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► **To cite this version:**

Anisse Kasraoui, Dennis Stanton, Jiang Zeng. The combinatorics of Al-Salam-Chihara  $q$ -Laguerre polynomials. 2008. hal-00332856

**HAL Id: hal-00332856**

**<https://hal.science/hal-00332856>**

Preprint submitted on 22 Oct 2008

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# THE COMBINATORICS OF AL-SALAM-CHIHARA $q$ -LAGUERRE POLYNOMIALS

ANISSE KASRAOUI, DENNIS STANTON, AND JIANG ZENG

ABSTRACT. We describe various aspects of the Al-Salam-Chihara  $q$ -Laguerre polynomials. These include combinatorial descriptions of the polynomials, the moments, the orthogonality relation and a combinatorial interpretation of the linearization coefficients.

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*Keywords:*  $q$ -Laguerre polynomials, Al-Salam-Chihara polynomials,  $y$ -version of  $q$ -Stirling numbers of the second kind, linearization coefficients.

**MR Subject Classifications:** Primary 05A18; Secondary 05A15, 05A30.

## 1. INTRODUCTION

The monic simple Laguerre polynomials  $L_n(x)$  may be defined by the explicit formula:

$$L_n(x) = \sum_{k=0}^n (-1)^{n-k} \frac{n!}{k!} \binom{n}{k} x^k, \quad (1)$$

or by the three-term recurrence relation

$$L_{n+1}(x) = (x - (2n + 1))L_n(x) - n^2 L_{n-1}(x). \quad (2)$$

The moments are

$$\mu_n = \mathcal{L}(x^n) = \int_0^\infty x^n e^{-x} dx = n!. \quad (3)$$

The linearization formula reads as follows:

$$L_{n_1}(x)L_{n_2}(x) = \sum_{n_3} C_{n_1 n_2}^{n_3} L_{n_3}(x),$$

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Version of October 11, 2008.

where

$$C_{n_1 n_2}^{n_3} = \sum_{s \geq 0} \frac{n_1! n_2! 2^{N_2+n_3-2s} s!}{(s-n_1)!(s-n_2)!(s-n_3)!(N_2+n_3-2s)!n_3!}.$$

Equivalently we have

$$\mathcal{L}(L_{n_1}(x)L_{n_2}(x)L_{n_3}(x)) = \sum_{s \geq 0} \frac{n_1! n_2! n_3! 2^{N_2+n_3-2s} s!}{(s-n_1)!(s-n_2)!(s-n_3)!(N_2+n_3-2s)!}. \quad (4)$$

Given positive integers  $n_1, n_2, \dots, n_k$  such that  $n = n_1 + \dots + n_k$ , let  $S_i$  be the consecutive integer segment  $\{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}$  with  $n_0 = 0$ , then  $S_1 \cup \dots \cup S_k = [n]$ . A permutation  $\sigma$  of  $[n]$  is said to be a *generalized derangement* if  $i$  and  $\sigma(i)$  do not belong to a same segment  $S_j$  for all  $i \in [n]$ . Let  $\mathcal{D}_n$  be the set of generalized derangements of  $[n]$  then we have

$$\mathcal{L}(L_{n_1}(x) \dots L_{n_k}(x)) = \sum_{\sigma \in \mathcal{D}_n} 1. \quad (5)$$

A  $q$ -version of (1) was studied by Garsia and Remmel [9] in 1980. Several  $q$ -analogues of the moments (2) and recurrence relation(3) were investigated in the last two decades (see [2, 18, 19]) in order to obtain new *mahonian* statistics on the symmetric groups. On the other hand, in view of the unified combinatorial interpretations of several aspects of Sheffer orthogonal polynomials (moments, polynomials, and the linearization coefficients)(see [14, 20, 22]) it is natural to seek for a  $q$ -version of this picture.

As one can expect, the first result in this direction was the linearization formula for  $q$ -Hermite polynomials due to Ismail, Stanton and Viennot [12], dated back to 1987. In particular, their formula provides a combinatorial evaluation of the Askey-Wilson integral. However, a similar formula for  $q$ -Charlier polynomials was discovered only recently by Anshelevich [1], who used the machinery of  $q$ -Levy stochastic processes. Short later, Kim, Stanton and Zeng [15] gave a combinatorial proof of Anshelevich's result.

The object of this paper is to give a  $q$ -version of all the above formulas for simple Laguerre polynomials.

## 2. AL-SALAM-CHIHARA POLYNOMIALS REVISITED

The Al-Salam-Chihara polynomials  $Q_n(x) := Q_n(x; \alpha, \beta|q)$  may be defined by the recurrence relation [16, Chapter 3]:

$$\begin{cases} Q_0(x) = 1, & Q_{-1}(x) = 0, \\ Q_{n+1}(x) = (2x - (\alpha + \beta)q^n)Q_n(x) - (1 - q^n)(1 - \alpha\beta q^{n-1})Q_{n-1}(x), & n \geq 0. \end{cases} \quad (6)$$

Let  $Q_n(x) = 2^n p_n(x)$  then

$$xp_n(x) = p_{n+1}(x) + \frac{1}{2}(\alpha + \beta)q^n p_n(x) + \frac{1}{4}(1 - q^n)(1 - \alpha\beta q^{n-1})p_{n-1}(x). \quad (7)$$

They also have the following explicit expressions:

$$\begin{aligned} Q_n(x; \alpha, \beta|q) &= \frac{(\alpha\beta; q)_n}{\alpha^n} {}_3\phi_2 \left( \begin{matrix} q^{-n}, \alpha u, \alpha u^{-1} \\ \alpha\beta, 0 \end{matrix} \middle| q; q \right) \\ &= (\alpha u; q)_n u^{-n} {}_2\phi_1 \left( \begin{matrix} q^{-n}, \beta u^{-1} \\ \alpha^{-1} q^{-n+1} u^{-1} \end{matrix} \middle| q; \alpha^{-1} q u \right) \\ &= (\beta u^{-1}; q)_n u^n {}_2\phi_1 \left( \begin{matrix} q^{-n}, \alpha u \\ \beta^{-1} q^{-n+1} u \end{matrix} \middle| q; \beta^{-1} q u^{-1} \right), \end{aligned}$$

where  $x = \frac{u+u^{-1}}{2}$  or  $x = \cos \theta$  if  $u = e^{i\theta}$ .

The Al-Salam-Chihara polynomials have the following generating function

$$G(t, x) = \sum_{n=0}^{\infty} Q_n(x; \alpha, \beta|q) \frac{t^n}{(q; q)_n} = \frac{(\alpha t, \beta t; q)_{\infty}}{(te^{i\theta}, te^{-i\theta}; q)_{\infty}}.$$

They are orthogonal with respect to the linear functional  $\hat{\mathcal{L}}_q$ :

$$\hat{\mathcal{L}}_q(x^n) = \frac{1}{2\pi} \int_0^{\pi} (\cos \theta)^n \frac{(q, \alpha\beta, e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, \beta e^{i\theta}, \beta e^{-i\theta}; q)_{\infty}} d\theta, \quad (8)$$

where  $x = \cos \theta$ . Note that

$$\hat{\mathcal{L}}_q(Q_n(x)^2) = (q; q)_n (\alpha\beta; q)_n.$$

**Theorem 1.** *We have*

$$Q_{n_1}(x)Q_{n_2}(x) = \sum_{n_3 \geq 0} C_{n_1, n_2}^{n_3}(\alpha, \beta; q) Q_{n_3}(x), \quad (9)$$

where

$$\begin{aligned} C_{n_1, n_2}^{n_3}(\alpha, \beta; q) &= (-1)^{N_2+n_3} \frac{(q; q)_{n_1} (q; q)_{n_2}}{(\alpha\beta; q)_{n_3}} \\ &\times \sum_{m_2, m_3} \frac{(\alpha\beta; q)_{n_1+m_3} \alpha^{m_2} \beta^{n_3+n_2-n_1-m_2-2m_3} q^{\binom{m_2}{2} + (n_3+n_2-n_1-m_2-2m_3)}}{(q; q)_{n_3+n_2-n_1-m_2-2m_3} (q; q)_{m_2} (q; q)_{m_3+n_1-n_3} (q; q)_{m_3+n_1-n_2} (q; q)_{m_3}}. \end{aligned}$$

*Proof.* Clearly  $C_{n_1, n_2}^{n_3}(\alpha, \beta; q) = \hat{\mathcal{L}}_q(Q_{n_1}(x)Q_{n_2}(x)Q_{n_3}(x)) / \hat{\mathcal{L}}_q(Q_{n_3}(x)Q_{n_3}(x))$ . Using the Askey-Wilson integral:

$$\frac{(q; q)_{\infty}}{2\pi} \int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_{\infty}} d\theta = \frac{(t_1 t_2 t_3 t_4; q)_{\infty}}{\prod_{1 \leq j < k \leq 4} (t_j t_k; q)_{\infty}},$$

one can prove [12, Theorem 3.5] that

$$\begin{aligned} &\hat{\mathcal{L}}_q(G(t_1, x)G(t_2, x)G(t_3, x)) \\ &= \frac{(\alpha t_1 t_2 t_3, \beta q t_1 t_2 t_3, \alpha\beta q; q)_{\infty}}{(t_1 t_2, t_1 t_3, t_2 t_3; q)_{\infty}} {}_3\phi_2 \left( \begin{matrix} t_1 t_2, & t_1 t_3, & t_2 t_3 \\ \alpha t_1 t_2 t_3, & \beta t_1 t_2 t_3 \end{matrix} \middle| q; \alpha\beta \right). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n_1, n_2, n_3} \hat{\mathcal{L}}_q(Q_{n_1}(x)Q_{n_2}(x)Q_{n_3}(x)) \frac{t_1^{n_1}}{(q; q)_{n_1}} \frac{t_2^{n_2}}{(q; q)_{n_2}} \frac{t_3^{n_3}}{(q; q)_{n_3}} \\ = \sum_{k \geq 0} \frac{(\alpha t_1 t_2 t_3 q^k, \beta t_1 t_2 t_3 q^k, \alpha \beta; q)_\infty}{(t_1 t_2 q^k, t_1 t_3 q^k, t_2 t_3 q^k; q)_\infty} \frac{(\alpha \beta)^k}{(q; q)_k}. \end{aligned} \quad (10)$$

