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# Clifford Fourier Transform for Color Image Processing

Thomas Batard, Michel Berthier and Christophe Saint-Jean

**Abstract** The aim of this paper is to define a Clifford Fourier transform that is suitable for color image spectral analysis. There have been many attempts to define such a transformation using quaternions or Clifford algebras. We focus here on a geometric approach using group actions. The idea is to generalize the usual definition based on the characters of abelian groups by considering group morphisms from  $\mathbb{R}^2$  to spinor groups  $Spin(3)$  and  $Spin(4)$ . The transformation we propose is parameterized by a bivector, the choice of which is related to the application to be treated. A general definition for 4D signal defined on the plane is also given; for particular choice of spinors it coincides with the definition of T. Bülöw.

## 1 Introduction

During the last years several attempts have been made to generalize the classical approach of scalar signal processing with the Fourier transform to higher dimensional signals. The reader will find a detailed overview of the related works at the beginning of [1]. We only mention in this introduction some of the approaches investigated by several authors.

Motivated by the spectral analysis of color images, S. Sangwine and T. Ell have proposed in [12] and [5] a generalization based on the use of quaternions: a color corresponds to an imaginary quaternion and the imaginary complex  $i$  is replaced by

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the unit quaternion  $\mu$  coding the grey axis. A quaternionic definition is also given by T. Bülöw and G. Sommer in the context of analytic signals, for signals defined on the plane and with values in the algebra  $\mathbb{H}$  of quaternions [3]. Concerning analytic signals, let us mention the work of M. Felsberg who makes use of the Clifford algebras  $\mathbb{R}_{2,0}$  and  $\mathbb{R}_{3,0}$  to define an appropriate Clifford Fourier transform [6].

A generalization in the Clifford algebras context appears also in J. Ebling and G. Scheuermann [4]. The authors mainly use their transformation to analyze frequencies of vector fields. Using the same Fourier kernel, B. Mawardi and E. Hitzer obtain an uncertainty principle for  $\mathbb{R}_{3,0}$  multivector functions [10]. The reader may find in [1] definitions of Clifford Fourier transform and Clifford Gabor filters based on the Dirac operator and Clifford analysis.

One could ask the reason why to propose a new generalization. An important thing when dealing with Fourier transform is its link with group representations. We then recall in section 2 the usual definition of the Fourier transform of a function defined on an abelian Lie group by means of the characters of the group. The definition we propose in section 4 relies mainly on the generalization of the notion of characters, that is why we study in section 3 the group morphisms from  $\mathbb{R}^2$  to  $Spin(3)$  and  $Spin(4)$ . These morphisms help to understand the behavior of the Fourier transform with respect to well chosen spinors. We treat in section 5 three applications corresponding to specific bivectors of  $\mathbb{R}_{3,0}$ . They consist in filtering frequencies according to color, hue and chrominance part of a given color. In section 6, a definition of the Clifford Fourier transform of a 4D signal on  $\mathbb{R}^2$  is proposed that coincides, for a particular choice of spinors, with the definition of T. Bülöw [2].

## 2 Fourier transform and group actions

Let us recall briefly some basic ideas related to the group approach of the definition of the Fourier transform. Details can be found in [15], see also [14] for examples of applications to Fourier descriptors.

Let  $G$  be a Lie group. The Pontryagin dual of  $G$ , denoted  $\widehat{G}$ , is the set of equivalence classes of unitary irreducible representations of  $G$ . It appears that if  $G$  is abelian, every irreducible unitary representation of  $G$  is of dimension 1, i. e. is a continuous group morphism from  $G$  in  $S^1$ . This is precisely the definition of a character. It is well known that the characters of  $\mathbb{R}$ , resp.  $\mathbb{R}^2$ , are given by

$$x \longmapsto e^{iax}$$

for  $a$  in  $\mathbb{R}$ , resp. by

$$(x, y) \longmapsto e^{i(ux+vy)}$$

with  $u$  and  $v$  real. This shows that  $\widehat{\mathbb{R}} = \mathbb{R}$  and  $\widehat{\mathbb{R}^2} = \mathbb{R}^2$ . The characters of  $SO(2)$  are the group morphisms

$$\theta \longmapsto e^{in\theta}$$

for  $n \in \mathbb{Z}$  and the corresponding Pontryagin dual is  $\mathbb{Z}$ . In the general case (provided that  $G$  is unimodular), the Fourier transform of a function  $f \in L^2(G; \mathbb{C})$  is defined on  $\widehat{G}$  by

$$\widehat{f}(\varphi) = \int_G f(x) \varphi(x^{-1}) d\nu(x)$$

(for  $\nu$  a well chosen invariant measure on  $G$ ). Applying this formula to the case  $G = \mathbb{R}$ , resp.  $G = SO(2)$ , leads to the usual definition of the Fourier transform, resp. Fourier coefficients.

Let us mention that if  $f$  is a real function on  $\mathbb{R}$ , then its Fourier transform can be written in the Clifford algebra language as

$$\begin{aligned} \widehat{f}(a) = \int_{\mathbb{R}} [\cos(ax/2)1 + \sin(ax/2)e_1e_2] (f(x)e_1 + 0e_2) \\ [\cos(-ax/2)1 + \sin(-ax/2)e_1e_2] dx \end{aligned}$$

using the fact that  $Spin(2) = \{a1 + be_1e_2, a^2 + b^2 = 1\}$  is isomorphic to the group  $S^1$  of unit complex numbers. We can write this last formula in the following form:

$$\widehat{f}(a) = \int_{\mathbb{R}} (f(x)e_1 + 0e_2) \perp \varphi_a(-x) dx$$

where  $\varphi_a$  is the morphism from  $\mathbb{R}$  to  $Spin(2)$  that sends  $x$  to  $\exp[(ax/2)(e_1e_2)]$  and  $\perp$  denotes the action  $v \perp s = s^{-1}vs$  of  $Spin(2)$  on  $\mathbb{R}^2$ , and more generally the action of  $Spin(n)$  on  $\mathbb{R}^n$ . Starting from these elementary observations, we now proceed to generalize this construction for  $\mathbb{R}^3$  valued functions of  $\mathbb{R}^2$ .

