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THROWING BALLS ON HOMOGENEOUS SURFACES

ANNE ESTRADE AND JACQUES ISTAS

ABSTRACT. Throwing balls on Euclidean spaces have been considered since a while. With suitable renormalization, it leads to fractional Brownian motion as limit object. We investigate in this paper the throw of balls on spheres and hyperbolic spaces. This leads to dramatically different behaviors. On spheres, we obtain a Gaussian limit that is no more a fractal Brownian motion, but is locally self-similar. On hyperbolic spaces, we prove that there is no any limit.

1. INTRODUCTION

Random balls models have been considered since a while and are known as germ-grain, shot-noise, or micropulses. The common point of those models consists in throwing balls at random in a d -dimensional space, eventually overlapping each other. Many random phenomena can be modeled through this procedure and many application fields are concerned with: Internet traffic in dimension one, communication network or imaging in dimension two, biology or material sciences in dimension three. A pioneer work in this direction is due to Wicksell [24] with the study of corpuscles. The heavy literature on germ-grain models can be splitted into two directions. Either one is interested in the geometrical or morphological aspect of the union of the random balls (see [20] or [21] and references therein), or one is interested in the number of balls covering each point. This second point of view is currently known as shot-noise or spot-noise (see [6] for existence). In dimension three, the shot-noise process is a natural candidate for modelling porous or granular media, and more generally heterogeneous media with irregularities at any scale. The idea is to build a microscopic model which yields a macroscopic field with self-similar properties. The same idea is expected in dimension one for Internet traffic for instance [25]. A usual way for catching self-similarity is to deal with scaling limits. Roughly speaking, the balls are dilated with a scaling parameter λ and one lets λ go either to 0 or to infinity. We quote for instance [4] and [15] for the case $\lambda \rightarrow 0^+$, [11] and [2] for the case $\lambda \rightarrow +\infty$, [5] and [3] where both cases are considered.

In the present paper, we follow a procedure which is similar to [2] and [3]. Let us describe it precisely. The authors consider a collection of random balls in \mathbb{R}^n whose centers and radii are randomly chosen according to a random Poisson measure on $\mathbb{R}^n \times \mathbb{R}^+$. The Poisson intensity is chosen as follows

$$\nu(dx, dr) = r^{-n-1+2H} dx dr.$$

⁰keywords: Euclidean spaces, spheres, hyperbolic spaces, balls, fractional Brownian motions

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Since the Lebesgue measure dx is invariant with respect to isometry, so is the random balls model, and so will be any (eventual) limit. As the distribution of the radii follows a homogeneous distribution $r^{-n-1+2H} dr$, it leaves hope for a self-similar scaling limit. Indeed, with additional technical conditions, the scaling limits of such random balls models are isometry invariant self-similar Gaussian fields. The self-similarity index depends on the parameter H . When $0 < H < 1/2$, the Gaussian field is nothing else than the well-known fractional Brownian motion [16, 18, 19].

Manifold indexed fields that share property with Euclidean self-similar fields have been studied and is still a lot (e.g. [8, 9, 12, 13, 14, 17, 22, 23]) In this paper, one wonders what happens when balls are thrown onto a surface, and no more onto an Euclidean space. More precisely, is there a scaling limit of random balls models, and, when exists, is this scaling limit a fractional Brownian field indexed by the surface for $0 < H < 1/2$?

The surfaces we are studying in this paper are, geometrically speaking, the closest to Euclidean spaces. This leads naturally to homogeneous surfaces, that is spheres and hyperbolic spaces. The random field is still obtained by throwing overlapping balls in a Poissonian way. The Poisson intensity is chosen as follows

$$\nu(dx, dr) = f(r) \sigma(dx) dr.$$

The Lebesgue measure dx has been replaced by the surface measure $\sigma(dx)$. The function f , that manages the distribution of the radii, is still equivalent to $r^{-n-1+2H}$, at least for small r , where n stands for the surface dimension.

It turns out that the results are completely different in the three cases (Euclidean, spherical and hyperbolic).

- (a) In the spherical case, there is a Gaussian scaling limit. But it is no more a fractional Brownian field, as defined by [12]. We then investigate the local behaviour, in the tangent bundle, of this scaling limit, in the spirit of local self-similarity [1, 7]. It is locally asymptotically self-similar, with an Euclidean fractional Brownian field as tangent field. Our microscopic model has led to a local self-similar macroscopic model.
- (b) In the hyperbolic case, there isn't any scaling limit! Let us give roughly the reason. For small radii r , hyperbolic balls have a volume of size $O(r^n)$, but for large r , this volume grows exponentially. Therefore, there is no way of rescaling simultaneously both small and large balls. Our model can not lead to a global (or even local) self-similar macroscopic model.

The paper is organized as follows. In Section 2, the spherical model is introduced and we prove the existence of a scaling limit. We prove in Section 3 that there isn't any scaling limit in the hyperbolic case. In Section 4, we study the locally self-similar property of the spherical model. Eventually, some technical computations are postponed in the Appendix.

2. SCALING LIMIT ON SPHERES

We work on \mathbb{S}_n the n -dimensional unit sphere, $n \geq 1$:

$$\mathbb{S}_n = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{R}^{n+1}; \sum_{1 \leq i \leq n+1} x_i^2 = 1\}.$$

2.1. Spherical caps. For $x, y \in \mathbb{S}_n$, let $d(x, y)$ denote the distance between x and y on \mathbb{S}_n , i.e. the non-oriented angle between Ox and Oy where O denotes the origin of \mathbb{R}^{n+1} . For $r \geq 0$, $B(x, r)$ denotes the closed ball on S centered at x with radius r :

$$B(x, r) = \{y \in \mathbb{S}_n ; d(x, y) \leq r\}.$$

Let us notice that for $r < \pi$, $B(x, r)$ is a spherical cap on the unit sphere \mathbb{S}_n , centered at x with opening angle r and that for $r \geq \pi$, $B(x, r) = \mathbb{S}_n$.