Using the Euler formulas:

$$(t; q)_\infty = \sum_{n \geq 0} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} t^n; \quad \frac{1}{(t; q)_\infty} = \sum_{n \geq 0} \frac{1}{(q; q)_n} t^n,$$

we can rewrite the sum in (10) as follows:

$$\begin{aligned} (\alpha \beta; q)_\infty \sum_{k \geq 0} \frac{(\alpha \beta)^k}{(q; q)_k} \sum_{l_1, l_2 \geq 0} \frac{\alpha^{l_1} \beta^{l_2} q^{k(l_1+l_2)} (-t_1 t_2 t_3)^{l_1+l_2} q^{\binom{l_1}{2} + \binom{l_2}{2}}}{(q; q)_{l_1} (q; q)_{l_2}} \\ \times \sum_{m_1, m_2, m_3 \geq 0} \frac{q^{(m_1+m_2+m_3)k} t_1^{m_1+m_2} t_2^{m_1+m_3} t_3^{m_1+m_3}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3}}. \end{aligned} \quad (11)$$

Substituting

$$\sum_{k \geq 0} \frac{(\alpha \beta q^{l_1+l_2+m_1+m_2+m_3})^k}{(q; q)_k} = \frac{1}{(\alpha \beta q^{l_1+l_2+m_1+m_2+m_3}; q)_\infty}$$

in (11), we get

$$\sum_{l_1, l_2, m_1, m_2, m_3} t_1^{n_1} t_2^{n_2} t_3^{n_3} \frac{(\alpha \beta)_{n_1+m_3} \alpha^{l_1} \beta^{l_2} q^{\binom{l_1}{2} + \binom{l_2}{2}}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{l_1} (q; q)_{l_2}} (-1)^{l_1+l_2}, \quad (12)$$

where  $l_1 + l_2 + m_1 + m_2 = n_1$ ,  $l_1 + l_2 + m_1 + m_3 = n_2$  and  $l_1 + l_2 + m_2 + m_3 = n_3$ .

Since  $l_1 + l_2 \equiv N_2 + n_3 \pmod{2}$ , extracting the coefficient of  $\frac{t_1^{n_1} t_2^{n_2} t_3^{n_3}}{(q; q)_{n_1} (q; q)_{n_2} (q; q)_{n_3}}$  in (12) and dividing by  $(q, \alpha \beta; q)_{n_3}$  we obtain (9) where  $l_1$  is replaced by  $m_2$ .  $\square$

### 3. THE NEW $q$ -LAGUERRE POLYNOMIALS

We define the new  $q$ -Laguerre polynomials  $L_n(x; q)$  by re-scaling Al-Salam-Chihara polynomials:

$$L_n(x; q) = \left( \frac{\sqrt{y}}{q-1} \right)^n Q_n \left( \frac{(q-1)x + y + 1}{2\sqrt{y}}; \frac{1}{\sqrt{y}}, \sqrt{y}q|q \right). \quad (13)$$

It follows from (7) that the polynomials  $L_n(x; q)$  satisfy the recurrence:

$$L_{n+1}(x; q) = (x - y[n+1]_q - [n]_q) L_n(x; q) - y[n]_q^2 L_{n-1}(x; q). \quad (14)$$

We derive then the explicit formula for  $L_n(x)$ :

$$L_n(x; q) = \sum_{k=0}^n (-1)^{n-k} \frac{n!_q}{k!_q} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} y^{n-k} \prod_{j=0}^{k-1} (x - (1 - yq^{-j})[j]_q). \quad (15)$$

Thus

$$\begin{aligned} L_1(x; q) &= x - y, \\ L_2(x; q) &= x^2 - (1 + 2y + qy)x + (1 + q)y^2, \\ L_3(x; q) &= x^3 - (q^2y + 3y + q + 2 + 2qy)x^2 \\ &\quad + (q^3y^2 + yq^2 + q + 2qy + 3q^2y^2 + 1 + 4qy^2 + 2y + 3y^2)x \\ &\quad - (2q^2 + 2q + q^3 + 1)y^3 \end{aligned}$$

A combinatorial interpretation of these  $q$ -Laguerres polynomials can be derived from the Simion and Stanton's combinatorial model for octabasic Laguerre polynomials [19]. For a subset  $A$  of  $[n]$ , the *functional digraph* of an injection  $f : A \rightarrow [n]$  consists of disjoint paths and cycles. Each path  $P$  is of the form  $a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_l$ , where  $f(a_j) = a_{j+1}$  for  $0 \leq j < l$ , with  $f^{-1}(a_0)$  empty, and  $a_l \in [n] - A$ . We put  $\text{last}(P) = a_l$  and if  $i = a_k \in P$  we write  $\text{ind}(i, P) = k$  for the index of  $i$  on the path  $P$ . For any path  $P$  in the digraph and two integers  $i < j$ , we put

$$n_P(i, j) = |\{a \in P : i < a < j\}|.$$

For  $p \in P$  and two integers  $i < j$ , we define

$$m_P(p; i, j) = |\{a \in P : i < a < j, \text{ind}(p, P) < \text{ind}(a, P)\}|,$$

that is, the number of points on the path "to the right" of  $p$ , whose values are strictly between  $i$  and  $j$ . And finally, for  $i \in A$ , we denote by  $F(i)$  the "first forward iterate" of  $f$  which is smaller than  $i$ , i.e.,

$$F(i) = \begin{cases} f^p(i), & \text{where } p = \min\{m \geq 1, f^m(i) < i \text{ if such } m \text{ exists}\}; \\ i, & \text{if } \{m \geq 1, f^m(i) < i\} \text{ is empty.} \end{cases}$$

For instance, suppose that the path  $P = 2 \rightarrow 7 \rightarrow 1 \rightarrow 5 \rightarrow 3$  is a connected component of the functional digraph of  $f$ . Then  $n_P(1, 4) = |\{2, 3\}| = 2$ ,  $m_P(7; 1, 4) = |\{3\}| = 1$ , and  $F(2) = F(7) = 1$ ,  $F(1) = 1$ , and  $F(5) = 3$ .

For any  $k \in [n]$ , let  $\alpha(k) = w(k) = 0$  if  $k \notin A$ , otherwise if  $k$  is on a cycle or a path  $P$  such that  $k > \text{last}(P)$ , then  $\alpha(k) = 1$  and

$$w(k) = F(k) - 1 - \sum_{\text{last}(Q) > k} n_Q(0, F(k));$$

if  $k$  is on a path  $P$  such that  $k < \text{last}(P)$ , then  $\alpha(k) = 0$  and

$$w(k) = k - 1 - m_P(k; 0, k) - \sum_{\text{last}(Q) > \text{last}(P)} n_Q(0, k),$$

where  $Q$  ranges over all paths in the functional digraphs of  $f$ . Let

$$w(A, f) = \sum_{k \in A} w(k) \quad \text{and} \quad \alpha(A, f) = \sum_{k \in A} \alpha(k).$$

**Example 1.** Let  $n = 9$ ,  $A = \{2, 9\}$  and  $\sigma = (6)(47)(3518)$  (in cycle notation with maximum at last). Then we have  $\text{cyc}(\sigma) = 3$  and

$$w(A, \sigma) = (3 - 1 - 1) + (5 - 1 - 1) + (1 - 1) + (4 - 1 - 2) = 5.$$

**Theorem 2.** *The  $q$ -Laguerre polynomials have the following interpretation:*

$$L_n(x; q) = \sum_{A \subset [n], f: A \rightarrow [n]} (-1)^{|A|} x^{n-|A|} y^{\alpha(A,f)} q^{w(A,f)},$$

where  $f$  is injective.

*Proof.* This is the  $a = 1$ ,  $s = u = 1$  and  $r = t = q$  special case of the quadrabasic Laguerre polynomials [19, p.313].  $\square$

**Remark 1.** *It is easy to see that the constant term  $L_n(0)$  is equal to*

$$L_n(0) = (-1)^n y^n n!_q.$$

*So the restriction of the statistic on permutations is a Mahonian statistic.*

#### 4. MOMENTS OF THE $q$ -LAGUERRE POLYNOMIALS

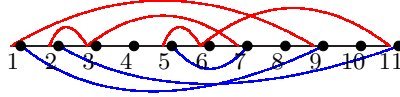
Let  $\mathcal{S}_n$  be the set of permutations of  $[n] := \{1, 2, \dots, n\}$ . For  $\sigma \in \mathcal{S}_n$  the *crossing number* of  $\sigma$  is defined by

$$cr(\sigma) = \sum_{i=1}^n \#\{j | j < i \leq \sigma(j) < \sigma(i)\} + \sum_{i=1}^n \#\{j | j > i > \sigma(j) > \sigma(i)\},$$

while the number of *weak excedances* of  $\sigma$  is defined by

$$wex(\sigma) = \#\{i | 1 \leq i \leq n \text{ and } i \leq \sigma(i)\}.$$

We can depict these statistics by associating with each permutation  $\sigma$  of  $[n]$  a diagram by drawing an arc  $i \rightarrow \sigma(i)$  above (resp. under) the segment  $1 \rightarrow 2 \rightarrow \dots \rightarrow n$  if  $i \leq \sigma(i)$  (resp.  $i > \sigma(i)$ ). For example, the permutation  $\sigma = 9374611581102$  can be depicted as follows:



Let  $\mu_n^{(\ell)}(y, q)$  be the enumerating polynomial of permutations in  $\mathcal{S}_n$  with respect to weak excedances and crossing numbers:

$$\mu_n^{(\ell)}(y, q) := \sum_{\sigma \in \mathcal{S}_n} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Randrianarivony [17] and Corteel [3] have proved the following continued fraction expansion:

$$E(y, q, t) := \sum_{n \geq 0} \mu_n^{(\ell)}(y, q) t^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{\ddots}}}, \quad (16)$$

where  $b_n = y[n+1]_q + [n]_q$  and  $\lambda_n = y[n]_q^2$ .

We derive then from the classical theory of orthogonal polynomials the following interpretation of the moments of the  $q$ -Laguerre polynomials.

