



# Scaling approach to existence of long cycles in Casimir boxes

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# Scaling approach to existence of long cycles in Casimir boxes

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## Abstract

We analyse a concept of generalized Bose-Einstein condensation (g-BEC) for the perfect Bose gas (PBG) in Casimir boxes via relation between concepts of long cycles and Off-Diagonal-Long-Range-Order (ODLRO) usually considered as an adequate way to describe the standard BEC in the fundamental state of the cubic box. We revise definitions of theses concepts, called *scaled* Bose-Einstein condensation (s-BEC), *scaled* short/long cycles (s-short/long cycles) and *scaled* ODLRO (s-ODLRO) which are based on some scaling arguments.

The principle result is that the classification of the g-BEC in three types (I,II,III) implies a hierarchy of long cycles (depending of their size) and a hierarchy of ODLRO (depending of the size of spatial delocalization of the condensate).

**Keywords:** Generalized Bose-Einstein condensation, (Long) Cycles, Off-Diagonal-Long-Range-Order (ODLRO), perfect Bose gas, Casimir boxes, Scaling.

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## 1 Introduction: BEC in Casimir boxes

In this section we give a list of the different concepts concerning the BEC such as generalized BEC, cycles, ODLRO permit to understand the original Bose-Einstein condensation phenomenon via those different complementary points of view.

### 1.1 About the concept of generalized Bose-Einstein condensation and ODLRO

The history of the generalized Bose-Einstein condensation (g-BEC) began with H.Casimir in 1968 who studied the ideal Bose gas in an anisotropic boxes. He showed that condensation may come from infinite number of levels macroscopically occupied. Later in 1982 M.van den Berg, J.Lewis [vdB-L] and in 1986 M.van den Berg, J.Lewis and J.Pulé [vdB-L-P] gave a definition of the concept of the generalized Bose-Einstein condensation.

Consider the grand-canonical  $(\beta, \mu)$  perfect Bose gas (PBG) in Casimir boxes  $\Lambda = L_1 \times L_2 \times L_3 \in \mathbb{R}^{d=3}$ , of volume  $|\Lambda| = V$  with sides length  $L_\nu = V^{\alpha_\nu}$ ,  $\nu = 1, 2, 3$ , where  $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$ ,  $\alpha_1 + \alpha_2 + \alpha_3 = 1$ . For the single-particle hamiltonian  $H_\Lambda^{(N=1)} = T_\Lambda^{(1)} := -(\hbar^2/2m)\Delta$  with periodic boundary conditions, we get the dual vector-spaces  $\Lambda^*$  defined by:

$$\Lambda^* = \left\{ k \in \mathbb{R}^3 : k = \left( \frac{2\pi n_1}{V^{\alpha_1}}, \frac{2\pi n_2}{V^{\alpha_2}}, \frac{2\pi n_3}{V^{\alpha_3}} \right); n_\nu \in \mathbb{Z}^1 \right\}. \quad (1.1)$$

In the grand-canonical ensemble, the mean values of the  $k$ -mode particle densities  $\{\rho_\Lambda(k)\}_{k \in \Lambda^*}$  are:

$$\rho_\Lambda(k) := \frac{1}{V} \langle N_\Lambda(k) \rangle_\Lambda(\beta, \mu) = \frac{1}{V} \frac{1}{e^{\beta(\epsilon_\Lambda(k) - \mu)} - 1}, \quad (1.2)$$

where  $\epsilon_\Lambda(k) = \hbar^2 k^2 / 2m$ ,  $k \in \Lambda^*$  are the eigenvalue of the Laplacian with periodic boundary conditions,  $\langle N_\Lambda(k) \rangle_\Lambda(\beta, \mu)$  is the Gibbs expectation of the particles number operator  $N_\Lambda(k)$  in the mode  $k$ . The total density of particles is  $\rho_\Lambda(\beta, \mu) := \sum_{k \in \Lambda^*} \rho_\Lambda(k)$

Recall that the eigenfunctions  $\{\Psi_{\Lambda,k}^{(N=1)}(x)\}_{k \in \Lambda^*}$  of the single particle hamiltonian  $T_{\Lambda}^{(1)}$  are:

$$\Psi_{\Lambda,k}^{(1)}(x) = \frac{1}{\sqrt{V}} e^{ik \cdot x} . \quad (1.3)$$

We consider now the fixed density of particles. The chemical potential is then a unique solution of the equation  $\rho = \rho_{\Lambda}(\beta, \mu)$ , which we denote by  $\bar{\mu}_{\Lambda} := \bar{\mu}_{\Lambda}(\beta, \rho)$ .

The van den Berg-Lewis-Pulé formulation of the concept of g-BEC in Casimir boxes is:

**Definition 1.1** *We say that for the grand-canonical PBG there is g-BEC with a fixed total density of particle  $\rho$ , if we have:*

$$\rho_0(\beta, \rho) := \lim_{\epsilon \downarrow 0} \lim_{V \uparrow \infty} \sum_{\{k \in \Lambda^* : \|k\| \leq \epsilon\}} \rho_{\Lambda}(k) > 0 , \quad (1.4)$$

where  $\rho_{\Lambda}(k)$  are defined by (1.2) for  $\mu = \bar{\mu}_{\Lambda}(\beta, \rho)$ .

This criterion of condensation means that all single particle states  $\Psi_{\Lambda,k}^{(1)} \in \mathcal{H}_{\Lambda}^{(1)}$  in the vicinity of the ground state form a macroscopic "atom".

This motivate the following classification:

**Definition 1.2** • *One gets g-BEC of type I if a finite number of single particle states are macroscopically occupied.*

• *There is g-BEC of type II imply that an infinite number of single particle states are macroscopically occupied.*

• *The g-BEC is called type III, if all single particle states are microscopically occupied, although  $\rho_0(\beta, \rho) > 0$ .*

This concept allows the study PBG in Casimir boxes and to give an illustration of the different types of g-BEC.

As usually one introduces for the PBG the critical density,  $\rho_c(\beta)$  defined by:

$$\rho_c(\beta) := \sup_{\mu < 0} \lim_{V \uparrow \infty} \rho_{\Lambda}(\beta, \mu) = g_{3/2}(1)/\lambda_{\beta}^3, \quad (1.5)$$

where  $g_s(z) := \sum_{j=1}^{\infty} z^j/j^s$  is related to the Riemann zeta-function  $\zeta(s) := g_s(1)$ . Here  $\lambda_{\beta} = \hbar\sqrt{2\pi\beta/m}$  is the thermal De Broglie length.

Due to [vdB-L-P] to we have the following proposition:

**Proposition 1.1** *If one consider the PBG in Casimir boxes with  $1/2 > \alpha_1$ , then for a fixed particles density  $\rho > \rho_c(\beta)$  the chemical potential is  $\bar{\mu}_{\Lambda} = -A/\beta V + o(1/V)$ , with  $A > 0$  and there is g-BEC of the type I in the zero mode  $k = 0$ . Here  $A$  is the solution of the equation :*

$$\rho - \rho_c(\beta) = \frac{1}{A} . \quad (1.6)$$

*If  $1/2 = \alpha_1$ , then the chemical potential is  $\bar{\mu}_{\Lambda} = -B/\beta V + o(1/V)$ , with  $B > 0$  and there is g-BEC of the type II in the infinite number of modes:*

$$\begin{aligned} \lim_{V \uparrow \infty} \rho_{\Lambda}(k) &= \frac{1}{B + \lambda_{\beta}^2 n_1^2}, \text{ for } k = (2\pi n_1/V^{\alpha_1}, 0, 0), \ n_1 \in \mathbb{Z}^1 , \\ &= 0, \text{ for } k \neq (2\pi n_1/V^{\alpha_1}, 0, 0), \ n_1 \in \mathbb{Z}^1 . \end{aligned}$$

Here  $B$  is the solution of the equation:

$$\rho - \rho_c(\beta) = \sum_{n_1 \in \mathbb{Z}^1} \frac{1}{B + \lambda_{\beta}^2 n_1^2} . \quad (1.7)$$

If  $1/2 < \alpha_1$ , then the chemical potential  $\bar{\mu}_\Lambda = -C/\beta V^\delta + o(1/V^\delta)$ , with  $\delta = 2(1 - \alpha_1)$  and  $C > 0$ . The corresponding g-BEC is the type III: for all  $k \in \Lambda^*$  we have  $\lim_{V \uparrow \infty} \rho_\Lambda(k) = 0$  although  $\rho_0(\beta, \rho) > 0$  and  $C$  is the solution of the equation:

$$\rho - \rho_c(\beta) = \frac{\pi}{\lambda_\beta C^{1/2}}. \quad (1.8)$$

In 1951 O.Penrose and later in 1956 O.Penrose and L.Onsager [P-O] proposed a criterion of BEC well adapted to both ideal or interacting gases: there is Bose-Einstein condensation if and only if there is Off-Diagonal Long-Range Order (ODLRO) for the reduce statistical operator. Here we recall the definition for the two-point correlation function of ODLRO for particular case of the PBG with periodic boundary conditions:

**Definition 1.3** We say that for the PBG there is an ODLRO for a fixed total density of particles  $\rho$  if we have:

$$\sigma(\beta, \rho) := \lim_{\|x-x'\| \uparrow \infty} \lim_{V \uparrow \infty} \sigma_\Lambda(\beta, \rho; x, x') > 0, \quad (1.9)$$

where  $\sigma(\beta, \rho; x, x') := \lim_{V \uparrow \infty} \sigma_\Lambda(\beta, \rho; x, x')$  is the two-correlation function, defined by:

$$\sigma_\Lambda(\beta, \rho; x, x') := \sum_{k \in \Lambda^*} \langle N_\Lambda(k) \rangle_\Lambda(\beta, \rho) \Psi_{\Lambda, k}^{(1)}(x) \Psi_{\Lambda, k}^{(1)}(x')^* = \sum_{k \in \Lambda^*} \rho_\Lambda(k) e^{ik \cdot (x - x')}, \quad (1.10)$$

for periodic boundary conditions. Here  $\rho_\Lambda(k)$  is defined by (1.2).