Denoting by $\sigma(dx)$ the surface measure on \mathbb{S}_n , we prescribe $\phi(r)$ as the surface of any ball on \mathbb{S}_n with radius r ,

$$\phi(r) := \sigma(B(x, r)) , \quad x \in \mathbb{S}_n, r \geq 0 .$$

We also introduce the following function defined for z and z' two points in \mathbb{S}_n and $r \in \mathbb{R}^+$,

$$(1) \quad \Psi(z, z', r) := \int_{\mathbb{S}_n} \mathbf{1}_{d(x, z) < r} \mathbf{1}_{d(x, z') < r} \sigma(dx) .$$

Actually $\Psi(z, z', r)$ denotes the surface measure of the set of all points in \mathbb{S}_n that belong to both balls $B(z, r)$ and $B(z', r)$. Clearly it only depends on the distance $d(z, z')$ between z and z' . We write

$$(2) \quad \psi(d(z, z'), r) = \Psi(z, z', r)$$

and note that it satisfies the following: $\forall (u, r) \in [0, \pi] \times \mathbb{R}^+$

- $0 \leq \psi(u, r) \leq \sigma(\mathbb{S}_n) \wedge \phi(r)$
- if $r < u/2$ then $\psi(u, r) = 0$ and if $r > \pi$ then $\psi(u, r) = \sigma(\mathbb{S}_n)$
- $\psi(0, r) = \phi(r) \sim cr^n$ as $r \rightarrow 0^+$.

In the sequel, we consider a family of balls $B(X_j, R_j)$ generated at random following a strategy described in the next section.

2.2. Poisson point process. We consider a Poisson point process $(X_j, R_j)_j$ in $\mathbb{S}_n \times \mathbb{R}^+$, or equivalently $N(dx, dr)$ a Poisson random measure on $\mathbb{S}_n \times \mathbb{R}^+$, with intensity

$$\nu(dx, dr) = f(r) \sigma(dx) dr$$

where f satisfies the following assumption $\mathbf{A}(H)$ for some $H > 0$:

- $\text{supp}(f) \subset [0, \pi)$
- f is bounded on any compact subset of $(0, \pi)$
- $f(r) \sim r^{-n-1+2H}$ as $r \rightarrow 0^+$.

Remarks :

- 1) The first condition implies that no ball of radius R_j on the sphere will overlap itself.
- 2) Since $\phi(r) \sim cr^n$, $r \rightarrow 0^+$, the last condition implies that $\int_{\mathbb{R}^+} \phi(r) f(r) dr < +\infty$, which means that the mean surface of the balls $B(X_j, R_j)$ is finite.

2.3. Random field. Let \mathcal{M} denote the space of signed measures μ on \mathbb{S}_n with finite total variation $|\mu|(\mathbb{S}_n)$, with $|\mu|$ the total variation measure of μ . For any $\mu \in \mathcal{M}$, we define

$$(3) \quad X(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r)) N(dx, dr) .$$

Note that the stochastic integral in (3) is well defined since

$$\begin{aligned} \int_{\mathbb{S}_n \times \mathbb{R}^+} |\mu(B(x, r))| f(r) \sigma(dx) dr &\leq \int_{\mathbb{S}_n} \int_{\mathbb{S}_n} \int_{\mathbb{R}^+} \mathbf{1}_{d(x, y) < r} f(r) \sigma(dx) |\mu|(dy) dr \\ &= |\mu|(\mathbb{S}_n) \left(\int_{\mathbb{R}^+} \phi(r) f(r) dr \right) < +\infty . \end{aligned}$$

In the particular case where μ is a Dirac measure δ_z for some point $z \in \mathbb{S}_n$ we simply denote

$$(4) \quad X(z) = X(\delta_z) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mathbf{1}_{B(x, r)}(z) N(dx, dr) .$$

The pointwise field $\{X(z); z \in \mathbb{S}_n\}$ corresponds to the number of random balls (X_j, R_j) covering each point of \mathbb{S}_n . Each random variable $X(z)$ has a Poisson distribution with mean $\int_{\mathbb{R}^+} \phi(r) f(r) dr$.

Furthermore for any $\mu \in \mathcal{M}$,

$$\mathbb{E}(X(\mu)) = \mu(\mathbb{S}_n) \left(\int_{\mathbb{R}^+} \phi(r) f(r) dr \right)$$

and

$$\text{Var}(X(\mu)) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 f(r) \sigma(dx) dr \in (0, +\infty] .$$

2.4. Key lemma. For $H > 0$, we would like to compute the integral

$$\int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H} \sigma(dx) dr$$

which is a candidate for the variance of an eventually scaling limit. We first introduce \mathcal{M}^H the set of measures for which the above integral does converge:

$$\mathcal{M}^H = \mathcal{M} \text{ if } 2H < n ; \mathcal{M}^H = \{\mu \in \mathcal{M} ; \mu(\mathbb{S}_n) = 0\} \text{ if } 2H > n .$$

The following lemma deals with the function ψ prescribed by (2).

Lemma 2.1. *Let $H > 0$ with $2H \neq n$. We introduce*

$$\psi^{(H)} = \psi \text{ if } 0 < 2H < n ; \psi^{(H)} = \psi - \sigma(\mathbb{S}_n) \text{ if } 2H > n .$$

Then for all $u \in [0, \pi]$,

$$\int_{\mathbb{R}^+} |\psi^{(H)}(u, r)| r^{-n-1+2H} dr < +\infty .$$

Furthermore, denoting

$$(5) \quad K_H(u) = \int_{\mathbb{R}^+} \psi^{(H)}(u, r) r^{-n-1+2H} dr$$

for any u in $[0, \pi]$, we have for all $\mu \in \mathcal{M}^H$,

$$0 \leq \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H} \sigma(dx) dr = \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz) \mu(dz') < +\infty.$$

Remark 2.2.

- 1) For x, y in \mathbb{S}_n , the difference of Dirac measures $\delta_x - \delta_y$ belongs to \mathcal{M}^H for any H .
- 2) In the case $2H > n$, since any $\mu \in \mathcal{M}^H$ is centered, the rhs integral is not changed when a constant is added to the kernel K_H .
- 3) This lemma proves that the kernel K_H defines a covariance function on \mathcal{M}^H .

Proof. Using the properties of ψ we get in the case $0 < 2H < n$,

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^+} \psi(u, r) r^{-n-1+2H} dr &\leq \int_{(0, \pi)} \phi(r) r^{-n-1+2H} dr + \sigma(\mathbb{S}_n) \int_{(\pi, \infty)} r^{-n-1+2H} dr \\ &< +\infty. \end{aligned}$$

In the same vein, in the case $2H > n$,

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^+} (\sigma(\mathbb{S}_n) - \psi(u, r)) r^{-n-1+2H} dr &\leq \sigma(\mathbb{S}_n) \int_{(0, \pi)} r^{-n-1+2H} dr \\ &< +\infty. \end{aligned}$$

We have just established that there exists a finite constant C_H such that

$$(6) \quad \forall u \in [0, \pi], \quad \int_{\mathbb{R}^+} |\psi^{(H)}(u, r)| r^{-n-1+2H} dr \leq C_H.$$

The first statement is proved.