**Theorem 3.** *The  $n$ -th moment of the  $q$ -Laguerre polynomials is equal to  $\mu_n^{(\ell)}(y, q)$ . More precisely, let  $\mathcal{L}_q$  be the linear functional defined by  $\mathcal{L}_q(x^n) = \mu_n^{(\ell)}(y, q)$ , then*

$$\mathcal{L}_q(L_{n_1}(x; q)L_{n_2}(x; q)) = y^{n_1}(n_1!_q)^2\delta_{n_1 n_2}. \quad (17)$$

The first values of the moment sequence are as follows:

$$\begin{aligned} \mu_1^{(\ell)}(y, q) &= y, \\ \mu_2^{(\ell)}(y, q) &= y + y^2, \\ \mu_3^{(\ell)}(y, q) &= y + (3 + q)y^2 + y^3, \\ \mu_4^{(\ell)}(y, q) &= y + (6 + 4q + q^2)y + (6 + 4q + q^2)y^3 + y^4. \end{aligned}$$

Combining the results of Corteel [3], Williams [21, Proposition 4.11] and the classical theory of orthogonal polynomials, one can write the moments of the above  $q$ -Laguerre polynomials as a finite double sum (cf. (28)). Here we propose a direct proof of this result. Actually we shall give such a formula for the moments of Al-Salam-Chihara polynomials.

**Definition 4.** *Define the  $y$ -versions of the  $q$ -Stirling numbers of the second kind by*

$$X^n = \sum_{k=1}^n S_q(n, k, y) \prod_{j=0}^{k-1} (X - [j]_q(1 - yq^{-j})). \quad (18)$$

*The  $y$ -versions of  $q$ -Stirling numbers of the first kind can be defined by the inverse matrix or equivalently*

$$\prod_{j=0}^{n-1} (X - [j]_q(1 - yq^{-j})) = \sum_{k=1}^n s_q(n, k, y) X^k.$$

**Remark 2.** *We have*

$$S_q(n, k, y)|_{q=1} = S(n, k)(1 - y)^{n-k}, \quad S_q(n, k, 0) = S_q(n, k),$$

*where  $S(n, k)$  and  $S_q(n, k)$  are, respectively, the Stirling numbers of the second kind and their well-known  $q$ -analogues, see [11].*

Consider the rescaled Al-Salam-Chihara polynomials  $P_n(x)$ :

$$\begin{aligned} P_n(X) &= Q_n(((q-1)X + 1/\alpha^2 + 1)\alpha/2; \alpha, \beta|q) \\ &= \alpha^{-n} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} q^k (\alpha\beta q^k; q)_{n-k} (1-q)^k q^{\binom{k}{2}} \alpha^{2k} \\ &\quad \times \prod_{j=0}^{k-1} (X - [j]_q(1 - q^{-j}/\alpha^2)). \end{aligned} \quad (19)$$

**Theorem 1.** *The moments of the rescaled Al-Salam-Chihara polynomials  $P_n(X)$  are*

$$\mu_n(\alpha, \beta) = \sum_{k=1}^n S_q(n, k, 1/\alpha^2) (\alpha\beta; q)_k q^{-\binom{k}{2}} (1-q)^{-k} \alpha^{-2k}.$$



*Proof.* Let  $L : X^n \mapsto \mu_n(\alpha, \beta)$  be the linear functional. We check that these moments do satisfy  $L(P_n(X)) = 0$  for  $n > 0$ . Let  $a_k$  be the coefficients in front of the product in (19), then we have, using  $y$ -Stirling orthogonality,

$$\begin{aligned} L(P_n(X)) &= \sum_{k=0}^n a_k \sum_{j=1}^k s_q(k, j, 1/\alpha^2) \sum_{t=1}^j S_q(j, t, 1/\alpha^2) (\alpha\beta; q)_t q^{-\binom{t}{2}} (1-q)^{-t} \alpha^{-2t} \\ &= \sum_{k=0}^n a_k (\alpha\beta; q)_k q^{-\binom{k}{2}} (1-q)^{-k} \alpha^{-2k} \\ &= \alpha^{-n} (\alpha\beta; q)_n \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} q^k = 0. \end{aligned}$$

Note that the last equality follows by applying the  $q$ -binomial formula.  $\square$

**Theorem 2.** *The generating function for the moments  $\mu_n(\alpha, \beta)$  is*

$$\sum_{n=0}^{\infty} \mu_n(\alpha, \beta) t^n = \sum_{k=0}^{\infty} \frac{(\alpha\beta; q)_k q^{-\binom{k}{2}} (1-q)^{-k} \alpha^{-2k} t^k}{\prod_{i=1}^k (1 - [i]_q t (1 - q^{-i}/\alpha^2))}. \quad (20)$$

*Proof.* By definition (18) we have

$$S_q(n, k, y) = S_q(n-1, k-1, y) + [k]_q (1 - yq^{-k}) S_q(n-1, k, y).$$

It follows that (18) is equivalent to

$$\sum_{n \geq k} S_q(n, k, y) t^n = \frac{t^k}{\prod_{i=1}^k (1 - [i]_q t (1 - q^{-i}y))}, \quad (21)$$

which yields immediately (20) in view of Theorem 1.  $\square$

The moment of  $q$ -Charlier polynomials corresponds to the  $\beta = 0$ ,  $\alpha = -1/\sqrt{a(1-q)}$  case, while that of  $q$ -Laguerre polynomials corresponds to the  $\alpha = 1/\sqrt{y}$ ,  $\alpha\beta = q$  case. Therefore,

$$\sum_{n=0}^{\infty} \mu_n^{(c)}(a, q) t^n = \sum_{k=0}^{\infty} \frac{a(qt)^k}{\prod_{i=1}^k (q^i - q^i [i]_q t + a(1-q)[i]_q t)}, \quad (22)$$

$$\sum_{n=0}^{\infty} \mu_n^{(\ell)}(y, q) t^n = \sum_{k=0}^{\infty} \frac{k!_q (qty)^k}{\prod_{i=1}^k (q^i - q^i [i]_q t + [i]_q qty)}. \quad (23)$$

**Theorem 3.** *Let  $p = 1/q$ . We have*

$$\sum_{k=0}^{\infty} \frac{(\alpha\beta; q)_k q^{-\binom{k}{2}} (1-q)^{-k} \alpha^{-2k} t^k}{\prod_{i=1}^k (1 - [i]_q t (1 - q^{-i}/\alpha^2))} = \sum_{i \geq 0} \frac{c_i(\alpha, \beta)}{1 - [i]_q t (1 - q^{-i}/\alpha^2)}, \quad (24)$$

where

$$c_i(\alpha, \beta) = \frac{(\alpha\beta; q)_i}{(q; q)_i} \frac{q^{i-i^2} \alpha^{-2i}}{(q^{1-2i}/\alpha^2; q)_i} \frac{(p^{1+i} \alpha \beta / \alpha^2; p)_{\infty}}{(p^{1+2i} / \alpha^2; p)_{\infty}}.$$

*Proof.* Note the following partial fraction decomposition formula:

$$\frac{t^k}{(1-a_1t)(1-a_2t)\dots(1-a_kt)} = \frac{(-1)^k}{a_1 \cdots a_k} + \sum_{i=1}^k \frac{a_i^{-1} \prod_{j=1, j \neq i}^k (a_i - a_j)^{-1}}{1 - a_i t}.$$

Therefore

$$\frac{t^k}{\prod_{i=1}^k (1 - [i]_q t (1 - q^{-i}/\alpha^2))} = \sum_{i=0}^k \frac{\gamma_k(i)}{1 - [i]_q t (1 - q^{-i}/\alpha^2)}, \quad (25)$$

where

$$\gamma_k(i) = \frac{1}{k!_q} \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{\alpha^{2(k-i)} q^{\binom{k}{2} + k - i^2}}{(q^{1-2i}/\alpha^2; q)_i (q^{1+2i}\alpha^2; q)_{k-i}} \quad (0 \leq i \leq k).$$

Substituting this in (24) yields

$$\begin{aligned} c_i(\alpha, \beta) &= \sum_{k \geq i} \frac{(\alpha\beta; q)_k}{(q; q)_k} \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{q^{k-i^2} \alpha^{-2i}}{(q^{1-2i}/\alpha^2; q)_i (q^{1+2i}\alpha^2; q)_{k-i}} \\ &= \frac{(\alpha\beta; q)_i}{(q; q)_i} \frac{q^{i-i^2} \alpha^{-2i}}{(q^{1-2i}/\alpha^2; q)_i} \sum_{k \geq 0} \frac{(\alpha\beta q^i; q)_k}{(q; q)_k} \frac{q^k}{(q^{1+2i}\alpha^2; q)_k}. \end{aligned}$$

The theorem follows then by applying the  ${}_1\Phi_1$  summation formula (see [10, II.5]).  $\square$

By partial fraction decomposition (see [21, Theorem 4.12]), we get

$$\sum_{n=0}^{\infty} \mu_n^{(c)}(a, q) t^n = \sum_{i \geq 0} \frac{a^i (1 - a(1-q)p^{2i}) / (a(1-q)p^i; p)_{\infty}}{i!_q q^{i^2} (q^i - q^i [i]_q t + a [i]_q t (1-q))}, \quad (26)$$

$$\sum_{n=0}^{\infty} \mu_n^{(\ell)}(y, q) t^n = \sum_{i \geq 0} \frac{y^i (q^{2i} - y)}{q^{i^2} (q^i - q^i [i]_q t + [i]_q t y)}. \quad (27)$$

Note that (27) yields the following polynomial formula in  $y$  for  $\mu_n^{(\ell)}(y, q)$ :

$$\mu_n^{(\ell)}(y, q) = \sum_{k=1}^n \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n q^{k(i-k)} \left( \binom{n}{i} q^{k-i} + \binom{n}{i-1} \right) y^k, \quad (28)$$

while (26) does not yield such a polynomial formula in  $a$  for  $\mu_n^{(c)}(a, q)$ .

On the other hand, it follows from (25) and (21) that

$$S_q(n, k, y) = \frac{q^{-\binom{k}{2}}}{k!_q} \sum_{i=1}^k \begin{bmatrix} k \\ i \end{bmatrix}_q y^{i-k} q^{k^2 - i^2} \frac{([i]_q (1 - q^{-i} y))^n}{(q^{1-2i} y; q)_i (q^{1+2i}/y; q)_{k-i}}. \quad (29)$$

Using Theorem 1 and the above explicit formula for  $q$ -Stirling numbers we can also write the moments  $\mu_n(\alpha, \beta)$  as a double sum.

5. LINEARIZATION COEFFICIENTS OF THE  $q$ -LAGUERRE POLYNOMIALS

The following is our main result of this section.

**Theorem 5.** *The linearization coefficients of the  $q$ -Laguerre polynomials are*

$$\mathcal{L}_q(L_{n_1}(x; q) \dots L_{n_k}(x; q)) = \sum_{\sigma \in \mathcal{D}(n_1, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}. \quad (30)$$

A proof à la Viennot (cf. [12, 15]) of (30) would use the combinatorial interpretations for the moments and  $q$ -Laguerre polynomials to rewrite the left-hand side of (30) and then construct an adequate *killing involution* on the resulting set. For the time being we do not have such a proof to offer, instead we provide an inductive proof.

