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# On the parity of generalized partition functions III

by

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**Abstract.** Improving on some results of J.-L. Nicolas [15], the elements of the set  $\mathcal{A} = \mathcal{A}(1 + z + z^3 + z^4 + z^5)$ , for which the partition function  $p(\mathcal{A}, n)$  (i.e. the number of partitions of  $n$  with parts in  $\mathcal{A}$ ) is even for all  $n \geq 6$  are determined. An asymptotic estimate to the counting function of this set is also given.

**Key words :** Partitions, periodic sequences, order of a polynomial, orbits, 2-adic numbers, counting function, Selberg-Delange formula.

**2000 MSC :** 11P81, 11N25, 11N37.

## 1 Introduction.

Let  $\mathbb{N}$  (resp.  $\mathbb{N}_0$ ) be the set of positive (resp. non-negative) integers. If  $\mathcal{A} = \{a_1, a_2, \dots\}$  is a subset of  $\mathbb{N}$  and  $n \in \mathbb{N}$  then  $p(\mathcal{A}, n)$  is the number of partitions of  $n$  with parts in  $\mathcal{A}$ , i.e., the number of solutions of the diophantine equation

$$a_1x_1 + a_2x_2 + \dots = n, \quad (1.1)$$

in non-negative integers  $x_1, x_2, \dots$ . As usual we set  $p(\mathcal{A}, 0) = 1$ .

The counting function of the set  $\mathcal{A}$  will be denoted by  $A(x)$ , i.e.,

$$A(x) = |\{n \leq x, n \in \mathcal{A}\}|. \quad (1.2)$$

Let  $\mathbb{F}_2$  be the field with 2 elements,  $P = 1 + \epsilon_1 z^1 + \dots + \epsilon_N z^N \in \mathbb{F}_2[z]$ ,  $N \geq 1$ . Although it is not difficult to prove (cf. [14], [5]) that there is a unique subset

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$\mathcal{A} = \mathcal{A}(P)$  of  $\mathbb{N}$  such that the generating function  $F(z)$  satisfies

$$F(z) = F_{\mathcal{A}}(z) = \prod_{a \in \mathcal{A}} \frac{1}{1 - z^a} = \sum_{n \geq 0} p(\mathcal{A}, n) z^n \equiv P(z) \pmod{2}, \quad (1.3)$$

the determination of the elements of such sets for general  $P$ 's seems to be hard.

Let the decomposition of  $P$  into irreducible factors over  $\mathbb{F}_2$  be

$$P = P_1^{\alpha_1} P_2^{\alpha_2} \dots P_l^{\alpha_l}. \quad (1.4)$$

We denote by  $\beta_i = \text{ord}(P_i)$ ,  $1 \leq i \leq l$ , the order of  $P_i$ , that is the smallest positive integer  $\beta_i$  such that  $P_i(z)$  divides  $1 + z^{\beta_i}$  in  $\mathbb{F}_2[z]$ . It is known that  $\beta_i$  is odd (cf. [13]). We set

$$\beta = \text{lcm}(\beta_1, \beta_2, \dots, \beta_l). \quad (1.5)$$

Let  $\mathcal{A} = \mathcal{A}(P)$  satisfy (1.3) and  $\sigma(\mathcal{A}, n)$  be the sum of the divisors of  $n$  belonging to  $\mathcal{A}$ , i.e.,

$$\sigma(\mathcal{A}, n) = \sum_{d|n, d \in \mathcal{A}} d = \sum_{d|n} d \chi(\mathcal{A}, d), \quad (1.6)$$

where  $\chi(\mathcal{A}, \cdot)$  is the characteristic function of the set  $\mathcal{A}$ , i.e,  $\chi(\mathcal{A}, d) = 1$  if  $d \in \mathcal{A}$  and  $\chi(\mathcal{A}, d) = 0$  if  $d \notin \mathcal{A}$ . It was proved in [6] (see also [4], [12]) that for all  $k \geq 0$ , the sequence  $(\sigma(\mathcal{A}, 2^k n) \bmod 2^{k+1})_{n \geq 1}$  is periodic with period  $\beta$  defined by (1.5), in other words,

$$n_1 \equiv n_2 \pmod{\beta} \Rightarrow \forall k \geq 0, \sigma(\mathcal{A}, 2^k n_1) \equiv \sigma(\mathcal{A}, 2^k n_2) \pmod{2^{k+1}}. \quad (1.7)$$

Moreover, the proof of (1.7) in [6] allows to calculate  $\sigma(\mathcal{A}, 2^k n) \bmod 2^{k+1}$  and to deduce the value of  $\chi(\mathcal{A}, n)$  where  $n$  is any positive integer. Indeed, let

$$S_{\mathcal{A}}(m, k) = \chi(\mathcal{A}, m) + 2\chi(\mathcal{A}, 2m) + \dots + 2^k \chi(\mathcal{A}, 2^k m). \quad (1.8)$$

If  $n$  writes  $n = 2^k m$  with  $k \geq 0$  and  $m$  odd, (1.6) implies

$$\sigma(\mathcal{A}, n) = \sigma(\mathcal{A}, 2^k m) = \sum_{d|m} d S_{\mathcal{A}}(d, k), \quad (1.9)$$

which, by Möbius inversion formula, gives

$$m S_{\mathcal{A}}(m, k) = \sum_{d|m} \mu(d) \sigma(\mathcal{A}, \frac{n}{d}) = \sum_{d|\overline{m}} \mu(d) \sigma(\mathcal{A}, \frac{n}{d}), \quad (1.10)$$

where  $\overline{m} = \prod_{p|m} p$  denotes the radical of  $m$  with  $\overline{1} = 1$ .

In the above sums,  $\frac{n}{d}$  is always a multiple of  $2^k$ , so that, from the values of  $\sigma(\mathcal{A}, \frac{n}{d})$ , by (1.10), one can determine the value of  $S_{\mathcal{A}}(m, k) \pmod{2^{k+1}}$  and by (1.8), the value of  $\chi(\mathcal{A}, 2^i m)$  for all  $i, i \leq k$ .

Let  $\beta$  be an odd integer  $\geq 3$  and  $(\mathbb{Z}/\beta\mathbb{Z})^*$  be the group of invertible elements modulo  $\beta$ . We denote by  $\langle 2 \rangle$  the subgroup of  $(\mathbb{Z}/\beta\mathbb{Z})^*$  generated by 2 and consider its action  $\star$  on the set  $\mathbb{Z}/\beta\mathbb{Z}$  given by  $a \star x = ax$  for all  $a \in \langle 2 \rangle$  and  $x \in \mathbb{Z}/\beta\mathbb{Z}$ . The quotient set will be denoted by  $(\mathbb{Z}/\beta\mathbb{Z})/\langle 2 \rangle$  and the orbit of some  $n$  in  $\mathbb{Z}/\beta\mathbb{Z}$  by  $O(n)$ . For  $P \in \mathbb{F}_2[z]$  with  $P(0) = 1$  and  $\text{ord}(P) = \beta$ , let  $\mathcal{A} = \mathcal{A}(P)$  be the set obtained from (1.3). Property (1.7) shows (after [3]) that if  $n_1$  and  $n_2$  are in the same orbit then

$$\sigma(\mathcal{A}, 2^k n_1) \equiv \sigma(\mathcal{A}, 2^k n_2) \pmod{2^{k+1}}, \quad \forall k \geq 0. \quad (1.11)$$

Consequently, for fixed  $k$ , the number of distinct values that  $(\sigma(\mathcal{A}, 2^k n) \pmod{2^{k+1}})_{n \geq 1}$  can take is at most equal to the number of orbits of  $\mathbb{Z}/\beta\mathbb{Z}$ .

Let  $\varphi$  be the Euler function and  $s$  be the order of 2 modulo  $\beta$ , i.e., the smallest positive integer  $s$  such that  $2^s \equiv 1 \pmod{\beta}$ . If  $\beta = p$  is a prime number then  $(\mathbb{Z}/p\mathbb{Z})^*$  is cyclic and the number of orbits of  $\mathbb{Z}/p\mathbb{Z}$  is equal to  $1 + r$  with  $r = \frac{\varphi(p)}{s} = \frac{p-1}{s}$ . In this case, we have

$$(\mathbb{Z}/p\mathbb{Z})/\langle 2 \rangle = \{O(g), O(g^2), \dots, O(g^r) = O(1), O(p)\}, \quad (1.12)$$

where  $g$  is some generator of  $(\mathbb{Z}/p\mathbb{Z})^*$ . For  $r = 2$ , the sets  $\mathcal{A} = \mathcal{A}(P)$  were completely determined by N. Baccar, F. Ben Saïd and J.-L. Nicolas ([2], [8]). Moreover, N. Baccar proved in [1] that for all  $r \geq 2$ , the elements of  $\mathcal{A}$  of the form  $2^k m$ ,  $k \geq 0$  and  $m$  odd, are determined by the 2-adic development of some root of a polynomial with integer coefficients. Unfortunately, his results are not explicit and do not lead to any evaluation of the counting function of the set  $\mathcal{A}$ . When  $r = 6$ , J.-L. Nicolas determined (cf. [15]) the odd elements of  $\mathcal{A} = \mathcal{A}(1 + z + z^3 + z^4 + z^5)$ . His results (which will be stated in Section 2, Theorem 0) allowed to deduce a lower bound for the counting function of  $\mathcal{A}$ . In this paper, we will consider the case  $p = 31$  which satisfies  $r = 6$ . In  $\mathbb{F}_2[z]$ , we have

$$\frac{1 - z^{31}}{1 - z} = P^{(1)} P^{(2)} \dots P^{(6)}, \quad (1.13)$$

with

$$\begin{aligned} P^{(1)} &= 1 + z + z^3 + z^4 + z^5, \quad P^{(2)} = 1 + z + z^2 + z^4 + z^5, \quad P^{(3)} = 1 + z^2 + z^3 + z^4 + z^5, \\ P^{(4)} &= 1 + z + z^2 + z^3 + z^5, \quad P^{(5)} = 1 + z^2 + z^5, \quad P^{(6)} = 1 + z^3 + z^5. \end{aligned}$$

In fact, there are other primes  $p$  with  $r = 6$ . For instance,  $p = 223$  and  $p = 433$ .

In Section 2, for  $\mathcal{A} = \mathcal{A}(P^{(1)})$ , we evaluate the sum  $S_{\mathcal{A}}(m, k)$  which will lead to results of Section 3 determining the elements of the set  $\mathcal{A}$ . Section 4 will be devoted to the determination of an asymptotic estimate to the counting function  $A(x)$  of  $\mathcal{A}$ . Although, in this paper, the computations are only carried out for  $P = P^{(1)}$ , the results could probably be extended to any  $P^{(i)}$ ,  $1 \leq i \leq 6$ , and more generally, to any polynomial  $P$  of order  $p$  and such that  $r = 6$ .