The interpretation of this criterion is very intuitive. Since ODLRO measures the correlation between two points at the infinity after thermodynamical limit. Recall that according to Definition 1.1 for non-zero BEC we catch in thermodynamical limit the density of particles in the modes null i.e. with de Broglie wave-length of those particles equals infinity. This allows communication in the condensate through the whole space (in  $\mathbb{R}^3$ ). The heuristic arguments can be rigorous by checked showing that g-BEC and ODLRO are equivalent for the PBG in Casimir boxes:

**Theorem 1.1** Consider a PBG in Casimir boxes  $\Lambda$ . Then we have the following result:

$$\begin{aligned} \sigma(\beta, \rho) &= 0, \text{ for } \rho < \rho_c, \\ &= \rho - \rho_c(\beta), \text{ for } \rho > \rho_c, \end{aligned} \quad (1.11)$$

therefore ODLRO is non-zero if and only if there is g-BEC.

*Proof*

We split the correlation function in two part, the first one is the correlation due to the condensate and the second one correspond to the particles outside the condensate:

$$\begin{aligned} \sigma_\Lambda(\beta, \rho; x, x') &:= \lim_{\epsilon \downarrow 0} \lim_{V \uparrow \infty} \sum_{k \in \{\Lambda_\Lambda^* : \|k\| \leq \epsilon\}} \rho_\Lambda(k) e^{ik \cdot (x - x')} + \lim_{\epsilon \downarrow 0} \lim_{V \uparrow \infty} \sum_{k \in \{\Lambda_\Lambda^* : \|k\| > \epsilon\}} \rho_\Lambda(k) e^{ik \cdot (x - x')} \\ &= \lim_{\epsilon \downarrow 0} \lim_{V \uparrow \infty} \sum_{k \in \{\Lambda_\Lambda^* : \|k\| \leq \epsilon\}} \rho_\Lambda(k) + \frac{1}{(2\pi)^3} \int_{k \in \mathbb{R}^3} dk \frac{e^{ik \cdot (x - x')}}{e^{\beta(\hbar^2 k^2 / 2m - \mu)} - 1}, \end{aligned}$$

since  $\forall k \in \{\Lambda_\Lambda^* : \|k\| \leq \epsilon\}$ ,  $k \cdot (x - x') \rightarrow 0$  when  $\epsilon \rightarrow 0$ . Here  $\mu := \lim_{V \uparrow \infty} \bar{\mu}_\Lambda$ .

Then by virtue of Proposition 1.1 and by the Riemann-Lebesgue Theorem we obtain the result.  $\square$

## 1.2 Feynman theory of cycles (loops)

Here we would like to recall the Feynman concept of cycles introduced in 1953 [F]. It is related to the Bose statistic and Feynman-Kac representation for the partition function.

Recall that for the PBG in boxes  $\Lambda = L_1 \times L_2 \times L_3 \subset \mathbb{R}^3$  with *periodic* boundary conditions, (i.e. with the dual vector-spaces  $\Lambda^* = \{k \in \mathbb{R}^3 : k = (2\pi n_1/L_1, 2\pi n_2/L_2, 2\pi n_3/L_3), n_\nu \in \mathbb{Z}\}$ , see(1.1)) the grand-canonical pressure can be represented in the form:

$$p_\Lambda(\beta, \mu) = \frac{1}{\beta|\Lambda|} \log(\Xi_\Lambda^0(\beta, \mu)) = \frac{1}{\beta|\Lambda|} \sum_{k \in \Lambda^*} \ln \left[ \left( 1 - e^{-\beta(\epsilon_\Lambda(k) - \mu)} \right)^{-1} \right] = \frac{1}{\beta|\Lambda|} \sum_{j=1}^{\infty} \frac{1}{j} e^{j\beta\mu} \text{Tr}_{\mathcal{H}_\Lambda^{(1)}} \left( e^{-j\beta T_\Lambda^{(1)}} \right), \quad (1.12)$$

where  $\Xi_\Lambda^0(\beta, \mu)$  is the grand-canonical partition function and we used that  $\text{Tr}_{\mathcal{H}_{\Lambda,1}} \left( e^{-j\beta T_\Lambda^{(1)}} \right) = \sum_{k \in \Lambda^*} e^{-j\beta \epsilon_\Lambda(k)}$ .

The Feynman-Kac representation naturally appears from (1.12) if we consider the representation of the trace of the Gibbs semigroup  $\left\{ e^{-\beta T_\Lambda^{(1)}} \right\}_{\beta > 0}$  via its kernel  $K_\Lambda(\beta; x, x') = \left( e^{-\beta T_\Lambda^{(1)}} \right)(x, x')$ :

$$\text{Tr}_{\mathcal{H}_\Lambda^{(1)}} \left( e^{-\beta T_\Lambda^{(1)}} \right) = \int_\Lambda dx K_\Lambda(\beta; x, x). \quad (1.13)$$

It is known [G] that the kernel  $K_\Lambda(\beta; x, x')$  can be represented as an integral over Wiener trajectories starting at point  $x$  and finishing at point  $x'$ . Thus  $K_\Lambda(\beta; x, x)$  can be represent as a Wiener integral over close trajectories (loops) starting and finishing at the same point. The order of the size of the trajectories coincide with the size of the quantum fluctuations  $\lambda_\beta$ , known as the thermal De Broglie length [M].

By virtue of the Gibbs semigroup properties, and by expressions (1.12), (1.13), for the grand-canonical pressure we get:

$$p_\Lambda(\beta, \mu) = \sum_{j=1}^{\infty} \frac{1}{j} e^{j\beta\mu} \int_\Lambda dx_1 \int_\Lambda dx_2 \dots \int_\Lambda dx_j K_\Lambda(\beta; x_1, x_2) K_\Lambda(\beta; x_2, x_3) \dots K_\Lambda(\beta; x_j, x_1), \quad (1.14)$$

In (1.14) each integral over  $(x_1, \dots, x_j) \in \Lambda^j$  correspond to the impact of the Wiener's loops of the length  $j\lambda_\beta$  [G], [M].

Notice that the total density of particles in this ensemble is given by:

$$\rho_\Lambda(\beta, \mu) := \frac{\langle N_\Lambda \rangle_\Lambda(\beta, \mu)}{V} = \partial_\mu p_\Lambda(\beta, \mu) = \frac{1}{V} \sum_{j=1}^{\infty} e^{j\beta\mu} \text{Tr}_{\mathcal{H}_\Lambda^{(1)}} e^{-j\beta T_\Lambda^{(1)}}. \quad (1.15)$$

One can use this representation to identify the *repartition* of the total density (1.15) over densities of particles involved into loops of the length  $j\lambda_\beta$ :

$$\rho_{\Lambda,j}(\beta, \mu) := \frac{1}{V} e^{j\beta\mu} \text{Tr}_{\mathcal{H}_\Lambda^{(1)}} e^{-j\beta T_\Lambda^{(1)}}. \quad (1.16)$$

The  $j$ -loops particles density (1.16) and representation (1.14) are the keys to the concept of the *short/long* cycles. In fact after thermodynamical limit one can obtain loops of finite sizes or infinite sizes. Using this point of view one can relate BEC to appearance of loops of infinite size as an explanation of the long range order and the macroscopic size of the quantum fluctuations [F], [P-O], [U].

### 1.3 Bose statistic and cycles

To develop the mathematical approach of Feynman cycles, we introduce the permutation group via Bose statistics is related to the permutation group.

Consider a system of  $N$  identical bosons enclose in a region  $\Lambda \in \mathbb{R}^3$ . Let they be in the thermal equilibrium at inverse temperature  $\beta$ . The Hamiltonian of the system in general is given by  $H_\Lambda^{(N)} := T_\Lambda^{(N)} + U_\Lambda^{(N)}$ , where  $T_\Lambda^{(N)}$  is the usual kinetic-energy operator and  $U_\Lambda^{(N)}$  is the interacting-operator term with the condition that of the trace of the operator  $e^{-\beta H_\Lambda^{(N)}}$  exist.

Let us introduce the symmetrization projector  $P_N := \frac{1}{N!} \sum_{\pi \in S_N} U_\pi$ , where  $U_\pi = \mathcal{H}_\Lambda^{(N)} \mapsto \mathcal{H}_\Lambda^{(N)}$  is the unitary representation of the permutation group  $S_N$  defined by  $U_\pi \Psi(x_1, \dots, x_N) = \Psi(x_{\pi(1)}, \dots, x_{\pi(N)})$ .

