the set $\Psi(p)(\eta_1(p))$, for $p \in D$:

$$\eta_2 = \sum_{l=1}^m df_{j_l} \otimes \left(\frac{e_l}{\sqrt{h_l}} \right)$$

From η_2 , we aim at constructing a symmetric tensor of rank 2 with values in $CT_2(D)$ that could be interpreted as the first fundamental form of a 2-dimensional surface embedded into \mathbb{R}^{m+2} equipped with a Riemannian metric.

From the bimodule structure of $\Gamma(T^*D \otimes CT_2(D))$ over $\Gamma(CT_2(D))$ given by isomorphisms (4) and (5), we have

$$\sum_{l=1}^m \frac{1}{2} \left(\frac{e_l}{\sqrt{h_l}} \eta_2 + \eta_2 \frac{e_l}{\sqrt{h_l}} \right) e_l = \sum_{l=1}^m df_{j_l} \otimes e_l \quad (8)$$

If we denote \mathcal{B} the ring $\Gamma(CT_2(D))$, then we may show that

$$\Gamma(T^*D \otimes CT_2(D)) \otimes_{\mathcal{B}} \Gamma(T^*D \otimes CT_2(D)) \simeq \Gamma(T^*D \otimes T^*D \otimes CT_2(D))$$

the isomorphism being given by the two following morphisms

$$\begin{aligned}\gamma &: (\omega_1 \otimes s_1) \otimes (\omega_2 \otimes s_2) \longmapsto (\omega_1 \otimes \omega_2) \otimes s_1 s_2 \\ \delta &: (\omega_1 \otimes \omega_2 \otimes s) \longmapsto (\omega_1 \otimes s) \otimes (\omega_2 \otimes 1)\end{aligned}$$

From this tensor product, we define a symmetrized tensor product by

$$(\omega_1 \otimes s_1)(\omega_2 \otimes s_2) := \frac{1}{2} \left[(\omega_1 \otimes s_1) \otimes (\omega_2 \otimes s_2) + (\omega_2 \otimes s_1) \otimes (\omega_1 \otimes s_2) \right]$$

Applying this latter to (8) with itself gives

$$\left(\sum_{l=1}^m df_{j_l} \otimes e_l \right) \left(\sum_{l=1}^m df_{j_l} \otimes e_l \right) = \sum_{l=1}^m (df_{j_l})^2 \otimes h_l \quad (9)$$

Then we can consider the following $CT_2(D)$ -valued symmetric tensor of rank 2

$$dx^2 \otimes 1 + dy^2 \otimes 1 + (df_{j_1})^2 \otimes h_1 + (df_{j_2})^2 \otimes h_2 + \cdots + (df_{j_m})^2 \otimes h_m$$

Identifying \mathbb{R} and its injection into each fiber, this $CT_2(D)$ -valued tensor may be viewed as the first fundamental form of the surface S parametrized by

$$\varphi : (x, y) \longmapsto (x, y, f_{j_1}(x, y), f_{j_2}(x, y), \dots, f_{j_m}(x, y))$$

embedded into \mathbb{R}^{m+2} equipped with the metric

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & h_m \end{pmatrix}$$

Indeed, the first fundamental form of S is the symmetric tensor of rank 2

$$dS^2 = dx^2 + dy^2 + h_1(df_{j_1})^2 + h_2(df_{j_2})^2 + \cdots + h_m(df_{j_m})^2$$

4. Applications

The aim of this part is to show that by the choices of the vector subspaces $F_1(p)$ (of dimension m), the 1-form ω and the section we derive with the connection $\nabla_1 = \tilde{\nabla}_0 + \omega$, we may perform several kinds of edge detection.

Let us remark that the vector spaces $E_1(p)$ (of dimension n) are fixed in the sense that they are acquisition spaces of the images. Moreover, we have seen that the first fundamental form we construct is independant of the metric g of (E_1, π_1, D) .

In what follows, we denote by $\| \|$ the function mapping a blade x to $\sqrt{x t(x)}$, where t is the main anti-automorphism of the corresponding Clifford algebra.

4.1. The usual color edge detection: $n=3$

In this part, we show that we may obtain the metric (1) of the usual method of color edge detection following the Clifford bundles framework described above. However, in this very particular case, the general method of the previous section may be simplified to compute (1) since this one does not depend on any information on the image that needs to be computed.

Let $(E_1, \pi_1, D) \in \mathcal{F}_3$ of metric

$$g := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let I be the considered color image, seen as a section of the Clifford bundle $(CT_1(D), \tilde{\pi}_1, D)$ generated by (E_1, π_1, D) , i.e.

$$I(p) = r(p) e_1(p) + g(p) e_2(p) + b(p) e_3(p)$$

where $e_1(p)^2 = e_2(p)^2 = e_3(p)^2 = 1$.

Let ∇_1 be the connection on $(CT_1(D), \tilde{\pi}_1, D)$ defined by $\nabla_1 = \tilde{\nabla}_0$. In other words, we choose $\omega \equiv 0$. Then,

$$\nabla_1(I) = dr \otimes e_1 + dg \otimes e_2 + db \otimes e_3$$

We apply (9) to $\nabla_1(I)$ to get the following $CT_1(D)$ -valued symmetric tensor of rank 2

$$dr^2 \otimes 1 + dg^2 \otimes 1 + db^2 \otimes 1$$

At last, we have only to remark that

$$dx^2 \otimes 1 + dy^2 \otimes 1 + dr^2 \otimes 1 + dg^2 \otimes 1 + db^2 \otimes 1$$

may indeed be viewed as the first fundamental form of the surface parametrized by

$$\varphi : (x, y) \longmapsto (x, y, r(x, y), g(x, y), b(x, y))$$

embedded into $(\mathbb{R}^5, \|\cdot\|_2)$.

Figure 2. shows the result of such an edge detection. We see that it consists in detecting the strongest color variations of the image.

4.2. Color edge detection with respect to a given hue: $n=3, m=3$

The aim of this application is to detect edges on a color image without taking into account those located in a region of hue similar to a given hue h_0 .