**Notation.** We write  $a \bmod b$  for the remainder of the euclidean division of  $a$  by  $b$ . The ceiling of the real number  $x$  is denoted by

$$\lceil x \rceil = \inf\{n \in \mathbb{Z}, x \leq n\}.$$

## 2 The sum $S_{\mathcal{A}}(m, k)$ , $\mathcal{A} = \mathcal{A}(1 + z + z^3 + z^4 + z^5)$ .

From now on, we take  $\mathcal{A} = \mathcal{A}(P)$  with

$$P = P^{(1)} = 1 + z + z^3 + z^4 + z^5. \quad (2.1)$$

The order of  $P$  is  $\beta = 31$ . The smallest primitive root modulo 31 is 3 that we shall use as a generator of  $(\mathbb{Z}/31\mathbb{Z})^*$ . The order of 2 modulo 31 is  $s = 5$  so that

$$(\mathbb{Z}/31\mathbb{Z})/\langle 2 \rangle = \{O(3), O(3^2), \dots, O(3^6) = O(1), O(31)\}, \quad (2.2)$$

with

$$O(3^j) = \{2^k 3^j, 0 \leq k \leq 4\}, \quad 1 \leq j \leq 6 \quad (2.3)$$

and

$$O(31) = \{31n, n \in \mathbb{N}\}. \quad (2.4)$$

For  $k \geq 0$  and  $0 \leq j \leq 5$ , we define the integers  $u_{k,j}$  by

$$u_{k,j} = \sigma(\mathcal{A}, 2^k 3^j) \bmod 2^{k+1}. \quad (2.5)$$

**The Graeffe transformation.** Let  $\mathbb{K}$  be a field and  $\mathbb{K}[[z]]$  be the ring of formal power series with coefficients in  $\mathbb{K}$ . For an element

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

of this ring, the product

$$f(z)f(-z) = b_0 + b_1 z^2 + b_2 z^4 + \dots + b_n z^{2n} + \dots$$

is an even power series. We shall call  $\mathcal{G}(f)$  the series

$$\mathcal{G}(f)(z) = b_0 + b_1z + b_2z^2 + \dots + b_nz^n + \dots \quad (2.6)$$

It follows immediately from the above definition that for  $f, g \in \mathbb{K}[[z]]$ ,

$$\mathcal{G}(fg) = \mathcal{G}(f)\mathcal{G}(g). \quad (2.7)$$

Moreover if  $q$  is an odd integer and  $f(z) = 1 - z^q$ , we have  $\mathcal{G}(f) = f$ . We shall use the following notation for the iterates of  $f$  by  $\mathcal{G}$  :

$$f_{(0)} = f, \quad f_{(1)} = \mathcal{G}(f), \quad \dots, \quad f_{(k)} = \mathcal{G}(f_{(k-1)}) = \mathcal{G}^{(k)}(f). \quad (2.8)$$

More details about the Graeffe transformation are given in [6]. By making the logarithmic derivative of formula (1.3), we get (cf. [14]) :

$$\sum_{n=1}^{\infty} \sigma(\mathcal{A}, n) z^n = z \frac{F'(z)}{F(z)} \equiv z \frac{P'(z)}{P(z)} \pmod{2}, \quad (2.9)$$

which, by Propositions 2 and 3 of [6], leads to

$$\sum_{n=1}^{\infty} \sigma(\mathcal{A}, 2^k n) z^n \equiv z \frac{P'_{(k)}(z)}{P_{(k)}(z)} = \frac{z}{1 - z^{31}} (P'_{(k)}(z) W_{(k)}(z)) \pmod{2^{k+1}}, \quad (2.10)$$

with  $P'_{(k)}(z) = \frac{d}{dz}(P_{(k)}(z))$  and

$$W(z) = (1 - z)P^{(2)}(z) \dots P^{(6)}(z). \quad (2.11)$$

Formula (2.10) proves (1.11) with  $\beta = 31$ , and the computation of the  $k$ -th iterates  $P_{(k)}$  and  $W_{(k)}$  by the Graeffe transformation yields the value of  $\sigma(\mathcal{A}, 2^k n) \pmod{2^{k+1}}$ . For instance, for  $k = 11$ , we obtain :

$$u_{k,0} = 1183, \quad u_{k,1} = 1598, \quad u_{k,2} = 1554, \quad u_{k,3} = 845, \quad u_{k,4} = 264, \quad u_{k,5} = 701.$$

A divisor of  $2^k 3^j$  is either a divisor of  $2^{k-1} 3^j$  or a multiple of  $2^k$ . Therefore, from (2.5) and (1.6),  $u_{k,j} \equiv u_{k-1,j} \pmod{2^k}$  holds and the sequence  $(u_{k,j})_{k \geq 0}$  defines a 2-adic integer  $U_j$  satisfying for all  $k$ 's :

$$U_j \equiv u_{k,j} \pmod{2^{k+1}}, \quad 0 \leq j \leq 5. \quad (2.12)$$

It has been proved in [1] that the  $U_j$ 's are the roots of the polynomial

$$R(y) = y^6 - y^5 + 3y^4 - 11y^3 + 44y^2 - 36y + 32.$$

Note that  $R(y)^5$  is the resultant in  $z$  of  $\phi_{31}(z) = 1 + z + \dots + z^{30}$  and  $y + z + z^2 + z^4 + z^8 + z^{16}$ .

Let us set

$$\theta = U_0 = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^7 + 2^{10} + \dots$$

It turns out that the Galois group of  $R(y)$  is cyclic of order 6 and therefore the other roots  $U_1, \dots, U_5$  of  $R(y)$  are polynomials in  $\theta$ . With Maple, by factorizing  $R(y)$  on  $\mathbb{Q}[\theta]$  and using the values of  $u_{11,j}$ , we get

$$\begin{aligned} U_0 &= \theta \equiv 1183 \pmod{2^{11}} \\ U_1 &= \frac{1}{32}(3\theta^5 + 5\theta^3 - 36\theta^2 + 84\theta) \equiv 1598 \pmod{2^{11}} \\ U_2 &= \frac{1}{32}(-3\theta^5 - 5\theta^3 + 20\theta^2 - 100\theta) \equiv 1554 \pmod{2^{11}} \\ U_3 &= \frac{1}{32}(-\theta^5 - 7\theta^3 + 12\theta^2 - 44\theta + 32) \equiv 845 \pmod{2^{11}} \\ U_4 &= \frac{1}{32}(-\theta^5 + 4\theta^4 + \theta^3 + 24\theta^2 - 68\theta + 96) \equiv 264 \pmod{2^{11}} \\ U_5 &= \frac{1}{16}(\theta^5 - 2\theta^4 + 3\theta^3 - 10\theta^2 + 48\theta - 48) \equiv 701 \pmod{2^{11}}. \end{aligned} \quad (2.13)$$

For convenience, if  $j \in \mathbb{Z}$ , we shall set

$$U_j = U_{j \bmod 6}. \quad (2.14)$$

We define the completely additive function  $\ell : \mathbb{Z} \setminus 31\mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z}$  by

$$\ell(n) = j \quad \text{if } n \in O(3^j), \quad (2.15)$$

so that  $\ell(n_1 n_2) \equiv \ell(n_1) + \ell(n_2) \pmod{6}$ . We split the odd primes different from 31 into six classes according to the value of  $\ell$ . More precisely, for  $0 \leq j \leq 5$ ,

$$p \in \mathcal{P}_j \iff \ell(p) = j \iff p \equiv 2^k 3^j \pmod{31}, \quad k = 0, 1, 2, 3, 4. \quad (2.16)$$

We take  $L : \mathbb{N} \setminus 31\mathbb{N} \rightarrow \mathbb{N}_0$  to be the completely additive function defined on primes by

$$L(p) = \ell(p). \quad (2.17)$$

We define, for  $0 \leq j \leq 5$ , the additive function  $\omega_j : \mathbb{N} \longrightarrow \mathbb{N}_0$  by

$$\omega_j(n) = \sum_{p|n, p \in \mathcal{P}_j} 1 = \sum_{p|n, \ell(p)=j} 1, \quad (2.18)$$

and  $\omega(n) = \omega_0(n) + \dots + \omega_5(n) = \sum_{p|n} 1$ . We remind that additive functions vanish on 1.

From (2.5), (2.3), (1.11) and (2.12), it follows that if  $n = 2^k m \in O(3^j)$  (so that  $j = \ell(n) = \ell(m)$ ),

$$\sigma(\mathcal{A}, n) = \sigma(\mathcal{A}, 2^k m) \equiv U_{\ell(m)} \pmod{2^{k+1}}. \quad (2.19)$$

We may consider the 2-adic number

$$S(m) = S_{\mathcal{A}}(m) = \chi(\mathcal{A}, m) + 2\chi(\mathcal{A}, 2m) + \dots + 2^k \chi(\mathcal{A}, 2^k m) + \dots \quad (2.20)$$

satisfying from (1.8),

$$S(m) \equiv S_{\mathcal{A}}(m, k) \pmod{2^{k+1}}. \quad (2.21)$$

Then (1.10) implies for  $(m, 31) = 1$ ,

$$mS(m) = \sum_{d|\overline{m}} \mu(d) U_{\ell(\frac{m}{d})}. \quad (2.22)$$

If 31 divides  $m$ , it was proved in [3, (3.6)] that, for all  $k$ 's,

$$\sigma(\mathcal{A}, 2^k m) \equiv -5 \pmod{2^{k+1}}. \quad (2.23)$$

**Remark 1.** No element of  $\mathcal{A}$  has a prime factor in  $\mathcal{P}_0$ . This general result has been proved in [3], but we recall the proof on our example : let us assume that  $n = 2^k m \in \mathcal{A}$ , where  $m$  is an odd integer divisible by some prime  $p$  in  $\mathcal{P}_0$ , in other words  $\omega_0(m) \geq 1$ . (1.10) gives

$$\begin{aligned} mS_{\mathcal{A}}(m, k) &= \sum_{d|m} \mu(d) \sigma\left(\mathcal{A}, \frac{n}{d}\right) = \sum_{d|\overline{m}} \mu(d) \sigma\left(\mathcal{A}, 2^k \frac{m}{d}\right) \\ &= \sum_{d|\frac{\overline{m}}{p}} \mu(d) \sigma\left(\mathcal{A}, 2^k \frac{m}{d}\right) + \sum_{d|\frac{\overline{m}}{p}} \mu(pd) \sigma\left(\mathcal{A}, 2^k \frac{m}{pd}\right) \\ &= \sum_{d|\frac{\overline{m}}{p}} \mu(d) \left( \sigma\left(\mathcal{A}, 2^k \frac{m}{d}\right) - \sigma\left(\mathcal{A}, 2^k \frac{m}{pd}\right) \right). \end{aligned}$$



In the above sum, both  $\frac{m}{d}$  and  $\frac{m}{pd}$  are in the same orbit, so that from (1.11),  $\sigma(\mathcal{A}, 2^k \frac{m}{d}) \equiv \sigma(\mathcal{A}, 2^k \frac{m}{pd}) \pmod{2^{k+1}}$  and therefore  $mS_{\mathcal{A}}(m, k) \equiv 0 \pmod{2^{k+1}}$ . Since  $m$  is odd and (cf. (1.8))  $0 \leq S_{\mathcal{A}}(m, k) < 2^{k+1}$  then  $S_{\mathcal{A}}(m, k) = 0$ , so that by (1.8),  $2^h m \notin \mathcal{A}$ , for all  $0 \leq h \leq k$ .