Then the grand-canonical partition function is equal to:

$$\Xi_{\Lambda}(\beta, \mu) = \sum_{N=0}^{\infty} \text{Tr}_{\mathcal{H}_{\Lambda}^{(N), \text{Sym}}} \left( e^{-\beta(H_{\Lambda}^{(N)} - \mu N)} \right) = \sum_{N=0}^{\infty} \frac{1}{N!} e^{\mu N} \sum_{\pi} \text{Tr}_{\mathcal{H}_{\Lambda}^{(N)}} \left( U_{\pi} e^{-\beta H_{\Lambda}^{(N)}} \right), \quad (1.17)$$

To make a contact with Feynman's idea of cycles, we recall the relation between Bose statistics and cycles [D-M-P]:

- Each permutation  $\pi \in S_N$  can be decomposed into a number of cyclic permutations of sizes  $q_1, q_2, \dots, q_r$  with  $r \leq N$  and  $\sum q_i = N$ . We consider the set  $\Omega = \bigcup_{r \in \mathbb{N}} \Omega_r$  of unordered  $r$ -tuples of natural numbers  $q = (q_1, q_2, \dots, q_r) \in \Omega_r$  for  $r = 0, 1, 2, \dots$  and let  $|q| = \sum_{i=1}^r q_i$  for  $q \in \Omega$ . Then a decomposition of  $\pi \in S_N$  into cycles is labeled by  $q \in \Omega$  with  $|q| = N$ .
- The decomposition into cycles leads to a partition of  $S_N$  into equivalence classes of permutations  $C_q$ ,  $|q| = N$ .
- Two permutations  $\pi'$  and  $\pi$  belong to the same class if and only if they are conjugate in  $S_N$  i.e. there exists  $\pi \in S_N$  such that  $\pi' = \pi^{-1} \pi' \pi$ .
- The number of permutations belonging to the class  $C_q$ , noted  $|C_q|$  is  $N! / \prod_{p=1}^r n_p! q_p$ , where  $n_q$  is the number of cycles of size  $p$  in  $q$ .
- If  $[H_{\Lambda}^{(N)}, U_{\pi}] = 0 \ \forall \pi \in S_N$ , for  $\pi', \pi'' \in C_q$  one has:

$$\text{Tr}_{\mathcal{H}_{\Lambda}^{(N)}} \left( U_{\pi'} e^{-\beta H_{\Lambda}^{(N)}} \right) = \text{Tr}_{\mathcal{H}_{\Lambda}^{(N)}} \left( U_{\pi}^{-1} U_{\pi'} U_{\pi} e^{-\beta H_{\Lambda}^{(N)}} \right) = \text{Tr}_{\mathcal{H}_{\Lambda}^{(N)}} \left( U_{\pi}^{-1} U_{\pi''} e^{-\beta H_{\Lambda}^{(N)}} U_{\pi} \right) = \text{Tr}_{\mathcal{H}_{\Lambda}^{(N)}} \left( U_{\pi''} e^{-\beta H_{\Lambda}^{(N)}} \right),$$

consequently  $\text{Tr}_{\mathcal{H}_{\Lambda}^{(N)}} \left( U_{\pi} e^{-\beta H_{\Lambda}^{(N)}} \right)$  depend only on the cycle decomposition  $q$  of  $\pi$

Under those assumptions we obtain the *magic formula* [M], [M-B]:

$$\Xi_{\Lambda}(\beta, \mu) = 1 + \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{q \in \mathbb{N}^r} \prod_{i=1}^r \frac{1}{q_i} e^{\beta \mu q_i} \text{Tr}_{\mathcal{H}_{\Lambda}^{(N)}} \left( U_q e^{-\beta H_{\Lambda}^{(N)}} \right). \quad (1.18)$$

For the PBG in Casimir boxes  $\Lambda$  we have  $\mathcal{H}_{\Lambda}^{(N)} = \otimes_{i=1}^N \mathcal{H}_{\Lambda}^{(1)}$ , where  $\mathcal{H}_{\Lambda}^{(1)}$  is the single particle Hilbert space  $\mathcal{H}_{\Lambda}^{(1)} = \mathcal{L}^2(\Lambda)$ . Consequently we have:

$$\begin{aligned} \text{Tr}_{\mathcal{H}_{\Lambda}^{(N)}} \left( U_q e^{-\beta T_{\Lambda}^{(N)}} \right) &= \sum_{k_1, \dots, k_N \in \Lambda^*} \left( \Psi_{\Lambda, k_1}^{(1)} \otimes \dots \otimes \Psi_{\Lambda, k_N}^{(1)}, e^{-\beta T_{\Lambda}^{(N)}} U_q \Psi_{\Lambda, k_1}^{(1)} \otimes \dots \otimes \Psi_{\Lambda, k_N}^{(1)} \right) \\ &= \prod_{p=1}^r \sum_{k_1, \dots, k_{q_i} \in \Lambda^*} \left( \Psi_{\Lambda, k_1}^{(1)} \otimes \dots \otimes \Psi_{\Lambda, k_{q_i-1}}^{(1)} \otimes \Psi_{\Lambda, k_{q_i}}^{(1)}, e^{-\beta T_{\Lambda}^{(q_i)}} \Psi_{\Lambda, k_2}^{(1)} \otimes \dots \otimes \Psi_{\Lambda, k_{q_i}}^{(1)} \otimes \Psi_{\Lambda, k_1}^{(1)} \right) \\ &= \prod_{i=1}^r \text{Tr}_{\mathcal{H}_{\Lambda}^{(1)}} \left( e^{-q_i \beta T_{\Lambda}^{(1)}} \right), \end{aligned}$$

since  $U_q = \prod_{i=1}^r U_{q_i}$  act on independent parts of the state  $\Psi_{\Lambda, k_1}^{(1)} \otimes \dots \otimes \Psi_{\Lambda, k_N}^{(1)}$ , where  $U_{q_i}$  is the unitary representation of the cyclic permutation  $\pi_{q_i} \in S_{q_i}$  defined by  $U_{q_i} \Psi_{\Lambda, k_1}^{(1)} \otimes \dots \otimes \Psi_{\Lambda, k_{q_i-1}}^{(1)} \otimes \Psi_{\Lambda, k_{q_i}}^{(1)} = \Psi_{\Lambda, k_2}^{(1)} \otimes \dots \otimes \Psi_{\Lambda, k_{q_i}}^{(1)} \otimes \Psi_{\Lambda, k_1}^{(1)}$  and because the single particles states are orthogonal.

Inserting this equation into (1.18) we obtain for the PBG grand canonical partition function  $\Omega_{\Lambda}^0(\beta, \mu)$  c.f. (1.12) in Casimir boxes:

$$\Xi_{\Lambda}^{(0)}(\beta, \mu) = 1 + \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{q \in \mathbb{N}^r} \prod_{i=1}^r \frac{1}{q_i} e^{\beta \mu q_i} \text{Tr}_{\mathcal{H}_{\Lambda}^{(1)}} \left( e^{-q_i \beta T_{\Lambda}^{(1)}} \right). \quad (1.19)$$

We can establish the contact with (1.12) rewriting the precedent formula:

$$\begin{aligned}
\Xi_{\Lambda}^{(0)}(\beta, \mu) &= \sum_{N \geq 0} \sum_{\{n_j\}_{j=1}^N: \sum j n_j = N} \frac{1}{n_j!} \prod_{j=1}^N \left( \frac{1}{j} e^{\beta \mu j} \text{Tr}_{\mathcal{H}_{\Lambda}^{(1)}} \left( e^{-j \beta T_{\Lambda}^{(1)}} \right) \right)^{n_j} . \\
&= \prod_{j=1}^{\infty} \text{Exp} \left( \frac{1}{j} e^{\beta \mu j} \text{Tr}_{\mathcal{H}_{\Lambda}^{(1)}} \left( e^{-j \beta T_{\Lambda}^{(1)}} \right) \right) .
\end{aligned}$$

One can use this representation to identify the *repartition* of the total density over mean densities of particles involved into cycles of the size  $j$ :

$$\rho_{\Lambda, j}(\beta, \mu) := \left\langle \frac{1}{V} \sum_{i=1}^r j \delta_{j, q_i} \right\rangle_{\Lambda}(\beta, \mu) = \frac{1}{V} e^{j \beta \mu} \text{Tr}_{\mathcal{H}_{\Lambda}^{(1)}} e^{-j \beta T_{\Lambda}^{(1)}} . \quad (1.20)$$

The  $j$ -cycles particle density (1.20) is the keys to the concept of the *short/long* cycles.

**Definition 1.4** *We say that the representation (1.19) for the grand-canonical PBG contains only short cycles if the limit:*

$$\rho_{\text{short}}(\beta, \mu) := \lim_{M \rightarrow \infty} \left\{ \lim_{V \uparrow \infty} \sum_{j=1}^M \rho_{\Lambda, j}(\beta, \mu) \right\} = \rho(\beta, \mu) := \lim_{V \uparrow \infty} \rho_{\Lambda}(\beta, \mu) , \quad (1.21)$$

*i.e. it coincides with limit of the total particle density. Since in general the limits in (1.21) are not interchangeable, we say that the representation (1.19) for the grand-canonical PBG contains macroscopic number of long cycles for a total density of particles  $\rho$ , if we have  $\rho > \rho_{\text{short}}(\beta, \rho)$ , or equivalently if:*

$$\rho_{\text{long}}(\beta, \rho) := \lim_{M \rightarrow \infty} \left\{ \lim_{V \uparrow \infty} \sum_{j=M+1}^{\infty} \rho_{\Lambda, j}(\beta, \rho) \right\} > 0 . \quad (1.22)$$

Since Feynman [F] the presence of the non-zero density of the long cycles is usually connected with the existence of zero-mode BEC, but a rigorous proof of this conjecture for a certain class of models has been obtained only recently. There we noticed that even for the PBG type I BEC the mathematical proof of this connection was not straightforward and appealed to a non-trivial analysis, see [S], [D-M-P], [U].

**Theorem 1.2** *Consider a PBG in Casimir boxes  $\Lambda$ . Then we have:*

$$\rho_{\text{short}}(\beta, \mu) = \frac{g_{3/2}(e^{\beta \mu})}{\lambda_{\beta}^3} < \rho_c(\beta) , \quad (1.23)$$

*where  $\rho_c(\beta)$  is the critical density defined by (1.5). Therefore, for  $\rho > \rho_c(\beta)$  the grand-canonical representation of the partition function (1.19) for the PBG contains a macroscopic number of particles in long cycles.*

*Proof*

By virtue of (1.20) and (1.21) we have:

$$\begin{aligned}
\rho_{\text{short}}(\beta, \mu) &= \lim_{M \rightarrow \infty} \left\{ \lim_{V \uparrow \infty} \sum_{j=1}^M e^{j \beta \mu} \prod_{\nu=1}^3 \frac{1}{V^{\alpha_{\nu}}} \sum_{n_{\nu} \in \mathbb{Z}^1} e^{-j \beta \pi \lambda_{\beta}^2 (n_{\nu} / V^{\alpha_{\nu}})^2} \right\} \\
&= \sum_{j=1}^{\infty} e^{j \beta \mu} \prod_{\nu=1}^3 \int_{\xi_{\nu} \in \mathbb{R}} e^{-j \beta \pi \lambda_{\beta}^2 \xi_{\nu}^2} = \sum_{j=1}^{\infty} \frac{e^{\beta \mu}}{\lambda_{\beta}^3 j^{3/2}} =: \frac{g_{3/2}(e^{\beta \mu})}{\lambda_{\beta}^3} ,
\end{aligned}$$

c.f. definition of  $g_s(z)$ .