### 3 Morphisms of $\mathbb{R}^2$ with values in $Spin(3)$ and $Spin(4)$

We are looking for a generalization of the action of a character on the values of a real function. This can be done by replacing morphisms

$$\varphi : \mathbb{R}^2 \longrightarrow S^1 = Spin(2)$$

by morphisms

$$\varphi : \mathbb{R}^2 \longrightarrow Spin(3)$$

More precisely, let  $\mathbb{R}_{n,0}$ , resp.  $\mathbb{R}_{n,0}^k$ , denote the Clifford algebra of  $\mathbb{R}^n$  with the euclidean quadratic form, resp. the subspace of elements of degree  $k$  of  $\mathbb{R}_{n,0}$ . As usual we identify  $\mathbb{R}^n$  with  $\mathbb{R}_{n,0}^1$ . Let also  $\alpha$  denote the unique automorphism of  $\mathbb{R}_{n,0}$  such that  $\alpha(v) = -v$  for all  $v$  in  $\mathbb{R}_{n,0}^1$ . The group  $Spin(n)$  is defined as

$$Spin(n) = \{x \in \mathbb{R}_{n,0}, \alpha(x) = x, \|x\| = 1, xv x^{-1} \in \mathbb{R}_{n,0}^1, \forall v \in \mathbb{R}_{n,0}^1\}$$

(see [11] for details). It can be shown that  $Spin(n)$  is a connected compact Lie group that universally covers  $SO(n)$  (for  $n \geq 3$ ). Moreover its Lie algebra is  $\mathbb{R}_{n,0}^2$  with Lie bracket

$$\frac{1}{2}(A \times B) = \frac{1}{2}(AB - BA)$$

As the exponential map from  $\mathbb{R}_{n,0}^2$  to  $Spin(n)$  is onto (see [7] for a proof), every spinor can be written as

$$S = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$$

for some bivector  $A$ . In the following  $\mathbb{S}_{n,0}^2$  denotes the set of elements of  $\mathbb{R}_{n,0}^2$  of norm 1. One can verify that  $Spin(3)$  is the group

$$Spin(3) = \{a1 + be_1e_2 + ce_2e_3 + de_3e_1, a^2 + b^2 + c^2 + d^2 = 1\}$$

and is isomorphic to the group of unit quaternions.

**Proposition 1.** *The group morphisms from  $\mathbb{R}^2$  to  $Spin(3)$  are given by*

$$(x, y) \longmapsto e^{(ux+vy)B}$$

where  $B$  belongs to  $\mathbb{S}_{3,0}^2$  and  $u$  and  $v$  are real.

*Proof.* We have to determine the abelian subalgebras of the Lie algebra  $\mathfrak{spin}(3) = \mathbb{R}_{3,0}^2$  of the Lie group  $Spin(3)$ . More precisely, as the exponential map of  $\mathbb{R}^2$  is onto, group morphisms from  $\mathbb{R}^2$  to  $Spin(3)$  are given by Lie algebra morphisms from the abelian Lie algebra  $\mathfrak{A}^2$  of  $\mathbb{R}^2$  to  $\mathfrak{spin}(3)$ . Taking two generators  $(f_1, f_2)$  of  $\mathfrak{A}^2$ , any morphism  $\varphi$  from  $\mathfrak{A}^2$  to  $\mathfrak{spin}(3)$  satisfies:

$$\varphi(f_1) \times \varphi(f_2) = 0.$$

We deduce that  $Im(\varphi)$  is an abelian subalgebra of  $\mathbb{R}_{3,0}^2$  of dimension  $\leq 2$ . If  $a = a_1e_1e_2 + a_2e_3e_1 + a_3e_2e_3$  and  $b = b_1e_1e_2 + b_2e_3e_1 + b_3e_2e_3$  satisfy  $a \times b = 0$ , then the structure relations of  $\mathbb{R}_{3,0}^2$ , i. e.

$$e_1e_2 \times e_3e_1 = e_2e_3, \quad e_3e_1 \times e_2e_3 = e_1e_2, \quad e_2e_3 \times e_1e_2 = e_3e_1,$$

imply

$$(a_1b_2 - a_2b_1)e_2e_3 - (a_1b_3 - a_3b_1)e_3e_1 + (a_2b_3 - a_3b_2)e_1e_2 = 0.$$