In [15], J.-L. Nicolas has described the odd elements of  $\mathcal{A}$ . In fact, he obtained the following :

**Theorem 0.** ([15]) (a) *The odd elements of  $\mathcal{A}$  which are primes or powers of primes are of the form  $p^\lambda$ ,  $\lambda \geq 1$ , satisfying one of the following four conditions :*

$$\begin{aligned} p \in \mathcal{P}_1 \quad \text{and} \quad \lambda &\equiv 1, 3, 4, 5 \pmod{6} \\ p \in \mathcal{P}_2 \quad \text{and} \quad \lambda &\equiv 0, 1 \pmod{3} \\ p \in \mathcal{P}_4 \quad \text{and} \quad \lambda &\equiv 0, 1 \pmod{3} \\ p \in \mathcal{P}_5 \quad \text{and} \quad \lambda &\equiv 0, 2, 3, 4 \pmod{6}. \end{aligned}$$

(b) *No odd element of  $\mathcal{A}$  is a multiple of  $31^2$ . If  $m$  is odd,  $m \neq 1$ , and not a multiple of 31, then*

$$m \in \mathcal{A} \quad \text{if and only if} \quad 31m \in \mathcal{A}.$$

(c) *An odd element  $n \in \mathcal{A}$  satisfies  $\omega_0(n) = 0$  and  $\omega_3(n) = 0$  or 1 ; in other words,  $n$  is free of prime factor in  $\mathcal{P}_0$  and has at most one prime factor in  $\mathcal{P}_3$ .*

(d) *The odd elements of  $\mathcal{A}$  different from 1, not divisible by 31, which are not primes or powers of primes are exactly the odd  $n$ 's,  $n \neq 1$ , such that (where  $\bar{n} = \prod_{p|n} p$ ) :*

1.  $\omega_0(n) = 0$  and  $\omega_3(n) = 0$  or 1.
2. If  $\omega_3(n) = 1$  then  $\ell(n) + \ell(\bar{n}) \equiv 0$  or 1  $\pmod{3}$ .
3. If  $\omega_3(n) = 0$  and  $\omega_1(n) + \ell(n) - \ell(\bar{n})$  is even then

$$2\ell(n) - \ell(\bar{n}) \equiv 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \pmod{6}.$$

4. If  $\omega_3(n) = 0$  and  $\omega_1(n) + \ell(n) - \ell(\bar{n})$  is odd then

$$2\ell(n) - \ell(\bar{n}) \equiv 0 \text{ or } 4 \pmod{6}.$$

**Remark 2.** Point (b) of Theorem 0 can be improved in the following way : No element of  $\mathcal{A}$  is a multiple of  $31^2$ . Indeed, from (1.10), we have for  $m$  odd,  $k \geq 0$  and  $\tau \geq 2$ ,

$$31^\tau m S_{\mathcal{A}}(31^\tau m, k) = \sum_{d|31^\tau m} \mu(d) \sigma\left(\mathcal{A}, 2^k 31^\tau \frac{m}{d}\right) = \sum_{d|31^\tau m} \mu(d) \sigma\left(\mathcal{A}, 2^k 31^\tau \frac{m}{d}\right)$$

$$= \sum_{d|\overline{m}} \mu(d) \left\{ \sigma \left( \mathcal{A}, 2^k 31^\tau \frac{m}{d} \right) - \sigma \left( \mathcal{A}, 2^k 31^{\tau-1} \frac{m}{d} \right) \right\}.$$

Since  $31^\tau \frac{m}{d}$  and  $31^{\tau-1} \frac{m}{d}$  are in the same orbit  $O(31)$  then (1.11) and (2.23) give  $\sigma(\mathcal{A}, 2^k 31^\tau \frac{m}{d}) \equiv \sigma(\mathcal{A}, 2^k 31^{\tau-1} \frac{m}{d}) \equiv -5 \pmod{2^{k+1}}$ , so that we get  $S_{\mathcal{A}}(31^\tau m, k) \equiv 0 \pmod{2^{k+1}}$ . Hence, from (1.8),  $S_{\mathcal{A}}(31^\tau m, k) = 0$  and for all  $0 \leq h \leq k$  and all  $\tau \geq 2$ ,  $2^h 31^\tau m$  does not belong to  $\mathcal{A}$ .

In view of stating Theorem 1 which will extend Theorem 0, we shall need some notation. The radical  $\overline{m}$  of an odd integer  $m \neq 1$ , not divisible by 31 and free of prime factors belonging to  $\mathcal{P}_0$  will be written

$$\overline{m} = p_1 \dots p_{\omega_1} p_{\omega_1+1} \dots p_{\omega_1+\omega_2} p_{\omega_1+\omega_2+1} \dots \dots p_{\omega_1+\omega_2+\omega_3+\omega_4+1} \dots p_\omega, \quad (2.24)$$

where  $\ell(p_i) = j$  for  $\omega_1 + \dots + \omega_{j-1} + 1 \leq i \leq \omega_1 + \dots + \omega_j$ ,  $\omega_j = \omega_j(m) = \omega_j(\overline{m})$  and  $\omega = \omega(m) = \omega(\overline{m}) \geq 1$ . We define the additive functions from  $\mathbb{Z} \setminus 31\mathbb{Z}$  into  $\mathbb{Z}/12\mathbb{Z}$  :

$$\alpha = \alpha(m) = 2\omega_5 - 2\omega_1 + \omega_4 - \omega_2 \pmod{12}, \quad (2.25)$$

$$a = a(m) = \omega_5 - \omega_1 + \omega_2 - \omega_4 \pmod{12}. \quad (2.26)$$

Let  $(v_i)_{i \in \mathbb{Z}}$  be the periodic sequence of period 12 defined by

$$v_i = \begin{cases} \frac{2}{\sqrt{3}} \cos(i\frac{\pi}{6}) & \text{if } i \text{ is odd} \\ 2 \cos(i\frac{\pi}{6}) & \text{if } i \text{ is even.} \end{cases} \quad (2.27)$$

The values of  $(v_i)_{i \in \mathbb{Z}}$  are given by :

$i =$	0	1	2	3	4	5	6	7	8	9	10	11
$v_i =$	2	1	1	0	-1	-1	-2	-1	-1	0	1	1

Note that

$$v_{i+6} = -v_i, \quad (2.28)$$

$$v_i + v_{i+2} = \begin{cases} v_{i+1} & \text{if } i \text{ is odd} \\ 3v_{i+1} & \text{if } i \text{ is even,} \end{cases} \quad (2.29)$$

$$v_{2i} \equiv -2^i \pmod{3} \quad (2.30)$$

and

$$v_i \equiv v_{i+3} \equiv v_{2i} \pmod{2}. \quad (2.31)$$

From the  $U_j$ 's (cf. (2.12) and (2.13)), we introduce the following 2-adic integers :

$$E_i = \sum_{j=0}^5 v_{i+2j} U_j, \quad i \in \mathbb{Z}, \quad (2.32)$$

$$F_i = \sum_{j=0}^5 v_{i+4j} U_j, \quad i \in \mathbb{Z}, \quad (2.33)$$

$$G = \sum_{j=0}^5 (-1)^j U_j. \quad (2.34)$$

From (2.28), we have

$$E_{i+6} = -E_i, \quad E_{i+12} = E_i, \quad F_{i+6} = -F_i, \quad F_{i+12} = F_i. \quad (2.35)$$

From (2.29), it follows that, if  $i$  is odd,

$$E_i + E_{i+2} = E_{i+1}, \quad F_i + F_{i+2} = F_{i+1}, \quad (2.36)$$

while, if  $i$  is even,

$$E_i + E_{i+2} = 3E_{i+1}, \quad F_i + F_{i+2} = 3F_{i+1}, \quad (2.37)$$

The values of these numbers are given in the following array :

$Z$		$Z \bmod 2^{11}$
$E_0 =$	$\frac{1}{32}(11\theta^5 - 8\theta^4 + 29\theta^3 - 124\theta^2 + 500\theta - 256)$	1157
$E_1 =$	$\frac{1}{16}(3\theta^5 - 2\theta^4 + 9\theta^3 - 26\theta^2 + 136\theta - 64)$	1533
$E_2 =$	$3E_1 - E_0$	1394
$E_3 =$	$2E_1 - E_0$	1909
$E_4 =$	$3E_1 - 2E_0$	237
$E_5 =$	$E_1 - E_0$	376
$F_0 =$	$\frac{1}{32}(-3\theta^5 - 21\theta^3 + 36\theta^2 - 36\theta + 64)$	1987
$F_1 =$	$\frac{1}{32}(-3\theta^5 - 4\theta^4 - 13\theta^3 + 24\theta^2 - 28\theta - 64)$	166
$F_2 =$	$3F_1 - F_0$	559
$F_3 =$	$2F_1 - F_0$	393
$F_4 =$	$3F_1 - 2F_0$	620
$F_5 =$	$F_1 - F_0$	227
$G =$	$\frac{1}{4}(-\theta^5 + \theta^4 - \theta^3 + 11\theta^2 - 34\theta + 20)$	1905

TABLE 1

**Lemma 1.** *The polynomials  $(U_j)_{0 \leq j \leq 5}$  (cf. (2.13)) form a basis of  $\mathbb{Q}[\theta]$ . The polynomials  $E_0, E_1, F_0, F_1, G, U_0$  form an other basis of  $\mathbb{Q}[\theta]$ . For all  $i$ 's,  $E_i$  and  $F_i$  are linear combinations of respectively  $E_0$  and  $E_1$  and  $F_0$  and  $F_1$ .*

**Proof.** With Maple, in the basis  $1, \theta, \dots, \theta^5$ , we compute determinant  $(U_0, \dots, U_5) = \frac{1}{1024}$ . From (2.32), (2.33) and (2.34), the determinant of  $(E_0, E_1, F_0, F_1, G, U_0)$  in the basis  $U_0, U_1, \dots, U_5$  is equal to 12. The last point follows from (2.36) and (2.37).  $\square$

We have

**Theorem 1.** *Let  $m \neq 1$  be an odd integer not divisible by 31 with  $\overline{m}$  of the form (2.24). Under the above notation and the convention*

$$0^\omega = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{if } \omega > 0, \end{cases} \quad (2.38)$$

we have :

1) *The 2-adic integer  $S(m)$  defined by (2.20) satisfies*

$$\begin{aligned} mS(m) &= 2^{\omega_3-1} 3^{\lceil \frac{\omega_2+\omega_4}{2} - 1 \rceil} E_{\alpha-2\ell(m)} + \frac{0^{\omega_3}}{2} 3^{\lceil \frac{\omega}{2} - 1 \rceil} F_{\alpha-4\ell(m)} \\ &\quad + \frac{0^{\omega_2+\omega_4}}{3} 2^{\omega-1} (-1)^{\ell(m)} G. \end{aligned} \quad (2.39)$$

2) *The 2-adic integer  $S(31m)$  satisfies*

$$S(31m) = -31^{-1} S(m), \quad (2.40)$$

where  $31^{-1}$  is the inverse of 31 in  $\mathbb{Z}_2$ . In particular, for all  $k \in \{0, 1, 2, 3, 4\}$ , we have

$$2^k m \in \mathcal{A} \iff 31 \cdot 2^k m \in \mathcal{A},$$

since the inverse of 31 modulo  $2^{k+1}$  is  $-1$  for  $k \leq 4$ .