□



## 1.4 What is the problem ?

In fact it was F.London [L] who for the first time used implicitly the *scaling* approach to solve the controversy between the G.Uhlenbeck mathematical arguments against condensation of the perfect Bose-gas for high densities and Einstein's intuitive reasoning in favor of this phenomenon. Recall that his line of reasoning was based on the following observations: since the explicit formula for the total particle density  $\rho_\Lambda(\beta, \mu)$  in the box  $\Lambda$  is known only in grand-canonical ensemble  $(\beta, \mu)$ , to ensure fixed density  $\rho$  in this box one has first to solve equation  $\rho = \rho_\Lambda(\beta, \mu)$  which determine the corresponding value of the chemical potential  $\bar{\mu}_\Lambda := \bar{\mu}_\Lambda(\beta, \rho)$ .

Concept of the g-BEC does not show up explicitly. The natural question: is the scaling approach relevant for the analysis of the long cycles? There is the same question for concept of ODLRO.

In the next section we propose to revisit the concept of g-BEC and to introduce a concept of *scaled* BEC. The concept of *scaled* long cycles and the concept of *scaled* ODLRO are considered in section 3 and 4.

## 2 Generalized BEC concept: revisited

In this section we propose a new concept of generalized BEC which we call *scaled* BEC (noted s-BEC). This imply the corresponding concept for cycles (*scaled* short/long cycles noted s-short/long cycles) via similar scaling arguments. Moreover we introduce a classification of the s-BEC (type I, II, III) and the hierarchy of the s-long cycles (long-microscopic/macrosopic-cycles). We illustrate our concepts by some remarks and propositions.

### 2.1 Generalized condensation and scaled condensation

The original van den Berg-Lewis-Pulé concept of the g-BEC [vdB-L-P] has not been explicitly addressed to detect a *fine* structure of the condensate: *a priori* it does not distinguish between generalized BEC of types I, II or III. In fact one can do this analysis, as it was done for the first time in [vdB-L] for the case of the Casimir boxes. To make this facet more evident we introduce here a *new* definition of *generalized* BEC, a *scaled* BEC (s-BEC). Take for simplicity the PBG in Casimir boxes  $\Lambda$  with *periodic* boundary conditions, i.e. with the dual vector-spaces  $\Lambda^*$  defined by (1.1), and with the  $k$ -modes mean particle densities  $\{\rho_\Lambda(k)\}_{k \in \Lambda}$  defined by (1.2).

**Definition 2.1** *We say that the perfect Bose gas manifests a s-BEC in boxes  $\Lambda$ , for a fixed total density  $\rho$  if there exists a positive decreasing function  $\eta : V \mapsto \mathbb{R}_+$ , such that  $\lim_{V \uparrow \infty} \eta(V) = 0$  and we have:*

$$\rho_\eta(\beta, \rho) := \liminf_{V \uparrow \infty} \sum_{\{k \in \Lambda_\Lambda^* : \|k\| \leq \eta(V)\}} \rho_\Lambda(k) > 0 . \quad (2.24)$$

**Remark 2.1** *Recall that the van den Berg-Lewis-Pulé definition of g-BEC is formulated for this case as:*

$$\rho_0(\beta, \rho) := \lim_{\epsilon \downarrow 0} \lim_{V \uparrow \infty} \sum_{\{k \in \Lambda_\Lambda^* : \|k\| \leq \epsilon\}} \rho_\Lambda(k) > 0 , \quad (2.25)$$

*for the total particle densities  $\rho$ . Hence the two Definitions 1.1 and 2.1 are evidently not equivalent. Moreover, we shall show that it allows also to connect this fine mode-structure of the condensate the type I, II or III with the long-cycles hierarchy, and to show that there exist a relation between the structure of the condensate and the size of cycles.*

The following statement is an evident consequence of Definitions 1.1 and 2.1:

**Lemma 2.1** *For any function  $\eta(V)$  we have:*

$$0 \leq \rho_\eta(\beta, \rho) \leq \rho_0(\beta, \rho) . \quad (2.26)$$

This lemma means that s-BEC imply g-BEC and that the contraposition is true (i.e. that if there is no g-BEC, then there is no s-BEC).

A simple example of application of the s-BEC approach is the possibility to distinguish the type I, II or III condensations of the PBG in Casimir boxes  $\Lambda$  with *periodic* boundary conditions.

**Proposition 2.1** *If the function  $\eta(V)$  decreases for  $V \rightarrow \infty$  faster than  $1/V^\gamma$  for any power  $\gamma > 0$ , then for  $\rho > \rho_c(\beta)$  one obtains the limits:*

$$\begin{aligned}\rho_\eta(\beta, \rho) &= \rho_0(\beta, \rho) \equiv \rho - \rho_c(\beta) , & \text{for } \alpha_1 < 1/2 , \\ \rho_\eta(\beta, \rho) &= \frac{1}{B} < \rho_0(\beta, \rho) , & \text{for } \alpha_1 = 1/2 , \\ \rho_\eta(\beta, \rho) &= 0 < \rho_0(\beta, \rho) , & \text{for } \alpha_1 > 1/2 ,\end{aligned}$$

related correspondingly to the g-BEC of types I, II and III. Recall that  $B$  is the unique solution of equation (1.7).

*Proof:* The proof follows straightforwardly from Definition 2.1 and the explicit form of the mean value particle density  $\rho_\Lambda(k)$  asymptotics for the three values of the exponent  $\alpha_1$ , see [vdB-L] and Introduction.  $\square$

**Proposition 2.2** *The rate  $\eta(V) = O(1/V^{1/2})$  is an important threshold to refine a discrimination between different types of g-BEC. If for example, one takes  $\eta_\delta(V) = O(1/V^{(1/2-\delta)})$  such that  $\delta > 0$ , then the first and second equations of the Proposition 2.1 rest unchangeable, but for the type III condensation we obtain:*

$$\begin{aligned}\rho_{\eta_\delta}(\beta, \rho) &= 0 , & \text{for } \alpha_1 > 1/2 + \delta , \\ \rho_{\eta_\delta}(\beta, \rho) &= \rho_0(\beta, \rho) , & \text{for } 1/2 + \delta > \alpha_1 > 1/2 .\end{aligned}$$

On the other hand, for  $\alpha_1 = 1/2$  and  $\eta_\Gamma(V) := \Gamma/V^{1/2}$  one gets a modification of the density of the type II condensation:

$$\rho_{\eta_\Gamma}(\beta, \rho) = \sum_{|n_1| < \Gamma/2\pi} \frac{1}{\lambda_\beta^2 n_1^2 + B} < \rho_0(\beta, \rho) .$$

For  $\alpha_1 > 1/2$  and  $\eta_{\Gamma'}(V) := \Gamma'/V^{1-\alpha_1}$  one gets a modification of the density of the type III condensation:

$$\rho_{\eta_{\Gamma'}}(\beta, \rho) = \int_{\zeta \in \mathbb{R}^+} \frac{e^{-C\zeta}}{\zeta^{1/2} \lambda_\beta} \text{erf}(\Gamma' \lambda_\beta \zeta / \sqrt{4\pi}) < \rho_0(\beta, \rho) ,$$

where  $\text{erf}(\cdot)$  stands for error function and where  $C$  is the unique solution of equation (1.8).

*Proof:*

By virtue of the proof of the Theorems 3.1, 3.2 and 3.3, with the different choices of  $\eta(V)$ , we can obtain the results.  $\square$

## 2.2 Classification of s-BEC versus g-BEC

Introduced by van der Berg and Lewis [vdB-L] as a geometric effect due to anisotropy of  $\Lambda$ , the g-BEC is was discovered then as the result of an interparticle interaction [Br-Z].

Here we want to justify formally the classification of g-BEC via our concept of s-BEC. It is important for our future theorems related to s-BEC.

By Definition 2.1 the value of s-BEC for a function  $\eta(V)$  is given by 2.24.

A question is what is the structure of  $\rho_\eta(\beta, \rho)$  versus the structure of  $\rho_0(\beta, \rho)$  imposed by Definition 1.2 ? We introduce a possible classification:

**Definition 2.2** *If for any functions  $\eta(V)$  such that  $\rho_\eta(\beta, \rho) > 0$ , only a finite number of modes in  $\Lambda_\eta^* := \{k \in \Lambda^* : \|k\| \leq \eta(V)\}$  are macroscopically occupied then the PBG manifests the s-BEC of type I.*

*If there exist a function  $\eta(V)$  such that  $\rho_\eta(\beta, \rho) > 0$  and an infinite number of modes in  $\Lambda_\eta^*$  are macroscopically occupied then the PBG manifests the s-BEC of type II.*

*If for any a function  $\eta(V)$  no modes in  $\Lambda^*$  are macroscopically occupied, lout there exist  $\eta(V)$  such that  $\rho_\eta(\beta, \rho) > 0$  then the PBG manifests the s-BEC of type III.*

It is clear that classification due to Definition 2.2 is more refined than those of Definition 1.2 show clearly that the first imply the second. Consequently to study the classification of the g-BEC we can use below the Definition 2.2.

### 2.3 Concept of the short/long cycles

One of the aim of this note is to generalize the concept of cycles to be able to control the size of short and long cycles keeping the trace thermodynamic limit scaling. Why do we want to do this? It is for two reasons: first to adapt the concepts of short and long cycles to the new definition of g-BEC via s-BEC, and second to find the typical size of long cycles responsible for the g-BEC condensate: i.e. the hierarchy of long cycles.