This shows that two commuting elements of  $\mathbb{R}_{3,0}^2$  are colinear and that the abelian subalgebras of  $\mathbb{R}_{3,0}^2$  are of dimension 1. If we write  $\varphi(f_1) = uB$  and  $\varphi(f_2) = vB$  for some  $u, v \in \mathbb{R}$  and  $B \in \mathbb{S}_{3,0}^2$ , we see that the morphisms from  $\mathfrak{A}^2$  to  $\mathbb{R}_{3,0}^2$  are parameterized by two real numbers and one unit bivector, and are given by

$$\varphi_{u,v,B} : (x, y) \longmapsto (ux + vy)B$$

Consequently, the group morphisms from  $\mathbb{R}^2$  to  $Spin(3)$  are the morphisms  $\tilde{\varphi}_{u,v,B}$  with

$$\tilde{\varphi}_{u,v,B} : (x, y) \mapsto e^{(ux+vy)B}$$

In section 4, we define a Fourier transform starting from these group morphisms. As we want to split this transformation to be able to apply the FFT algorithm, we need to read the action of  $Spin(3)$  in  $Spin(4)$ . Let us recall what group is  $Spin(4)$ . Every  $\tau$  in  $Spin(4)$  is of the form

$$\begin{aligned} \tau &= u + Iv \\ &= (a1 + be_1e_2 + ce_2e_3 + de_3e_1) + I(d'1 + b'e_1e_2 + c'e_2e_3 + d'e_3e_1) \end{aligned}$$

where  $I$  denotes the pseudoscalar of  $\mathbb{R}_{4,0}$  and the following relations hold:

$$u\bar{u} + v\bar{v} = 1 \quad u\bar{v} + v\bar{u} = 0$$

The morphism  $\chi : Spin(4) \rightarrow Spin(3) \times Spin(3)$  with

$$\chi(u + Iv) = (u + v, u - v)$$

is an isomorphism. An alternative description of  $Spin(4)$  relies on the following fact: the morphism  $\psi : \mathbb{H}_1 \times \mathbb{H}_1 \rightarrow SO(4)$  defined by

$$(\tau, \rho) \mapsto (v \mapsto \tau v \bar{\rho})$$

(where  $v$  is a vector of  $\mathbb{R}^4$  considered as a quaternion) is a universal covering of  $SO(4)$  (see [11]). This means that  $Spin(4)$  is isomorphic to  $\mathbb{H}_1 \times \mathbb{H}_1$ . We will use this remark later on to compare our transform to Bülow's one. We consider every spinor  $\tau$  of  $Spin(3)$  as an element of  $Spin(4)$  by associating to  $\tau$  the spinor  $\chi^{-1}(\tau, 1)$ .

**Proposition 2.**<sup>1</sup> *The group morphism  $\tilde{\varphi}_{u,v,B}$  from  $\mathbb{R}^2$  to  $Spin(3)$  defines a group morphism from  $\mathbb{R}^2$  to  $Spin(4)$  that can be written*

$$\tilde{\varphi}_{u,v,B,0,0,C} : (x, y) \mapsto e^{\frac{1}{2}(ux+vy)(B+IB)}$$

*Proof.* The group law of  $Spin(3) \times Spin(3)$  being

$$\left( (a, b), (c, d) \right) \rightarrow (ac, bd)$$

the group morphisms from  $\mathbb{R}^2$  to  $Spin(3) \times Spin(3)$  are the morphisms  $\tilde{\varphi}_{u,v,B,w,z,C}$  defined by

$$\tilde{\varphi}_{u,v,B,w,z,C} : (x, y) \mapsto \left( e^{(ux+vy)B}, e^{(wx+zy)C} \right)$$

with  $u, v, w, z$  real and  $B, C$  two elements of  $\mathbb{S}_{3,0}^2$ . Let us verify that  $\chi^{-1}(\tilde{\varphi}_{u,v,B}, 1) = \tilde{\varphi}_{u,v,B,0,0,C}$  for any  $C$ , that maps  $(x, y)$  to

<sup>1</sup> Notations are justified by the more general result stated in proposition 5

$$\frac{1}{2}[\cos(ux + vy) + \sin(ux + vy)B + 1 + \cos(ux + vy)I + \sin(ux + vy)IB - I],$$

is precisely  $\tilde{\phi}_{u,v,B,0,0,C}$ . An easy computation shows that

$$\begin{aligned} \chi^{-1}(\tilde{\phi}_{u,v,B}, 1) &= \cos\left(\frac{ux + vy}{2}\right)\cos\left(\frac{ux + vy}{2}\right) + \sin\left(\frac{ux + vy}{2}\right)\cos\left(\frac{ux + vy}{2}\right)B \\ &+ \cos\left(\frac{ux + vy}{2}\right)\sin\left(\frac{ux + vy}{2}\right)IB - \sin\left(\frac{ux + vy}{2}\right)\sin\left(\frac{ux + vy}{2}\right)I. \end{aligned}$$

Clearly

$$\chi^{-1}(\tilde{\phi}_{u,v,B}, 1) = e^{\frac{1}{2}(ux+vy)B} e^{\frac{1}{2}(ux+vy)IB} = e^{\frac{1}{2}(ux+vy)(B+IB)}$$

since  $B$  and  $IB$  commute. •

Proposition 5 of section 6 describes the general form of a group morphism from  $\mathbb{R}^2$  to  $Spin(4)$  written as a product of exponentials. We are now ready to give the definition of the transformation we propose.

## 4 Fourier transform of color images

A color image is considered as a function  $f \in L^2(\mathbb{R}^2; \mathbb{R}_{4,0}^1)$  defined by:

$$f(x, y) = r(x, y)e_1 + g(x, y)e_2 + b(x, y)e_3 + 0e_4$$

where  $r$ ,  $g$  and  $b$  correspond to the red, green and blue levels.

There is an other way to decompose a color, that is with respect to its luminance and chrominance parts. The first corresponds to the projection of the color to the axis generated by  $e_1 + e_2 + e_3$ , and the latter to its projection on the orthogonal plane, called the chrominance plane, represented by the unit bivector  $(e_1e_2 - e_1e_3 + e_2e_3)/3$ . In what follows we make use of the following fact too: every hue can be represented in the RGB cube as an equivalence class of bivectors of  $\mathbb{R}_{3,0}$ . More precisely, we have the following result.