Let us denote for $\mu \in \mathcal{M}^H$

$$I_H(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H} \sigma(dx) dr$$

and start with proving that $I_H(\mu)$ is finite. We will essentially use Fubini's theorem to write the following lines.

$$\begin{aligned} I_H(\mu) &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}_n} \mu(B(x, r))^2 \sigma(dx) \right) r^{-n-1+2H} dr \\ &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}_n \times \mathbb{S}_n} \Psi(z, z', r) \mu(dz) \mu(dz') \right) r^{-n-1+2H} dr. \end{aligned}$$

Since $\mu \in \mathcal{M}^H$ is centered in the case $2H > n$, one can change ψ into $\psi^{(H)}$ within the previous integral. Then

$$\begin{aligned} I_H(\mu) &\leq \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}_n \times \mathbb{S}_n} |\psi^{(H)}(d(z, z'), r)| |\mu|(dz) |\mu|(dz') \right) r^{-n-1+2H} dr \\ &\leq \int_{\mathbb{S}_n \times \mathbb{S}_n} \left(\int_{\mathbb{R}^+} |\psi^{(H)}(d(z, z'), r)| r^{-n-1+2H} dr \right) |\mu|(dz) |\mu|(dz') \\ &\leq C_H |\mu|(\mathbb{S}_n)^2 < +\infty. \end{aligned}$$

Following the same lines (except for the last one) without the “|” allows the computation of $I_H(\mu)$ and concludes the proof. \square

An explicit value for the kernel K_H is available starting from its definition. The point is to compute $\psi^{(H)}$. We give in the Appendix a recurrence formula for $\psi^{(H)}$, based on the dimension n of the unit sphere \mathbb{S}_n (see Lemma 4.3).

2.5. Scaling. Let $\rho > 0$ and λ be any positive function on $(0, +\infty)$. We consider the scaled Poisson measure N_ρ obtained from the original Poisson measure N by taking the image under the map $(x, r) \in S \times \mathbb{R}^+ \mapsto (x, \rho r)$ and multiplying the intensity by $\lambda(\rho)$. Hence N_ρ is still a Poisson random measure with intensity

$$\nu_\rho(dx, dr) = \lambda(\rho)\rho^{-1}f(\rho^{-1}r)\sigma(dx)dr.$$

We also introduce the scaled random field X_ρ defined on \mathcal{M} by

$$(7) \quad X_\rho(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r)) N_\rho(dx, dr).$$

Theorem 2.3. *Let $H > 0$ with $2H \neq n$ and let f satisfying $\mathbf{A}(H)$. For all positive functions λ such that $\lambda(\rho)\rho^{n-2H} \xrightarrow{\rho \rightarrow +\infty} +\infty$, the limit*

$$\frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{\sqrt{\lambda(\rho)\rho^{n-2H}}} \xrightarrow[\rho \rightarrow +\infty]{fdd} W_H(\mu)$$

holds for all $\mu \in \mathcal{M}^H$, in the sense of finite dimensional distributions of the random functionals. Here W_H is the centered Gaussian random linear functional on \mathcal{M}^H with

$$(8) \quad \text{Cov}(W_H(\mu), W_H(\nu)) = \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz)\nu(dz'),$$

where K_H is the kernel introduced in Lemma 2.1.

The theorem can be rephrased in term of the pointwise field $\{X(z); z \in \mathbb{S}_n\}$ defined in (4).

Corollary 2.4. *Let $H > 0$ with $2H \neq n$ and let f satisfying $\mathbf{A}(H)$. For all positive functions λ such that $\lambda(\rho)\rho^{n-2H} \xrightarrow{\rho \rightarrow +\infty} +\infty$,*

- if $0 < 2H < n$ then

$$\left\{ \frac{X_\rho(z) - \mathbb{E}(X_\rho(z))}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; z \in \mathbb{S}_n \right\} \xrightarrow[\rho \rightarrow +\infty]{fdd} \{W_H(z); z \in \mathbb{S}_n\}$$

where W_H is the centered Gaussian random field on \mathbb{S}_n with

$$\text{Cov}(W_H(z), W_H(z')) = K_H(d(z, z')).$$

- if $2H > n$ then for any fixed point $z_0 \in \mathbb{S}_n$,

$$\left\{ \frac{X_\rho(z) - X_\rho(z_0)}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; z \in \mathbb{S}_n \right\} \xrightarrow[\rho \rightarrow +\infty]{fdd} \{W_{H, z_0}(z); z \in \mathbb{S}_n\}$$

where W_{H,z_0} is the centered Gaussian random field on \mathbb{S}_n with

$$\text{Cov}(W_{H,z_0}(z), W_{H,z_0}(z')) = K_H(d(z, z')) - K_H(d(z, z_0)) - K_H(d(z', z_0)) + K_H(0) .$$

Proof. of Theorem 2.3.

Let us denote $n(\rho) := \sqrt{\lambda(\rho)\rho^{n-2H}}$. The characteristic function of the normalized field $(X_\rho(\cdot) - \mathbb{E}(X_\rho(\cdot)))/n(\rho)$ is then given by

$$(9) \quad \mathbb{E} \left(\exp \left(i \frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)} \right) \right) = \exp \left(\int_{\mathbb{S}_n \times \mathbb{R}^+} G_\rho(x, r) dr \sigma(dx) \right)$$

where

$$(10) \quad G_\rho(x, r) = \left(e^{i \frac{\mu(B(x, r))}{n(\rho)}} - 1 - i \frac{\mu(B(x, r))}{n(\rho)} \right) \lambda(\rho) \rho^{-1} f(\rho^{-1}r) .$$

We will make use of Lebesgue's Theorem in order to get the limit of $\int_{\mathbb{S}_n \times \mathbb{R}^+} G_\rho(x, r) dr \sigma(dx)$ as $\rho \rightarrow +\infty$.