We first show that the above result is true for  $(n_1, \dots, n_k) = (1, \dots, 1)$ .

**Lemma 6.** *Let  $d_n(y, q) = \sum_{\sigma \in \mathcal{D}_n} y^{wex(\sigma)} q^{cr(\sigma)}$ . Then  $\mathcal{L}_q((x - y)^n) = d_n(y, q)$ .*

*Proof.* Note that

$$\mathcal{L}_q((x - y)^n) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} y^{n-k} \mu_k^{(\ell)}(y, q).$$

By binomial inversion, it suffices to prove that

$$\mu_n^{(\ell)}(y, q) = \sum_{k=0}^n \binom{n}{k} y^k d_{n-k}(y, q).$$

But the latter identity is obvious. □

The invariance of  $\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}$  by permutating the  $n'_i$ 's is a direct consequence of Theorem 5, but for our proof we need to first establish this property.

**Theorem 7.** *For any permutation  $\gamma \in \mathcal{S}_k$  we have*

$$\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in \mathcal{D}(n_{\gamma(1)}, n_{\gamma(2)}, \dots, n_{\gamma(k)})} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Since the two cyclic permutations  $(1, 2)$  and  $(1, 2, 3, \dots, k)$  generate the symmetric group  $\mathcal{S}_k$ , Theorem 7 is a corollary of the following two lemmas (proved in the next two sections).

**Lemma 8.**

$$\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in \mathcal{D}(n_2, n_3, \dots, n_k, n_1)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

**Lemma 9.**

$$\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in \mathcal{D}(n_2, n_1, n_3, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

**Proof of Theorem 4.** Writing (14) as

$$(x - y)L_n(x) = L_{n+1}(x) + (yq + 1)[n]_q L_n(x) + y[n]_q^2 L_{n-1}(x),$$

we derive that

$$\sum_{\sigma \in \mathcal{D}(1, n, n_2, \dots, n_k)} w(\pi) = \sum_{\sigma \in \mathcal{D}(n+1, n_2, \dots, n_k)} w(\pi) + (yq+1)[n]_q \sum_{\sigma \in \mathcal{D}(n, n_2, \dots, n_k)} w(\pi) + y[n]_q^2 \sum_{\sigma \in \mathcal{D}(n-1, n_2, \dots, n_k)} w(\pi), \quad (31)$$

where  $w(\pi) = y^{wex(\sigma)} q^{cr(\sigma)}$ . In view of Lemma 6 it suffices to prove (31).

We distinguish four cases for permutations  $\pi \in \mathcal{D}(1, n, n_2, \dots, n_k)$ .

a)  $\pi(1), \pi^{-1}(1) \in \{2, \dots, n+1\}$ . Let  $\pi(1) = i$  and  $\pi(j) = 1$  with  $i, j \in \{2, \dots, n+1\}$ .

Then we define the mapping  $\pi \rightarrow \pi' \in \mathcal{D}(n-1, n_2, \dots, n_k)$  by deleting 1 and  $j$  and adding the edge  $\pi^{-1}(j) \rightarrow i$  if  $i \neq j$ . Clearly

$$w(\pi) = yq^{(i-1)+(j-1)-2} w(\pi').$$

Summing over all  $i, j \in \{2, \dots, n+1\}$  yields the generating function:

$$y[n]_q^2 \sum_{\sigma \in \mathcal{D}(n-1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

b)  $\pi(1) \in \{2, \dots, n+1\}$  and  $\pi^{-1}(1) > n+1$ . We define the mapping  $\pi \rightarrow \pi' \in \mathcal{D}(n, n_2, \dots, n_k)$  by deleting  $i := \pi(1)$  and replace the two edges  $1 \rightarrow \pi(1) \rightarrow \pi^2(1)$  by  $1 \rightarrow \pi^2(1)$ . Clearly  $w(\pi) = yq^{i-1} w(\pi')$ . Summing over all  $i = 2, \dots, n+1$  yields the generating function:

$$qy[n]_q \sum_{\sigma \in \mathcal{D}(n, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

c)  $\pi^{-1}(1) \in \{2, \dots, n+1\}$  and  $\pi(1) > n+1$ . We define the mapping  $\pi \rightarrow \pi' \in \mathcal{D}(n, n_2, \dots, n_k)$  by deleting  $i := \pi^{-1}(1)$  and replace the two edges  $1 \leftarrow \pi^{-1}(1) \leftarrow \pi^{-2}(1)$  by  $1 \leftarrow \pi^{-2}(1)$ . Clearly  $w(\pi) = q^{i-2} w(\pi')$ . Summing over all  $i = 2, \dots, n+1$  yields the generating function:

$$[n]_q \sum_{\sigma \in \mathcal{D}(n, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

d)  $\pi(1) > n+1$  and  $\pi^{-1}(1) > n+1$ . Clearly we can consider  $\pi$  as a permutation in  $\mathcal{D}(n+1, n_2, \dots, n_k)$ . The generating function is

$$\sum_{\sigma \in \mathcal{D}(n+1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Summing up we obtain (31).  $\square$

When  $k = 2$  Theorem 4 reduces to the orthogonality of the  $q$ -Laguerre polynomials (17). When  $k = 3$  we can derive the following explicit formula from Theorem 1.

**Theorem 10.** *We have*

$$\begin{aligned} \mathcal{L}_q(L_{n_1}(x; q)L_{n_2}(x; q)L_{n_3}(x; q)) &= \sum_s \frac{n_1!_q n_2!_q n_3!_q s!_q y^s}{(n_1 + n_2 + n_3 - 2s)!_q (s - n_3)!_q (s - n_2)!_q (s - n_1)!_q} \\ &\quad \times \sum_k \begin{bmatrix} n_1 + n_2 + n_3 - 2s \\ k \end{bmatrix}_q y^k q^{\binom{k+1}{2} + (n_1 + n_2 + n_3 - 2s - k)}. \end{aligned}$$

*Proof.* By Theorem 1 with  $a = \frac{1}{\sqrt{y}}$  and  $b = \sqrt{y}q$  we have

$$\begin{aligned} & \mathcal{L}_q(L_{n_1}(x; q)L_{n_2}(x; q)L_{n_3}(x; q)) \\ &= \mathcal{L}_q(L_{n_3}(x; q)^2) \left( \frac{\sqrt{y}}{q-1} \right)^{n_1+n_2-n_3} C_{n_1, n_2}^{n_3}(a, b; q) \\ &= \sum_{m_2, m_3} \frac{n_1!_q n_2!_q n_3!_q (n_1 + m_3)!_q y^{n_2+n_3-m_2-m_3} q^{\binom{m_2}{2} + (n_3+n_2-n_1-\frac{m_2}{2}-2m_3+1)}}{(n_3 + n_2 - n_1 - m_2 - 2m_3)!_q m_2!_q (m_3 + n_1 - n_3)!_q (m_3 + n_1 - n_2)!_q m_3!_q}. \end{aligned}$$

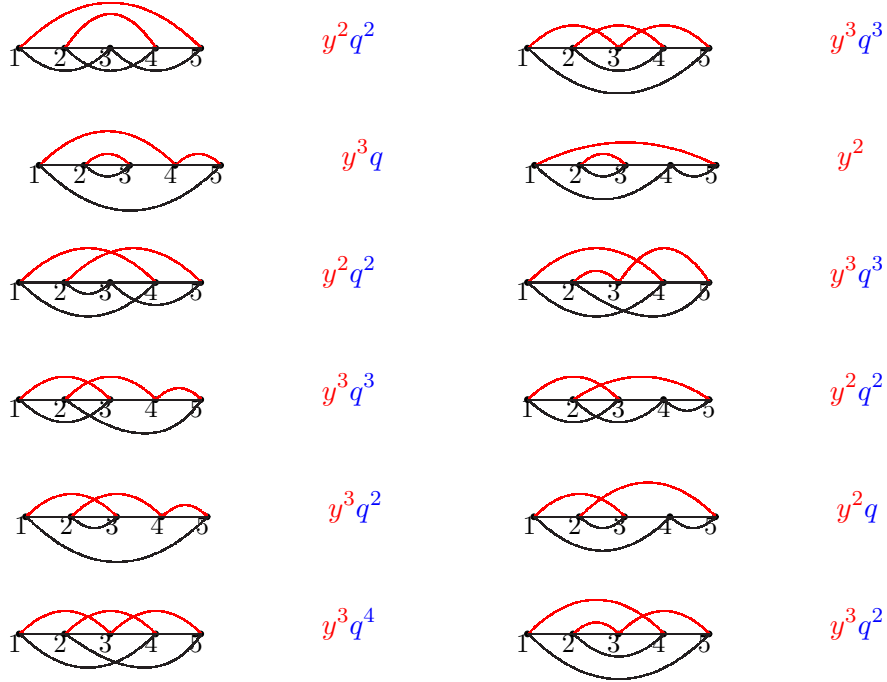
Substituting  $s = n_1 + m_3$  and  $k = n_3 + n_2 - n_1 - m_2 - 2m_3$  in the last sum yields the desired formula.  $\square$

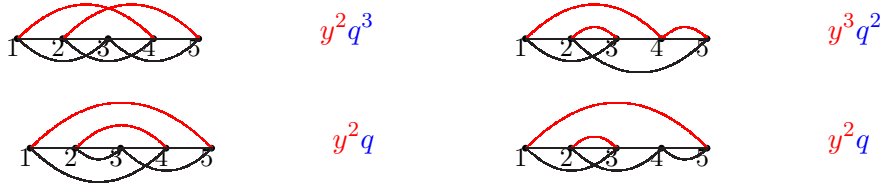
**Remark 3.** *It would be interesting to give a combinatorial proof of the above result as in [12, 15]. When  $q = 1$  such a proof was given in [23].*

We end this section with an example. If  $\mathbf{n} = (2, 2, 1)$ , by Theorem 8 we have

$$\begin{aligned} \mathcal{L}_q(L_2(x; q)L_2(x; q)L_1(x; q)) &= \sum_s \frac{2!_q 2!_q 1!_q s!_q y^s}{(5-2s)!_q (s-1)!_q (s-2)!_q (s-2)!_q} \\ &\quad \times \sum_{k \geq 0} \begin{bmatrix} 5-2s \\ k \end{bmatrix}_q y^k q^{\binom{k+1}{2} + (5-2s-k)} \\ &= (1+q)^3 (1+qy)y^2. \end{aligned} \tag{32}$$

On the other hand, the sixteen derangements, depicted by their diagrams and the corresponding weights are tabulated as follows:





Summing up we get  $\sum_{\sigma \in D(2,2,1)} y^{w_{ex} \sigma} q^{cr \sigma} = y^2(1 + qy)(1 + q)^3$ , which coincides with (32).