**Proof of Theorem 1, 1).** From (2.22), we have

$$mS(m) = \sum_{d|\overline{m}} \mu(d) U_{\ell(\frac{m}{d})} = \sum_{d|\overline{m}} \mu(d) U_{\ell(m)-\ell(d)}. \quad (2.41)$$

Further, (2.41) becomes

$$mS(m) = \sum_{j=0}^5 T(m, j) U_{\ell(m)-j} = \sum_{j=0}^5 T(m, \ell(m) - j) U_j, \quad (2.42)$$

with

$$T(m, j) = T(\overline{m}, j) = \sum_{d|\overline{m}, \ell(d) \equiv j \pmod{6}} \mu(d). \quad (2.43)$$

Therefore (2.39) will follow from (2.42) and from the following lemma :

**Lemma 2.** *The integer  $T(m, j)$  defined in (2.43) with the convention (2.38) and the definitions (2.18) and (2.24)-(2.27), for  $m \neq 1$ , is equal to*

$$\begin{aligned} T(m, j) = & 2^{\omega_3-1} 3^{\lceil \frac{\omega_2+\omega_4}{2} - 1 \rceil} v_{\alpha-2j} + \frac{0^{\omega_3}}{2} 3^{\lceil \frac{\omega}{2} - 1 \rceil} v_{a-4j} \\ & + 0^{\omega_2+\omega_4} \frac{(-1)^j}{3} 2^{\omega-1}. \end{aligned} \quad (2.44)$$

**Proof.** Let us introduce the polynomial

$$f(X) = (1 - X)^{\omega_1} (1 - X^2)^{\omega_2} \dots (1 - X^5)^{\omega_5} = \sum_{\nu \geq 0} f_\nu X^\nu. \quad (2.45)$$

If the five signs were plus instead of minus,  $f(X)$  would be the generating function of the partitions in at most  $\omega_1$  parts equal to 1, ..., at most  $\omega_5$  parts equal to 5. More generally, the polynomial

$$\tilde{f}(X) = \prod_{i=1}^{\omega} (1 + a_i X^{b_i}) = \sum_{\nu \geq 0} \tilde{f}_\nu X^\nu$$

is the generating function of

$$\tilde{f}_\nu = \sum_{\epsilon_1, \dots, \epsilon_\omega \in \{0,1\}, \sum_{i=1}^{\omega} \epsilon_i b_i = \nu} \prod_{i=1}^{\omega} a_i^{\epsilon_i}.$$

To the vector  $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_\omega) \in \mathbb{F}_2^\omega$ , we associate

$$d = \prod_{i=1}^{\omega} p_i^{\epsilon_i}, \quad \mu(d) = \prod_{i=1}^{\omega} (-1)^{\epsilon_i}, \quad L(d) = \sum_{i=1}^{\omega} \epsilon_i \ell(p_i)$$

where  $L$  is the arithmetic function defined by (2.17) and we get

$$f_\nu = \sum_{d|\overline{m}, L(d)=\nu} \mu(d), \quad (2.46)$$

Consequently, by setting  $\xi = \exp(\frac{i\pi}{3})$ , (2.43), (2.45) and (2.46) give

$$T(m, j) = \sum_{\nu, \nu \equiv j \pmod{6}} \sum_{d|\overline{m}, L(d)=\nu} \mu(d)$$

$$\begin{aligned}
&= \sum_{\nu \equiv j \pmod{6}} f_\nu = \frac{1}{6} \sum_{i=0}^5 \xi^{-ij} f(\xi^i) = \frac{1}{6} \sum_{i=1}^5 \xi^{-ij} f(\xi^i) \\
&= \frac{1}{6} \sum_{i=1}^5 \xi^{-ij} (1 - \xi^i)^{\omega_1} (1 - \xi^{2i})^{\omega_2} (1 - \xi^{3i})^{\omega_3} (1 - \xi^{4i})^{\omega_4} (1 - \xi^{5i})^{\omega_5}. \quad (2.47)
\end{aligned}$$

By observing that

$$\begin{aligned}
1 - \xi = \xi^5, \quad 1 - \xi^2 = \varrho = \sqrt{3}(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}), \quad 1 - \xi^3 = 2, \quad 1 - \xi^4 = \bar{\varrho}, \quad 1 - \xi^6 = 0,
\end{aligned}$$

the sum of the terms in  $i = 1$  and  $i = 5$  in (2.47), which are conjugate, is equal to

$$\frac{2}{6} \mathcal{R}(\xi^{-j} \xi^{5\omega_1} \varrho^{\omega_2} 2^{\omega_3} \bar{\varrho}^{\omega_4} \xi^{\omega_5}) = \frac{2^{\omega_3}}{3} \sqrt{3}^{\omega_2 + \omega_4} \cos \frac{\pi}{6} (2\omega_5 - 2\omega_1 + \omega_4 - \omega_2 - 2j). \quad (2.48)$$

Now, the contribution of the terms in  $i = 2$  and  $i = 4$  is

$$\begin{aligned}
\frac{2}{6} \mathcal{R}(\xi^{-2j} \varrho^{\omega_1} \bar{\varrho}^{\omega_2} 0^{\omega_3} \varrho^{\omega_4} \bar{\varrho}^{\omega_5}) &= 0^{\omega_3} \frac{\sqrt{3}^{\omega_1 + \omega_2 + \omega_4 + \omega_5}}{3} \cos \frac{\pi}{6} (\omega_2 + \omega_5 - \omega_1 - \omega_4 - 4j) \\
&= 0^{\omega_3} \frac{\sqrt{3}^\omega}{3} \cos \frac{\pi}{6} (\omega_2 + \omega_5 - \omega_1 - \omega_4 - 4j) \quad (2.49)
\end{aligned}$$

Finally, the term corresponding to  $i = 3$  in (2.47) is equal to

$$\frac{1}{6} (-1)^j 2^{\omega_1} 0^{\omega_2} 2^{\omega_3} 0^{\omega_4} 2^{\omega_5} = 0^{\omega_2 + \omega_4} \frac{(-1)^j}{6} 2^{\omega_1 + \omega_3 + \omega_5} = 0^{\omega_2 + \omega_4} \frac{(-1)^j}{6} 2^\omega. \quad (2.50)$$

Consequently, by using our notation (2.24)-(2.26), (2.47) becomes

$$\begin{aligned}
T(m, j) &= \frac{2^{\omega_3}}{3} \sqrt{3}^{\omega_2 + \omega_4} \cos \frac{\pi}{6} (\alpha - 2j) + 0^{\omega_3} \frac{\sqrt{3}^\omega}{3} \cos \frac{\pi}{6} (a - 4j) \\
&\quad + 0^{\omega_2 + \omega_4} \frac{(-1)^j}{6} 2^\omega. \quad (2.51)
\end{aligned}$$

Observing that  $\alpha - 2j$  has the same parity than  $\omega_2 + \omega_4$  and similarly for  $a - 4j$  and  $\omega$  (when  $\omega_0 = \omega_3 = 0$ ), via (2.27), we get (2.44).

**Proof of Theorem 1, 2).** For all  $k \geq 0$ , from (1.10), we have

$$\begin{aligned}
31mS_{\mathcal{A}}(31m, k) &= \sum_{d|31m} \mu(d) \sigma(\mathcal{A}, 31 \cdot 2^k \frac{m}{d}) = \sum_{d|31\bar{m}} \mu(d) \sigma(\mathcal{A}, 31 \cdot 2^k \frac{m}{d}) \\
&= \sum_{d|\bar{m}} \mu(d) \sigma(\mathcal{A}, 31 \cdot 2^k \frac{m}{d}) - \sum_{d|\bar{m}} \mu(d) \sigma(\mathcal{A}, 2^k \frac{m}{d}) \\
&= \sum_{d|\bar{m}} \mu(d) \sigma(\mathcal{A}, 31 \cdot 2^k \frac{m}{d}) - mS_{\mathcal{A}}(m, k). \quad (2.52)
\end{aligned}$$

Since for all  $d$  dividing  $\overline{m}$ ,  $31 \cdot 2^k \frac{m}{d} \in O(31)$  then, from (2.23),  $\sigma(\mathcal{A}, 31 \cdot 2^k \frac{m}{d}) \equiv \sigma(\mathcal{A}, 31 \cdot 2^k) \equiv -5 \pmod{2^{k+1}}$ , so that (2.52) gives

$$31mS_{\mathcal{A}}(31m, k) + mS_{\mathcal{A}}(m, k) \equiv -5 \sum_{d|\overline{m}} \mu(d) \pmod{2^{k+1}}. \quad (2.53)$$

Since  $\overline{m} \neq 1$ ,  $31mS_{\mathcal{A}}(31m, k) + mS_{\mathcal{A}}(m, k) \equiv 0 \pmod{2^{k+1}}$ . Recalling that  $m$  is odd, by using (2.20), (2.21) and their similar for  $S(31m)$ , we obtain the desired result.  $\square$

### 3 Elements of the set $\mathcal{A} = \mathcal{A}(1 + z + z^3 + z^4 + z^5)$ .

In this section, we will determine the elements of the set  $\mathcal{A}$  of the form  $n = 2^k 31^\tau m$ , where  $\overline{m} \neq 1$  satisfies (2.24) and  $\tau \in \{0, 1\}$ , since from Remark 2,  $2^k 31^\tau m \notin \mathcal{A}$  for all  $\tau \geq 2$ . The elements of the set  $\mathcal{A}(1 + z + z^3 + z^4 + z^5)$  of the form  $31^\tau 2^k$ ,  $\tau = 0$  or  $1$ , were shown in [1] to be solutions of 2-adic equations. More precisely, the following was proved in that paper.

1) The elements of the set  $\mathcal{A}(1 + z + z^3 + z^4 + z^5)$  of the form  $2^k$ ,  $k \geq 0$ , are given by the 2-adic solution

$$\sum_{k \geq 0} \chi(\mathcal{A}, 2^k) 2^k = S(1) = U_0 = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^7 + 2^{10} + 2^{11} + \dots$$

of the equation

$$y^6 - y^5 + 3y^4 - 11y^3 + 44y^2 - 36y + 32 = 0.$$

Note that  $S(1) = U_0$  follows from (2.22).

2) The elements of the set  $\mathcal{A}(1 + z + z^3 + z^4 + z^5)$  of the form  $31 \cdot 2^k$ ,  $k \geq 0$ , are given by the solution

$$\sum_{k \geq 0} \chi(\mathcal{A}, 31 \cdot 2^k) 2^k = S(31) = y = 2^2 + 2^5 + 2^{11} + \dots$$

of the equation

$$31^5 y^6 + 31^5 y^5 + 13 \cdot 31^4 y^4 + 91 \cdot 31^3 y^3 + 364 \cdot 31^2 y^2 + 796 \cdot 31 y + 752 = 0,$$

since, from (2.53) with  $m = 1$ , we have  $31S(31) = -5 - U_0$ , so that

$$S(31) = \frac{5 + U_0}{1 - 32} = (1 + 4 + U_0)(1 + 2^5 + 2^{10} + \dots) = 2^2 + 2^5 + 2^{11} + \dots$$

**Theorem 2.** Let  $m \neq 1$  be an odd integer not divisible by any prime  $p \in \mathcal{P}_0$  (cf. (2.16)) neither by  $31^2$ . Then the sum  $S(m)$  defined by (2.20) does not vanish. So we may introduce the 2-adic valuation of  $S(m)$  :

$$\gamma = \gamma(m) = v_2(S(m)). \quad (3.1)$$

Then, if 31 does not divide  $m$ , we have

$$\gamma(31m) = \gamma(m). \quad (3.2)$$

Let us assume now that  $m$  is coprime with 31. We shall use the quantities  $\omega_i = \omega_i(m)$  defined by (2.18),  $\ell(m)$ ,  $\alpha = \alpha(m)$ ,  $a = a(m)$  defined by (2.15), (2.25) and (2.26),

$$\alpha' = \alpha'(m) = \alpha - 2\ell(m) \pmod{12} = 2\omega_5 - 2\omega_1 + \omega_4 - \omega_2 - 2\ell(m) \pmod{12}, \quad (3.3)$$

$$a' = a'(m) = a - 4\ell(m) \pmod{12} = \omega_5 - \omega_1 + \omega_2 - \omega_4 - 4\ell(m) \pmod{12}, \quad (3.4)$$

$$\begin{aligned} t = t(m) &= \left\lfloor \frac{\omega_1 + \omega_5 + \omega_2 + \omega_4}{2} - 1 \right\rfloor - \left\lfloor \frac{\omega_2 + \omega_4}{2} - 1 \right\rfloor \\ &= \begin{cases} \left\lceil \frac{\omega_1 + \omega_5}{2} \right\rceil & \text{if } \omega_1 + \omega_5 \equiv \omega_2 + \omega_4 \equiv 1 \pmod{2} \\ \left\lceil \frac{\omega_1 + \omega_5}{2} \right\rceil - 1 & \text{if not.} \end{cases} \end{aligned} \quad (3.5)$$