On one hand, $n(\rho)$ tends to $+\infty$ so that $\left(e^{i \frac{\mu(B(x, r))}{n(\rho)}} - 1 - i \frac{\mu(B(x, r))}{n(\rho)} \right)$ behaves like $-\frac{1}{2} \left(\frac{\mu(B(x, r))}{n(\rho)} \right)^2$. Together with the assumption $\mathbf{A}(H)$ yield the following asymptotic. For all $(x, r) \in \mathbb{S}_n \times \mathbb{R}^+$,

$$(11) \quad G_\rho(x, r) \xrightarrow{\rho \rightarrow +\infty} -\frac{1}{2} \mu(B(x, r))^2 r^{-n-1+2H} .$$

On the other hand, since $\frac{|\mu(B(x, r))|}{n(\rho)} \leq |\mu|(\mathbb{S}_n)$ for ρ large enough, we note that there exists some positive constant K such that for all x, r, ρ ,

$$\left| e^{i \frac{\mu(B(x, r))}{n(\rho)}} - 1 - i \frac{\mu(B(x, r))}{n(\rho)} \right| \leq K \left(\frac{\mu(B(x, r))}{n(\rho)} \right)^2 .$$

Therefore

$$|G_\rho(x, r)| \leq K \mu(B(x, r))^2 \rho^{-n-1+2H} f(\rho^{-1}r) .$$

There exists $C > 0$ such that for all $r \in \mathbb{R}^+$, $f(r) \leq Cr^{-n-1+2H}$. Then we get

$$(12) \quad |G_\rho(x, r)| \leq KC \mu(B(x, r))^2 r^{-n-1+2H}$$

where the right hand side is integrable on $\mathbb{S}_n \times \mathbb{R}^+$ by Lemma 2.1.

Applying Lebesgue's Theorem yields

$$\begin{aligned} \int_{\mathbb{S}_n \times \mathbb{R}^+} G_\rho(x, r) \sigma(dx) dr &\xrightarrow{\rho \rightarrow +\infty} -\frac{1}{2} \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H} \sigma(dx) dr \\ &= -\frac{1}{2} \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz) \mu(dz') . \end{aligned}$$

Hence $(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)))/n(\rho)$ converges in distribution to the centered Gaussian random variable $W(\mu)$ whose variance is equal to

$$\mathbb{E}(W(\mu)^2) = C \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz) \mu(dz') .$$

By linearity, the covariance of W satisfies (8).

□

Remark 2.5.

- 1) The increments of the pointwise limit field $\{W_H(z); z \in \mathbb{S}_n\}$ in Corollary 2.4 are stationary, i.e. their distributions are invariant under the isometry group of \mathbb{S}_n , whereas the increments of W_{H,z_0} are distribution invariant under the group of all isometries of \mathbb{S}_n which keep invariant the point z_0 .
- 2) When $0 < H < 1/2$ the Gaussian field W_H does not coincide with the field introduced in [12] as the spherical fractional Brownian motion on \mathbb{S}_n .

Actually the covariance kernel of W_H has not the prescribed shape $|u|^{2H}$. Indeed in the case $n = 1$, it's easy to obtain the following piecewise expression for $\psi = \psi_1$: $\forall(u, r) \in [0, \pi] \times \mathbb{R}^+$,

$$\begin{aligned} \psi_1(u, r) &= 0 \quad \text{for } 0 \leq r < u/2 \\ &= 2r - u \quad \text{for } u/2 \leq r \leq \pi - u/2 \\ &= 4r - 2\pi \quad \text{for } \pi - u/2 \leq r \leq \pi \\ &= 2\pi \quad \text{for } r > \pi \end{aligned}$$

and to compute

$$K_H(u) = \frac{1}{H(1-2H)2^{2H}} (2(2H)^{2H} - u^{2H} - (2\pi - u)^{2H}) ,$$

which is not proportional to u^{2H} .

At the opposite, a Euclidean theorem equivalent to Theorem 2.3 above is stated in [3]. The corresponding field does coincide with the Euclidean fractional Brownian motion.

3. NO SCALING LIMIT ON HYPERBOLIC SPACES

Let \mathbb{H}_n be the n -dimensional hyperbolic space

$$\mathbb{H}_n = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{R}^{n+1} ; x_{n+1}^2 - \sum_1^n x_i^2 = 1, x_{n+1} \geq 1\} .$$

We start by introducing the same tools as in the spherical case. The pseudo scalar product,

$$[M, M'] = x_{n+1}x'_{n+1} - \sum_1^n x_i x'_i ,$$

is used to define the geodesic distance between M and M'

$$d(M, M') = \operatorname{arccosh}[M, M'] .$$

We still denote by $B(M, r)$ the ball in \mathbb{H}_n centered at M with radius r

$$B(M, r) = \{N \in \mathbb{H}_n ; d(M, N) \leq r\} ,$$

by $\sigma(dx)$ the surface measure on \mathbb{H}_n and by $\phi(r)$ the surface of any ball on \mathbb{H}_n with radius r ,

$$\phi(r) := \sigma(B(x, r)) , \quad x \in \mathbb{H}_n, r \geq 0 .$$

3.1. Random field. As in the sphere case, we consider a family of balls $B(X_j, R_j)$ generated at random through a Poisson point process $(X_j, R_j)_j$ in $\mathbb{H}_n \times \mathbb{R}^+$. The intensity measure is given by

$$\nu(dx, dr) = f(r) \sigma(dx) dr$$

where f is bounded on any compact subset of \mathbb{R}^+ and $\int_{\mathbb{R}^+} \phi(r) f(r) dr < +\infty$.

We still denote by $N(dx, dr)$ the associated Poisson random measure on $\mathbb{H}_n \times \mathbb{R}^+$ and by \mathcal{M} the space of signed measures μ on \mathbb{H}_n with finite total variation $|\mu|(\mathbb{H}_n)$.

We also introduce the random field X defined on \mathcal{M} by

$$X(\mu) = \int_{\mathbb{H}_n \times \mathbb{R}^+} \mu(B(x, r)) N(dx, dr) .$$

Let us quote that the previous stochastic integral is well defined since

$$\int_{\mathbb{H}_n \times \mathbb{R}^+} |\mu(B(x, r))| f(r) \sigma(dx) dr \leq |\mu|(\mathbb{H}_n) \left(\int_{\mathbb{R}^+} \phi(r) f(r) dr \right) < +\infty .$$

3.2. Scaling limit? At the opposite of the spherical case, we prove that there isn't any non-degenerate scaling limit for the scaled field X_ρ defined by analogy from (7). We proceed by contradiction assuming a scaling normalization term $n(\rho)$. We expect $n(\rho) \rightarrow +\infty$ as $\rho \rightarrow +\infty$. The appropriate version of (9) and (10) are still in force. As $\rho \rightarrow +\infty$, $G_\rho(x, r)$ behaves as $-1/2\mu^2(B(x, r))A(\rho)f(r/\rho)$, where $A(\rho) = \lambda(\rho)/(\rho n(\rho))$. If there should exist a non-degenerate limit as $\rho \rightarrow +\infty$, then $A(\rho)f(r/\rho)$ would converge to a non-degenerate function $g(r)$. Moreover, by standard homogeneity arguments, g would be an homogeneous function. Indeed, take $k > 0$. Then as $\rho \rightarrow +\infty$,

$$A(\rho)f(kr/(k\rho)) \rightarrow g(r) \quad \text{and} \quad A(k\rho)f(kr/(k\rho)) \rightarrow g(kr) .$$