6. PROOF OF LEMMA 8

For each fixed  $k \in [n]$  define the two subsets of  $\mathcal{S}_n$ :

$${}^k\mathcal{S}_n = \{\sigma \in \mathcal{S}_n \mid \sigma(i) > k \text{ for } 1 \leq i \leq k\},$$

$$\mathcal{S}_n^k = \{\sigma \in \mathcal{S}_n \mid \sigma(n+1-i) < n+1-k \text{ for } 1 \leq i \leq k\}.$$

We first construct a simple bijection  $\Phi_k : {}^k\mathcal{S}_n \rightarrow \mathcal{S}_n^k$ . Let  $\sigma \in {}^k\mathcal{S}_n$ . For  $1 \leq i \leq n$  we define  $\sigma'(i) := \Phi_k(\sigma)(i)$  as follows:

$$\sigma'(i) = \begin{cases} \sigma(i+k) - k, & \text{if } 1 \leq i \leq n-k \text{ and } \sigma(i+k) > k; \\ \sigma(i+k) + n - k, & \text{if } 1 \leq i \leq n-k \text{ and } \sigma(i+k) \leq k; \\ \sigma(i+k-n) - k, & \text{if } n-k+1 \leq i \leq n. \end{cases}$$

We can illustrate the map by the diagrams of permutations.

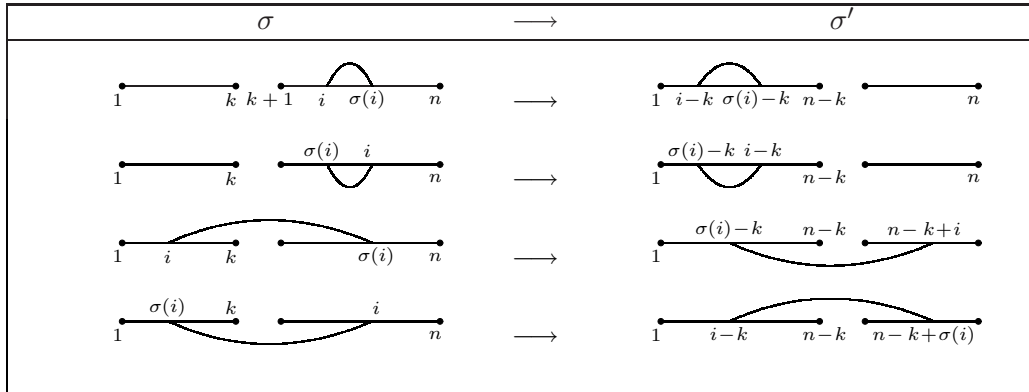
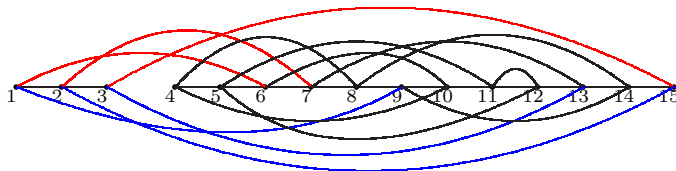
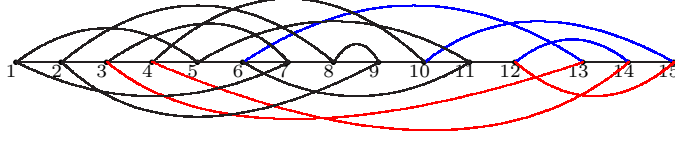


TABLE 1. The mapping  $\Phi_k : \sigma \rightarrow \sigma'$ .

For example, consider the permutation  $\sigma \in {}^3\mathcal{S}_{15}$ , whose diagram is given below.



Then the diagram of  $\Phi_3(\sigma)$  is given by



The main properties of  $\Phi_k$  are summarized in the following result.

**Lemma 11.** *For each positive integer  $k \in [n]$ , the map  $\Phi_k : {}^k\mathcal{S}_n \rightarrow \mathcal{S}_n^k$  is a bijection such that for any  $\sigma \in {}^k\mathcal{S}_n$  there holds*

$$(wex, cr)\Phi_k(\sigma) = (wex, cr)\sigma. \quad (33)$$

Now, Lemma 8 is an immediate consequence of Lemma 11. Let  $n = n_1 + n_2 + \dots + n_k$ . Then  $\mathcal{D}(n_1, n_2, \dots, n_k) \subseteq {}^{n_1}\mathcal{S}_n$ . By definition of  $\Phi_{n_1}$ , for any  $\sigma \in {}^{n_1}\mathcal{S}_n$  and  $i \in [n - n_1]$  satisfying  $\sigma(i + n_1) > n_1$ , we have  $i - \Phi_{n_1}(\sigma)(i) = i + n_1 - \sigma(i + n_1)$ , so  $\Phi_{n_1}(\mathcal{D}(n_1, n_2, \dots, n_k)) \subseteq \mathcal{D}(n_2, n_3, \dots, n_k, n_1)$ . Since the cardinality of  $\mathcal{D}(n_1, n_2, \dots, n_k)$  is invariant by permutations of the  $n_i$ 's and  $\Phi_{n_1}$  is bijective, we have  $\Phi_{n_1}(\mathcal{D}(n_1, n_2, \dots, n_k)) = \mathcal{D}(n_2, n_3, \dots, n_k, n_1)$ . The result follows then by applying (33).

$i$	$L_i(\sigma)$	$R_i(\sigma')$
1		
2		
3		

TABLE 2. Forms of crossings in  $L_i(\sigma)$  and  $R_i(\sigma')$ .

**Proof of Lemma 11** It is easy to see that  $\Phi_k$  is a bijection. Let  $\sigma \in {}^k\mathcal{S}_n$  and  $\sigma' = \Phi_k(\sigma)$ . The equality  $wex(\sigma') = wex(\sigma)$  follows directly from the definition of  $\Phi_k$ . It then remains to prove

that  $cr(\sigma') = cr(\sigma)$ . We first decompose the crossings of  $\sigma$  and  $\sigma'$  into three subsets. Set

$$L_1(\sigma) = \{(i, j) \mid k < i < j \leq \sigma(i) < \sigma(j) \text{ or } i > j > \sigma(i) > \sigma(j) > k\},$$

$$L_2(\sigma) = \{(i, j) \mid i < j \leq k < \sigma(i) < \sigma(j) \text{ or } i > j > k \geq \sigma(i) > \sigma(j)\},$$

$$L_3(\sigma) = \{(i, j) \mid i \leq k < j \leq \sigma(i) < \sigma(j) \text{ or } i > j > \sigma(i) > k \geq \sigma(j)\},$$

and

$$R_1(\sigma') = \{(i, j) \mid i < j \leq \sigma'(i) < \sigma'(j) \leq n - k \text{ or } n - k \geq i > j > \sigma'(i) > \sigma'(j)\},$$

$$R_2(\sigma') = \{(i, j) \mid i < j \leq n - k < \sigma'(i) < \sigma'(j) \text{ or } i > j > n - k \geq \sigma'(i) > \sigma'(j)\},$$

$$R_3(\sigma') = \{(i, j) \mid i < j \leq \sigma'(i) \leq n - k < \sigma'(j) \text{ or } i > n - k \geq j > \sigma(i) > \sigma(j)\}.$$

The "forms" of the crossings in the  $L_i$ 's and  $R_i$ 's is given in Table 2. Clearly, we have  $cr(\sigma) = \sum_{i=1}^3 |L_i(\sigma)|$  and  $cr(\sigma') = \sum_{i=1}^3 |R_i(\sigma')|$  since  $\sigma \in {}^k\mathcal{S}_n$  and  $\sigma' \in \mathcal{S}_n^k$ .

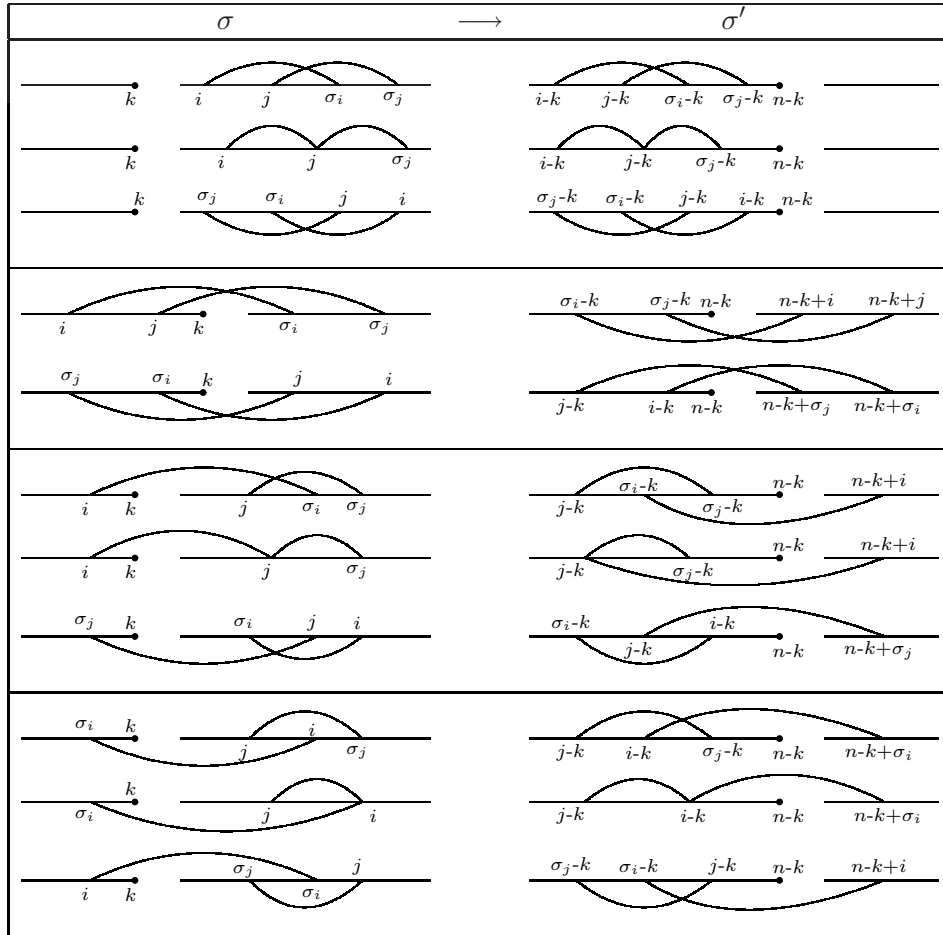


TABLE 3. Effects of the mapping  $\Phi_k$  on the crossings of  $\sigma$  and  $\sigma'$ .