Let us first introduce concepts of *scaled* short/long cycles (s-short/long cycles):

**Definition 2.3** *We say that a Bose gas manifests s-long cycles if there exists a positive increasing function of the volume  $\lambda : \mathbb{R}^+ \rightarrow \mathbb{N}^+$ , such that  $\lim_{V \uparrow \infty} \lambda(V) = \infty$  and:*

$$\rho_{\text{long}, \lambda}(\beta, \rho) := \lim_{V \uparrow \infty} \sum_{j \geq \lambda(V)} \rho_{\Lambda, j}(\beta, \rho) > 0, \quad (2.27)$$

where  $\rho_{\Lambda, j}(\beta, \rho)$  is given by (1.20).

The following statement is an evident consequence of Definitions 1.4 and 2.3:

**Lemma 2.2** *In the particular case of PBG in Casimir boxes we have:*

$$0 \leq \rho_{\text{long}, \lambda}(\beta, \rho) \leq \rho_{\text{long}}(\beta, \rho), \quad (2.28)$$

for any function  $\lambda(V)$ . Here  $\rho_{\text{long}}(\beta, \rho)$  is given by (1.22).

This lemma imply that the presence of s-long cycles imply the presence of the long cycles (c.f. Definition 1.4) and that the contraposition is true.

A simple example of application of the s-long cycles approach is a possibility to distinguish the type I, II or III condensations of the PBG in Casimir boxes  $\Lambda$  with *periodic* boundary conditions.

**Proposition 2.3** *If  $\lambda(V) = V^\delta$  then for  $\rho \geq \rho_c(\beta)$  we obtain:*

$$\begin{aligned} \rho_{\text{long}, \lambda}(\beta, \rho) &= 0, \text{ for } \delta > 1, \\ \rho_{\text{long}, \lambda}(\beta, \rho) &= \rho_0(\beta, \rho), \text{ for } \alpha_1 \leq 1/2, \ 0 < \delta < 1, \\ \rho_{\text{long}, \lambda}(\beta, \rho) &= \rho_0(\beta, \rho), \text{ for } \alpha_1 > 1/2, \ 0 < \delta < 2(1 - \alpha_1), \\ \rho_{\text{long}, \lambda}(\beta, \rho) &= 0, \text{ for } \alpha_1 > 1/2, \ 2(1 - \alpha_1) < \delta. \end{aligned}$$

*Proof:* Adapting the proof of the Theorems 3.1, 3.2 and 3.3 with the different choices of  $\delta$  for  $\lambda(V) = V^\delta$  we can obtain the results. □

We propose two definitions:

**Definition 2.4** *If  $j : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  is a bounded positive increasing function of the volume, i.e.  $\lim_{V \uparrow \infty} j(V) < \infty$ , then  $\rho_{\Lambda, j(V)}(\beta, \rho)$  is the density of particles in the s-short-cycles of size  $j(V)$ .*

**Definition 2.5** If  $j : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  is a positive increasing function of the volume such that  $\lim_{V \uparrow \infty} j(V) = \infty$ , then  $\rho_{\Lambda, j(V)}(\beta, \rho)$  is the density of particles in the  $s$ -long-cycles of size  $j(V)$ .

There is a natural classification of  $s$ -long-cycles:

- if  $\lim_{V \uparrow \infty} (j(V)/V) = 0$ , we say that  $\rho_{\Lambda, j(V)}(\beta, \rho)$  is the density of particles in the microscopic-long-cycles of size  $j(V)$ ,
- if  $0 < \lim_{V \uparrow \infty} (j(V)/V) < \infty$ , we say that  $\rho_{\Lambda, j(V)}(\beta, \rho)$  is the density of particles in the macroscopic-cycles of size  $j(V)$
- if  $\lim_{V \uparrow \infty} (j(V)/V) = \infty$ , we say that  $\rho_{\Lambda, j(V)}(\beta, \rho)$  is the density of particles in the large-cycles of size  $j(V)$ .

To understand the Definition 2.5, we make the following remark:

**Remark 2.2** We say that if  $j(V)$  in (1.20) is of the order  $V^\alpha$  if  $0 < \lim_{V \uparrow \infty} (j(V)/V^\alpha) < \infty$ , for example  $j(V) = xV^\alpha$ ,  $x > 0$ . If  $\alpha < 1$ , we are in the first case of the classification in Definition 2.5. This case is important because there is a question: can we have a macroscopic quantity of particles in the microscopic-long-cycles? If  $\alpha = 1$  we are in the second case of the classification of  $s$ -long-cycles and if  $\alpha > 1$  we are in the third case. Of course, one can take above any increasing function  $j(V)$  including e.g.  $\ln(V)$ .

## 2.4 Hierarchy of $s$ -long cycles

The Definitions 2.4, 2.5 and the Remark 2.2 allow to give a natural classification of scaled-long ( $s$ -long) cycles. We call this classification a *hierarchy* of cycles, depending on their size.

**Definition 2.6** We say that a Bose gas manifests long-microscopic-cycles if there exists two non-negative increasing functions of the volume  $\lambda_1 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  and  $\lambda_2 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  with  $\lim_{V \uparrow \infty} \lambda_1(V) = \infty$  and  $\lambda_2(V) > \lambda_1(V)$ , with the property:

$$\lim_{V \uparrow \infty} (\lambda_2(V)/V) = 0,$$

such that the density:

$$\rho_{\text{long}}(\beta, \rho | \text{micro}) := \lim_{V \uparrow \infty} \sum_{j=\lambda_1(V)}^{\lambda_2(V)} \rho_{\Lambda, j}(\beta, \rho) > 0.$$

**Definition 2.7** We say that a Bose gas manifests macroscopic-cycles if there exists two non-negative increasing functions of the volume  $\lambda_1 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  and  $\lambda_2 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  with  $\lim_{V \uparrow \infty} (\lambda_1(V)) = \infty$  and  $\lambda_2(V) > \lambda_1(V)$  with the properties:

$$\begin{aligned} 0 &< \lim_{V \uparrow \infty} (\lambda_1(V)/V) < \infty, \\ 0 &< \lim_{V \uparrow \infty} (\lambda_2(V)/V) < \infty, \end{aligned}$$

such that the density:

$$\rho_{\text{long}}(\beta, \rho | \text{macro}) := \lim_{V \uparrow \infty} \sum_{j=\lambda_1(V)}^{\lambda_2(V)} \rho_{\Lambda, j}(\beta, \rho) > 0.$$

In general there are long cycles of any size:

**Definition 2.8** We say that a Bose gas manifests the  $s$ -long cycles of the order  $g(V)$  where  $g : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  is a positive increasing function of the volume, if there exists two non-negative increasing functions of the volume  $\lambda_1 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  and  $\lambda_2 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  such that  $\lim_{V \uparrow \infty} \lambda_1(V) = \infty$ ,  $\lim_{V \uparrow \infty} \lambda_1(V) = \infty$  and  $\lambda_2(V) > \lambda_1(V)$ , with the properties:

$$\begin{aligned} 0 &< \lim_{V \uparrow \infty} (\lambda_1(V)/g(V)) < \infty, \\ 0 &< \lim_{V \uparrow \infty} (\lambda_2(V)/g(V)) < \infty, \end{aligned}$$

and the  $s$ -long cycles particle density:

$$\rho_{\text{long}}(\beta, \rho | O(g)) := \lim_{V \uparrow \infty} \sum_{j=\lambda_1(V)}^{\lambda_2(V)} \rho_{\Lambda, j}(\beta, \rho) > 0.$$

For the suitable it is important to establish this lemma:

**Lemma 2.3** *Consider  $g : \mathbb{R}^+ \rightarrow \mathbb{N}^+$ . If for any non-negative increasing functions of the volume  $\lambda_1 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  and  $\lambda_2 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  such that  $\lim_{V \uparrow \infty} \lambda_1(V) = \infty$ ,  $\lim_{V \uparrow \infty} \lambda_2(V) = \infty$  and  $\lambda_2(V) > \lambda_1(V)$ , with the properties:*

$$\begin{aligned} \lim_{V \uparrow \infty} (\lambda_1(V)/g(V)) &= 0, \\ \lim_{V \uparrow \infty} (\lambda_2(V)/g(V)) &= \infty, \end{aligned}$$

and the  $s$ -long cycles particle density:

$$\lim_{\Lambda} \sum_{j=\lambda_1(V)}^{\lambda_2(V)} \rho_{\Lambda, j}(\beta, \rho) = \rho_{\text{long}}(\beta, \rho),$$

then the Bose gas manifest the  $s$ -long cycles only of the order  $g(V)$ .

*Proof:*

By our assumptions, for any function  $f : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  such that  $\lim_{V \uparrow \infty} (f(V)/g(V)) = 0$ , taking any functions with the hypothesis of Definition 2.8 one gets:

$$\rho_{\text{long}}(\beta, \rho | O(f)) = 0,$$

moreover for any function  $h : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  such that  $\lim_{V \uparrow \infty} (f(V)/g(V)) = 0$ , taking any functions with the hypothesis of Definition 2.8 one gets:

$$\rho_{\text{long}}(\beta, \rho | O(h)) = 0,$$

this conclude the proof.  $\square$

In the next section we shall see that in the particular case of the PBG in Casimir boxes, the classification of s-BEC induces a hierarchy in classification of  $s$ -long cycles. By virtue of Theorem 3.1, 3.2 and 3.3 we obtain:

**Proposition 2.4** *If  $\lambda_1(V) = xV^\delta$ ,  $x > 0$ ,  $\delta > 0$  and  $\lambda_2(V) = yV^\delta$ ,  $y > x$ , then we have:*

$$\begin{aligned} \rho_{\text{long}}(\beta, \rho | \text{macro}) &= (e^{-xA} - e^{-yA})\rho_0(\beta, \rho), \text{ for } \alpha_1 < 1/2, \delta = 1, \\ \rho_{\text{long}}(\beta, \rho | \text{macro}) &= (e^{-xB} - e^{-yB})\rho_0(\beta, \rho), \text{ for } \alpha_1 = 1/2, \delta = 1, \\ \rho_{\text{long}}(\beta, \rho | \text{micro}) &= (e^{-xC} - e^{-yC})\rho_0(\beta, \rho) \text{ for } \alpha_1 > 1/2, \delta = 2(1 - \alpha_1). \end{aligned}$$

where  $A$  is the unique solution of the equation (1.6),  $B$  is the unique solution of the equation (1.7) and  $C$  is the unique solution of the equation (1.8).