It follows that, as $\rho \rightarrow +\infty$

$$\frac{A(\rho)}{A(k\rho)} \rightarrow \frac{g(kr)}{g(r)} .$$

Therefore g is an homogeneous function: there exist two constants α and β such that $g(r) = \alpha r^\beta$. By Fatou's Lemma

$$\begin{aligned} \liminf_{\rho \rightarrow +\infty} \int_{\mathbb{H}_n \times \mathbb{R}^+} -2G_\rho(x, r) \sigma(dx) dr &\geq \int_{\mathbb{H}_n \times \mathbb{R}^+} \liminf_{\rho \rightarrow +\infty} -2G_\rho(x, r) \sigma(dx) dr \\ &= \alpha \int_{\mathbb{H}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^\beta \sigma(dx) dr \end{aligned}$$

By Fubini's Theorem

$$\int_{\mathbb{H}_n} \mu(B(x, r))^2 \sigma(dx) = \int_{\mathbb{H}_n \times \mathbb{H}_n} \Psi(z, z', r) \mu(dz) \mu(dz')$$

where

$$\Psi(z, z', r) := \int_{\mathbb{H}_n} \mathbf{1}_{d(x, z) < r} \mathbf{1}_{d(x, z') < r} \sigma(dx) .$$

Since \mathbb{H}_n is a two-points homogeneous space, function Ψ only depends on the distance $d(z, z')$. We denote

$$\Psi(z, z', r) = \psi(d(z, z'), r) ,$$

and by Fubini's Theorem again,

$$\begin{aligned} \int_{\mathbb{R}^+} \left(\int_{\mathbb{H}_n \times \mathbb{H}_n} \psi(d(z, z'), r) \mu(dz) \mu(dz') \right) r^\beta dr \\ = \int_{\mathbb{H}_n \times \mathbb{H}_n} \left(\int_{\mathbb{R}^+} \psi(d(z, z'), r) r^\beta dr \right) \mu(dz) \mu(dz'). \end{aligned}$$

We now prove that $\int_{\mathbb{R}^+} \psi(d(z, z'), r) r^\beta dr$ is a divergent integral for all z, z' . Indeed, for any given $d(z, z')$ and r large enough

$$\psi(d(z, z'), r) \geq \psi(0, r/2).$$

Using the exponential growth of the hyperbolic balls (cf. Appendix, Lemma 4.4)

$$\psi(d(z, z'), r) \geq ce^{r/2}.$$

The integral $\int_0^{+\infty} e^{r/2} r^\beta dr$ is clearly divergent for all β . Hence the normalized field

$$(X_\rho(\cdot) - \mathbb{E}(X_\rho(\cdot))) / n(\rho)$$

cannot converge to a non-degenerate limit as $\rho \rightarrow +\infty$.

4. LOCAL SELF-SIMILAR BEHAVIOUR ON SPHERES

We study an eventually lass property of the limit field W_H obtained in the previous section. More precisely we will let a ‘‘dilation’’ of order ε act on W_H near a fixed point A in \mathbb{S}_n and, up to a renormalization factor, we look for an asymptotic behavior as ε goes to 0. A H -self-similar tangent field T_H is expected. Recall that W_H is defined on a subspace \mathcal{M}^H of measures on \mathbb{S}_n , so that T_H will be defined on a subspace of measures on the tangent space $\mathcal{T}_A \mathbb{S}_n$ of \mathbb{S}_n .

4.1. Dilation. Let us fix a point A in \mathbb{S}_n and consider $\mathcal{T}_A \mathbb{S}_n$ the tangent space of \mathbb{S}_n at A . It can be identified with \mathbb{R}^n and A with the null vector of \mathbb{R}^n .

Let $1 < \delta < \pi$. The exponential map at point A , denoted by \exp , is a diffeomorphism between the Euclidean ball $\{y \in \mathbb{R}^n, \|y\| < \delta\}$ and $\mathring{B}(A, \delta) \subset \mathbb{S}_n$, where $\|\cdot\|$ denotes the Euclidean norm in \mathbb{R}^n and $\mathring{B}(A, \delta)$ the open ball with center A and radius δ in \mathbb{S}_n . Furthermore for all $y, y' \in \mathbb{R}^n$ such that $\|y\|, \|y'\| < \delta$,

$$d(A, \exp y) = \|y\| \quad \text{and} \quad d(\exp y, \exp y') \leq \|y - y'\|.$$

We refer to [10] for precisions on the exponential map.

Let τ be a signed measure on \mathbb{R}^n . We define the dilated measure τ_ε by

$$\forall B \in \mathcal{B}(\mathbb{R}^n) \quad \tau_\varepsilon(B) = \tau(B/\varepsilon)$$

and then map it by the application \exp , defining the measure $\mu_\varepsilon = \exp^* \tau_\varepsilon$ on $\mathring{B}(A, \delta)$ by

$$(13) \quad \forall C \in \mathcal{B}(\mathring{B}(A, \delta)) \quad \mu_\varepsilon(C) = \exp^* \tau_\varepsilon(C) = \tau_\varepsilon(\exp^{-1}(C)).$$

We then consider the measure μ_ε as a measure on the whole sphere \mathbb{S}_n with support included in $\overset{\circ}{B}(A, \delta)$.

At last, we define the dilation of W_H within a “neighborhood of A ” by the following procedure. For any finite measure τ on \mathbb{R}^n , we consider $\mu_\varepsilon = \exp^* \tau_\varepsilon$ as defined by (13) and compute $W_H(\mu_\varepsilon)$. We will establish the convergence in distribution of $\varepsilon^{-H} W_H(\exp^* \tau_\varepsilon)$ for any τ in an appropriate space of measures on \mathbb{R}^n . Since $W_H(\mu_\varepsilon)$ is Gaussian, we will focus on its variance.

4.2. Asymptotic of the kernel K_H . For $0 < H < 1/2$, we already mentioned that the kernel $K_H(u)$ is not proportional to u^{2H} . As a consequence, one cannot expect W_H to be self-similar. Nevertheless, as we are looking for an asymptotic local self-similarity, only the behaviour of K_H near zero is relevant. Actually we will establish that, up to a constant, $K_H(u)$ behaves like u^{2H} when $u \rightarrow 0^+$.