**Proposition 3.** *Let  $T$  be the set of bivectors*

$$T = \{(e_1 + e_2 + e_3) \wedge \alpha, \alpha \in RGB\}$$

with the following equivalence relation:

$$B \simeq C \iff B = \lambda C \quad \text{for } \lambda > 0$$

Then, there is a bijection between  $T / \simeq$  and the set of hues.

*Proof.* We have

$$(e_1 + e_2 + e_3) \wedge \alpha = (e_1 + e_2 + e_3)v_\alpha$$

where  $v_\alpha$  is the projection of  $\alpha$  on the chrominance. Then, there is a bijection between  $T/\simeq$  and the set  $(e_1 + e_2 + e_3)v$  for  $v$  a unit vector in the chrominance plane. This latter being in bijection with the set of different hues, we conclude that there exists a bijection between  $T/\simeq$  and the set of hues. •

**Definition 1.** Clifford Fourier transform of a color image. Let  $f$  be a color image and  $B$  be an element of  $\mathbb{S}_{3,0}^2$ . The Clifford Fourier transform of  $f$  with respect to  $B$  is the  $\mathbb{R}_{4,0}^1$  valued function

$$\widehat{f}_B(u, v) = \int_{\mathbb{R}^2} f(x, y) \perp \widetilde{\Phi}_{u,v,B,0,0,C}(-x, -y) dx dy$$

Given a bivector  $B$  of  $\mathbb{S}_{3,0}^2$ , every color image  $f$  can be decomposed as

$$f = f_{B\parallel} + f_{B\perp}$$

where  $f_{B\parallel}$ , resp.  $f_{B\perp}$ , is the  $B$ -parallel part, resp. the  $IB$ -parallel part (i.e. the  $B$ -orthogonal part in  $\mathbb{R}^4$ ) of  $f$ . This leads to

$$\widehat{f}_B(u, v) = \int_{\mathbb{R}^2} (f_{B\parallel}(x, y) + f_{IB\parallel}(x, y)) \perp \widetilde{\Phi}_{u,v,B,0,0,C}(-x, -y) dx dy$$

that can be written

$$\widehat{f}_B(u, v) = \widehat{f}_{B\parallel}(u, v) + \widehat{f}_{B\perp}(u, v)$$

where

$$\begin{aligned} \widehat{f}_{B\parallel}(u, v) &= \int_{\mathbb{R}^2} f_{B\parallel}(x, y) \perp \widetilde{\Phi}_{u,v,B,0,0,C}(-x, -y) dx dy \\ &= \int_{\mathbb{R}^2} f_{B\parallel}(x, y) \perp e^{\frac{-(ux+vy)B}{2}} dx dy \end{aligned}$$

and

$$\begin{aligned} \widehat{f}_{B\perp}(u, v) &= \int_{\mathbb{R}^2} f_{IB\parallel}(x, y) \perp \widetilde{\Phi}_{u,v,B,0,0,C}(-x, -y) dx dy \\ &= \int_{\mathbb{R}^2} f_{IB\parallel}(x, y) \perp e^{\frac{-(ux+vy)IB}{2}} dx dy. \end{aligned}$$

Let us remark that each of the two integrals may be identified with the Fourier transform of a function from  $\mathbb{R}^2$  to  $\mathbb{C}$ . Then, we deduce that there exists an inversion formula for the Clifford-Fourier transform  $\widehat{f}_B$  given by

$$f(x, y) = \int_{\mathbb{R}^2} \widehat{f}_B(u, v) \perp \widetilde{\Phi}_{u,v,B,0,0,C}(x, y) dudv$$

Indeed, the right term equals

$$\int_{\mathbb{R}^2} (\widehat{f}_{B\parallel}(u, v) + \widehat{f}_{IB\parallel}(u, v)) \perp \widetilde{\Phi}_{u,v,B,0,0,C}(x, y) dudv$$



$$= \int_{\mathbb{R}^2} \widehat{f}_{B_{\parallel}}(u, v) \perp e^{\frac{(ux+vy)B}{2}} dudv + \int_{\mathbb{R}^2} \widehat{f}_{IB_{\parallel}}(u, v) \perp e^{\frac{(ux+vy)IB}{2}} dudv \quad (1)$$

Each of these integrals may be identified with the inversion formula of the Fourier transform of a function from  $\mathbb{R}^2$  to  $\mathbb{C}$ , hence

$$\begin{aligned} (1) &= f_{B_{\parallel}}(x, y) + f_{IB_{\parallel}}(x, y) \\ &= f(x, y). \end{aligned}$$

The following proposition will be useful for the applications to the filtering of frequencies developed in the next section. It justifies the fact that the filters used are symmetric with respect to the transformation  $(u, v) \mapsto (-u, -v)$ .

**Proposition 4.** *With the previous notations, if  $B$  and  $D$  are elements of  $\mathbb{S}_{3,0}^2$ , we have*

$$\begin{aligned} &\widehat{f}_B(u, v) \perp \widetilde{\phi}_{u,v,B,0,0,C}(x, y) + \widehat{f}_B(-u, -v) \perp \widetilde{\phi}_{-u,-v,B,0,0,C}(x, y) \\ &= \widehat{f}_D(u, v) \perp \widetilde{\phi}_{u,v,D,0,0,C}(x, y) + \widehat{f}_D(-u, -v) \perp \widetilde{\phi}_{-u,-v,D,0,0,C}(x, y) \end{aligned}$$

Moreover, the  $e_4$  component of this expression is null.