Let us first mention how to proceed with the method based on a surfacic approach. We consider a color image as a 2-dimensional surface S parametrized by

$$\varphi : (x, y) \longmapsto (x, y, r(x, y), g(x, y), b(x, y))$$

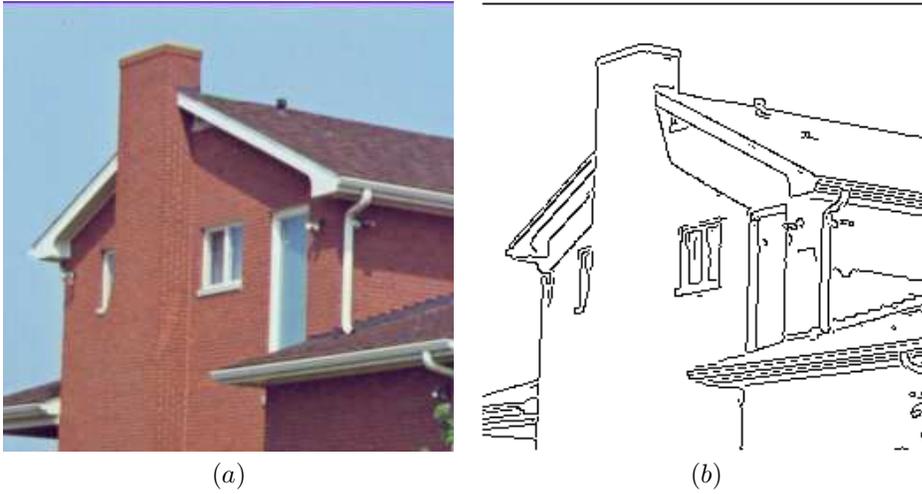


FIGURE 2. a. original image b. Color edge detection

embedded into \mathbb{R}^5 equipped with the metric

$$g := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

where we define λ as follows. We consider the domain

$$\Omega(p) = \{(x, y), \|(x, y) - p\|_\infty \leq 1\}$$

We set

$$\lambda(p) = 1 \quad \text{if} \quad \max_{(x,y) \in \Omega(p)} d(h_0, h[r(x, y), g(x, y), b(x, y)]) > d_0 \quad (10)$$

where d_0 is a threshold that determines if there is similarity or not, and d is the angular distance on S^1 . Then, we extend λ into a derivable strictly positive function on $[0, \pi]$ such that λ is negligible on $[0, d_0 - \epsilon]$, where $\epsilon \ll 1$ is fixed, the aim being to not take into account color variations in the domain of similarity $\overline{B}(h_0, d_0)$ of the hue h_0 . Then, from the first fundamental form of S :

$$dS^2 = dx^2 + dy^2 + \lambda dr^2 + \lambda dg^2 + \lambda db^2$$

we derive the "edge detector" of Section 2.1.

Let us now explain how to construct such a metric in the Clifford bundles framework. We take $(F_1, \pi_1, D) = (E_1, \pi_1, D) \in \mathcal{F}_3$ as subbundle of (E_1, π_1, D) . Let I

be the section of $(CT_1(D), \tilde{\pi}_1, D)$ representing the considered color image, i.e.

$$I(p) = r(p) \frac{e_1(p)}{\sqrt{g_1(p)}} + g(p) \frac{e_2(p)}{\sqrt{g_2(p)}} + b(p) \frac{e_3(p)}{\sqrt{g_3(p)}}$$

For each $p \in D$, we denote $A(p)$ the endomorphism

$$A(p)(x) = \begin{cases} \left(\left(\frac{e_1(p)}{\sqrt{g_1(p)}} + \frac{e_2(p)}{\sqrt{g_2(p)}} + \frac{e_3(p)}{\sqrt{g_3(p)}} \right) \wedge x \right) B(p) & \text{if } x \text{ is a vector} \\ 0 & \text{otherwise} \end{cases}$$

where $B(p)$ is the bivector

$$B(p) = \left(\frac{e_1(p)}{\sqrt{g_1(p)}} + \frac{e_2(p)}{\sqrt{g_2(p)}} + \frac{e_3(p)}{\sqrt{g_3(p)}} \right) \wedge \mu(p)$$

and

$$\mu(p) = \mu_1 \frac{e_1(p)}{\sqrt{g_1(p)}} + \mu_2 \frac{e_2(p)}{\sqrt{g_2(p)}} + \mu_3 \frac{e_3(p)}{\sqrt{g_3(p)}}$$

is a vector of hue h_0 . Remark that $A(p)$ makes the geometric product between a representant of the current hue and a representant of the hue h_0 .

Let ∇_1 be the connection on $(CT_1(D), \tilde{\pi}_1, D)$ defined by

$$\nabla_1 = \tilde{\nabla}_0 + (dx + dy) \otimes A$$

Then,

$$\nabla_1(I) = \tilde{\nabla}_0(I) + (dx + dy) \otimes \left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I \right) B$$

We see that the vector part, the scalar part and the bivector part of $\nabla_1(I)$ are respectively given by

$$\begin{aligned} & \tilde{\nabla}_0(I) \\ & (dx + dy) \otimes \left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I \right) \cdot B \\ & (dx + dy) \otimes \left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I \right) \wedge B \end{aligned}$$

Denoting by $\Delta(h)$ the function measuring for each p the hue difference between $I(p)$ and μ , we have the following result:

Proposition

Let X be the constant vector field on D of coordinates $(1, 0)$, then

$$|\tan(\Delta(h))| = \frac{\| \langle \nabla_{1X}(I) \rangle_2 \|}{| \langle \nabla_{1X}(I) \rangle_0 |}$$

Proof. We have

$$\langle \nabla_{1x}(I) \rangle_2 = \left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I \right) \wedge B$$

and

$$\langle \nabla_{1x}(I) \rangle_0 = \left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I \right) \cdot B$$

The proof is trivial once the expression of

$$\left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I \right) B$$

is simplified.