The proof of this statement follows straightforwardly from (3.35), (3.42) and (3.49).

This proposition gives an illustration of the hierarchy of cycles, that we are going to discuss in details in the next Section 3.

### 3 Does generalized BEC of the types I, II, III imply a hierarchy of long cycles?

In this section we apply our concept of s-BEC and the notion of the  $s$ -long cycles to study the Bose gas in Casimir boxes. This allows to relate the fine structure of g-BEC and s-BEC (type I, II or III) with hierarchy of long cycles. Recall that we are dealing with Casimir boxes  $\Lambda = V^{\alpha_1} \times V^{\alpha_2} \times V^{\alpha_3}$ ,  $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$ ,  $\alpha_1 + \alpha_2 + \alpha_3 = 1$ ,  $|\Lambda| = V$ .

### 3.1 Generalized BEC in the case : $\alpha_1 < 1/2$

In this case the geometry is close to the usual cubic box. Our main result here is that the g-BEC of type I implies the macroscopic-cycles in the fundamental state.

**Theorem 3.1** *If one takes the Casimir boxes  $\Lambda$  with  $1/2 > \alpha_1$ , then for a fixed density of particles  $\rho > \rho_c(\beta)$  the chemical potential is  $\bar{\mu}_\Lambda = -A/\beta V + o(1/V)$ , with  $A > 0$ . This implies the s-BEC of type I as well as the g-BEC in the zero mode (ground state) together only with macroscopic-cycles in this mode. Here  $A$  is the unique solution of equation (1.6).*

*Proof:*

Taking into account (1.1) we denote the family of Casimir boxes by  $\Lambda_I$  with the corresponding dual space:

$$\Lambda_I^* := \left\{ k \in \mathbb{R}^3 : k = \left( \frac{2\pi n_1}{V^{\alpha_1}}, \frac{2\pi n_2}{V^{\alpha_2}}, \frac{2\pi n_3}{V^{\alpha_3}} \right); n_\nu \in \mathbb{Z}^1; 1/2 > \alpha_1 \right\}. \quad (3.29)$$

Let  $\Lambda_{0,I}^*$  be a subset of  $\Lambda_I^*$  define by:

$$\Lambda_{0,I}^* = \{k \in \Lambda_I^* : \|k\| \leq \eta_I(V)\}, \quad (3.30)$$

where  $\eta_I(V) = 1/V$ . Then we have  $\Lambda_{0,I}^* = \{k = 0\}$ .

Consider the total density of particles:

$$\rho := \lim_{V \uparrow \infty} \rho_\Lambda(\beta, \bar{\mu}_\Lambda) = \rho_{\text{short}}(\beta, \rho) + \rho_{\text{long}}(\beta, \rho), \quad (3.31)$$

c.f. (1.21), (1.22).

We can decompose the density of particles in long-cycles into two parts defined by:

$$\rho_{\text{long}}(\beta, \rho) := \rho_{\text{long}}(\Lambda_I^* \setminus \Lambda_{0,I}^*) + \rho_{\text{long}}(\Lambda_{0,I}^*), \quad (3.32)$$

where  $\rho_{\text{long}}(\Lambda_I^* \setminus \Lambda_{0,I}^*)$  is the limiting density of particles in long-cycles outside  $\Lambda_{0,I}^*$ :

$$\rho_{\text{long}}(\Lambda_I^* \setminus \Lambda_{0,I}^*) := \lim_{M \rightarrow \infty} \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_I^* \setminus \Lambda_{0,I}^*} \sum_{j=M}^{\infty} \rho_{\Lambda,j}(k) \right), \quad (3.33)$$

where the spectral repartition of particles density in  $j$ -cycles is:

$$\rho_{\Lambda,j}(k) := \frac{1}{V} e^{j\beta \bar{\mu}_\Lambda} e^{-j\beta \epsilon_\Lambda(k)}, \quad (3.34)$$

and  $\bar{\mu}_\Lambda$  is the solution of equation  $\rho = \rho_\Lambda(\beta, \mu)$ .

First we shall estimate the density of particles in long-cycles of  $\Lambda_I^* \setminus \Lambda_{0,I}^*$  by (3.33) and asymptotic for  $\bar{\mu}_\Lambda$  we get:

$$\begin{aligned} \rho_{\text{long}}(\Lambda_I^* \setminus \Lambda_{0,I}^*) &= \lim_{M \rightarrow \infty} \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_I^* \setminus \Lambda_{0,I}^*} \sum_{j=M}^{\infty} \frac{1}{V} e^{j\beta \bar{\mu}_\Lambda} e^{-j\beta \epsilon_\Lambda(k)} \right), \\ &= \lim_{M \rightarrow \infty} \sum_{j=M}^{\infty} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dk e^{-j\pi \lambda_\beta^2 k^2}, \\ &= 0. \end{aligned}$$

Consequently there is no long cycles in  $\Lambda_I^* \setminus \Lambda_{0,I}^*$  and since our last estimate is valid for any  $M \rightarrow \infty$  we conclude that there are no s-long cycles in  $\Lambda_I^* \setminus \Lambda_{0,I}^*$  (see Definition 2.3).

Now consider the modes in  $\Lambda_{0,I}^*$ , we would like to apply Lemma 2.3 to prove that the PBG manifests s-long cycles of the order  $O(V)$ , i.e. macroscopic-cycles (see Definition 2.7).

For any positive increasing functions  $\lambda_1 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  and  $\lambda_2 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  such that  $\lim_{V \uparrow \infty} \lambda_1(V)/V = 0$  and  $\lim_{V \uparrow \infty} \lambda_2(V)/V = \infty$ , since  $\bar{\mu}_\Lambda = -A/\beta V + o(1/V)$ , with  $A > 0$  we have:

$$\begin{aligned} \rho_{\text{long}}(\Lambda_{0,I}^* | \text{macro}) &:= \lim_{V \uparrow \infty} \sum_{j=\lambda_1(V)}^{\lambda_2(V)} \frac{1}{V} e^{j\beta\bar{\mu}_\Lambda} \\ &= \lim_{V \uparrow \infty} \frac{(e^{\beta\lambda_1(V)\bar{\mu}_\Lambda} - e^{\beta\lambda_2(V)\bar{\mu}_\Lambda})}{e^{-\beta\bar{\mu}_\Lambda} - 1} \\ &= \lim_{V \uparrow \infty} \frac{1}{e^{-\beta\bar{\mu}_\Lambda} - 1} = \lim_{V \uparrow \infty} \rho_\Lambda(\Lambda_{0,I}^*), \end{aligned} \quad (3.35)$$

where  $\rho_\Lambda(\Lambda_{0,I}^*) := \sum_{k \in \Lambda_{0,I}^*} \rho_\Lambda(k)$  is the density of particles in  $\Lambda_{0,I}^*$ .

We can easily calculate  $\rho_\Lambda(\Lambda_{0,I}^*)$ :

$$\rho_\Lambda(\Lambda_{0,I}^*) = \frac{1}{V} \frac{1}{e^{-\beta\bar{\mu}_\Lambda} - 1} = \frac{1}{V} \frac{1}{e^{\beta \frac{A}{\beta V} + o(\frac{1}{V})} - 1} = \frac{1}{A} + o\left(\frac{1}{V}\right). \quad (3.36)$$

So by virtue of (3.32), (3.35) and (3.36):

$$\rho_{\text{long}}(\Lambda_{0,I}^* | \text{macro}) = \rho_{\text{long}}(\beta, \rho) = \frac{1}{A}, \quad (3.37)$$

We know by Theorem 1.2 that the density of particles in short-cycles is equal to the critical density. Consequently by virtue of (3.31), (3.37) and by Lemma 2.3 we can conclude the proof of the theorem.  $\square$

### 3.2 Generalized BEC in the case : $\alpha_1 = 1/2$

In this particular case of geometry the boxes are anisotropic with  $\alpha_1 = 1/2$ . Our main result is the theorem about generalized Bose-Einstein condensation of type II due to presence of macroscopic-cycles in an infinite (in thermodynamical limit) number of modes.

**Theorem 3.2** *If one takes the Casimir boxes with  $1/2 = \alpha_1$ , then for a fixed density of particles  $\rho > \rho_c(\beta)$  the chemical potential is  $\bar{\mu}_\Lambda = -B/\beta V + o(1/V)$ , with  $B > 0$ . This implies the s-BEC of type II as well as the g-BEC in an infinite (in thermodynamical limit) number of modes together only with macroscopic-cycles in these modes. Here  $B$  is the unique solution of equation (1.7).*

*Proof:*

Taking into account (1.1) we denote the family of Casimir boxes by  $\Lambda_{II}$  and the dual space is:

$$\Lambda_{II}^* = \left\{ k \in \mathbb{R}^3 : k = \left( \frac{2\pi n_1}{V^{\alpha_1}}, \frac{2\pi n_2}{V^{\alpha_2}}, \frac{2\pi n_3}{V^{\alpha_3}} \right); n_\nu \in \mathbb{Z}^1; 1/2 = \alpha_1 \right\}, \quad (3.38)$$

Let  $\Lambda_{0,II}^*$  be a subset of  $\Lambda_{II}^*$

$$\Lambda_{0,II}^* = \{k \in \Lambda_{II}^* : \|k\| \leq \eta_{II}(V)\}, \quad (3.39)$$

where:

$$\eta_{II}(V) := \frac{2\pi[cV^\epsilon]}{V^{1/2}}, 0 < \epsilon < 1/2, c \in \mathbb{R}^+ \setminus \{0\}, \quad (3.40)$$

here  $[X]$  denotes an integer part of  $X$ . We introduce  $c$  in (3.40) to make combination in numerator dimensionless (e.g.  $c = 1/\lambda_\beta^{3\epsilon}$ ).