By the definition of  $\Phi_k$ , it is readily seen (see Row 1 in Table 3) that  $(i, j) \in L_1(\sigma)$  if and only if  $(i - k, j - k) \in R_1(\sigma')$ , and thus  $|L_1(\sigma)| = |R_1(\sigma')|$ . Similarly, we have (see Row 2 in Table 3) that  $|L_2(\sigma)| = |R_2(\sigma')|$ . It then remains to prove that  $|L_3(\sigma)| = |R_3(\sigma')|$ . Let

$$L_4(\sigma) = \{(i, j) \mid \sigma(i) \leq k < j < i \leq \sigma(j) \text{ or } i \leq k < \sigma(j) < \sigma(i) < j\}.$$

Then it is not difficult to show (see Row 4 of Table 3) that  $|R_3(\sigma')| = |L_4(\sigma)|$ . It then suffices to prove that  $|L_3(\sigma)| = |L_4(\sigma)|$ .

Suppose  $\sigma([1, k]) = \{i_1, i_2, \dots, i_k\}$  and  $\sigma^{-1}([1, k]) = \{j_1, j_2, \dots, j_k\}$ . Then by definition of  $L_3(\sigma)$  and  $L_4(\sigma)$  we have

$$|L_3(\sigma)| = \sum_{s=1}^k |\{\ell \mid k < \ell \leq i_s < \sigma(\ell)\}| + |\{\ell \mid \ell > j_s > \sigma(\ell) > k\}|, \quad (34)$$

$$|L_4(\sigma)| = \sum_{s=1}^k |\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| + |\{\ell \mid k < \ell < j_s \leq \sigma(\ell)\}|. \quad (35)$$

For any integer  $i \in [n]$  set  $A_i(\sigma) = \{j \mid j < i < \sigma(j)\}$ . Note that it is easily seen that

$$|A_i(\sigma)| = |\{j \mid j < i < \sigma(j)\}| = |\{j \mid j > i > \sigma(j)\}| = |A_i(\sigma^{-1})|. \quad (36)$$

Let  $s \in [k]$ . By elementary manipulations we get

$$\begin{aligned} |\{\ell \mid k < \ell \leq i_s < \sigma(\ell)\}| &= |\{\ell \mid \ell \leq i_s < \sigma(\ell)\}| - |\{\ell \mid \ell \leq k < i_s < \sigma(\ell)\}| \\ &= |A_{i_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{\ell \mid \ell \leq k < i_s < \sigma(\ell)\}| \\ &= |A_{i_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{t \mid i_t > i_s\}|. \end{aligned} \quad (37)$$

By a similar reasoning, we obtain the following identities:

$$|\{\ell \mid \ell > j_s > \sigma(\ell) > k\}| = |A_{j_s}(\sigma^{-1})| - |\{t \mid j_t > j_s\}|. \quad (38)$$

$$|\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| = |A_{i_s}(\sigma^{-1})| - |\{t \mid j_t > i_s\}| \quad (39)$$

$$|\{\ell \mid k \leq \ell < j_s \leq \sigma(\ell)\}| = |A_{j_s}(\sigma)| + \chi(k < \sigma^{-1}(j_s) < j_s) - |\{t \mid i_t > j_s\}|. \quad (40)$$

Inserting (37) and (38) in (34) and using (36), we get

$$|L_3(\sigma)| = \sum_{s=1}^k |A_{i_s}(\sigma)| + |A_{j_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{t \mid i_t > i_s\}| - |\{t \mid j_t > j_s\}|. \quad (41)$$

Similarly, inserting (39) and (40) in (35) and using (36), we get

$$|L_4(\sigma)| = \sum_{s=1}^k |A_{i_s}(\sigma)| + |A_{j_s}(\sigma)| + \chi(k < \sigma^{-1}(j_s) < j_s) - |\{t \mid j_t > i_s\}| - |\{t \mid i_t > j_s\}|. \quad (42)$$

Since the  $i_t$ 's and  $j_t$ 's are distinct we have  $\sum_{s=1}^k |\{t \mid i_t > i_s\}| = \sum_{s=1}^k |\{t \mid j_t > j_s\}| = \binom{k}{2}$  and thus

$$\sum_{s=1}^k |\{t \mid i_t > i_s\}| + |\{t \mid j_t > j_s\}| = k(k-1). \quad (43)$$

On the other hand,

$$\begin{aligned}
\sum_{s=1}^k |\{t \mid j_t > i_s\}| + |\{t \mid i_t > j_s\}| &= \sum_{s=1}^k |\{t \mid i_t \neq j_s\}| \\
&= k^2 - \sum_{s=1}^k \chi(j_s \in \{i_1, i_2, \dots, i_k\}) \\
&= k^2 - \sum_{s=1}^k \chi(\sigma^{-1}(j_s) \leq k). \tag{44}
\end{aligned}$$

Also, it follows from the definition of the  $j_t$ 's that for any  $s \in [k]$ , we have  $j_s > k$  and  $\sigma^{-1}(j_s) \neq j_s$ , and thus

$$\chi(k < \sigma^{-1}(j_s) < j_s) + \chi(\sigma^{-1}(j_s) > j_s) = \chi(\sigma^{-1}(j_s) > k). \tag{45}$$

Inserting (53), (54) and (45) in (41) and (42) lead to  $|L_3(\sigma)| = |L_4(\sigma)|$  as desired.

## 7. PROOF OF LEMMA 9

For any two integers  $n_1, n_2$  satisfying  $N_2 := n_1 + n_2 \leq n$  we denote by  $\mathcal{S}_n^{(n_1, n_2)}$  the set of permutations  $\sigma$  in  $\mathcal{S}_n$  such that

$$(i, \sigma(i)) \notin [1, n_1]^2 \cup [n_1 + 1, N_2]^2.$$

In other words, any two integers in  $[1, n_1]$  or  $[n_1 + 1, N_2]$  are not connected by an arc in its graph.

We now construct a map  $\Gamma^{(n_1, n_2)} : \sigma \mapsto \sigma'$  from  $\mathcal{S}_n^{(n_1, n_2)}$  to  $\mathcal{S}_n^{(n_2, n_1)}$  as follows. For  $i = 1, \dots, n$ ,

- (1) If  $i > N_2$  and  $\sigma(i) > N_2$ , set  $\sigma'(i) = \sigma(i)$ .
- (2) Suppose

$$\begin{aligned}
\{(i, \sigma(i)) \mid i < \sigma(i) \leq N_2\} &= \{(i_1, N_2 + 1 - j_1), (i_2, N_2 + 1 - j_2), \dots, (i_p, N_2 + 1 - j_p)\} \\
\{(\sigma(i), i) \mid \sigma(i) < i \leq N_2\} &= \{(k_1, N_2 + 1 - \ell_1), (k_2, N_2 + 1 - \ell_2), \dots, (k_q, N_2 + 1 - \ell_q)\}.
\end{aligned}$$

Then set  $\sigma'(j_s) = N_2 + 1 - i_s$  and  $\sigma'(N_2 + 1 - k_t) = \ell_t$  for any  $s \in [p]$  and  $t \in [q]$ .

- (3) Let

$$C = \{i \in [1, N_2] ; \sigma(i) > N_2\} \quad \text{and} \quad D = \{i \in [1, N_2] ; \sigma^{-1}(i) > N_2\}.$$

It is easy to see that  $|C| = |D|$ . Suppose  $C = \{c_1, c_2, \dots, c_u\}_<$ ,  $D = \{d_1, d_2, \dots, d_u\}_<$ ,  $\sigma(C) = \{r_1, r_2, \dots, r_u\}_<$  and  $\sigma^{-1}(D) = \{s_1, s_2, \dots, s_u\}_<$ . Let  $\alpha, \beta \in \mathcal{S}_u$  be the (unique) permutations satisfying  $\sigma(c_i) = r_{\alpha(i)}$  and  $\sigma^{-1}(d_i) = s_{\beta(i)}$  for each  $1 \leq i \leq u$ . Let

$$\begin{aligned}
E &= [1, N_2] \setminus \{j_1, \dots, j_p, N_2 + 1 - k_1, \dots, N_2 + 1 - k_q\} \\
F &= [1, N_2] \setminus \{N_2 + 1 - i_1, \dots, N_2 + 1 - i_p, \ell_1, \dots, \ell_q\}.
\end{aligned}$$

Clearly, we have  $|E| = |C|$  and  $|F| = |D|$ . Suppose  $E = \{e_1, \dots, e_u\}_<$  and  $F = \{f_1, \dots, f_u\}_<$ . Then set  $\sigma'(e_i) = r_{\alpha(i)}$  and  $\sigma'(s_i) = f_{\beta(i)}$  for each  $1 \leq i \leq u$ .

We can illustrate the map through the diagrams of permutations. See Table 4.