*Proof.* Simple computations show that

$$\begin{aligned} &\widehat{f}_B(u, v) \perp \widetilde{\phi}_{u,v,B,0,0,C}(x, y) + \widehat{f}_B(-u, -v) \perp \widetilde{\phi}_{-u,-v,B,0,0,C}(x, y) \quad (2) \\ &= \int_{\mathbb{R}^2} e^{-\frac{xu+yv}{2}(B+IB)} e^{\frac{\lambda u+\mu v}{2}(B+IB)} f(\lambda, \mu) e^{-\frac{\lambda u+\mu v}{2}(B+IB)} e^{\frac{xu+yv}{2}(B+IB)} d\lambda d\mu \\ &\quad + \int_{\mathbb{R}^2} e^{\frac{xu+yv}{2}(B+IB)} e^{-\frac{\lambda u+\mu v}{2}(B+IB)} f(\lambda, \mu) e^{\frac{\lambda u+\mu v}{2}(B+IB)} e^{-\frac{xu+yv}{2}(B+IB)} d\lambda d\mu \\ &= \int_{\mathbb{R}^2} 2\cos(u(x-\lambda) + v(y-\mu)) f_{B_{\parallel}}(\lambda, \mu) d\lambda d\mu \\ &\quad + \int_{\mathbb{R}^2} 2\cos(u(x-\lambda) + v(y-\mu)) f_{IB_{\parallel}}(\lambda, \mu) d\lambda d\mu. \end{aligned}$$

Hence

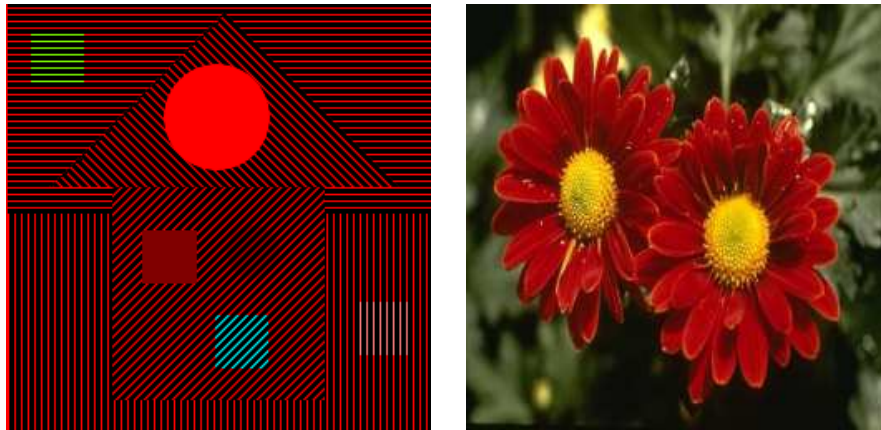
$$(2) = \int_{\mathbb{R}^2} 2\cos(u(x-\lambda) + v(y-\mu)) f(\lambda, \mu) d\lambda d\mu. \quad \bullet$$

## 5 Applications

We now present applications to color image filtering. The use of the Fourier transform is motivated by the well known fact that non trivial filters in the spatial domain may be implemented efficiently with masks in the Fourier domain. Although it seems natural to believe that the results on grey level images may be generalized,

there are not so many works dedicated to the specific case of color images. Let us mention the reference [13] where an attempt is made through the use of an adhoc quaternionic transform. The mathematical construction we propose appears to be well founded since it explains the fundamental role of bivectors in terms of group actions. As explained before, the possibility to choose the bivector  $B$  is an asset allowing a wider range of applications. Indeed, Sangwine and al. proposal can be written in our formalism by considering an appropriate  $B$ .

Recall that  $\mu = (e_1 + e_2 + e_3)/\sqrt{3}$  is the unit vector coding the grey axis and  $\alpha$  denotes a given color in the RGB cube. In what follows, we treat three different cases and consider the bivector  $B$  of  $\mathbb{S}_{3,0}^2$  proportional to  $B_c = \alpha \wedge e_4$ ,  $B_h = \alpha \wedge \mu$  and  $B_r = v_\alpha \wedge e_4$  ( $v_\alpha$  is the projection of  $\alpha$  on the chrominance)<sup>2</sup>. The corresponding Clifford Fourier transforms are denoted respectively by  $\hat{f}_c$  (w.r.t to color),  $\hat{f}_h$  (w.r.t to hue) and  $\hat{f}_r$  (w.r.t to the chrominance part of  $\alpha$ ). In this paper, we restrict ourselves to apply low pass, high pass and directional filters only on the  $B$ -parallel part leaving the  $IB$ -parallel part unmodified.