For this, we orthogonalize each one of the two bivectors (strictly speaking, they are bivector-valued sections). Let us treat the example of B . We have

$$B = \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge \left(\mu - k \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \right)$$

for any $k \in \mathbb{R}$. Then, we search k_0 such that

$$\left[\left(\mu_1 \frac{e_1}{\sqrt{g_1}} + \mu_2 \frac{e_2}{\sqrt{g_2}} + \mu_3 \frac{e_3}{\sqrt{g_3}} \right) - k_0 \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \right] \cdot \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) = 0$$

A simple computation leads to

$$k_0 = \frac{\mu_1 + \mu_2 + \mu_3}{3}$$

Therefore, B may be written

$$\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \tilde{\mu}$$

where $\tilde{\mu} =$

$$\left[\left(\mu_1 - \frac{\mu_1 + \mu_2 + \mu_3}{3} \right) \frac{e_1}{\sqrt{g_1}} + \left(\mu_2 - \frac{\mu_1 + \mu_2 + \mu_3}{3} \right) \frac{e_2}{\sqrt{g_2}} + \left(\mu_3 - \frac{\mu_1 + \mu_2 + \mu_3}{3} \right) \frac{e_3}{\sqrt{g_3}} \right]$$

We see that $\tilde{\mu}$ is in fact the chrominance vector v_μ of μ .

In the same way, we show that

$$\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I = \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) P_{Ch}(I)$$

where P_{Ch} denotes the orthogonal projection on the chrominance plane.

Consequently,

$$\begin{aligned} \left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I \right) B &= \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) P_{Ch}(I) \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) v_\mu \\ &= -3 \left(P_{Ch}(I) v_\mu \right) \end{aligned} \tag{11}$$

At last, let us remark that the (non-oriented) angle between $P_{Ch}(I)$ and v_μ is in fact Δh . Then we have

$$\frac{\|\langle \nabla_{1x} I \rangle_2\|}{|\langle \nabla_{1x} I \rangle_0|} = \frac{\|\langle P_{Ch}(I) v_\mu \rangle_2\|}{|\langle P_{Ch}(I) v_\mu \rangle_0|} = \frac{\|P_{Ch}(I)\| \|v_\mu\| |\sin(\pm \Delta h)|}{\|P_{Ch}(I)\| \|v_\mu\| |\cos(\pm \Delta h)|} = |\tan(\pm \Delta h)|$$

□

From the sign of the scalar part $-3 \left(P_{Ch}(I) \cdot v_\mu \right)$ of $\nabla_{1x} I$, we know if $\pm \Delta h$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ or not. Moreover, as the tangent map is odd, we have $|\tan(\pm \Delta h)| = |\tan(\Delta h)|$. Therefore, if the scalar part is negativ, we obtain

$$\Delta h = \arctan(|\tan(\pm \Delta h)|)$$

In a similar way, we may obtain Δh when the sign of the scalar part is positiv.

We construct the function λ mentioned above from Δh , just replacing (10) by

$$\lambda(p) = 1 \quad \text{if} \quad \max_{(x,y) \in \Omega(p)} \Delta h(x,y) > d_0$$

Then, we apply the rest of the method described in Section 3.3, i.e. (F_1, π_1, D) generates a vector bundle (E_2, π_2, D) of rank 3 equipped with a metric h given by

$$h := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

and a Clifford bundle $(CT_2(D), \tilde{\pi}_2, D)$. Then, we construct a 1-form η_2 with values in $CT_2(D)$

$$\eta_2 = dr \otimes \frac{e_1}{\sqrt{h_1}} + dg \otimes \frac{e_2}{\sqrt{h_2}} + db \otimes \frac{e_3}{\sqrt{h_3}}$$

from the 1-form

$$\nabla_1(I) = dr \otimes \frac{e_1}{\sqrt{g_1}} + dg \otimes \frac{e_2}{\sqrt{g_2}} + db \otimes \frac{e_3}{\sqrt{g_3}}$$

Applying formulas (8) and (9) to η_2 , we get the symmetric tensor of rank 2

$$dx^2 \otimes 1 + dy^2 \otimes 1 + dr^2 \otimes \lambda + dg^2 \otimes \lambda + db^2 \otimes \lambda$$

which can be assimilated to the metric of S .

Figure 3. shows an example of such an application. We aim at not detecting edges corresponding to the ornaments of the hat. The picture (c) is the result of the computations detailed above for $h_0 = \frac{5\pi}{3}$ (the magenta hue) and $d_0 = 0.43$. We observe that almost all the color variations inside this region are not detected, and the frontier of the region is globally well-detected (in particular transitions with the background and with the hat). It is more difficult to detect transitions with the hair since their hue is close to the magenta hue.



FIGURE 3. a. original image b. usual color edge detection c. color edge detection with respect to the magenta hue

4.3. Color edge classification: $n=3$, $m_i = 3$

Let I be the section of $(CT_1(D), \tilde{\pi}_1, D) \in \mathcal{G}_3$ representing a color image

$$I = r \frac{e_1}{\sqrt{g_1}} + g \frac{e_2}{\sqrt{g_2}} + b \frac{e_3}{\sqrt{g_3}}$$

4.3.1. Measure of variations with respect to a hue. We say that I is varying at a given point in the direction X with respect to the hue h_0 if its usual derivative in the direction X , $\tilde{\nabla}_{0_X}(I)$, or in the direction $-X$, $-\tilde{\nabla}_{0_X}(I)$ belongs to RGB and if the corresponding hue is h_0 . We aim at characterizing such vectors in the Clifford algebras (bundles) framework. Let B be a bivector-valued section coding the hue h_0 .

First, we suppose that

$$[\tilde{\nabla}_{0_X}(I)(p) \wedge B(p)]B(p)^{-1} = 0$$

i.e. $\tilde{\nabla}_{0_X}(I)(p)$ belongs to the plane generated by $B(p)$. We distinguish four cases:

- (a) Neither $\tilde{\nabla}_{0_X}(I)(p)$ nor $-\tilde{\nabla}_{0_X}(I)(p)$ belongs to RGB .
- (b) $\tilde{\nabla}_{0_X}(I)(p)$ or $-\tilde{\nabla}_{0_X}(I)(p)$ is along the grey axis.
- (c) $\tilde{\nabla}_{0_X}(I)(p)$ or $-\tilde{\nabla}_{0_X}(I)(p)$ belongs to RGB and the corresponding hue is the complementary of h_0 .
- (d) $\tilde{\nabla}_{0_X}(I)(p)$ or $-\tilde{\nabla}_{0_X}(I)(p)$ belongs to RGB and the corresponding hue is h_0 .