We have :

(i) if  $\omega_3 \neq 0$  and  $\omega_2 + \omega_4 \neq 0$ , the value of  $\gamma = \gamma(m)$  is given by

$$\gamma = \begin{cases} \omega_3 - 1 & \text{if } \alpha' \equiv 0, 1, 3, 4 \pmod{6} \\ \omega_3 & \text{if } \alpha' \equiv 2 \pmod{6} \\ \omega_3 + 2 & \text{if } \alpha' \equiv 5 \pmod{6}. \end{cases}$$

(ii) If  $\omega_2 + \omega_4 = 0$  and  $\omega_3 \geq 1$ , we set  $\alpha'' = \alpha' + 6\ell(m) \pmod{12}$  and  $\delta(i) = v_2(E_i + 2^{v_2(E_i)}G)$  and we have

$$\begin{aligned} \text{if } \omega_1 + \omega_5 < v_2(E_{\alpha''}), & \quad \text{then } \gamma = \omega_3 - 1 + \omega_1 + \omega_5, \\ \text{if } \omega_1 + \omega_5 = v_2(E_{\alpha''}), & \quad \text{then } \gamma = \omega_3 - 1 + \delta(\alpha''), \\ \text{if } \omega_1 + \omega_5 > v_2(E_{\alpha''}), & \quad \text{then } \gamma = \omega_3 - 1 + v_2(E_{\alpha'').} \end{aligned}$$

(iii) If  $\omega_3 = 0$  and  $\omega_2 + \omega_4 \neq 0$ , we have

$$\gamma = -1 + v_2(E_{\alpha'} + 3^t F_{\alpha'}).$$



(iv) If  $\omega_3 = \omega_2 = \omega_4 = 0$  and  $\omega_1 + \omega_5 \neq 0$ , we have

$$\gamma = -1 + v_2(E_{\alpha'} + 3^t F_{\alpha'} + 2^{\omega_1 + \omega_5} (-1)^{\ell(m)} G).$$

**Proof.** We shall prove that  $S(m) \neq 0$  in each of the four cases above. Assuming  $S(m) \neq 0$ , it follows from Theorem 1, 2) that  $S(31m) \neq 0$  and that  $\gamma(31m) = \gamma(m)$ , which sets (3.2).

**Proof of Theorem 2 (i).** In this case, formula (2.39) reduces to

$$mS(m) = 2^{\omega_3 - 1} 3^{\lceil \frac{\omega_2 + \omega_4}{2} - 1 \rceil} E_{\alpha'}.$$

Since  $E_{\alpha'} \neq 0$ ,  $S(m)$  does not vanish; we have

$$\gamma = v_2(S(m)) = \omega_3 - 1 + v_2(E_{\alpha'})$$

and the result follows from the values of  $E_{\alpha'}$  modulo  $2^{11}$  given in Table 1.

**Proof of Theorem 2 (ii).** If  $\omega_2 + \omega_4 = 0$  and  $\omega_3 \neq 0$ , formula (2.39) becomes (since, cf. (2.35),  $E_{i+6} = -E_i$  holds)

$$mS(m) = \frac{2^{\omega_3 - 1}}{3} (E_{\alpha'} + 2^{\omega_1 + \omega_5} (-1)^{\ell(m)} G) = (-1)^{\ell(m)} \frac{2^{\omega_3 - 1}}{3} (E_{\alpha''} + 2^{\omega_1 + \omega_5} G).$$

As displayed in Table 1,  $E_i$  is a linear combination of  $E_0$  and  $E_1$  so that, from Lemma 1,  $S(m)$  does not vanish and  $\gamma = \omega_3 - 1 + v_2(E_{\alpha''} + 2^{\omega_1 + \omega_5} G)$ , whence the result. The values of  $v_2(E_i)$  and  $\delta(i)$  calculated from Table 1 are given below.

$i$	0	1	2	3	4	5	6	7	8	9	10	11
$v_2(E_i)$	0	0	1	0	0	3	0	0	1	0	0	3
$\delta(i)$	1	1	2	1	1	8	2	2	4	2	2	4

**Proof of Theorem 2 (iii).** If  $\omega_3 = 0$  and  $\omega_2 + \omega_4 \neq 0$  it follows, from (2.39) and the definition of  $t$  above, that

$$mS(m) = \frac{1}{2} 3^{\lceil \frac{\omega_2 + \omega_4}{2} - 1 \rceil} (E_{\alpha'} + 3^t F_{\alpha'}).$$

But  $E_i$  and  $F_i$  are non-zero linear combinations of, respectively,  $E_0$  and  $E_1$  and  $F_0$  and  $F_1$ ; by Lemma 1,  $E_{\alpha'} + 3^t F_{\alpha'}$  does not vanish and  $\gamma = -1 + v_2(E_{\alpha'} + 3^t F_{\alpha'})$ .

**Proof of Theorem 2 (iv).** If  $\omega_3 = \omega_2 = \omega_4 = 0$  and  $m \neq 1$ , formula (2.39) gives

$$mS(m) = \frac{1}{6} (E_{\alpha'} + 3^t F_{\alpha'} + 2^{\omega_1 + \omega_5} (-1)^{\ell(m)} G).$$

From Lemma 1, we obtain  $E_{\alpha'} + 3^t F_{\alpha'} + 2^{\omega_1 + \omega_5} (-1)^{\ell(m)} G \neq 0$ , which implies  $S(m) \neq 0$  and  $\gamma = -1 + v_2(E_{\alpha'} + 3^t F_{\alpha'} + 2^{\omega_1 + \omega_5} (-1)^{\ell(m)} G)$ .  $\square$

**Theorem 3.** *Let  $m$  be an odd integer satisfying  $m \neq 1$ ,  $(m, 31) = 1$ , and with  $\overline{m}$  of the form (2.24). Let  $\gamma = \gamma(m)$  as defined in Theorem 2 and  $Z(m)$  be the odd part of the right hand-side of (2.39), so that*

$$mS(m) = 2^{\gamma(m)} Z(m). \quad (3.6)$$

- (i) *If  $k < \gamma$ , then  $2^k m \notin \mathcal{A}$  and  $2^k 31m \notin \mathcal{A}$ .*
- (ii) *If  $k = \gamma$ , then  $2^k m \in \mathcal{A}$  and  $2^k 31m \in \mathcal{A}$ .*
- (iii) *If  $k = \gamma + r$ ,  $r \geq 1$ , then we set  $\mathcal{S}_r = \{2^r + 1, 2^r + 3, \dots, 2^{r+1} - 1\}$  and we have*

$$2^{\gamma+r} m \in \mathcal{A} \iff \exists l \in \mathcal{S}_r, m \equiv l^{-1} Z(m) \pmod{2^{r+1}},$$

$$2^{\gamma+r} 31m \in \mathcal{A} \iff \exists l \in \mathcal{S}_r, m \equiv -(31l)^{-1} Z(m) \pmod{2^{r+1}}.$$

**Proof of Theorem 3, (i).** We remind that  $m$  is odd and (cf. 2.21)  $S(m) \equiv S_{\mathcal{A}}(m, k) \pmod{2^{k+1}}$ . It is obvious from (3.6) that if  $\gamma > k$  then  $S_{\mathcal{A}}(m, k) \equiv 0 \pmod{2^{k+1}}$ . So that from (1.8),  $S_{\mathcal{A}}(m, k) = 0$  and  $2^h m \notin \mathcal{A}$ , for all  $h$ ,  $0 \leq h \leq k$ . To prove that  $2^k 31m \notin \mathcal{A}$ , it suffices to use this last result and (2.40) modulo  $2^{k+1}$ .

**Proof of Theorem 3, (ii).** If  $\gamma = k$  then the same arguments as above show that

$$mS_{\mathcal{A}}(m, k) \equiv 2^k Z(m) \pmod{2^{k+1}}.$$

So that, by using Theorem 3, (i) and (1.8), we obtain

$$2^k m \chi(\mathcal{A}, 2^k m) \equiv 2^k Z(m) \pmod{2^{k+1}}.$$

Since both  $m$  and  $Z(m)$  are odd, we get  $\chi(\mathcal{A}, 2^k m) \equiv 1 \pmod{2}$ , which shows that  $2^k m \in \mathcal{A}$ . Once again, to prove that  $2^k 31m \in \mathcal{A}$ , it suffices to use this last result and (2.40) modulo  $2^{k+1}$ .

**Proof of Theorem 3, (iii).** Let us set  $k = \gamma + r$ ,  $r \geq 1$ . (3.6) and (2.21) give

$$mS_{\mathcal{A}}(m, k) \equiv 2^{\gamma} Z(m) \pmod{2^{\gamma+r+1}}. \quad (3.7)$$

So that, by using Theorem 3, (i) and (ii), we get

$$m(2^{\gamma} + 2^{\gamma+1} \chi(\mathcal{A}, 2^{\gamma+1} m) + \dots + 2^{\gamma+r} \chi(\mathcal{A}, 2^{\gamma+r} m)) \equiv 2^{\gamma} Z(m) \pmod{2^{\gamma+r+1}},$$

which reduces to

$$m(1 + 2\chi(\mathcal{A}, 2^{\gamma+1} m) + \dots + 2^r \chi(\mathcal{A}, 2^{\gamma+r} m)) \equiv Z(m) \pmod{2^{r+1}}.$$

By observing that  $2^{\gamma+r}m \in \mathcal{A}$  if and only if  $l = 1 + 2\chi(\mathcal{A}, 2^{\gamma+1}m) + \dots + 2^r\chi(\mathcal{A}, 2^{\gamma+r}m)$  is an odd integer in  $\mathcal{S}_r$ , we obtain

$$2^{\gamma+r}m \in \mathcal{A} \iff m \equiv l^{-1}Z(m) \pmod{2^{r+1}}, \quad l \in \mathcal{S}_r.$$

To prove the similar result for  $2^{\gamma+r}31m$ , one uses the same method and (2.40) modulo  $2^{k+1}$ .  $\square$

## 4 The counting function.

In Theorem 4 below, we will determine an asymptotic estimate to the counting function  $A(x)$  (cf. (1.2)) of the set  $\mathcal{A} = \mathcal{A}(1+z+z^3+z^4+z^5)$ . The following lemmas will be needed.

**Lemma 3.** *Let  $K$  be any positive integer and  $x \geq 1$  be any real number. We have*

$$|\{n \leq x : \gcd(n, K) = 1\}| \leq 7 \frac{\varphi(K)}{K} x,$$

where  $\varphi$  is the Euler function.