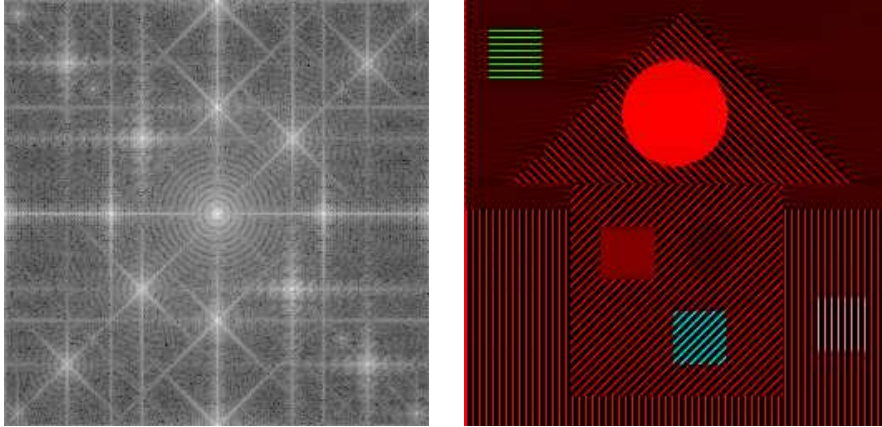
**Fig. 1** Original images : Fourier house (left), Flowers (right)

#### Comments on Figure 1

Figure 1 shows the original images used for these experiments<sup>3</sup>. Left image is a modified color version of the Fourier house containing red, desaturated red, green, cyan stripes in various directions, a uniform red circle and a red square with lower luminance. Right image is a natural image taken from the berkeley image segmentation database [9].

<sup>2</sup> Replacing  $B$  by  $IB$  when we deal with  $B_c$  and  $B_r$  we can suppose that  $B \in \mathbb{S}_{3,0}^2$

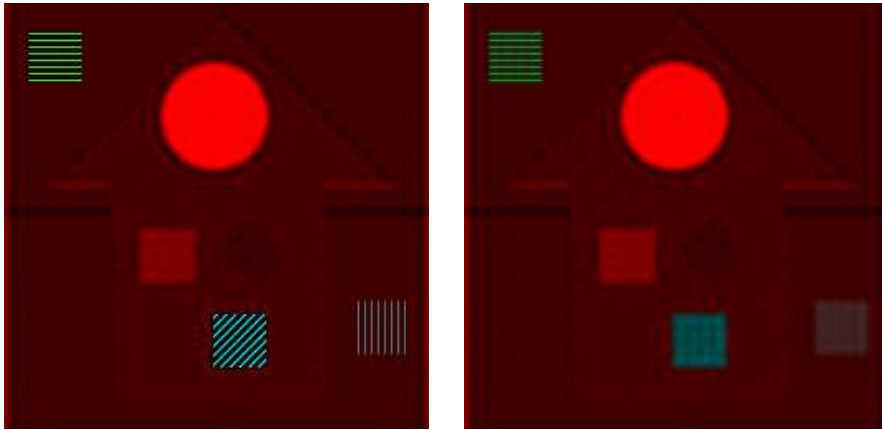
<sup>3</sup> Available at <http://mia.univ-larochelle.fr/> → Production → D emos



**Fig. 2**  $\hat{f}_c$  : Log-modulus (left) and directional cut filtering (right) for the red color

### Comments on Figure 2

Figure 2 (left) is the centered log-modulus of the parallel part of  $\hat{f}_c$  taking  $\alpha = 255e_1$ . Figure 2 (right) is the result of a directional cut filter around  $\pi/2$  which removes of vertical frequencies. Let us point out that horizontal green stripes are not modified since green color  $255e_2$  belongs to  $IB_c$ .



**Fig. 3** Low pass filterings on  $\hat{f}_c$  (left) and on  $\hat{f}_h$  (right) for respectively red color and red hue

### Comments on Figure 3

Figure 3 shows the difference between  $\hat{f}_c$  (left) and  $\hat{f}_h$  (right) via a low pass filter with  $\alpha = 255e_1$ . As the bivector  $B_h$  is built from  $\mu$ , the luminance of all colors of the right image is lowered. In contrast, the luminance of green and cyan pixels of the left image remains untouched since these colors are orthogonal to the red one. The

cyan stripes on the right image are blurred since the cyan and red hues are colinear.



**Fig. 4**  $\hat{f}_r$  : Low pass (left) and high pass (right) filters with  $\alpha = 96e_1 + 109e_2 + 65e_3$

Comments on Figure 4

Let us remark first that the transform  $\hat{f}_r$  preserves the luminance of all colors. The color  $\alpha$  has been chosen to match with the color of the green leaves. As low pass filter (left) removes green high frequencies, the center of flowers containing yellow high frequencies turns red. On the right image, background pixels corresponding to green low frequencies appear almost grey.

## 6 Clifford Fourier transform for 4D signals defined on the plane

The aim of this section is to give an analog of definition 1 for 4D signals defined on the plane. This requires an analog of proposition 2.

**Proposition 5.** *The group morphisms from  $\mathbb{R}^2$  to  $Spin(4)$  are the morphisms  $\tilde{\phi}_{u,v,B,w,z,C}$  that send  $(x,y)$  to*

$$e^{\frac{1}{4}[x(u+w)+y(v+z)][B+C+I(B-C)]} e^{\frac{1}{4}[x(u-w)+y(v-z)][B-C+I(B+C)]}$$

with  $u, v, w, z$  real and  $B, C$  two elements of  $\mathbb{S}_{3,0}^2$ .