For example, if we consider the permutation in  $\mathcal{S}_{15}^{(3,4)}$  whose diagram is given by

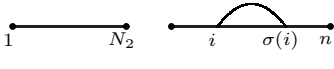
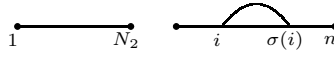
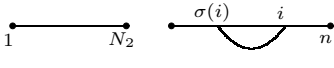
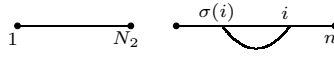
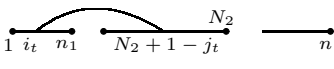
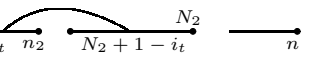
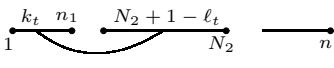
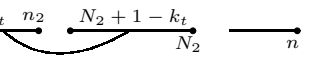
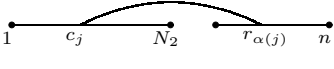
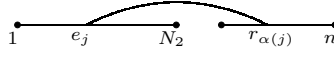
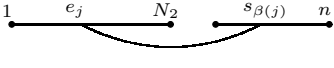
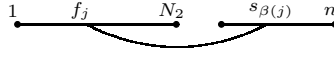
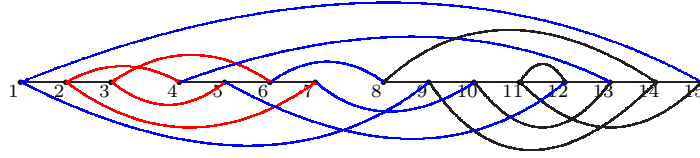
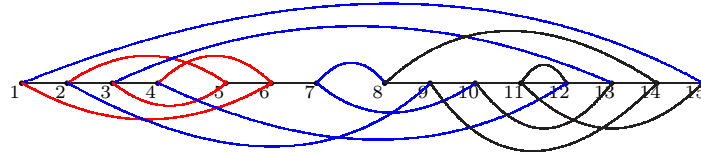
$\sigma$	$\longrightarrow$	$\sigma'$
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	

TABLE 4. The mapping  $\Gamma^{(n_1, n_2)} : \sigma \mapsto \sigma'$ 

then the diagram of  $\Gamma^{(n_1, n_2)}(\sigma)$  is given by



It is not hard to check that  $\Gamma^{(n_1, n_2)}$  is well defined from  $\mathcal{S}_n^{(n_1, n_2)}$  to  $\mathcal{S}_n^{(n_2, n_1)}$ . Since each step of the construction of  $\Gamma^{(n_1, n_2)}$  is reversible, the map  $\Gamma^{(n_1, n_2)}$  is bijective. Actually we can prove, the details are left to the reader, that  $(\Gamma^{(n_1, n_2)})^{-1} = \Gamma^{(n_2, n_1)}$ .

**Lemma 12.** *For each positive integers  $n_1, n_2, n$ , with  $N_2 \leq n$ , the map  $\Gamma^{(n_1, n_2)}$  is a bijection from  $\mathcal{S}_n^{(n_1, n_2)}$  to  $\mathcal{S}_n^{(n_2, n_1)}$  such that for each  $\sigma \in \mathcal{S}_n^{(n_1, n_2)}$ , we have*

$$(wex, cr)\Gamma^{(n_1, n_2)}(\sigma) = (wex, cr)\sigma. \quad (46)$$

As an immediate consequence, we obtain Lemma 9. Let  $n = n_1 + n_2 + \dots + n_k$ . Then  $\mathcal{D}(n_1, n_2, \dots, n_k) \subseteq \mathcal{S}_n^{(n_1, n_2)}$ . By definition of  $\Gamma^{(n_1, n_2)}$ , for any  $\sigma \in \mathcal{S}_n^{(n_1, n_2)}$  and  $i > N_2$  satisfying

$\sigma(i) > N_2$ , we have

$$i - \Gamma^{(n_1, n_2)}(\sigma)(i) = i - \sigma(i),$$

so  $\Gamma^{(n_1, n_2)}(\mathcal{D}(n_1, n_2, \dots, n_k)) \subseteq \mathcal{D}(n_2, n_3, \dots, n_k, n_1)$ . Since the cardinality of  $\mathcal{D}(n_1, n_2, \dots, n_k)$  doesn't depend of the order of the  $n_i$ 's and  $\Gamma^{(n_1, n_2)}$  is a bijection, we have

$$\Gamma^{(n_1, n_2)}(\mathcal{D}(n_1, n_2, \dots, n_k)) = \mathcal{D}(n_2, n_3, \dots, n_k, n_1).$$

Lemma 9 then follows from (46).

$i$	$G_i^{(n_1, n_2)}(\gamma)$	$G_i^{(n_2, n_1)}(\gamma)$
1		
2		
3		
4		
5		

TABLE 5. Forms of the crossings in  $G_i^{(n_1, n_2)}(\gamma)$  and  $G_i^{(n_2, n_1)}(\gamma)$ .

**Proof of Lemma 12** It was shown above that  $\Gamma^{(n_1, n_2)}$  is bijective. Let  $\sigma \in \mathcal{S}_n^{(n_1, n_2)}$  and  $\sigma' := \Gamma^{(n_1, n_2)}(\sigma)$ . The equality  $wex(\sigma') = wex(\sigma)$  is an immediate consequence of the definition

of  $\Gamma^{(n_1, n_2)}$ . It then remains to prove that  $cr(\sigma') = cr(\sigma)$ . The idea is the same than for the proof of Lemma 8. We first decompose the number of crossings of  $\sigma$  and  $\sigma'$ . For each permutation  $\gamma \in \mathcal{S}_n$ , set

$$\begin{aligned} G_1^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid N_2 < i < j \leq \gamma(i) < \gamma(j) \text{ or } i > j > \gamma(i) > \gamma(j) > N_2\}, \\ G_2^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid i < j < \gamma(i) < \gamma(j) \leq N_2 \text{ or } N_2 \geq i > j > \gamma(i) > \gamma(j)\}, \\ G_3^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid i < j \leq N_2 < \gamma(i) < \gamma(j) \text{ or } i > j > N_2 \geq \gamma(i) > \gamma(j)\}, \\ G_4^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid i \leq N_2 < j \leq \gamma(i) < \gamma(j) \text{ or } i > j > \gamma(i) > N_2 \geq \gamma(j)\}, \\ G_5^{(n_1, n_2)}(\gamma) &= \{(i, j) \mid i < j \leq \gamma(i) \leq N_2 < \gamma(j) \text{ or } i > N_2 \geq j > \gamma(i) > \gamma(j)\}. \end{aligned}$$

Clearly, for any  $\gamma \in \mathcal{S}_n^{(n_1, n_2)}$ , we have  $cr(\gamma) = \sum_{i=1}^5 |G_i^{(n_1, n_2)}(\gamma)|$ . In particular,

$$cr(\sigma) = \sum_{i=1}^5 |G_i^{(n_1, n_2)}(\sigma)| \quad \text{and} \quad cr(\sigma') = \sum_{i=1}^5 |G_i^{(n_2, n_1)}(\sigma')|. \quad (47)$$

The "forms" of the crossings in the  $G_i^{(n_1, n_2)}$ 's and  $G_i^{(n_2, n_1)}$ 's are given in Table 5. By the definition of  $\Gamma^{(n_1, n_2)}$ , it is readily seen (see Row 1 in Table 6) that  $G_1^{(n_1, n_2)}(\sigma) = G_1^{(n_2, n_1)}(\sigma')$  and thus  $|G_1^{(n_1, n_2)}(\sigma)| = |G_1^{(n_2, n_1)}(\sigma')|$ . By similar considerations we can prove (see Table 6) that  $|G_i^{(n_1, n_2)}(\sigma)| = |G_i^{(n_2, n_1)}(\sigma')|$  for  $i = 2, 3, 4$ . It then suffices to prove that  $|G_5^{(n_1, n_2)}(\sigma)| = |G_5^{(n_2, n_1)}(\sigma')|$  which will follow from the following lemma.

**Lemma 13.** *Let  $n_1, n_2, n$  be positive integers with  $N_2 \leq n$  and  $\gamma \in \mathcal{S}_n^{(n_1, n_2)}$ . Suppose that*

$$\begin{aligned} B(\gamma) &:= \{(i, \gamma(i)) \mid i < \gamma(i) \leq N_2\} = \{(i_1, j_1), (i_2, j_2), \dots, (i_p, j_p)\} \\ B(\gamma^{-1}) &= \{(\gamma(i), i) \mid \gamma(i) < i \leq N_2\} = \{(k_1, \ell_1), (k_2, \ell_2), \dots, (k_q, \ell_q)\}, \end{aligned}$$

with  $i_1 < i_2 < \dots < i_p$  and  $k_1 < k_2 < \dots < k_q$ . Then we have

$$|G_5^{(n_1, n_2)}(\gamma)| = \sum_{r=1}^p (j_r - i_r) + \sum_{r=1}^q (\ell_r - k_r - 1) - \binom{p+q}{2}. \quad (48)$$

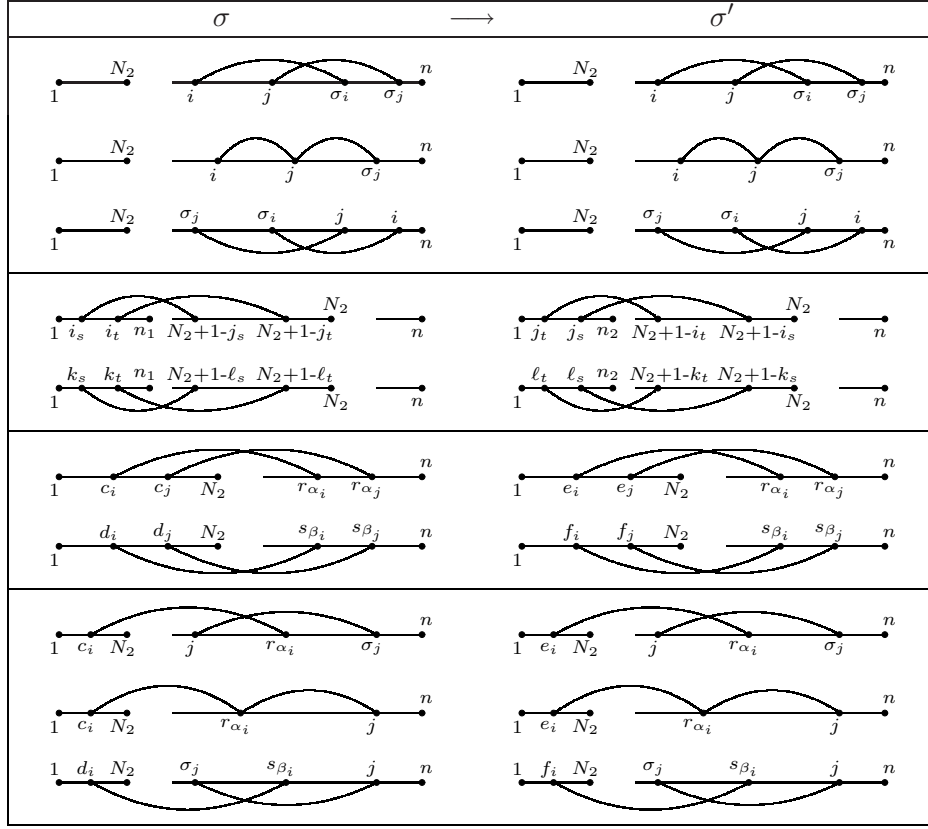
Suppose

$$\begin{aligned} B(\sigma) &= \{(i_1, N_2 + 1 - j_1), \dots, (i_p, N_2 + 1 - j_p)\} \\ B(\sigma^{-1}) &= \{(k_1, N_2 + 1 - \ell_1), \dots, (k_q, N_2 + 1 - \ell_q)\}, \end{aligned}$$

then, by construction of  $\sigma'$ , we have

$$\begin{aligned} B(\sigma') &= \{(j_1, N_2 + 1 - i_1), \dots, (j_p, N_2 + 1 - i_p)\} \\ B(\sigma'^{-1}) &= \{(\ell_1, N_2 + 1 - k_1), \dots, (\ell_q, N_2 + 1 - k_q)\}. \end{aligned}$$

By symmetry, the identity (48) is also valid on  $\mathcal{S}_n^{(n_2, n_1)}$ . Applying (48) to  $\sigma'$  and  $\sigma$  lead to  $|G_5^{(n_1, n_2)}(\sigma)| = |G_5^{(n_1, n_2)}(\sigma')|$ . This concludes the proof of Lemma 12. It then remains to prove Lemma 13.