**Proof.** This is a classical result from sieve theory : see Theorems 3 – 5 of [11].  $\square$

**Lemma 4.** (Mertens's formula) *Let  $\theta$  and  $\eta$  be two positive coprime integers. There exists an absolute constant  $C_1$  such that, for all  $x > 1$ ,*

$$\pi(x; \theta, \eta) = \prod_{p \leq x, p \equiv \theta \pmod{\eta}} \left(1 - \frac{1}{p}\right) \leq \frac{C_1}{(\log x)^{\frac{1}{\varphi(\eta)}}}.$$

**Proof.** For  $\theta$  and  $\eta$  fixed, Mertens's formula follows from the Prime Number Theorem in arithmetic progressions. It is proved in [9] that the constant  $C_1$  is absolute.  $\square$

**Lemma 5.** *For  $i \in \{2, 3, 4\}$ , let*

$$K_i = K_i(x) = \prod_{p \leq x, \ell(p) \in \{0, i\}} p = \prod_{p \leq x, p \in \mathcal{P}_0 \cup \mathcal{P}_i} p,$$

where  $\ell$ ,  $\mathcal{P}_0$  and  $\mathcal{P}_i$  are defined by (2.15)-(2.16). Then for  $x$  large enough,

$$|\{n : 1 \leq n \leq x, \gcd(n, K_i) = 1\}| = \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right).$$

**Proof.** By Lemma 3 and (2.16), we have

$$\begin{aligned} & |\{n : n \leq x, \gcd(n, K_i) = 1\}| \leq 7x \frac{\varphi(K_i)}{K_i} \\ & = 7x \prod_{0 \leq j \leq 4, \tau \in \{0, i\}} \prod_{p \leq x, p \equiv 2^j 3^\tau \pmod{31}} \left(1 - \frac{1}{p}\right). \end{aligned}$$

So that by Lemma 4, for all  $i \in \{2, 3, 4\}$  and  $x$  large enough,

$$|\{n : n \leq x, \gcd(n, K_i) = 1\}| \leq \frac{7C_1^{10}x}{(\log x)^{\frac{10}{\varphi(31)}}} = \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right). \quad \square$$

**Lemma 6.** Let  $r, u \in \mathbb{N}_0$ ,  $\ell$  and  $\alpha'$  be the functions defined by (2.15) and (3.3),  $\omega_j$  be the additive function given by (2.18). We take  $\xi$  to be a Dirichlet character modulo  $2^{r+1}$  with  $\xi_0$  as principal character and we let  $\varrho$  be the completely multiplicative function defined on primes  $p$  by

$$\varrho(p) = \begin{cases} 0 & \text{if } \ell(p) = 0 \text{ or } p = 31 \\ 1 & \text{otherwise.} \end{cases} \quad (4.1)$$

If  $y$  and  $z$  are respectively some  $2^u$ -th and 12-th roots of unity in  $\mathbb{C}$ , and if  $x$  is a real number  $> 1$ , we set

$$S_{y,z,\xi}(x) = \sum_{2^{\omega_3(n)}n \leq x} \varrho(n) \xi(n) y^{\omega_2(n) + \omega_4(n)} z^{\alpha'(n)}. \quad (4.2)$$

Then, when  $x$  tends to infinity, we have

- If  $\xi \neq \xi_0$ ,

$$S_{y,z,\xi}(x) = \mathcal{O}\left(x \frac{\log \log x}{(\log x)^2}\right). \quad (4.3)$$

- If  $\xi = \xi_0$ ,

$$S_{y,z,\xi_0}(x) = \frac{x}{(\log x)^{1-f_{y,z}(1)}} \left( \frac{H_{y,z,\xi_0}(1)C_{y,z}}{\Gamma(f_{y,z}(1))} + \mathcal{O}\left(\frac{\log \log x}{\log x}\right) \right), \quad (4.4)$$

where  $\Gamma$  is the Euler gamma function,

$$f_{y,z}(s) = \frac{1}{\varphi(31)} \sum_{1 \leq j \leq 5} \sum_{p, \ell(p)=j} g_{j,y,z}(s), \quad (4.5)$$

$$g_{1,y,z}(s) = z^8, \quad g_{2,y,z}(s) = yz^7, \quad g_{3,y,z}(s) = \frac{z^6}{2s}, \quad g_{4,y,z}(s) = yz^5, \quad g_{5,y,z}(s) = z^4, \quad (4.6)$$

$$H_{y,z,\xi}(s) = \prod_{1 \leq j \leq 5} \prod_{p, \ell(p)=j} \left( 1 + \frac{g_{j,y,z}(s)\xi(p)}{p^s - z^{-2j}\xi(p)} \right) \left( 1 - \frac{\xi(p)}{p^s} \right)^{g_{j,y,z}(s)}, \quad (4.7)$$

$$C_{y,z} = \prod_{1 \leq j \leq 5} \left\{ \prod_{p, \ell(p)=j} \left( 1 - \frac{1}{p} \right)^{-g_{j,y,z}(1)} \prod_p \left( 1 - \frac{1}{p} \right)^{\frac{g_{j,y,z}(1)}{30}} \right\}. \quad (4.8)$$

**Proof.** The evaluation of such sums is based, as we know, on the Selberg-Delange method. In [7], one finds an application towards direct results on such problems. In our case, to apply Theorem 1 of that paper, one should start with expanding, for complex number  $s$  with  $\Re s > 1$ , the Dirichlet series

$$F_{y,z,\xi}(s) = \sum_{n \geq 1} \frac{\varrho(n)\xi(n)y^{\omega_2(n)+\omega_4(n)}z^{\alpha'(n)}}{(2^{\omega_3(n)}n)^s}$$

in an Euler product given by

$$\begin{aligned} F_{y,z,\xi}(s) &= \prod_{1 \leq j \leq 5} \prod_{p, \ell(p)=j} \left( 1 + \sum_{m=1}^{\infty} \frac{\xi(p^m)y^{\omega_2(p^m)+\omega_4(p^m)}z^{\alpha'(p^m)}}{(2^{\omega_3(p^m)}p^m)^s} \right) \\ &= \prod_{1 \leq j \leq 5} \prod_{p, \ell(p)=j} \left( 1 + \frac{g_{j,y,z}(s)\xi(p)}{p^s - z^{-2j}\xi(p)} \right), \end{aligned}$$

which can be written

$$F_{y,z,\xi}(s) = H_{y,z,\xi}(s) \prod_{1 \leq j \leq 5} \prod_{p, \ell(p)=j} \left( 1 - \frac{\xi(p)}{p^s} \right)^{-g_{j,y,z}(s)},$$

where  $g_{j,y,z}(s)$  and  $H_{y,z,\xi}(s)$  are defined by (4.6) and (4.7). To complete the proof of Lemma 6, one has to show that  $H_{y,z,\xi}(s)$  is holomorphic for  $\Re s > \frac{1}{2}$  and, for  $y$  and  $z$  fixed, that  $H_{y,z,\xi}(s)$  is bounded for  $\Re s \geq \sigma_0 > \frac{1}{2}$ , which can be done by adapting the method given in [7] (Preuve du Théorème 2, p. 235).  $\square$

**Lemma 7.** *We keep the above notation and we let  $\mathcal{G}$  be the set of integers of the form  $n = 2^{\omega_3(m)}m$  with the following conditions :*

- $m$  odd and  $\gcd(m, 31) = 1$ ,
- $m = m_1 m_2 m_3 m_4 m_5$ , where all prime factors  $p$  of  $m_i$  satisfy  $\ell(p) = i$ .

If  $G(x)$  is the counting function of the set  $\mathcal{G}$  then, when  $x$  tends to infinity,

$$G(x) = \frac{Cx}{(\log x)^{1/4}} \left( 1 + \mathcal{O} \left( \frac{\log \log x}{\log x} \right) \right), \quad (4.9)$$

where

$$C = \frac{H_{1,1,\xi_0}(1)C_{1,1}}{\Gamma(f_{1,1}(1))} = 0.61568378..., \quad (4.10)$$

$H_{1,1,\xi_0}(1)$ ,  $C_{1,1}$  and  $f_{1,1}(1)$  are defined by (4.7), (4.8) and (4.5).

**Proof of Lemma 7.** We apply Lemma 6 with  $y = z = 1$ ,  $\xi = \xi_0$  and remark that  $G(x) = S_{1,1,\xi_0}(x)$ . By observing that  $(1 + \frac{1}{p-1})(1 - \frac{1}{p}) = 1$ , we have

$$\begin{aligned} H_{1,1,\xi_0}(1) &= \prod_{p \in \mathcal{P}_3} \left( 1 + \frac{1}{2(p-1)} \right) \left( 1 - \frac{1}{p} \right)^{\frac{1}{2}} = \prod_{p \in \mathcal{P}_3} \left( 1 - \frac{1}{2p} \right) \left( 1 - \frac{1}{p} \right)^{-\frac{1}{2}} \\ &\asymp 1.000479390466, \\ C_{1,1} &= \lim_{x \rightarrow \infty} \prod_{p \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_4 \cup \mathcal{P}_5, p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \in \mathcal{P}_3, p \leq x} \left( 1 - \frac{1}{p} \right)^{\frac{-1}{2}} \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{\frac{3}{4}} \\ &\asymp 0.75410767606. \end{aligned}$$

The numerical value of the above Eulerian products has been computed by the classical method already used and described in [7]. Since  $\Gamma(f_{1,1}(1)) = \Gamma(\frac{3}{4}) = 1.225416702465...$ , we get (4.10).  $\square$

**Lemma 8.** We keep the notation introduced in Lemmas 6 and 7. If  $(y, z) \in \{(1, 1), (-1, -1)\}$ , we have

$$S_{y,z,\xi_0}(x) = \frac{Cx}{(\log x)^{1/4}} \left( 1 + \mathcal{O} \left( \frac{\log \log x}{\log x} \right) \right), \quad (4.11)$$

while, if  $(y, z, \xi) \notin \{(1, 1, \xi_0), (-1, -1, \xi_0)\}$ , we have

$$S_{y,z,\xi}(x) = \mathcal{O}_r \left( \frac{x}{(\log x)^{1/4+2^{-2u-3}}} \right). \quad (4.12)$$

**Proof.** For  $y = z = 1$ , Formula (4.11) follows from Lemma 7. For  $y = z = -1$  (which does not occur for  $u = 0$ ), it follows from (4.4) and by observing that the values of  $g_{j,y,z}(s)$ ,  $f_{y,z}(s)$ ,  $H_{y,z,\xi}(s)$ ,  $C_{y,z}$  do not change when replacing  $y$  by  $-y$  and  $z$  by  $-z$ .

Let us define

$$M_{y,z} = \Re(f_{y,z}(1)) = \frac{1}{6} \Re(z^6(z^2 + z^{-2} + \frac{1}{2} + y(z + z^{-1}))).$$

When  $\xi \neq \xi_0$ , (4.3) implies (4.12) while, if  $\xi = \xi_0$ , it follows from (4.4) and from the inequality to be proved

$$M_{y,z} \leq \frac{3}{4} - \frac{1}{2^{2u+3}}, \quad (y, z) \notin \{(1, 1), (-1, -1)\}. \quad (4.13)$$

To show (4.13), let us first recall that  $z$  is a twelfth root of unity.

If  $z \neq \pm 1$ ,  $6f_{y,z}(1)$  is equal to one of the numbers  $-3/2 \pm y\sqrt{3}$ ,  $-1/2 \pm y$ ,  $3/2$  so that

$$M_{y,z} \leq |f_{y,z}(1)| \leq \frac{1}{6} \left( \frac{3}{2} + \sqrt{3} \right) < 0.55 \leq \frac{3}{4} - \frac{1}{2^{2u+3}}$$

for all  $u \geq 0$ , which proves (4.13).