Lemma 4.1. *Let $0 < H < 1/2$. The kernel K_H defined by (5) satisfies*

$$K_H(u) = K_1 - K_2 u^{2H} + o(u^{2H}), \quad u \rightarrow 0^+$$

where $K_1 = K_H(0)$ and K_2 are nonnegative constants.

Proof. Let us quote that the assumption $H < 1/2$ implies $H < n/2$ so that in that case K_H is prescribed by

$$K_H(u) = \int_{\mathbb{R}^+} \psi(u, r) r^{-n-1+2H} dr, \quad u \in [0, \pi].$$

We note that $K_H(0) < +\infty$ since $\psi(0, r) \sim cr^n$ as $r \rightarrow 0^+$ and $\psi(0, r) = \sigma(\mathbb{S}_n)$ for $r > \pi$. Then, subtracting $K_H(0)$ and quoting that $\psi(0, r) = \psi(u, r) = \sigma(\mathbb{S}_n)$ for $r > \pi$, we write

$$\begin{aligned} K_H(0) - K_H(u) &= \int_0^\pi (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr \\ &= \int_0^\delta (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr \\ &\quad + \int_\delta^\pi (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr \end{aligned}$$

where we recall that $\delta \in (1, \pi)$ is such that the exponential map is a diffeomorphism between $\{\|y\| < \delta\} \subset \mathbb{R}^n$ and $\overset{\circ}{B}(A, \delta) \subset \mathbb{S}_n$.

The second term is of order u , and therefore is negligible with respect to u^{2H} , since ψ is clearly Lipschitz on the compact interval $[\delta, \pi]$.

We focus now on the first term. Performing the change of variable $r \mapsto r/u$, we write it as

$$\int_0^\delta (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr = u^{2H} \int_{\mathbb{R}^+} \Delta(u, r) r^{-n-1+2H} dr,$$

where

$$\Delta(u, r) := \mathbf{1}_{ur < \delta} u^{-n} (\psi(u, ur) - \psi(0, ur)).$$

We only have to prove that $\int_{\mathbb{R}^+} \Delta(u, r) r^{-n-1+2H} dr$ admits a finite limit K_2 as $u \rightarrow 0^+$. We will use Lebesgue's Theorem and start with establishing the simple convergence of $\Delta(u, r)$ for any given $r \in \mathbb{R}^+$.

We fix a unit vector \mathbf{v} in \mathbb{R}^n and a point $A' = \exp \mathbf{v}$ in \mathbb{S}_n . We then consider for any $u \in (0, \delta)$, the point $A'_u := \exp(u\mathbf{v}) \in \mathbb{S}_n$ located on the geodesic between A and A' such that $d(A, A'_u) = \|u\mathbf{v}\| = u$. We can then use (1) and (2) to write

$$\psi(u, \cdot) = \Psi(A, A'_u, \cdot) = \int_{\mathbb{S}_n} \mathbf{1}_{d(A, z) < u} \mathbf{1}_{d(A'_u, z) < u} d\sigma(z)$$

and

$$\psi(0, \cdot) = \Psi(A, A, \cdot) = \int_{\mathbb{S}_n} \mathbf{1}_{d(A, z) < u} d\sigma(z)$$

in order to express $\Delta(u, r)$ as

$$\Delta(u, r) = \mathbf{1}_{ur < \delta} u^{-n} \int_{\mathbb{S}_n} \mathbf{1}_{d(A, z) < ur} \mathbf{1}_{d(A'_u, z) > ur} d\sigma(z) .$$

Since $ur < \delta$ the above integral runs on $\overset{\circ}{B}(A, ur) \subset \overset{\circ}{B}(A, \delta)$ and we can perform the exponential change of variable to get

$$\begin{aligned} \Delta(u, r) &= \mathbf{1}_{ur < \delta} u^{-n} \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < ur} \mathbf{1}_{d(\exp(u\mathbf{v}), \exp(y)) > ur} d\sigma(\exp(y)) \\ &= \mathbf{1}_{ur < \delta} \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < r} \mathbf{1}_{d(\exp(u\mathbf{v}), \exp(uy)) > ur} \tilde{\sigma}(uy) dy . \end{aligned}$$

In the last integral, the image by \exp of the surface measure $d\sigma(\exp(y))$ is written as $\tilde{\sigma}(y)dy$ where dy denotes the Lebesgue measure on \mathbb{R}^n .

We use the fact that $d(\exp(ux), \exp(ux')) \sim u\|x - x'\|$ as $u \rightarrow 0^+$ to get the following limit for the integrand

$$\mathbf{1}_{d(\exp(u\mathbf{v}), \exp(uy)) < ur} \tilde{\sigma}(uy) \longrightarrow \mathbf{1}_{\|\mathbf{v}-y\| > r} \tilde{\sigma}(0) .$$

Since the integrand is clearly dominated by

$$\|\sigma\|_\infty := \sup\{\tilde{\sigma}(y), \|y\| \leq \delta\} ,$$

Lebesgue's Theorem yields for all $r \in \mathbb{R}^+$,

$$\Delta(u, r) \longrightarrow \tilde{\sigma}(0) \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < r} \mathbf{1}_{\|\mathbf{v}-y\| > r} dy .$$

We recall that $d(\exp x, \exp x') \leq \|x - x'\|$ for all $x, x' \in \mathbb{R}^n$ with norm less than δ . Therefore for all u ,

$$\Delta(u, r) \leq \|\sigma\|_\infty \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < r} \mathbf{1}_{\|\mathbf{v}-y\| > r} dy$$

where the right hand side belongs to $L^1(\mathbb{R}^+, r^{-n-1+2H} dr)$ (see [2] Lemma A.2).

Using Lebesgue's Theorem for the last time, we obtain

$$\int_{\mathbb{R}^+} \Delta(u, r) r^{-n-1+2H} dr \xrightarrow{u \rightarrow 0^+} K_2$$

where

$$K_2 = \tilde{\sigma}(0) \int_{\mathbb{R}^n \times \mathbb{R}^+} \mathbf{1}_{\|y\| < r} \mathbf{1}_{\|v-y\| > r} r^{-n-1+2H} dy dr \in (0, +\infty) .$$

□

Let us remark that the proof makes clear that the case $H > 1/2$ is dramatically different. The kernel K_H behaves like u near zero and loses its $2H$ power.