We show that this set contains the whole value of the condensate as well as particles involved in the long-cycles. With the definition of  $\eta_{II}(V)$  we see that the number of modes in this set is of the order

$O(V^\epsilon)$  that goes to infinity, with increasing volume.

Again we decompose the density of particles in long cycles into two parts:

$$\rho_{\text{long}}(\beta, \rho) = \rho_{\text{long}}(\Lambda_{II}^* \setminus \Lambda_{0,II}^*) + \rho_{\text{long}}(\Lambda_{0,II}^*). \quad (3.41)$$

By the same argument as in the proof of Theorem 3.1, one find that there are no long-cycles out of  $\Lambda_{0,II}^*$ .

Now consider the modes in  $\Lambda_{0,II}^*$ . We would like to apply the same strategy than the proof of the Theorem 3.1: we use Lemma 2.3 to prove that the PBG manifests s-long cycles of the order  $O(V)$ , i.e. macroscopic-cycles (see Definition 2.7).

For any positive increasing functions  $\lambda_1 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  and  $\lambda_2 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  such that  $\lim_{V \uparrow \infty} \lambda_1(V)/V = 0$  and  $\lim_{V \uparrow \infty} \lambda_2(V)/V = \infty$ , since  $\bar{\mu}_\Lambda = -B/\beta V + o(1/V)$ , with  $B > 0$  we have:

$$\begin{aligned} \rho_{\text{long}}(\Lambda_{0,II}^* | \text{macro}) &:= \lim_{V \uparrow \infty} \frac{1}{V} \sum_{k \in \Lambda_{0,II}^*} \sum_{j=\lambda_1(V)}^{\lambda_2(V)} \rho_{\Lambda,j}(k) \\ &= \lim_{V \uparrow \infty} \sum_{k \in \Lambda_{0,II}^*} \frac{(e^{\beta \lambda_1(V) \bar{\mu}_\Lambda} - e^{\beta \lambda_2(V) \bar{\mu}_\Lambda})}{e^{\beta(\epsilon_\Lambda(k) - \bar{\mu}_\Lambda)} - 1} \\ &= \lim_{V \uparrow \infty} \sum_{k \in \Lambda_{0,II}^*} \frac{1}{e^{\beta(\epsilon_\Lambda(k) - \bar{\mu}_\Lambda)} - 1} \\ &= \lim_{V \uparrow \infty} \left( \sum_{\mathbf{k} \in \Lambda_{0,II}^*} \rho_\Lambda(\mathbf{k}) \right) =: \rho(\Lambda_{0,II}^*). \end{aligned} \quad (3.42)$$

We can easily calculate  $\rho(\Lambda_{0,II}^*)$ :

$$\begin{aligned} \rho(\Lambda_{0,II}^*) &= \lim_{V \uparrow \infty} \left( \sum_{n_1=-[cV^\epsilon]}^{[cV^\epsilon]} \frac{1}{V} \frac{1}{e^{\beta(4\pi^2 n_1^2/V + B/V + O(1/V))} - 1} \right) \\ &= \sum_{n_1 \in \mathbb{Z}^1} \frac{1}{B + \lambda_\beta^2 n_1^2}. \end{aligned} \quad (3.43)$$

So by virtue of (3.41), (3.42) and (3.43):

$$\rho_{\text{long}}(\Lambda_{0,II}^* | \text{macro}) = \rho_{\text{long}}(\beta, \rho) = \sum_{n_1 \in \mathbb{Z}^1} \frac{1}{B + \lambda_\beta^2 n_1^2}, \quad (3.44)$$

We know by Theorem 1.2 that the density of particles in short-cycles is equal to the critical density. Consequently by virtue of (3.31), (3.44) and by Lemma 2.3 we can conclude the proof of the theorem.  $\square$

### 3.3 Generalized BEC in the case : $\alpha_1 > 1/2$

We study the anisotropic boxes with  $\alpha_1 > 1/2$ . Our main result is the theorem about generalized Bose-Einstein condensation of type III due to presence of long-microscopic-cycles in an infinite (in thermodynamical limit) number of modes.

**Theorem 3.3** *If one takes the Casimir boxes  $\Lambda = V^{\alpha_1} \times V^{\alpha_2} \times V^{\alpha_3}$  with  $1/2 > \alpha_1$ , then for a fixed density of particles  $\rho > \rho_c(\beta)$  the chemical potential is  $\bar{\mu}_\Lambda = -C/\beta V^\delta + o(1/V^\delta)$ , with  $\delta = 2(1 - \alpha_1)$  and  $C > 0$ . This implies the s-BEC of type III as well as the g-BEC in an infinite (in thermodynamical limit) number of modes together only with long-microscopic-cycles of the order  $V^\delta$  in these modes. Here  $C$  is the unique solution of equation (1.8).*



*Proof:*

Taking into account (1.1) we denote the family of Casimir boxes  $\Lambda_{III}$  and the dual spaces are defined by:

$$\Lambda_{III}^* = \left\{ k \in \mathbb{R}^3 : k = \left( \frac{2\pi n_1}{V^{\alpha_1}}, \frac{2\pi n_2}{V^{\alpha_2}}, \frac{2\pi n_3}{V^{\alpha_3}} \right); n_i \in \mathbb{Z}^1; \alpha_1 > 1/2 \right\}. \quad (3.45)$$

Let  $\Lambda_{0,III}^*$  be a subset of  $\Lambda_{III}^*$ :

$$\Lambda_{0,III}^* = \{k \in \Lambda_{III}^* : \|k\| \leq \eta_{III}(V)\}, \quad (3.46)$$

where:

$$\eta_{III}(V) := \frac{2\pi[c'V^{2\alpha_1-1+\epsilon}]}{V^{\alpha_1}} = O\left(\frac{V^\epsilon}{V^{\delta/2}}\right), \quad 0 < \epsilon < \delta/2, \quad c' \in \mathbb{R}^{+,*}, \quad (3.47)$$

here  $\delta = 2(1 - \alpha_1) < 1$  and  $[X]$  denotes an integer part of  $X$ . We introduce  $c'$  in (3.47) to make combination in numerator dimensionless (e.g.  $c' = 1/\lambda_\beta^{3(2\alpha_1-1+\epsilon)}$ ).

We show that  $\Lambda_{0,III}^*$  contains whole value of the condensate as well as the particles involved into long cycles. Before to start the formal proof let us make a remark about qualitative difference between the cases  $\alpha_1 > 1/2$  and  $\alpha_1 \leq 1/2$ . With the definition of  $\eta_{III}(V)$  (3.47), we see that the number of states in  $\Lambda_{0,III}^*$  is of the order  $O(V^{2\alpha_1-1+\epsilon})$  that goes to infinity, with increasing volume but there are much more states in the condensate than in  $\Lambda_{0,III}^*$  (defined by 3.39). Heuristically one can say that we have long-cycles of size of the order  $O(V^\delta)$  in a number of modes of the order  $O(V^{2\alpha_1-1})$ . Indeed the number of particles in those s-long cycles is of the order  $O(V^\delta)O(V^{2\alpha_1-1}) = O(V)$ , which is macroscopic. For this reason there is a macroscopic condensate (for which order of particles is  $O(V)$ ) because there is accumulation of microscopic condensates (for which order of particles is smaller than  $O(V)$ , precisely here of the order  $O(V^{2\alpha_1-1})$ ) as well as accumulation of long-microscopic-cycles in  $\Lambda_{0,III}^*$  (on each modes of the condensate).

Again we decompose the density of particles in long cycles into two parts:

$$\rho_{\text{long}}(\beta, \rho) = \rho_{\text{long}}(\Lambda_{III}^* \setminus \Lambda_{0,III}^*) + \rho_{\text{long}}(\Lambda_{0,III}^*). \quad (3.48)$$

By the same argument as in the proof of Theorem 3.1, one find that there are no particles involved in long cycles out of  $\Lambda_{0,III}^*$ .

Now consider the modes in  $\Lambda_{0,III}^*$ . In this case, we would like to apply Lemma 2.3 to prove that the PBG manifests s-long cycles of the order  $O(V^\delta)$ ,  $\delta = 2(1 - \alpha_1) < 1$ , i.e. microscopic-cycles (see Definition 2.6 and 2.8).