The following result gives a necessary and sufficient condition for the case (d) to occur. We set by convention $\text{sign}(0) = \pm$. Let us denote

$$\Sigma_i = \left(\tilde{\nabla}_{0_X}(I) \cdot \frac{e_i}{\sqrt{g_i}} \right)$$

for $i = 1, 2, 3$, and

$$\Sigma = \left[\left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge \tilde{\nabla}_{0_X}(I) \right) \cdot B \right]$$

Let also σ_i , for $i = 1, 2, 3$, and σ be defined by $\sigma_i = \text{sign}(\Sigma_i)$ and $\sigma = \text{sign}(\Sigma)$.

Proposition

With the previous notations, I is varying at p in the direction X with respect to the hue h_0 if and only if the following conditions are fulfilled:

- (i) The derivative $\tilde{\nabla}_{0_X}(I)(p)$ is not zero;
- (ii) $\sigma_1 = \sigma_2 = \sigma_3$;
- (iii) Σ is not null and $-\sigma$ is equal to the sign of one of the non zero Σ_i .

Proof. Let us first remark that (i) ensures that one of the Σ_i is not zero. The second assertion means that one of two vectors $\tilde{\nabla}_{0_X}(I)(p)$ or $-\tilde{\nabla}_{0_X}(I)(p)$ belongs to RGB . As $\Sigma \neq 0$, the case (b) does not occur.

If $\tilde{\nabla}_{0_X}(I)(p) \in RGB$, then the sign of one of the non zero Σ_i is $+$, and we know from (11) that σ is negative if and only if $h(\tilde{\nabla}_{0_X}(I)(p)) = h_0$.

If $-\tilde{\nabla}_{0_X}(I)(p) \in RGB$, then the sign of one of the non zero Σ_i is $-$, and the orthogonal projection of $\tilde{\nabla}_{0_X}(I)(p)$ onto the chrominance plane (see (11)) belongs to the axis generated by the chrominance vector of the complementary hue of the hue of $-\tilde{\nabla}_{0_X}(I)(p)$. Therefore σ is positive if and only if the hue of $-\tilde{\nabla}_{0_X}(I)(p)$ is h_0 . \square

From this proposition we deduce the following corollary: I is varying at p in the direction X with respect to the *complementary* hue of h_0 if and only if the three conditions (i), (ii) and (iv) are fulfilled, where the condition (iv) is obtained by replacing $-\sigma$ by σ in (iii).

We now suppose that $\tilde{\nabla}_{0_X}(I)(p)$ does not necessary belong to the plane generated by $B(p)$. The derivative $\tilde{\nabla}_{0_X}(I)(p)$ is said to be measurable with respect to h_0 if the conditions (i), (ii) and (iii) of the preceding proposition hold. Such a definition makes sense since we deduce from these relations that $\tilde{\nabla}_{0_X}(I)(p)$ is varying with respect to a hue that is closer of h_0 than of its complementary. In the same way, we can say if $\tilde{\nabla}_{0_X}(I)(p)$ is measurable with respect to the complementary of h_0 or not. We then measure this variation by the expression

$$\|\tilde{\nabla}_{0_X}(I)(p) \cdot B(p) B(p)^{-1}\|$$

4.3.2. Application to shadows and highlights detection. The aim of this application is to analyse the nature of the edges of a color image in function of the variation of the light. We refer to [4] and [5] for the corresponding classification in shadows, highlights and material edges. The segmentation we deduce from the general framework described before requires the hypothesis of white illumination and neutral interface reflection (see [4],[5]).

Let φ be the parametrization of the surface S representing a color image embedded into $(\mathbb{R}^5, \|\cdot\|_2)$. The analysis of edges we propose relies on the study of the derivative $\tilde{\nabla}_{0x}(I)$ where X is such that $d\varphi(X)$ is an eigenvector associated to the highest eigenvalue of the first fundamental form of S .

Before detailing the parameters of (3.3) we use for such an application, let us make the following crucial remark. Due to the nature of the objects we use to deal with an image (surfaces or fiber bundles), we only have to consider, in practice, points of D with integer coordinates. Therefore, there exists a certain freedom in the way to compute derivatives of functions defined on D . In particular, derivatives of the components red, green, blue and temperature at such points are given by discrete approximations (Sobel, Canny-Deriche, etc).

Let

$$B = \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge \left(u_1 \frac{e_1}{\sqrt{g_1}} + u_2 \frac{e_2}{\sqrt{g_2}} + u_3 \frac{e_3}{\sqrt{g_3}} \right)$$

be a bivector-valued section of $(CT_1(D), \tilde{\pi}_1, D)$ such that $h(u_1(p), u_2(p), u_3(p)) = h(I(p))$ and $d_p u_1 = d_p u_2 = d_p u_3 = 0$ for p with integer coordinates. In other words, B is a section coding the current hue and satisfying

$$\begin{aligned} \tilde{\nabla}_0(B) &= \tilde{\nabla}_0 \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \left(u_1 \frac{e_1}{\sqrt{g_1}} + u_2 \frac{e_2}{\sqrt{g_2}} + u_3 \frac{e_3}{\sqrt{g_3}} \right) \\ &\quad - \left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \tilde{\nabla}_0 \left(u_1 \frac{e_1}{\sqrt{g_1}} + u_2 \frac{e_2}{\sqrt{g_2}} + u_3 \frac{e_3}{\sqrt{g_3}} \right) = 0 \end{aligned}$$

on integer coordinates points.

Let ∇_1 be the connection with $\omega \equiv 0$, i.e. $\nabla_1 = \tilde{\nabla}_0$. We have at such points

$$\nabla_1 \left[I + \left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I \right) \cdot B \right] = \nabla_1(I) + \left[\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge \nabla_1(I) \right] \cdot B$$

We obtain a $CT_1(D)$ -valued 1-form on D , which may be decomposed into a scalar and a vector part.