4.3. Main result.

Let $0 < H < 1/2$. We consider the following space of measures on $\mathcal{T}_A \mathbb{S}_n \cong \mathbb{R}^n$

$$\begin{aligned} \mathfrak{M}^H &= \{ \text{measures } \tau \text{ on } \mathbb{R}^n \text{ with finite total variation such that} \\ &\quad \tau(\mathbb{R}^n) = 0 \text{ and } \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\|^{2H} |\tau|(dx) |\tau|(dx') < +\infty \} . \end{aligned}$$

For any measure $\tau \in \mathfrak{M}^H$, we compute the variance of $W_H(\mu_\varepsilon)$ where $\mu_\varepsilon = \exp^* \tau_\varepsilon$ is defined by (13).

By Lemma 2.1, since μ_ε belongs to $\mathcal{M} = \mathcal{M}^H$ in the case $H < 1/2$,

$$\text{var}(W_H(\mu_\varepsilon)) = \int_{B(A,\delta) \times B(A,\delta)} K_H(d(z, z')) \mu_\varepsilon(dz) \mu_\varepsilon(dz') .$$

Performing an exponential change of variable followed by a dilation in \mathbb{R}^n , we get

$$\begin{aligned} \text{var}(W_H(\mu_\varepsilon)) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_{\|y\| < \delta} \mathbf{1}_{\|y'\| < \delta} K_H(d(\exp(y), \exp(y'))) \tau_\varepsilon(dy) \tau_\varepsilon(dy') \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_{\|x\| < \delta/\varepsilon} \mathbf{1}_{\|x'\| < \delta/\varepsilon} K_H(d(\exp(\varepsilon x), \exp(\varepsilon x'))) \tau(dx) \tau(dx') . \end{aligned}$$

Denoting $\widetilde{K}_H(u) = K_H(u) - K_H(0)$,

$$\begin{aligned} \text{var}(W_H(\mu_\varepsilon)) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_{\|x\| < \delta/\varepsilon} \mathbf{1}_{\|x'\| < \delta/\varepsilon} \widetilde{K}_H(d(\exp(\varepsilon x), \exp(\varepsilon x'))) \tau(dx) \tau(dx') \\ &\quad + K_H(0) \tau(\{\|x\| < \delta/\varepsilon\})^2 . \end{aligned}$$

Let us admit for a while that

$$(14) \quad \frac{\tau(\{\|x\| < \delta/\varepsilon\})^2}{\varepsilon^{2H}} \xrightarrow{\varepsilon \rightarrow 0^+} 0 .$$

Then, applying Lebesgue's Theorem with the convergence argument on \widetilde{K}_H obtained in Lemma 4.1, yields

$$(15) \quad \frac{\text{var}(W_H(\mu_\varepsilon))}{\varepsilon^{2H}} \xrightarrow{\varepsilon \rightarrow 0^+} -K_2 \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\|^{2H} \tau(dx) \tau(dx') .$$

Let us prove now (14) where we recall that τ is any measure in \mathfrak{M}^H . In particular, the

total mass of τ is zero so that

$$\begin{aligned} \frac{\tau(\{\|x\| < \delta/\varepsilon\})}{\varepsilon^H} &= - \frac{\tau(\{\|x\| > \delta/\varepsilon\})}{\varepsilon^H} \\ &= - \int_{\mathbb{R}^n} \varepsilon^{-H} \mathbf{1}_{\|x\| > \delta/\varepsilon} \tau(dx) . \end{aligned}$$

For any fixed $x \in \mathbb{R}^n$, $\varepsilon^{-H} \mathbf{1}_{\|x\| > \delta/\varepsilon}$ is zero when ε is small enough. Moreover $\varepsilon^{-H} \mathbf{1}_{\|x\| > \delta/\varepsilon}$ is dominated by $\delta^{-H} \|x\|^H$ which belongs to $L^1(\mathbb{R}^n, |\tau|(dx))$ since τ belongs to \mathfrak{M}^H . We conclude by Lebesgue's Theorem again.

We deduce from asymptotic (15) the following theorem.

Theorem 4.2. *Let $0 < H < 1/2$. The limit*

$$\frac{W_H(\exp^* \tau_\varepsilon)}{\varepsilon^H} \xrightarrow[\varepsilon \rightarrow 0^+]{fdd} T_H(\tau)$$

holds for all $\tau \in \mathfrak{M}^H$, in the sense of finite dimensional distributions of the random functionals. Here T_H is the centered Gaussian random linear functional on \mathfrak{M}^H with

$$(16) \quad \text{Cov}(T_H(\tau), T_H(\tau')) = -K_2 \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\|^{2H} \tau(dx) \tau'(dx'),$$

As for Theorem 2.3, Theorem 4.2 can be rephrased in terms of pointwise fields. Indeed, $\delta_x - \delta_O$ belongs to \mathfrak{M}^H for all x in \mathbb{R}^n . Let us apply Theorem 4.2 with $\tau = \delta_x - \delta_O$. Then $T_H(\delta_x - \delta_O)$ has the covariance

$$\text{Cov}(T_H(\delta_x - \delta_O), T_H(\delta_{x'} - \delta_O)) = K_2(\|x\|^{2H} + \|x'\|^{2H} - \|x - x'\|^{2H}),$$

and the field $\{T_H(\delta_x - \delta_O); x \in \mathbb{R}^n\}$ is an Euclidean fractional Brownian field.

APPENDIX

4.4. Recurrence formula for the ψ_n 's. Recall that the functions ψ_n 's are defined by (1) and (2)

$$\psi_n(u, r) = \Psi_n(M, M', r) = \int_{\mathbb{S}_n} \mathbf{1}_{d(M, N) < r} \mathbf{1}_{d(M', N) < r} d\sigma_n(N), \quad (u, r) \in [0, \pi] \times \mathbb{R}^+,$$

for any pair (M, M') in \mathbb{S}_n such that $d(M, M') = u$. Here σ_n stands for the surface measure on \mathbb{S}_n .