If  $z = 1$  and  $y \neq 1$  (which implies  $u \geq 1$ ), we have

$$\Re y \leq \cos \frac{2\pi}{2^u} = 1 - 2 \sin^2 \frac{\pi}{2^u} \leq 1 - 2 \left( \frac{2}{\pi} \frac{\pi}{2^u} \right)^2 = 1 - \frac{8}{2^{2u}},$$

and

$$M_{y,1} = \frac{5}{12} + \frac{1}{3} \Re y \leq \frac{3}{4} - \frac{8}{3 \cdot 2^{2u}} < \frac{3}{4} - \frac{1}{2^{2u+3}}.$$

If  $z = -1$  and  $y \neq -1$ , (4.13) follows from the preceding case by observing that  $f_{y,z}(1) = f_{-y,-z}(1)$ , which completes the proof of (4.13).  $\square$

**Lemma 9.** *Let  $\mathcal{G}$  be the set defined in Lemma 7,  $\omega_j$  and  $\alpha'$  be the functions given by (2.18) and (3.3). For  $0 \leq j \leq 11$ ,  $r, u, \lambda, t \in \mathbb{N}_0$  such that  $t$  is odd, we let  $\mathcal{G}_{j,r,u,\lambda,t}$  be the set of integers  $n = 2^{\omega_3(m)}m$  in  $\mathcal{G}$  with the following conditions :*

- $\alpha'(m) \equiv j \pmod{12}$ ,
- $\omega_2(m) + \omega_4(m) \equiv \lambda \pmod{2^u}$ ,
- $m \equiv t \pmod{2^{r+1}}$ .

*If  $\rho$  is the function given by (4.1), the counting function  $G_{j,r,u,\lambda,t}(x)$  of the set  $\mathcal{G}_{j,r,u,\lambda,t}$  is equal to*

$$G_{j,r,u,\lambda,t}(x) = \sum_{\substack{2^{\omega_3(m)}m \leq x, \ m \equiv t \pmod{2^{r+1}} \\ \alpha'(m) \equiv j \pmod{12}, \ \omega_2(m) + \omega_4(m) \equiv \lambda \pmod{2^u}}} \rho(m).$$

If  $u \geq 1$  and  $\lambda \not\equiv j \pmod{2}$ ,  $\mathcal{G}_{j,r,u,\lambda,t}$  is empty while, if  $\lambda \equiv j \pmod{2}$ , when  $x$  tends to infinity, we have

$$G_{j,r,u,\lambda,t}(x) = \frac{C}{6 \cdot 2^{r+u}} \frac{x}{(\log x)^{\frac{1}{4}}} \left( 1 + \mathcal{O} \left( \frac{1}{(\log x)^{2-2u-3}} \right) \right),$$

where  $C$  is the constant given by (4.10).

If  $u = 0$ , then

$$G_{j,r,0,0,t}(x) = \frac{C}{12 \cdot 2^r} \frac{x}{(\log x)^{\frac{1}{4}}} \left( 1 + \mathcal{O} \left( \frac{1}{(\log x)^{1/8}} \right) \right),$$

**Proof.** If  $u \geq 1$ , it follows from (3.3) that  $\alpha'(m) \equiv \omega_2(m) + \omega_4(m) \pmod{2}$ ; therefore, if  $j \not\equiv \lambda \pmod{2}$ , then  $\mathcal{G}_{j,r,u,\lambda,t}$  is empty. Let us set

$$\zeta = e^{\frac{2i\pi}{2^u}}, \quad \mu = e^{\frac{2i\pi}{12}}.$$

By using the relations of orthogonality :

$$\sum_{j_2=0}^{11} \mu^{j_2 \alpha'(m)} \mu^{-j j_2} = \begin{cases} 12 & \text{if } \alpha' \equiv j \pmod{12} \\ 0 & \text{if not,} \end{cases}$$

$$\sum_{j_1=0}^{2^u-1} \zeta^{-\lambda j_1} \zeta^{j_1(\omega_2(m)+\omega_4(m))} = \begin{cases} 2^u & \text{if } \omega_2(m) + \omega_4(m) \equiv \lambda \pmod{2^u} \\ 0 & \text{if not,} \end{cases}$$

$$\sum_{\xi \bmod 2^{r+1}} \bar{\xi}(t) \xi(m) = \begin{cases} \varphi(2^{r+1}) = 2^r & \text{if } m \equiv t \pmod{2^{r+1}} \\ 0 & \text{if not,} \end{cases}$$

we get

$$G_{j,r,u,\lambda,t}(x) = \frac{1}{12 \cdot 2^{r+u}} \sum_{\xi \bmod 2^{r+1}} \sum_{j_1=0}^{2^u-1} \sum_{j_2=0}^{11} \bar{\xi}(t) \zeta^{-\lambda j_1} \mu^{-j j_2} S_{\zeta^{j_1}, \mu^{j_2}, \xi}(x).$$

In the above triple sums, the main contribution comes from  $S_{1,1,\xi_0}(x)$  and  $S_{-1,-1,\xi_0}(x)$ , and the result follows from (4.11) and (4.12).

If  $u = 0$ , we have

$$G_{j,r,0,0,t}(x) = \frac{1}{12 \cdot 2^r} \sum_{\xi \bmod 2^{r+1}} \sum_{j_2=0}^{11} \bar{\xi}(t) \mu^{-j j_2} S_{1, \mu^{j_2}, \xi}(x)$$

and, again, the result follows from Lemma 8.



**Theorem 4.** Let  $\mathcal{A} = \mathcal{A}(1 + z + z^3 + z^4 + z^5)$  be the set given by (1.3) and  $A(x)$  be its counting function. When  $x \rightarrow \infty$ , we have

$$A(x) \sim \kappa \frac{x}{(\log x)^{\frac{1}{4}}},$$

where  $\kappa = \frac{74}{31}C = 1.469696766\dots$  and  $C$  is the constant of Lemma 7 defined by (4.10).

**Proof of Theorem 4.** Let us define the sets  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$  containing the elements  $n = 2^k m$  ( $m$  odd) of  $\mathcal{A}$  with the restrictions :

$$\begin{aligned} \mathcal{A}_1 : & \quad \omega_3(m) \neq 0 \text{ and } \omega_2(m) + \omega_4(m) \neq 0 \\ \mathcal{A}_2 : & \quad \omega_3(m) \neq 0 \text{ and } \omega_2(m) = \omega_4(m) = 0 \\ \mathcal{A}_3 : & \quad \omega_3(m) = 0 \text{ and } \omega_2(m) + \omega_4(m) \neq 0 \\ \mathcal{A}_4 : & \quad \omega_2(m) = \omega_3(m) = \omega_4(m) = 0. \end{aligned}$$

We have

$$A(x) = A_1(x) + A_2(x) + A_3(x) + A_4(x). \quad (4.14)$$

Further, for  $i = 2, 3, 4$ , it follows from Lemma 5 that  $A_i(x) = \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$  and therefore

$$A(x) = A_1(x) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right). \quad (4.15)$$

Now, we split  $\mathcal{A}_1$  in two parts  $\mathcal{B}$  and  $\widehat{\mathcal{B}}$  by putting in  $\mathcal{B}$  the elements  $n \in \mathcal{A}_1$  which are coprime with 31 and in  $\widehat{\mathcal{B}}$  the elements  $n \in \mathcal{A}_1$  which are multiples of 31. Let us recall that, from Remark 2, no element of  $\mathcal{A}$  is a multiple of  $31^2$ . Therefore,

$$A_1(x) = \mathcal{B}(x) + \widehat{\mathcal{B}}(x) \quad (4.16)$$

with

$$\mathcal{B}(x) = \sum_{n=2^k m \in \mathcal{A}_1, n \leq x} \rho(m), \quad \widehat{\mathcal{B}}(x) = \sum_{n=2^k 31m \in \mathcal{A}_1, n \leq x} \rho(m). \quad (4.17)$$

Let us consider  $\mathcal{B}(x)$ ; the case of  $\widehat{\mathcal{B}}$  will be similar. We define

$$\nu_i = v_2(E_i) - 1 = \begin{cases} -1 & \text{if } i \equiv 0, 1, 3, 4 \pmod{6} \\ 0 & \text{if } i \equiv 2 \pmod{6} \\ 2 & \text{if } i \equiv 5 \pmod{6} \end{cases} \quad (4.18)$$

so that, if  $\widehat{E}_i$  is the odd part of  $E_i$  (cf. (2.32) and Table 1), we have

$$\widehat{E}_i = 2^{-1-\nu_i} E_i. \quad (4.19)$$

In view of Theorem 2 (i), if  $i = \alpha'(m) \bmod 12$  then

$$\gamma(m) - \omega_3(m) = \nu_i. \quad (4.20)$$

Further, an element  $n = 2^k m$  ( $m$  odd) belonging to  $\mathcal{A}_1$  is said of index  $r \geq 0$  if  $k = \gamma(m) + r$ . For  $r \geq 0$  and  $0 \leq i \leq 11$ ,

$$T_r^{(i)}(x) = \sum_{\substack{n=2^{\gamma(m)+r} m \in \mathcal{A}_1, \\ \alpha'(m) \equiv i \pmod{12}, \\ n \leq x}} \rho(m) = \sum_{\substack{n=2^{\gamma(m)+r} m \in \mathcal{A}_1, \\ \alpha'(m) \equiv i \pmod{12}, \\ 2^{\omega_3(m)} m \leq 2^{-r-\nu_i} x}} \rho(m) \quad (4.21)$$

will count the number of elements of  $\mathcal{A}_1$  up to  $x$  of index  $r$  and satisfying  $\alpha'(m) \equiv i \pmod{12}$ , so that

$$\mathcal{B}(x) = \sum_{r \geq 0} \sum_{i=0}^{11} T_r^{(i)}(x). \quad (4.22)$$

Since  $\gamma(m) \geq 0$ , from the first equality in (4.21), each  $n$  counted in  $T_r^{(i)}(x)$  is a multiple of  $2^r$ , hence the trivial upper bound

$$\sum_{i=0}^{11} T_r^{(i)}(x) \leq \frac{x}{2^r}. \quad (4.23)$$

Since  $\nu_i \geq -1$ , the second equality in (4.21) implies

$$\sum_{i=0}^{11} T_r^{(i)}(x) \leq G(2^{1-r} x) \quad (4.24)$$

with  $G$  defined in Lemma 7. Moreover, from Lemma 7, there exists an absolute constant  $K$  such that, for  $x \geq 3$ ,

$$G(x) \leq K \frac{x}{(\log x)^{\frac{1}{4}}}. \quad (4.25)$$

Now, let  $R$  be a large but fixed integer;  $R'$  is defined in terms of  $x$  by  $2^{R'-1} \leq \sqrt{x} < 2^{R'}$  and  $R'' = \frac{\log x}{\log 2}$ . Since  $T_r^{(i)}(x)$  is a non-negative integer, (4.23) implies that  $T_r^{(i)}(x) = 0$  for  $r > R''$ . If  $x$  is large enough,  $R < R' < R''$  holds. Setting

$$\mathcal{B}_R(x) = \sum_{r=0}^R \sum_{i=0}^{11} T_r^{(i)}(x), \quad (4.26)$$

from (4.22), we have

$$\mathcal{B}(x) - \mathcal{B}_R(x) = S' + S'',$$

with

$$S' = \sum_{r=R+1}^{R'} \sum_{i=0}^{11} T_r^{(i)}(x), \quad S''' = \sum_{r=R'+1}^{R''} \sum_{i=0}^{11} T_r^{(i)}(x).$$

The definition of  $R'$  and (4.23) yield

$$S''' \leq \sum_{r=R'+1}^{R''} \frac{x}{2^r} \leq \sum_{r=R'+1}^{\infty} \frac{x}{2^r} = \frac{x}{2^{R'}} \leq \sqrt{x},$$

while (4.24), (4.25) and the definition of  $R'$  give

$$\begin{aligned} S' &\leq \sum_{r=R+1}^{R'} G\left(\frac{x}{2^{r-1}}\right) \leq \sum_{r=R+1}^{R'} \frac{2Kx}{2^r \left(\log \frac{x}{2^{R'-1}}\right)^{\frac{1}{4}}} \\ &\leq \frac{2^{\frac{5}{4}}Kx}{(\log x)^{\frac{1}{4}}} \sum_{r=R+1}^{R'} \frac{1}{2^r} \leq \frac{3Kx}{2^R(\log x)^{\frac{1}{4}}}, \end{aligned}$$

so that, for  $x$  large enough, we have

$$0 \leq \mathcal{B}(x) - \mathcal{B}_R(x) \leq \sqrt{x} + \frac{3Kx}{2^R(\log x)^{\frac{1}{4}}}. \quad (4.27)$$

We now have to evaluate  $T_r^{(i)}(x)$ ; we shall distinguish two cases,  $r = 0$  and  $r \geq 1$ .