From this 1-form, we may perform 6 edge detections: usual edge detection (see Section 4.1), material edge detection, non material edge detection, this latter being divided into shadows detection, highlights detection, and others. Therefore, we need to construct 6 vector bundles of rank 3 (E_{2_i}, π_{2_i}, D) equipped with metrics h_i , $i = 1 \dots 6$. The metrics h_i , $i = 2 \dots 6$ are given by the geometry of $\nabla_{1_x}(I)$ for $d\varphi(X)$ be an eigenvector associated to the highest eigenvalue of the first fundamental form of S .

Let us first compute the metric of S by considering the vector bundle (E_{2_1}, π_{2_1}, D) of metric $h_1 = I_3$. This allows us to perform the edge detection of Figure 4.b. and to obtain for each p the vector $\nabla_{1_y}(I)$ that maximizes the norm of $\nabla_{1_x}(I)$ for

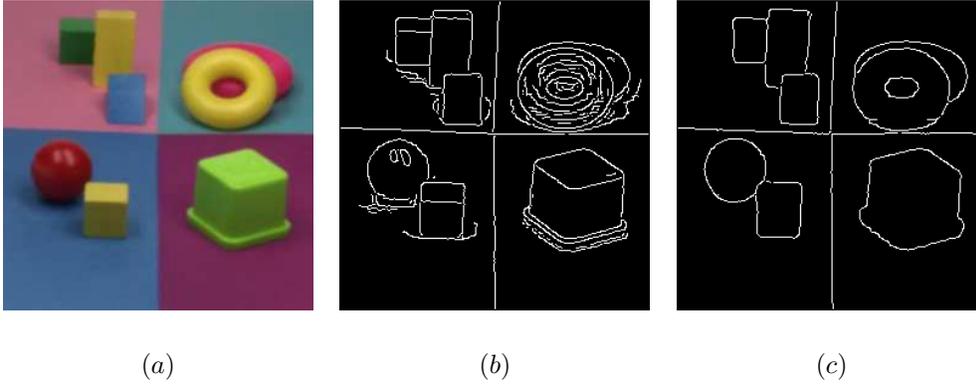


FIGURE 4. a. original image b. global segmentation c. material edges

$\|X\| = 1$ in $T_p D$. Following the arguments of [4], material edge detection arises from rejections of directional derivatives of the image on the planes generated by the light source direction (achromatic axis) and the current colors, which are the planes of the current hues. We now detail how to perform material and non material edge detection using (3.3).

We can decompose $\nabla_1(I)$ into

$$\nabla_1(I) = \nabla_1(I \wedge B B^{-1}) + \nabla_1(I \cdot B B^{-1})$$

This leads in particular for p with integer coordinates to

$$\nabla_1(I)(p) = \nabla_1(I)(p) \wedge B(p) B(p)^{-1} + \nabla_1(I)(p) \cdot B(p) B(p)^{-1}$$

Let λ_2 defined as follows. We set $\lambda_2(p) = 1$ if $\|\nabla_{1_Y}(I)(p) \wedge B(p) B(p)^{-1}\| \geq N$ for N a threshold to be determined, and we extend λ_2 into a derivable strictly positive function such that λ_2 is negligible on $[0, N - \epsilon]$, where $\epsilon \ll 1$ is fixed. We also define a function λ_3 by stating that $\lambda_3(p)$ is negligible if $\|\nabla_{1_Y}(I)(p) \cdot B(p) B(p)^{-1}\| \geq N$, and extending λ_3 into a derivable strictly positive function such that $\lambda_3 \equiv 1$ on $[0, N - \epsilon]$.

From λ_2 , we consider the metric $h_2 = \lambda_2 I_3$ generating a vector bundle (E_{2_2}, π_{2_2}, D) and a Clifford bundle $(CT_{2_2}, \tilde{\pi}_{2_2}, D)$. From the vector-valued 1-form $\nabla_1(I \wedge B B^{-1})$, we construct a vector-valued 1-form in $(CT_{2_2}, \tilde{\pi}_{2_2}, D)$ by (7). Following the process gives rise to a symmetric tensor of rank 2 that can be assimilated to the metric of a surface S_2 embedded into \mathbb{R}^5 equipped with the metric

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

In particular, for p with interger coordinates and $\lambda_2(p) = 1$, we have

$$(dS_2)^2(Z_1, Z_2)(p) = dx \otimes dx(Z_1, Z_2)(p) + dy \otimes dy(Z_1, Z_2)(p) \\ + \left(\nabla_{1_{Z_1}}(I)(p) \wedge B(p) B(p)^{-1} \right) \cdot \left(\nabla_{1_{Z_2}}(I)(p) \wedge B(p) B(p)^{-1} \right)$$

by the identification of \mathbb{R} with its injection into each fiber.

Replacing λ_2 by λ_3 and $\nabla_1(I \wedge B B^{-1})$ by $\nabla_1(I \cdot B B^{-1})$, the edge detection of Figure 4.c. is replaced by the edge detection of Figure 5.a.

Then, we aim at characterizing shadows and highlights. We say that variations of an image at p are measurable with respect to a hue h_0 if $\nabla_{1_Y}(I)(p)$ is measurable with respect to h_0 for Y be the tangent vector field mentioned above. We then distinguish variations measurable with respect to the current hue and variations measurable with respect to its complementary, the information being given by

$$\left[\left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge \nabla_{1_Y}(I) \right) \cdot B \right]$$

or equivalently

$$\nabla_{1_Y} \left[\left(\left(\frac{e_1}{\sqrt{g_1}} + \frac{e_2}{\sqrt{g_2}} + \frac{e_3}{\sqrt{g_3}} \right) \wedge I \right) \cdot B \right]$$

This is precisely the scalar part of the 1-form applied to the tangent vector field Y .

We construct three functions $\lambda_i, i = 4 \dots 6$, each one of them generating a Clifford bundle $(CT_{2_i}(D), \tilde{\pi}_{2_i}, D)$ associated to the metric $h_i = \lambda_i I_3$. Then, mapping the 1-form $\nabla_1(I \cdot B B^{-1})$ into each one of these Clifford bundles allows to perform different edge detections. Let us now explicit these functions.