*Proof.* From the proof of proposition 2, we know that the group morphisms from  $\mathbb{R}^2$  to  $Spin(4)$  are the  $\tilde{\phi}_{u,v,B,w,z,C}$  that send  $(x,y)$  to

$$\frac{e^{(ux+vy)B} + e^{(wx+zy)C}}{2} + I \frac{e^{(ux+vy)B} - e^{(wx+zy)C}}{2}$$

However, this writing is not convenient to determine representations in  $SO(4)$  since it doesn't provide explicitly the rotations in  $\mathbb{R}^4$  that  $\tilde{\phi}_{u,v,B,w,z,C}$  generates. The solution comes from an "orthogonalization" of the corresponding Lie algebras morphism from  $\mathfrak{X}^2$  to  $\mathbb{R}_{4,0}^2$ , namely the linear map

$$\phi_{u,v,B,w,z,C}(X, Y) = T_{(0,0)} \tilde{\phi}_{u,v,B,w,z,C}(X, Y)$$

where  $T$  denotes the linear tangent map. By definition,

$$\phi_{u,v,B,w,z,C}(X, Y) = \left. \frac{d}{dt} \left( \tilde{\phi}_{u,v,B,w,z,C}(\exp(t(X, Y))) \right) \right|_{t=0}$$

The exponential map of  $\mathbb{R}^2$  being the identity map, we get

$$\begin{aligned} \phi_{u,v,B,w,z,C}(X, Y) &= \left. \frac{d}{dt} \left( \tilde{\phi}_{u,v,B,w,z,C}(t(X, Y)) \right) \right|_{t=0} \\ &= \left. \frac{d}{dt} \left( \frac{e^{(uX+vY)B} + e^{(wX+zY)C}}{2} + I \frac{e^{(uX+vY)B} - e^{(wX+zY)C}}{2} \right) \right|_{t=0} \\ &= \frac{(uX + vY)B + (wX + zY)C}{2} + I \frac{(uX + vY)B - (wX + zY)C}{2} \end{aligned}$$

The "orthogonalization" of the morphism  $\phi_{u,v,B,w,z,C}$  consists in decomposing the bivector  $\phi_{u,v,B,w,z,C}(X, Y)$  for each  $X, Y$  into commuting bivectors whose squares are real. The corresponding spinor is written as a product of commuting spinors of the form  $e^{F_i}$  with  $F_i^2 < 0$ . These ones represent rotations of angle  $-F_i^2$  in the oriented planes given by the  $F_i$ 's. In our case, the bivector  $\phi_{u,v,B,w,z,C}(X, Y)$  is decomposed into  $F_1 + F_2$  where

$$F_1 = \frac{1}{4} \left[ (X(u+w) + Y(v+z))(B+C+I(B-C)) \right]$$

$$F_2 = \frac{1}{4} \left[ (X(u-w) + Y(v-z))(B-C+I(B+C)) \right]$$

(see [8] for details). The group morphisms  $\tilde{\phi}_{u,v,B,w,z,C}$  from  $\mathbb{R}^2$  to  $Spin(4)$  can then be written as

$$\begin{aligned} \tilde{\phi}_{u,v,B,w,z,C}(x, y) &= e^{\left[ \frac{(ux+vy)B+(wx+zy)C}{2} + I \frac{(ux+vy)B-(wx+zy)C}{2} \right]} \\ &= e^{\frac{1}{4} [(x(u+w)+y(v+z))(B+C+I(B-C))]} e^{\frac{1}{4} [(x(u-w)+y(v-z))(B-C+I(B+C))]} \end{aligned}$$

•

This is the convenient form to describe group morphisms from  $\mathbb{R}^2$  to  $SO(4)$ .

To conclude this paper, we show how to recover the quaternionic Fourier transform of T. Bülöw in the Clifford algebras context, using the appropriate morphism from

$\mathbb{R}^2$  to  $Spin(4)$ . First, let us recall the definition of this quaternionic Fourier transform.

The quaternionic Fourier transform of a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is the quaternion valued function  $\mathcal{F}(f)$  defined by

$$\mathcal{F}(f)(y_1, y_2) = \int_{\mathbb{R}^2} \exp(-2\pi i y_1 x_1) f(x_1, x_2) \exp(-2\pi j y_2 x_2) dx_1 dx_2.$$

The general definition we propose is the following one.

**Definition 2.** Let  $f \in L^2(\mathbb{R}^2; \mathbb{R}_{4,0}^1)$  and  $B$  and  $C$  be elements of  $\mathbb{S}_{3,0}^2$ . The Clifford Fourier transform of  $f$  with respect to  $B$  and  $C$  is defined by

$$\widehat{f}_{B,C}(u, v, w, z) = \int_{\mathbb{R}^2} f(x, y) \perp \widetilde{\Phi}_{u,v,B,w,z,C}(-x, -y) dx dy$$

•

The link between these two transforms is given by the next result.

**Proposition 6.** Let  $f \in L^2(\mathbb{R}^2; \mathbb{R})$ . The Clifford Fourier transform of  $f$  defined by

$$\widehat{f}_{e_2 e_3, e_1 e_3}(2\pi y_1, 0, 0, 2\pi y_2) =$$

$$\int_{\mathbb{R}^2} f(x_1, x_2) \perp \widetilde{\Phi}_{2\pi y_1, 0, e_2 e_3, 0, 2\pi y_2, e_1 e_3}(-x_1, -x_2) dx_1 dx_2$$

corresponds to the quaternionic Fourier transform of  $f$  seen as a  $\mathbb{H}$ -valued function under the following identification<sup>4</sup>

$$e_1 \leftrightarrow i \quad e_2 \leftrightarrow j \quad e_3 \leftrightarrow k \quad e_4 \leftrightarrow 1,$$

*Proof.* We have to determine one of the two elements of  $Spin(3) \times Spin(3)$  that generate the following rotation in  $\mathbb{H}$ :

$$f(x_1, x_2) \mapsto \exp(-2\pi i y_1 x_1) f(x_1, x_2) \exp(-2\pi j y_2 x_2)$$

Simple computations show that the rotation

$$f(x_1, x_2) \mapsto \exp(-2\pi i y_1 x_1) f(x_1, x_2)$$

can be written in  $\mathbb{R}_{4,0}^1$  as

$$f(x_1, x_2) \mapsto e^{\pi x_1 y_1 (e_4 e_1 + e_2 e_3)} f(x_1, x_2) e^{-\pi x_1 y_1 (e_4 e_1 + e_2 e_3)}$$