 TABLE 6. Effects of the mapping  $\Gamma^{(n_1, n_2)}$  on the crossings of  $\sigma$  and  $\sigma'$ .

**Proof of Lemma 13** By definition of  $G_5^{(n_1, n_2)}(\gamma)$ , we have

$$\begin{aligned}
 |G_5^{(n_1, n_2)}(\gamma)| &= |\{(i, j) \mid i < j < \gamma(i) \leq N_2 < \gamma(j)\}| + |\{(i, j) \mid \gamma(j) < \gamma(i) < j \leq N_2 < i\}| \\
 &\quad + |\{i \mid i < \gamma(i) \leq N_2 < \gamma^2(i)\}|.
 \end{aligned} \tag{49}$$

Now, by elementary manipulations and the definition of  $B(\gamma)$  we get

$$\begin{aligned}
 |\{(i, j) \mid i < j < \gamma(i) \leq N_2 < \gamma(j)\}| &= \sum_{r=1}^p |\{x \mid i_r < x < j_r \leq N_2 < \gamma(x)\}| \\
 &= \sum_{r=1}^p |\{x \mid i_r < x < j_r\}| - |\{x \mid i_r < x < j_r, \gamma(x) \leq N_2\}|.
 \end{aligned}$$

But for any  $r \in [1, p]$ , we have  $|\{x \mid i_r < x < j_r\}| = j_r - i_r - 1$  and

$$\begin{aligned} & |\{x \mid i_r < x < j_r, \gamma(x) \leq N_2\}| \\ &= |\{x \mid i_r < x < j_r, x < \gamma(x) \leq N_2\}| + |\{x \mid i_r < x < j_r, \gamma(x) < x \leq N_2\}| \\ &= |\{t \mid i_r < i_t < j_r\}| + |\{t \mid i_r < \ell_t < j_r\}| \quad \text{by definition of } B(\gamma) \text{ and } B(\gamma^{-1}), \\ &= |\{t \mid i_r < i_t\}| + |\{t \mid \ell_t < j_r\}|, \end{aligned}$$

since by definition of  $\mathcal{S}_n^{(n_1, n_2)}$  we have that for any integers  $r$  and  $t$ ,  $i_r \leq n_1$ ,  $k_t \leq n_1$ ,  $j_r > n_1$  and  $\ell_t > n_1$ , and thus  $i_r < j_t$  and  $i_r < \ell_t$ .

Summing over all  $r$  yields

$$|\{(i, j) \mid i < j < \gamma(i) \leq N_2 < \gamma(j)\}| = \sum_{r=1}^p (j_r - i_r - 1) - \sum_{r=1}^p |\{t \mid i_r < i_t\}| - \sum_{r=1}^p |\{t \mid \ell_t < j_r\}|. \quad (50)$$

Since  $|\{(i, j) \mid \gamma(j) < \gamma(i) < j \leq N_2 < i\}| = |\{(i, j) \mid i < j < \gamma^{-1}(i) \leq N_2 < \gamma^{-1}(j)\}|$ , it follows from (50) that

$$|\{(i, j) \mid \gamma(j) < \gamma(i) < j \leq N_2 < i\}| = \sum_{r=1}^q (\ell_r - k_r - 1) - \sum_{r=1}^q |\{t \mid k_r < k_t\}| - \sum_{r=1}^q |\{t \mid j_t < \ell_r\}|. \quad (51)$$

Remarking that  $|\{i \mid i < \gamma(i) \leq N_2 < \gamma^2(i)\}| = |\{t \mid \gamma(j_t) > N_2\}|$  and inserting (50) and (51) in (49) lead to

$$\begin{aligned} |G_5^{(n_1, n_2)}(\gamma)| &= \sum_{r=1}^p (j_r - i_r - 1) + \sum_{r=1}^q (\ell_r - k_r - 1) + |\{t \mid \gamma(j_t) > N_2\}| - \sum_{r=1}^p |\{t \mid i_r < i_t\}| \\ &\quad - \sum_{r=1}^p |\{t \mid \ell_t < j_r\}| - \sum_{r=1}^q |\{t \mid k_r < k_t\}| - \sum_{r=1}^q |\{t \mid j_t < \ell_r\}|. \end{aligned} \quad (52)$$

Since the  $i_r$ 's and  $k_r$ 's are distinct we have

$$\sum_{r=1}^p |\{t \mid i_r < i_t\}| = \binom{p}{2} \quad \text{and} \quad \sum_{r=1}^q |\{t \mid k_r < k_t\}| = \binom{q}{2}. \quad (53)$$

On the other hand,

$$\begin{aligned} \sum_{r=1}^p |\{t \mid \ell_t < j_r\}| + \sum_{r=1}^q |\{t \mid j_t < \ell_r\}| &= \sum_{r=1}^p |\{t \mid \ell_t \neq j_r\}| \\ &= pq - |\{t \mid j_t \in \{\ell_1, \ell_2, \dots, \ell_q\}\}| \\ &= pq - \sum_{s=1}^k |\{t \mid \gamma(j_t) \leq N_2\}|, \end{aligned} \quad (54)$$

where the last identity follows from the definitions of  $B(\gamma)$  and  $B(\gamma^{-1})$ . Inserting (53) and (54) in (52) lead to (48). This concludes the proof of Lemma 13.  $\square$

## REFERENCES

- [1] M. Anshelevich, Linearization coefficients for orthogonal polynomials using stochastic processes, *The Annals of Probability*, 2005, Vol. 33, No. 1, 114-136.
- [2] R. J. Clarke and E. Steingrímsson and J. Zeng, New Euler-Mahonian statistics on permutations and words, *Adv. in Appl. Math.*, 18(1997), 237-270.
- [3] S. Corteel, Crossings and alignements of permutations, *Adv. Appl. Math.*, 38(2007), 149-163.
- [4] A. de Médicis and X. G. Viennot, Moments des  $q$ -polynômes de Laguerre et la bijection de Foata-Zeilberger, *Adv. in Appl. Math.* 15 (1994), no. 3, 262-304.
- [5] M. de Saint-Catherine and G. Viennot, Combinatorial interpretation of integrals of products of Hermite, Laguerre and Tchebycheff polynomials. *Orthogonal polynomials and applications* (Bar-le-Duc, 1984), 120-128, *Lecture Notes in Math.*, 1171, Springer, Berlin, 1985.
- [6] S. Even and J. Gillis, Derangements and Laguerre polynomials, *Math. Proc. Cambridge Philos. Soc.*, 79(1976), 135-143.
- [7] D. Foata and D. Zeilberger, Laguerre polynomials, weighted derangements and positivity, *SIAM J. Disc. Math.* 1(1996), 425-433.
- [8] D. Foata and V. Strehl, *Combinatorics of Laguerre polynomials*, *Enumeration and design* (Waterloo, Ont., 1982), Academic Press, Toronto, 1984, 123-140.
- [9] A. M. Garsia, and J. Remmel, A combinatorial interpretation of  $q$ -derangement and  $q$ -Laguerre numbers, *European J. Combin.*, 1(1980), 47-59.
- [10] G. Gasper and M. Rahman, *Basic hypergeometric series*, Second edition. *Encyclopedia of Mathematics and its Applications*, 96. Cambridge University Press, Cambridge, 2004.
- [11] H. W. Gould, The  $q$ -Stirling Numbers of First and Second Kinds. *Duke Mathematical J.* 28 (1961), 281-289.
- [12] M. Ismail, D. Stanton and G. Viennot, The Combinatorics of  $q$ -Hermite polynomials and the Askey-Wilson integral, *Europ. J. Combinatorics* (1987) **8**, 379-392.
- [13] D. M. Jackson, Laguerre polynomials and derangements, *Math. Proc. Cambridge Philos. Soc.*, 80(1976), 213-214.
- [14] D. Kim and J. Zeng, A combinatorial formula for the linearization coefficients of general Sheffer polynomials, *European J. Combin.*, 22(2001), 313-332.
- [15] D. Kim, D. Stanton and J. Zeng, The combinatorics of the Al-Salam-Chihara  $q$ -Charlier polynomials, *Séminaire Lotharingien de Combinatoire* 54 (2006), Article B54i.
- [16] R. Koekoek and R. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its  $q$ -analogue*, Report 98-17, Delft University of Technology, 1998.
- [17] A. Randrianarivony, Moments des polynômes orthogonaux unitaires de Sheffer généralisés et spécialisations, *European J. Combin.* 19 (1998), no. 4, 507-518.
- [18] R. Simion and D. Stanton, Specializations of generalized Laguerre polynomials, *SIAM J. Math. Anal.*, 25(1994), 712-719.
- [19] R. Simion and D. Stanton, Octabasic Laguerre polynomials and permutation statistics, *J. Comp. Appl. Math.* 68 (1996), 297-329.
- [20] X. Viennot, Une théorie combinatoire des polynômes orthogonaux, *Notes de cours*, UQAM, Montréal 1988.
- [21] L. K. Williams, Enumeration of totally positive Grassmann cells, *Adv. Math.*, 190(2005), 319-342.
- [22] J. Zeng, Weighted derangements and the linearization coefficients of orthogonal Sheffer polynomials, *Proc. London Math. Soc.* (3), 65(1992), 1-22.
- [23] J. Zeng, Calcul Saalschützien des partitions et des dérangements colorés, *SIAM J. Disc. Math.*, Vol. 3, No. 1, pp. 149-156, 1990.



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