We set $\lambda_4(p)$ equals 1 if $\|\nabla_{1_Y}(I)(p) \wedge B(p) B(p)^{-1}\| \leq N$ and variations of I at p with respect to the hue given by $B(p)$, i.e. the current hue, are measurable. We set $\lambda_5(p)$ equals 1 if $\|\nabla_{1_Y}(I)(p) \wedge B(p) B(p)^{-1}\| \leq N$ and variations of I at p with respect to the complementary hue of $B(p)$ are measurable. At last, we set $\lambda_6(p)$ equals 1 if $\|\nabla_{1_Y}(I)(p) \wedge B(p) B(p)^{-1}\| \leq N$ and variations of I at p are neither measurable with respect to the current hue nor to its complementary. As usual, we extend these functions to strictly positive derivable functions that are negligible where the conditions are not respected.

Figure 5.a. shows edges of the image of Figure 4.a. which are non material edges (see Figure 4.c.). These edges are decomposed in function of the position of $\nabla_{1_Y}(I)$ in the cube RGB . We can see on Figure 5.b. the edges due to shadows. They correspond to high variations with respect to the current hue under the constraint $\|\nabla_{1_Y}(I)(p) \wedge B(p) B(p)^{-1}\| \leq N$. Highlights are given by Figure 5.c., they correspond to high variations with respect to the complementary of the current hue under the previous constraint. We do not represent edges corresponding to the

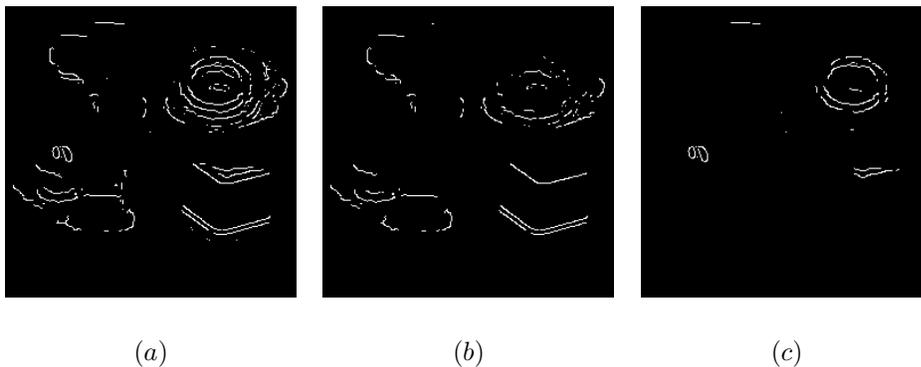


FIGURE 5. a. non material edges . b. edges arising from the measure of variations with respect to the current hue . c. edges arising from the measure of variations with respect to the complementary of the current hue

metric h_6 , since they do not represent significative edges.

4.4. Edge detection in the color domain with constraints on the temperature: $n=4, m=3$

The aim of this application is to detect hot objects in a room. We need both color and temperature information (see Figure 6.a. and 6.b.). Therefore we consider the color/infrared image associated to the scene.

Let λ be the function defined as follows. We set $\lambda(p)$ equals 1 if $t(p) \geq t_0$ for t_0 a given temperature, and we extend it to a derivable strictly positive function such that λ is negligible on $[0, t_0 - \epsilon]$, for $\epsilon \ll 1$ fixed. The role of λ is to not take into account color variations where $t < t_0$.

To perform the edge detection, we can either deal with the surface representing the color/infrared image or with the surface representing the color image. In the first case, we consider the surface parametrized by

$$\varphi : (x, y) \longmapsto (x, y, r(x, y), g(x, y), b(x, y), t(x, y))$$

embedded into \mathbb{R}^6 equipped with the metric

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In the second case, we consider the surface parametrized by

$$\varphi : (x, y) \mapsto (x, y, r(x, y), g(x, y), b(x, y))$$

embedded into \mathbb{R}^5 equipped with the metric

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

In both cases, the first fundamental form of the surface is

$$dx^2 + dy^2 + \lambda dr^2 + \lambda dg^2 + \lambda db^2$$

We detail now how to proceed to compute the metric corresponding to the second case. Let $(E_1, \pi_1, D) \in \mathcal{F}_4$ of metric g and let (F_1, π_1, D) be the subbundle of global frame

$$\left(\frac{e_1}{\sqrt{g_1}}, \frac{e_2}{\sqrt{g_2}}, \frac{e_3}{\sqrt{g_3}} \right)$$

We consider the section I of $(CT_1(D), \tilde{\pi}_1, D) \in \mathcal{G}_4$ generated by (E_1, π_1, D) representing the considered color/infrared image, i.e.

$$I(p) = r(p) \frac{e_1(p)}{\sqrt{g_1(p)}} + g(p) \frac{e_2(p)}{\sqrt{g_2(p)}} + b(p) \frac{e_3(p)}{\sqrt{g_3(p)}} + t(p) \frac{e_4(p)}{\sqrt{g_4(p)}}$$

The chosen connection ∇_1 on $(CT_1(D), \tilde{\pi}_1, D)$ is

$$\nabla_1 = \tilde{\nabla}_0 + (dx + dy) \otimes A$$

where for $p \in D$, $A(p)$ is the endomorphism

$$A(p)(x) = x \cdot \frac{e_4(p)}{\sqrt{g_4(p)}}$$

Then, we have

$$\nabla_1(I) = dr \otimes \frac{e_1}{\sqrt{g_1}} + dg \otimes \frac{e_2}{\sqrt{g_2}} + db \otimes \frac{e_3}{\sqrt{g_3}} + dt \otimes \frac{e_4}{\sqrt{g_4}} + (dx + dy) \otimes t$$

Here, $\nabla_1(I)$ contains only a scalar and a vector part. From the scalar part, we get the temperature component of the image and thus the function λ defined above. This one generates a metric $h = \lambda I_3$ of a vector bundle $(E_2, \pi_2, D) \in \mathcal{F}_3$.

From (7), we construct the following element of $\Gamma(CT_2(D))$

$$\eta_2 = dr \otimes \frac{e_1}{\sqrt{h_1}} + dg \otimes \frac{e_2}{\sqrt{h_2}} + db \otimes \frac{e_3}{\sqrt{h_3}}$$

As before we obtain from η_2 a $CT_2(D)$ -valued symmetric tensor of rank 2, namely

$$dx^2 \otimes 1 + dy^2 \otimes 1 + dr^2 \otimes \lambda + dg^2 \otimes \lambda + db^2 \otimes \lambda$$

This tensor corresponds to the first fundamental form we aimed at computing.