In the same way,

$$f(x_1, x_2) \mapsto f(x_1, x_2) \exp(-2\pi j y_2 x_2)$$

corresponds to

---

<sup>4</sup> The product law needs not to be respected since we just use an isomorphism of vector spaces

$$f(x_1, x_2) \mapsto e^{\pi x_2 y_2 (e_4 e_2 + e_1 e_3)} f(x_1, x_2) e^{-\pi x_2 y_2 (e_4 e_2 + e_1 e_3)}$$

By associativity, this shows that

$$\exp(-2\pi i y_1 x_1) f(x_1, x_2) \exp(-2\pi j y_2 x_2) = e^\tau e^\rho f(x_1, x_2) e^{-\rho} e^{-\tau}$$

where

$$\tau = \pi x_2 y_2 (e_4 e_2 + e_1 e_3)$$

and

$$\rho = \pi x_1 y_1 (e_4 e_1 + e_2 e_3).$$

By definition,

$$\chi(e^\tau e^\rho) = \chi(e^{\pi x_2 y_2 e_4 e_2}) \chi(e^{\pi x_2 y_2 e_1 e_3}) \chi(e^{\pi x_1 y_1 e_4 e_1}) \chi(e^{\pi x_1 y_1 e_2 e_3}).$$

From simple computations, we get

$$\chi(e^\tau e^\rho) = (e^{2\pi x_1 y_1 e_2 e_3}, e^{2\pi x_2 y_2 e_1 e_3})$$

and conclude therefore that

$$(x_1, x_2) \mapsto (e^{2\pi x_1 y_1 e_2 e_3}, e^{2\pi x_2 y_2 e_1 e_3})$$

is the morphism  $\tilde{\phi}_{2\pi y_1, 0, e_2 e_3, 0, 2\pi y_2, e_1 e_3}$ . •

## 7 Conclusion

We proposed in this paper a definition of Clifford Fourier transform that is motivated by group actions considerations. We showed how to generalize the notion of characters to obtain a transformation which is parameterized by one or two bivectors of the Clifford algebra  $\mathbb{R}_{3,0}$ . We have treated in this context applications to color image processing. The spectral analysis we have performed throws a new light on the work of S. Sangwine and al. Further work will be devoted to some mathematical improvements. We will show in particular how to choose a metric on the RGB cube so as to gain more efficiency in the filtering with respect to hue. Others applications to 4 channels images (like color/infrared images) are currently investigated.

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## References

1. Brackx, F., De Schepper, N., Sommen, F.: The Two-Dimensional Clifford-Fourier Transform. *J Math Imaging Vis* **26**, 5–18 (2006)
2. Bülow, T.: Hypercomplex spectral signal representations for the processing and analysis of images. Ph.D. thesis, Kiel (1999)
3. Bülow, T., Sommer, G.: The hypercomplex signal - a novel approach to the multidimensional analytic signal. *IEEE Transactions on Signal Processing* **49**, **11**, 2844–2852 (2001)
4. Ebling, J., Scheuermann, G.: Clifford Fourier Transform on Vector Fields. *IEEE Transactions on Visualization and Computer Graphics* **11**, **4**, 2844–2852 (July/August 2005)
5. Ell, T.A., Sangwine, S.: Hypercomplex Fourier Transforms of Color Images. *IEEE Transactions on Image Processing* **16**, **1**, 5–18 (2007)
6. Felsberg, M.: Low-level image processing with the structure multivector. Ph.D. thesis, Kiel (2002)
7. Helgason, S.: Differential geometry, Lie groups and symmetric spaces. Academic Press London (1978)
8. Hestenes, D., Sobczyk, G.: Clifford algebra to geometric calculus. D. Reidel, Dordrecht (1984)
9. Martin, D., Fowlkes, C., Tal, D., Malik, J.: A database of human segmented natural images and its application to evaluating segmentation algorithms and measuring ecological statistics. In: Proc. 8th Int'l Conf. Computer Vision, vol. 2, pp. 416–423 (2001)
10. Mawardi B. et Hitzer, E.: Clifford Fourier transformation and uncertainty principle for the Clifford algebra  $cl_{3,0}$ . *Adv. App. Cliff. Alg.* **16**, **1**, 41–61 (2006)
11. Postnikov, M.: *Leçons de Géométrie. Groupes et algèbres de Lie*. Mir (1982)
12. Sangwine, S., Ell, T.A.: Hypercomplex Fourier Transforms of Color Images. In: IEEE International Conference on Image Processing (ICIP), vol. 1, pp. 137–140. Thessaloniki, Greece (2001)
13. Sangwine, S., Ell, T.A., Gatsheni, B.: Colour-dependent linear vector image filtering. In: Twelfth European Signal Processing Conference, EUSIPCO 2004, pp. 585–588. Vienna, Austria (2004)
14. Smach, F., Lemaire, C., Gauthier, J.P., Miteran, J., Atri, M.: Generalized Fourier Descriptors with Applications to Objects Recognition in SVM Context. *J Math Imaging and Vision* **30**, **1**, 43–71 (2008)
15. Vilenkin, N.J.: Special functions and the theory of group representations, *Translations of Mathematical Monographs*, vol. 22. American Mathematical Society, Providence, Rhode Island (1968)