Lemma 4.3. *The family of functions $\psi_n, n \geq 2$ satisfies the following recursion:*

$\forall (u, r) \in [0, \pi] \times \mathbb{R}^+,$

$$\psi_n(u, r) = \int_{-\sin r}^{\sin r} (1 - a^2)^{n/2} \psi_{n-1} \left(u, \arccos \left(\frac{\cos r}{\sqrt{1 - a^2}} \right) \right) da .$$

Proof. An arbitrary point of \mathbb{S}_n is parameterized either in Cartesian coordinates, $(x_i)_{1 \leq i \leq n+1}$, or in spherical one

$$(\phi_i)_{1 \leq i \leq n} \in [0, \pi]^{n-1} \times [0, 2\pi)$$

with

$$\begin{aligned}
 x_1 &= \cos \phi_1 \\
 x_2 &= \sin \phi_1 \cos \phi_2 \\
 x_3 &= \sin \phi_1 \sin \phi_2 \cos \phi_3 \\
 &\dots \\
 x_n &= \sin \phi_1 \sin \phi_2 \dots \sin \phi_{n-1} \cos \phi_n \\
 x_{n+1} &= \sin \phi_1 \sin \phi_2 \dots \sin \phi_{n-1} \sin \phi_n
 \end{aligned}$$

Let M be the point $(\phi_i)_{1 \leq i \leq n} = (0, \dots, 0)$. One can write as follows the ball $B_n(M, r)$ of radius r , which is a spherical cap on \mathbb{S}_n with opening angle r ,

$$B_n(M, r) = \{(\phi_i)_{1 \leq i \leq n} \in \mathbb{S}_n ; \phi_1 \leq r\}$$

or in Cartesian coordinates

$$B_n(M, r) = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{S}_n ; x_1 \geq \cos r\}.$$

Let $a \in (-1, 1)$ and let P_a be the hyperplane of \mathbb{R}^{n+1} defined by $x_{n+1} = a$. Let us consider the intersection $P_a \cap B_n(M, r)$.

- If $1 - a^2 < \cos^2 r$ then $P_a \cap B_n(M, r) = \emptyset$.
- If $1 - a^2 \geq \cos^2 r$ then

$$\begin{aligned}
 P_a \cap B_n(M, r) &= \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{S}_n ; x_1 \geq \cos r \text{ and } x_{n+1} = a\} \\
 &= \{(x_i)_{1 \leq i \leq n} \in \mathbb{R}^n ; x_1 \geq \cos r \text{ and } \sum_{1 \leq i \leq n} x_i^2 = 1 - a^2\} \times \{a\}.
 \end{aligned}$$

In other words, denoting by $\mathbb{S}_{n-1}(R)$ the $(n-1)$ -dimensional sphere of radius R ,

$$P_a \cap B_n(M, r) = B_{n-1, \sqrt{1-a^2}}(M(a), r(a)) \times \{a\}$$

where $B_{n-1, \sqrt{1-a^2}}(M(a), r(a))$ is the spherical cap on $\mathbb{S}_{n-1}(\sqrt{1-a^2})$, centered at $M(a) = (\sqrt{1-a^2}, 0, \dots, 0)$ and with opening angle $r(a) = \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)$.

Let now M' be defined in spherical coordinates by $(\phi_i)_{1 \leq i \leq n} = (u, 0, \dots, 0)$, so that $d(M, M') = u$. The intersection $P_a \cap B_n(M', r)$ is the map of $P_a \cap B_n(M, r)$ by the rotation of angle u and center C in the plane $x_3 = \dots = x_{n+1} = 0$. So

- if $1 - a^2 < \cos^2 r$ then $P_a \cap B_n(M', r) = \emptyset$.
- if $1 - x_0^2 \geq \cos^2 r$ then

$$P_a \cap B_n(M', r) = B_{n-1, \sqrt{1-a^2}}(M'(a), r(a)) \times \{a\}$$

where the $(n-1)$ -dimensional spherical cap $B_{n-1, \sqrt{1-a^2}}(M'(a), r(a))$ is now centered at $M'(a) = (\sqrt{1-a^2} \cos u, \sqrt{1-a^2} \sin u, 0, \dots, 0)$.

Let us define $\psi_{n-1, R}(u, r)$ as the intersection surface of two spherical caps on $\mathbb{S}_{n-1}(R)$, whose centers are at a distance Ru and with the same opening angle r .

By homogeneity, this leads to

$$\psi_{n-1, R}(u, r) = R^{n-1} \psi_{n-1, 1}(u, r) = R^n \psi_{n-1}(u, r).$$

The surface measure σ_n of \mathbb{S}_n can be written as

$$d\sigma_n(x_1, \dots, x_n, a) = \sqrt{1-a^2} d\sigma_{n-1, \sqrt{1-a^2}}(x_1, \dots, x_n) \times da$$

where $\sigma_{n-1, R}$ is the surface measure of $\mathbb{S}_{n-1}(R)$.

We then obtain

$$\begin{aligned} \psi_n(u, r) &= \int_{-1}^1 \mathbf{1}_{1-a^2 \geq \cos^2 r} \psi_{n-1, \sqrt{1-a^2}}(u, \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)) \sqrt{1-a^2} da \\ &= \int_{-\sin r}^{\sin r} (\sqrt{1-a^2})^n \psi_{n-1}(u, \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)) da, \end{aligned}$$

and Lemma 4.3 is proved. \square

4.5. Asymptotic behaviour of the hyperbolic balls.

Lemma 4.4. *We denote by $\phi(r)$ the surface measure of any ball with radius r within the hyperbolic space \mathbb{H}_n .*

There exists a constant $C_n > 0$ such that

$$\lim_{r \rightarrow +\infty} e^{-r/2} \phi(r) = C_n.$$

Proof. The hyperbolic space can be parameterized in hyperbolic coordinates

$$\begin{aligned} x_{n+1} &= \cosh \varphi_n \\ x_n &= \sinh \varphi_n \cos \varphi_{n-1} \\ x_{n-1} &= \sinh \varphi_n \sin \varphi_{n-1} \cos \varphi_{n-2} \\ &\dots = \dots \\ x_2 &= \sinh \varphi_n \sin \varphi_{n-1} \dots \cos \varphi_1 \\ x_1 &= \sinh \varphi_n \sin \varphi_{n-1} \dots \sin \varphi_1 \end{aligned}$$

with $\varphi_1 \in [0, 2\pi)$, $\varphi_2, \dots, \varphi_{n-1} \in [0, \pi)$ and $\varphi_n \geq 0$. The surface measure is then given by

$$d\sigma(N) = \sinh \varphi_n \left(\prod_{k=1}^{n-1} \sin^{k-1} \varphi_k \right) d\varphi_1 d\varphi_2 \dots d\varphi_n .$$

Let us consider the point M whose hyperbolic coordinates are $(\varphi_k)_{1 \leq k \leq n} = (0, 0, \dots, 0)$. The hyperbolic ball $B(M, r)$ is described as $B(M, r) = \{\varphi_n \leq r\}$ and writing $\phi(r)$ as the surface of $B(M, r)$, one gets

$$\phi(r) = \sigma(B(M, r)) = 2C_n (\cosh r - 1) ,$$

with

$$C_n = 2\pi \prod_2^{n-1} \left(\int_0^\pi \sin^{k-1} \varphi_k d\varphi_k \right)$$

and Lemma 4.4 is proved. \square

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