Figure 6.c. shows the result of such an edge detection on a scene representing

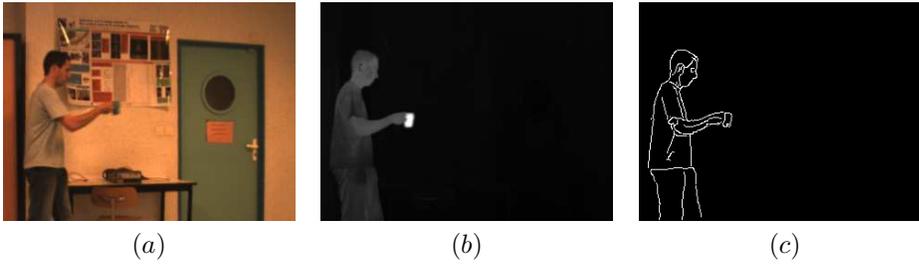


FIGURE 6. a. color information of the scene b. temperature information of the scene c. color edge detection with respect to temperature constraint.

a man standing in front of a wall and holding a cup of hot coffee. By a well-chosen t_0 , we may detect only the man and the cup.

4.5. Color/infrared edge detection with constraints on color and temperature: $n=4$, $m_\nu = 4$

The main goal of this application is to rewrite the color/infrared edge detection of [1] with the method described in (3.3). Note that they are not completely similar since in this paper we measure color variations in the RGB space, whereas we deal with the HSL color space defined in Section 2.2 in [1].

Let μ represent the skin color of the man (see Figure 7.a.) with (μ_1, μ_2, μ_3) as RGB coordinates. From μ we construct a section

$$\mu(p) = \mu_1 \frac{e_1(p)}{\sqrt{g_1(p)}} + \mu_2 \frac{e_2(p)}{\sqrt{g_2(p)}} + \mu_3 \frac{e_3(p)}{\sqrt{g_3(p)}}$$

of $(CT_1(D), \tilde{\pi}_1, D) \in \mathcal{G}_4$ generated by $(E_1, \pi_1, D) \in \mathcal{F}_4$ of metric g . For this application, we choose (E_1, π_1, D) itself as the subbundle (F_1, π_1, D) . We consider the color/infrared image I representing the scene as a section of this fiber bundle

$$I(p) = r(p) \frac{e_1(p)}{\sqrt{g_1(p)}} + g(p) \frac{e_2(p)}{\sqrt{g_2(p)}} + b(p) \frac{e_3(p)}{\sqrt{g_3(p)}} + t(p) \frac{e_4(p)}{\sqrt{g_4(p)}}$$

Then, we consider also the following connection

$$\nabla_1 = \tilde{\nabla}_0 + (dx + dy) \otimes A$$

where $A \in \Gamma(\text{End}(CT_1(D)))$ is defined for each p by

$$A(p)(x) = \mu(p)x$$

Therefore, we have

$$\nabla_1(I) = \tilde{\nabla}_0(I) + (dx + dy) \otimes \mu I$$

Let α denote the function measuring for each p the oriented angle between $\mu(p)$

and the color part of $I(p)$. We have the following result.

Lemma

Let X be the constant vector field on D of coordinates $(1, 0)$, then

$$|\tan(\alpha)| = \frac{\left\| \langle \nabla_{1_X}(I) \rangle_2 \cdot (e_1 e_2 e_3) (e_1 e_2 e_3)^{-1} \right\|}{\left| \langle \nabla_{1_X}(I) \rangle_0 \right|}$$

and

$$t = \frac{\left\| \langle \nabla_{1_X}(I) \rangle_2 - \langle \nabla_{1_X}(I) \rangle_2 \cdot (e_1 e_2 e_3) (e_1 e_2 e_3)^{-1} \right\|}{\|\mu\|}$$

Proof. We have

$$\begin{aligned} \langle \nabla_{1_X}(I) \rangle_2 &= \mu \wedge I = (\mu \wedge I) \cdot (e_1 e_2 e_3) (e_1 e_2 e_3)^{-1} + \mu \wedge t \frac{e_4}{\sqrt{g_4}} \\ &= \mu \wedge [I \cdot (e_1 e_2 e_3) (e_1 e_2 e_3)^{-1}] + \mu \wedge t \frac{e_4}{\sqrt{g_4}} \end{aligned}$$

and

$$\langle \nabla_{1_X}(I) \rangle_0 = \mu \cdot I = \mu \cdot [I \cdot (e_1 e_2 e_3) (e_1 e_2 e_3)^{-1}]$$

Therefore,

$$\frac{\left\| \langle \nabla_{1_X}(I) \rangle_2 \cdot (e_1 e_2 e_3) (e_1 e_2 e_3)^{-1} \right\|}{\left| \langle \nabla_{1_X}(I) \rangle_0 \right|} = \frac{\|\mu\| \|I \cdot (e_1 e_2 e_3) (e_1 e_2 e_3)^{-1}\| |\sin(\alpha)|}{\|\mu\| \|I \cdot (e_1 e_2 e_3) (e_1 e_2 e_3)^{-1}\| |\cos(\alpha)|}$$

and

$$\frac{\left\| \langle \nabla_{1_X}(I) \rangle_2 - \langle \nabla_{1_X}(I) \rangle_2 \cdot (e_1 e_2 e_3) (e_1 e_2 e_3)^{-1} \right\|}{\|\mu\|} = \frac{\|\mu\| \|t \frac{e_4}{\sqrt{g_4}}\|}{\|\mu\|}$$

□

From $|\tan(\alpha)|$ we may directly determine $|\alpha|$ since $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ (μ and the color part of I are vectors in the cube RGB).

We set $\lambda(p) = 1$ if

$$\min_{(x,y) \in \Omega(p)} |\alpha(x,y)| \leq \alpha_0 \quad \text{and} \quad \max_{(x,y) \in \Omega(p)} t(x,y) \geq t_0$$

where α_0 and t_0 are two thresholds that determine regions of interest. Then, we extend λ into a derivable strictly positive function on $[0, \pi/2] \times [0, 255]$ such that λ is negligible on $[\alpha_0 + \epsilon, \pi/2] \times [0, 255] \cup [0, \pi/2] \times [0, t_0 - \epsilon]$ where $\epsilon \ll 1$ is fixed.