**Calculation of  $T_0^{(i)}(x)$ .**

From (4.21), we have

$$T_0^{(i)}(x) = \sum_{\substack{n=2^{\gamma(m)}m \in \mathcal{A}_1, n \leq x \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m) = \sum_{\substack{n=2^{\gamma(m)}m \in \mathcal{A}, n \leq x, \omega_3 \neq 0, \omega_2 + \omega_4 \neq 0 \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m).$$

From Theorem 3, we know that  $2^{\gamma(m)}m \in \mathcal{A}$ . Hence,

$$T_0^{(i)}(x) = \sum_{\substack{2^{\gamma(m)}m \leq x, \omega_3 \neq 0, \omega_2 + \omega_4 \neq 0 \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m),$$

which, by use of (4.20), gives

$$T_0^{(i)}(x) = \sum_{\substack{2^{\omega_3(m)}m \leq 2^{-\nu_i}x, \omega_3 \neq 0, \omega_2 + \omega_4 \neq 0 \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m).$$

But, at the cost of an error term  $\mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$ , Lemma 5 allows us to remove the conditions  $\omega_3 \neq 0$ ,  $\omega_2 + \omega_4 \neq 0$ , and to get from the second part of Lemma 9,

$$\begin{aligned} T_0^{(i)}(x) &= G_{i,0,0,0,1}\left(\frac{x}{2^{\nu_i}}\right) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right) \\ &= \frac{C}{12} \frac{x}{2^{\nu_i} (\log x)^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{(\log x)^{1/12}}\right)\right). \end{aligned} \quad (4.28)$$

**Calculation of  $T_r^{(i)}(x)$  for  $r \geq 1$ .**

Under the conditions  $\omega_3 \neq 0$  and  $\omega_2 + \omega_4 \neq 0$ , from (3.6), (2.39), (3.3), (4.19) and (4.20), we get

$$Z(m) = 3^{\lceil \frac{\omega_2 + \omega_4}{2} - 1 \rceil} \widehat{E}_{\alpha'(m)}.$$

From (4.21), it follows that

$$T_r^{(i)}(x) = \sum_{\substack{n=2^{\gamma(m)+r} m \in \mathcal{A}, n \leq x, \omega_3 \neq 0, \omega_2 + \omega_4 \neq 0 \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m).$$

Now, by Theorem 3, we know that  $2^{\gamma(m)+r} m$  belongs to  $\mathcal{A}$  if there is some  $l \in \mathcal{S}_r = \{2^r + 1, \dots, 2^{r+1} - 1\}$  such that  $m \equiv l^{-1} Z(m) \pmod{2^{r+1}}$ . Note that the order of 3 modulo  $2^{r+1}$  is  $2^{r-1}$  if  $r \geq 2$  and  $2^r$  if  $r = 1$ . We choose

$$u = r + 1$$

so that  $\omega_2 + \omega_4 \equiv \lambda \pmod{2^{r+1}}$  implies  $3^{\lceil \frac{\lambda}{2} - 1 \rceil} \equiv 3^{\lceil \frac{\omega_2 + \omega_4}{2} - 1 \rceil} \pmod{2^{r+1}}$ . Therefore, we have

$$T_r^{(i)}(x) = \sum_{l \in \mathcal{S}_r} \sum_{\lambda=0}^{2^{r+1}-1} \sum_{\substack{2^{\omega_3(m)} m \leq 2^{-\nu_i-r} x, \omega_3 \neq 0, \omega_2 + \omega_4 \neq 0 \\ \alpha'(m) \equiv i \pmod{12}, \omega_2 + \omega_4 \equiv \lambda \pmod{2^{r+1}} \\ m \equiv l^{-1} 3^{\lceil \frac{\lambda}{2} - 1 \rceil} \widehat{E}_i \pmod{2^{r+1}}}} \rho(m).$$

As in the case  $r = 0$ , we can remove the conditions  $\omega_3 \neq 0$  and  $\omega_2 + \omega_4 \neq 0$  in the last sum by adding a  $\mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$  error term, and we get by Lemma 9 for  $r$  fixed

$$T_r^{(i)}(x) = \sum_{l \in \mathcal{S}_r} \sum_{\substack{\lambda=0 \\ \lambda \equiv i \pmod{2}}}^{2^{r+1}-1} G_{i,r,r+1,\lambda,l^{-1} 3^{\lceil \frac{\lambda}{2} - 1 \rceil} \widehat{E}_i} \left(\frac{x}{2^{\nu_i+r}}\right) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$$

$$= \frac{C}{24} \frac{x}{2^{\nu_i+r} (\log x)^{\frac{1}{4}}} \left( 1 + \mathcal{O} \left( \frac{1}{(\log x)^{2-2r-5}} \right) \right). \quad (4.29)$$

From (4.26), (4.28), (4.29) and (4.18), we have

$$\begin{aligned} \mathcal{B}_R(x) &= \frac{Cx}{12(\log x)^{\frac{1}{4}}} \left( \left( \sum_{i=0}^{11} \frac{1}{2^{\nu_i}} \right) \left( 1 + \frac{1}{2} \sum_{r=1}^R \frac{1}{2^r} \right) + \mathcal{O} \left( \frac{1}{(\log x)^{2-2R-5}} \right) \right) \\ &= \frac{37}{24} \frac{Cx}{(\log x)^{\frac{1}{4}}} \left( \frac{3}{2} - \frac{1}{2^R} \right) \left( 1 + \mathcal{O} \left( \frac{1}{(\log x)^{2-2R-5}} \right) \right). \end{aligned}$$

By making  $R$  going to infinity, the above equality together with (4.27) show that

$$\mathcal{B}(x) \sim \frac{37}{16} \frac{Cx}{(\log x)^{\frac{1}{4}}}, \quad x \rightarrow \infty. \quad (4.30)$$

In a similar way, we can show that  $\widehat{\mathcal{B}}(x)$  defined in (4.17) satisfies

$$\widehat{\mathcal{B}}(x) \sim \frac{1}{31} \mathcal{B}(x) \sim \frac{37}{16 \cdot 31} \frac{x}{(\log x)^{\frac{1}{4}}}$$

which, with (4.16) and (4.15), completes the proof of Theorem 4 with

$$\kappa = \frac{37}{16} \left( 1 + \frac{1}{31} \right) C = \frac{74}{31} C = 1.469696766\dots$$

### Numerical computation of $A(x)$ .

There are three ways to compute  $A(x)$ . The first one uses the definition of  $\mathcal{A}$  and simultaneously calculates the number of partitions  $p(\mathcal{A}, n)$  for  $n \leq x$ ; it is rather slow. The second one is based on the relation (1.10) and the congruences (2.19) and (2.23) satisfied by  $\sigma(\mathcal{A}, n)$ . The third one calculates  $\omega_j(n)$ ,  $0 \leq j \leq 5$ , in view of applying Theorem 1. The two last methods can be encoded in a sieving process

The following table displays the values of  $A(x)$ ,  $A_1(x)$ , ...,  $A_4(x)$  as defined in (4.14) and also

$$c(x) = \frac{A(x)(\log x)^{\frac{1}{4}}}{x}, \quad c_1(x) = \frac{A_1(x)(\log x)^{\frac{1}{4}}}{x}.$$

It seems that  $c(x)$  and  $c_1(x)$  converge very slowly to  $\kappa = 1.469696766\dots$ , which is impossible to guess from the table.

$x$	$A(x)$	$c(x)$	$A_1(x)$	$c_1(x)$	$A_2(x)$	$A_3(x)$	$A_4(x)$
$10^3$	480	0.7782	20	0.032	44	233	183
$10^4$	4543	0.7914	361	0.063	532	2294	1356
$10^5$	43023	0.7925	5087	0.094	5361	21810	10765
$10^6$	411764	0.7939	60565	0.117	52344	208633	90222
$10^7$	3981774	0.7978	680728	0.136	506199	2007168	787679
$10^8$	38719773	0.8022	7403138	0.153	4887357	19390529	7038749

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### References

- [1] N. Baccar, Sets with even partition functions and 2-adic integers, *Periodica Math. Hung.*, Vol. 55 (2), 2007, pp. 177-193.
- [2] N. Baccar and F. Ben Saïd, On sets such that the partition function is even from a certain point on, to appear in *International Journal of Number Theory*, 2009.
- [3] N. Baccar, F. Ben Saïd and A. Zekraoui, On the divisor function of sets with even partition functions, *Acta Math. Hungarica*, 112 (1-2) (2006), 25-37.
- [4] F. Ben Saïd, On a conjecture of Nicolas-Sárközy about partitions, *Journal of Number Theory*, 95 (2002), 209-226.
- [5] F. Ben Saïd, On some sets with even partition function, *The Ramanujan Journal*, 9, 2005, 63-75.
- [6] F. Ben Saïd and J.-L. Nicolas, Sets of parts such that the partition function is even, *Acta Arithmetica*, 106 (2003), 183-196.
- [7] F. Ben Saïd and J.-L. Nicolas, Sur une application de la formule de Selberg-Delange, *Colloquium Mathematicum*, vol. 98, n° 2, (2003), 223-247.
- [8] F. Ben Saïd and J.-L. Nicolas, Even partition functions, *Séminaire Lotharingien de Combinatoire* (<http://www.mat.univie.ac.at/~slc/>), 46 (2002), B 46i.

- [9] F. Ben Saïd, H. Lahouar and J.-L. Nicolas, On the counting function of the sets of parts such that the partition function takes even values for  $n$  large enough, *Discrete Mathematics*, **306** (2006), 1089-1096.
- [10] P. M. Cohn, *Algebra*, Volume 1, Second Edition, John Wiley and Sons Ltd, 1988.
- [11] H. Halberstam, H.-E. Richert, *Sieve methods*, Academic Press, New York, 1974.
- [12] H. Lahouar, Fonctions de partitions à parité périodique, *European Journal of Combinatorics*, **24** (2003), 1089-1096.
- [13] R. Lidl and H. Niederreiter, *Introduction to finite fields and their applications*, Cambridge University Press, revised edition, 1994.
- [14] J.-L. Nicolas, I.Z. Ruzsa and A. Sárközy, On the parity of additive representation functions, *J. Number Theory* **73** (1998), 292-317.
- [15] J.-L. Nicolas, On the parity of generalized partition functions II, *Periodica Mathematica Hungarica*, **43** (2001), 177-189.