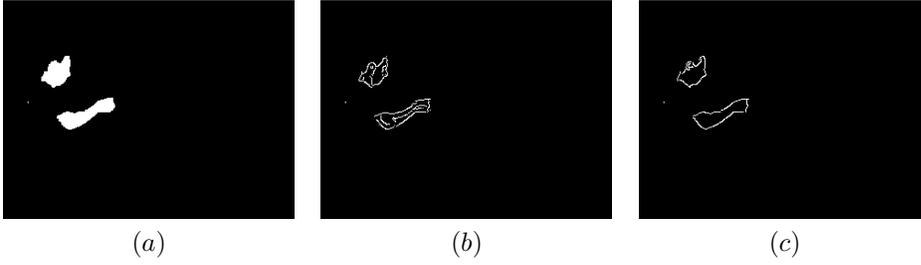


FIGURE 7. a. regions of interest b. edge detection for $k = 1$ c. edge detection for $k = 25$

From λ , we derive a family of vector bundles $(E_2, \pi_2, D) \in \mathcal{F}_4$ indexed by $\nu \in \mathbb{N}$ and equipped with the metric

$$h_\nu = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \nu\lambda \end{pmatrix}$$

The role of ν is to control the weight of temperature variations in the image. Each one of these vector bundles generates a Clifford bundle $(CT_2(D), \pi_2, D) \in \mathcal{G}_4$. Following (3.3), for a given ν we construct

$$\eta_2 = dr \frac{e_1}{\sqrt{h_1}} + dg \frac{e_2}{\sqrt{h_2}} + db \frac{e_3}{\sqrt{h_3}} + dt \frac{e_4}{\sqrt{h_4}}$$

from which we derive a $CT_2(D)$ -valued symmetric tensor of rank 2:

$$dx^2 \otimes 1 + dy^2 \otimes 1 + dr^2 \otimes \lambda + dg^2 \otimes \lambda + db^2 \otimes \lambda + dt^2 \otimes \nu\lambda$$

This tensor can be assimilated to the first fundamental form of the 2-dimensional surface parametrized by

$$\varphi : (x, y) \longmapsto (x, y, r(x, y), g(x, y), b(x, y), t(x, y))$$

embedded into \mathbb{R}^6 equipped with the metric

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \nu\lambda \end{pmatrix}$$

Figure 7. shows results of edge detections for different values of ν . We refer to ([1]) for detailed comments.

5. Conclusion

Defining an image as a section of a trivial Clifford bundle over its domain, we have developed in this paper a general method to construct first fundamental forms of surfaces whose characteristics (parametrization, metric of the ambient space) arise from the choices of the connection, the section to derive with this connection and the subbundle (F_1, π_1, D) . To construct such metrics, we take advantage of Clifford algebras framework by the efficiency of computations and their high dimensions. Applying the method of Section 2.1 to the metrics constructed, we perform edge detections on color and color/infrared images. By the genericity of the method we have presented, it could be applied to images of higher dimensions.

Such a formalism could appear superficial to detect edges, and in particular in the definitions of the coefficients of metrics. However, we refer the reader to some works on image smoothing which are entirely based on the metric of the surface representing the image. Indeed, in [12], the authors solve the heat equation associated to the Laplace-Beltrami operator on each component to denoise color images, the metric of the surface being the first fundamental form arising from the standard euclidean norm of the ambient space. The metrics we define and construct in this paper make possible for example to smooth only the regions of a color image that correspond to a given hue.

References

- [1] Batard, T., Saint-Jean, C., Berthier, M.: A metric approach to nD images edge detection with Clifford algebras. To appear in *Journal of Mathematical Imaging and Vision*
- [2] Blau, M.: Connections on Clifford bundles and the Dirac operator. *Letters in Mathematical Physics* **13** (1987) 83-92
- [3] Fairchild, M.: *Color Appearance Models*. John Wiley and Sons, New York (2005)
- [4] Gevers, T., Stokman, H.M.G.: Classifying color transitions into shadow-geometry, illumination, highlight or material edges. In: *Proceedings of the International Conference on Image Processing (ICIP)*, Volume **I**, 521-524 (2000)
- [5] Gevers, T., Smeulders, A.W.M.: Color based image segmentation. *Pattern Recognition* **32**, 453-464 (1999)
- [6] Graf, W.: Differentials forms as spinors. *Annales de l'Institut Henri Poincaré* **29**(1) (1978), 85-109
- [7] Hestenes, D., Sobczyk, G.: *Clifford algebra to geometric calculus*. D. Reidel, Dordrecht (1984)
- [8] Lenz, R.: Color edge detectors for conical color spaces. In: *Proceedings of CGIP 2000. First International Conference on Color in Graphics and Image Processing*, (2000) 284-289
- [9] Sangwine, S., Ell, T.: Colour image filters based on hypercomplex convolution. *IEEE Proceedings-Vision, Image and Signal Processing* **147**(2), (2000) 89-93

- [10] Evans, C., Ell, T., Sangwine, S.: Colour-sensitive edge detection using hypercomplex filters. In: Proceedings EUSIPCO 2000, pp. 107–110 (2000)
- [11] Sapiro, G.: Color snakes. *Computer Vision and Image understanding* **68**(2), (1997) 407–416
- [12] Sochen, N., Kimmel, R., Malladi, R.: A general framework for low level vision. *IEEE Transactions on Image Processing* **7**(3), 310–318 (1998)
- [13] Sommer, G.: *Geometric computing with Clifford Algebras. Theoretical Foundations and Applications in Computer Vision and Robotics*. Springer-Verlag, Berlin (2001)

Thomas Batard
Laboratoire Mathématiques, Image et Applications
Université de La Rochelle
Avenue Michel Crépeau, 17042 La Rochelle Cedex
France
e-mail: `thomas.batard01@univ-lr.fr`

Michel Berthier
Laboratoire Mathématiques, Image et Applications
Université de La Rochelle
Avenue Michel Crépeau, 17042 La Rochelle Cedex
France
e-mail: `michel.berthier@univ-lr.fr`