For any positive increasing functions  $\lambda_1 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  and  $\lambda_2 : \mathbb{R}^+ \rightarrow \mathbb{N}^+$  such that  $\lim_{V \uparrow \infty} (\lambda_1(V)/V^\delta) = 0$  and  $\lim_{V \uparrow \infty} (\lambda_2(V)/V^\delta) = \infty$ , since  $\bar{\mu}_\Lambda = -C/\beta V^\delta + o(1/V^\delta)$  with  $\delta = 2(1 - \alpha_1)$  and  $C > 0$  we have:

$$\begin{aligned} \rho_{\text{long}}(\Lambda_{0,III}^* | \text{micro}) &:= \lim_{V \uparrow \infty} \frac{1}{V} \sum_{k \in \Lambda_{0,III}^*} \sum_{j=\lambda_1(V)}^{\lambda_2(V)} \rho_{\Lambda,j}(k) \\ &= \lim_{V \uparrow \infty} \sum_{k \in \Lambda_{0,III}^*} \frac{(e^{\beta \lambda_1(V) \bar{\mu}_\Lambda} - e^{\beta \lambda_2(V) \bar{\mu}_\Lambda})}{e^{\beta(\epsilon_\Lambda(k) - \bar{\mu}_\Lambda)} - 1} \\ &= \lim_{V \uparrow \infty} \left( \sum_{\mathbf{k} \in \Lambda_{0,III}^*} \rho_\Lambda(\mathbf{k}) \right) =: \rho(\Lambda_{0,III}^*). \end{aligned} \quad (3.49)$$

We can easily calculate  $\rho(\Lambda_{0,III}^*)$ :

$$\rho(\Lambda_{0,III}^*) = \lim_{V \uparrow \infty} \left( \frac{1}{V^\delta} \frac{1}{V^{2\alpha_1-1}} \sum_{j=1}^{\infty} e^{-(j/V^\delta)C} \sum_{n_1=-[c'V^{2\alpha_1-1+\epsilon}]}^{[c'V^{2\alpha_1-1+\epsilon}]} e^{-\pi \lambda_\beta^2(j/V^\delta)(n_1/V^{2\alpha_1-1})^2} \right). \quad (3.50)$$

The precedent expression is the limit of double Darboux-Riemann sums. So in thermodynamical limit we obtain a double integral:

$$\rho(\Lambda_{0,III}^*) = \int_{\mathbb{R}^+} d\zeta e^{-\zeta C} \int_{\mathbb{R}} d\xi e^{-\zeta \pi \lambda_\beta^2 \xi^2} = \int_{\mathbb{R}^+} d\zeta \frac{e^{-\zeta C}}{\sqrt{\zeta} \lambda_\beta} = \frac{\pi}{C^{1/2} \lambda_\beta}. \quad (3.51)$$

Hence:

$$\rho_{\text{long}}(\Lambda_{0,III}^* | \text{micro}) = \rho_{\text{long}}(\beta, \rho) = \frac{\pi}{C^{1/2} \lambda_\beta}, \quad (3.52)$$

by virtue of (3.48), (3.49) and (3.51).

We know by Theorem 1.2 that the density of particles in short-cycles is equal to the critical density. Consequently by virtue of (3.31), (3.52) and by Lemma 2.3 we can conclude the proof of the theorem.  $\square$

## 4 Does generalized BEC I, II, III imply a hierarchy of ODLRO?

In the Introduction we presented three concepts related to the BEC: g-BEC, long cycles and ODLRO. We present in Sections 2 and 3 two new concepts: *scaled* BEC (s-BEC) and *scaled* short/long cycles (s-long cycles) associated with g-BEC and short/long cycles. Thus it seems consistent to introduce a concept of *scaled* ODLRO (s-ODLRO) via scaling argument to study the hierarchy of ODLRO.

### 4.1 Definition of ODLRO via scaling argument

Recall that the generalized criterion of ODLRO is that it exists ODLRO if and only if there is g-BEC, see Theorem 1.1. The standard definition of ODLRO is formulated in Definition 1.3:

$$\sigma(\beta, \rho) := \lim_{\|x-x'\| \uparrow \infty} \sigma(\beta, \rho; x, x'),$$

where  $\sigma(\beta, \rho; x, x')$  is the two-point correlation function between two points  $x$  and  $x'$  after thermodynamical limit. Notice that this definition can be not satisfactory when we use the definition of s-BEC, because we do not precise what are the *scaling* size of the long correlation. It seems to be interesting to take thermodynamical limit at the same time as we take the two point  $x$  and  $x'$  at infinite distance. This is analogous to the scaling approach to long cycles.

A natural question is whether we are able to detect different types of s-BEC (as well as g-BEC) with the help of a generalized criterion of ODLRO based on the scaling argument? We call it a *scaled* ODLRO (s-ODLRO).

**Definition 4.1** *The PBG manifests a s-ODLRO if there exist a vector-valued function of volume  $X : V \mapsto X(V) \in \Lambda$  such that  $\lim_{V \uparrow \infty} |X_\nu(V)| = \infty$ ,  $\nu = 1, 2, 3$  and:*

$$\sigma_X(\beta, \rho) := \lim_{V \uparrow \infty} (\sigma_{\Lambda, X})(V) > 0, \quad (4.53)$$

where  $(\sigma_{\Lambda, X})(V)$  is the scaled two-point correlation function (s-two-point correlation function) for  $x(V), x'(V) \in \Lambda$ , see (1.10):

$$(\sigma_{\Lambda, X})(V) := \sigma_\Lambda(\beta, \rho; x(V) - x'(V)) = \sum_{k \in \Lambda^*} \rho_\Lambda(k) e^{ik \cdot X(V)}, \quad (4.54)$$

here  $X(V) = (x - x')(V) \in \Lambda$ .

**Remark 4.1** *By (1.1) and (1.2) one can write (1.10) like:*

$$\sigma(\beta, \rho; x, x') = \sum_{j=1}^{\infty} e^{j\beta \bar{\mu}_\Lambda} \prod_{\nu=1}^3 \theta_3\left(\frac{\pi}{V^{2\alpha_\nu}}(x_\nu - x'_\nu), e^{-\frac{\lambda_\beta^2}{\pi V^{2\alpha_\nu}}}\right), \quad (4.55)$$

where  $\theta_3(u, q) := \sum_{n \in \mathbb{Z}} q^{n^2} e^{2inu}$  is the Elliptic Theta function.

This imply the following proposition:

**Proposition 4.1** *The two-point correlation function as well as the s-two-point correlation function by (4.54) (see Definition 4.1) is a non-negative symmetric and  $L_\nu$ -periodic function of  $x_\nu - x'_\nu$ ,  $\nu = 1, 2, 3$  on  $\mathbb{R}$ , decreasing/increasing on  $[nL_\nu, nL_\nu + L_\nu/2] \subset \mathbb{R}^+$ ,  $n \in \mathbb{N}$ , respectively on  $[nL_\nu + L_\nu/2, nL_\nu] \subset \mathbb{R}^+$ ,  $n \in \mathbb{N}$  (monotone on the semi-periods).*

*Proof:*

Those different properties become by the properties of the Elliptic Theta function [A-S].  $\square$

The following statement is an evident consequence of Definitions 1.3 and 4.1:

**Lemma 4.1** *For any vector-valued  $X(V)$  we have:*

$$0 \leq \sigma_X(\beta, \rho) \leq \sigma(\beta, \rho) . \quad (4.56)$$

This Lemma means that the s-ODLRO implies standard ODLRO and that the contraposition is true.

## 4.2 Hierarchy and anisotropy of ODLRO, delocalization of the condensate

Here we use Definition 4.1 to analysis the s-BEC and the s-long cycles in the Casimir boxes. Notice that the usual criterion of ODLRO is such that we have no indication of the scale of long correlations because we study them correlation after thermodynamical limit.

We introduce a classification of the s-OLDRO which is formally defined by:

**Definition 4.2** *The PBG manifests the macroscopic-ODLRO in the direction  $x_\nu$ , if there exist a vector  $X(V) = (X_1(V), X_2(V), X_3(V)) \in \Lambda$  such that:  $\lim_{V \uparrow \infty} |X_\nu(V)|/V^{\alpha_\nu} > 0$  and  $\sigma_X(\beta, \rho) > 0$ .*

**Definition 4.3** *If the PBG don't manifests the macroscopic-ODLRO in the direction  $x_\nu$  although the PBG manifests the s-ODLRO, then it manifests the microscopic-ODLRO in the direction  $x_\nu$ .*

With the periodic boundary conditions the system is homogeneous and so there is no localization of the condensate in the space contrary to the Dirichlet boundary conditions case. However there is the delocalization concept that appears naturally. To this end we study the correlations between a fixed point and an other moving to infinity at the same time as the volume (when we take thermodynamical limit). By virtue of the precedent Definitions 4.2 and 4.3 the knowledge of the vector  $X(V) \in \Lambda$  such that  $\sigma_X(\beta, \rho) > 0$  (4.53) give a geometric view of the condensate which can be associated with an *ellipsoid cloud* in three dimension.

The next theorem gives the decreasing laws of the correlation function on the *pseudo-frontiers* of the condensate *cloud*:

**Theorem 4.1** *Consider the grand-canonical PBG in Casimir boxes  $\Lambda = V^{\alpha_1} \times V^{\alpha_2} \times V^{\alpha_3}$  and the fixed density of particles  $\rho$ . Let  $X : V \in \mathbb{R}^+ \mapsto X(V) = (X_1(V), X_2(V), X_3(V)) \in \Lambda$ ,  $\lim_{V \uparrow \infty} X_\nu(V) = \infty$ ,  $0 < X_\nu(V) \leq V^{\alpha_\nu}/2$ ,  $\nu = 1, 2, 3$ . Then we have the following results concerning the s-ODLRO, see (4.53):*

$$\sigma_X(\beta, \rho) = 0, \text{ for } \rho < \rho_c(\beta), \quad (4.57)$$

*Whereas for  $\rho > \rho_c(\beta)$  we get:  
for  $\alpha_1 < 1/2$ :*

$$\sigma_X(\beta, \rho) = \rho_0(\beta) , \quad (4.58)$$

*for  $\alpha_1 = 1/2$ :*

$$\sigma_X(\beta, \rho) = \rho_0(\beta), \text{ for } \lim_{V \uparrow \infty} (X_1(V)/xV^{\alpha_1}) = 0 , \quad (4.59)$$

$$= \sum_{n_1 \in \mathbb{Z}^1} \frac{\cos \pi n_1 x}{\lambda_\beta^2 n_1^2 + B} < \rho_0(\beta), \text{ for } X_1(V) = xV^{\alpha_1}/2, \ 0 < x < 1, , \quad (4.60)$$

for  $\alpha_1 > 1/2$ :

$$\sigma_X(\beta, \rho) = \rho_0(\beta), \text{ for } \lim_{V \uparrow \infty} (X_1(V)/xV^\delta) = 0, \delta = 2(1 - \alpha_1), \quad (4.61)$$

$$= \frac{\lambda_\beta \sqrt{\pi}}{C} e^{-x\sqrt{C}/\sqrt{\pi}\lambda_\beta} < \rho_0(\beta), \text{ for } X_1(V) = xV^{\delta/2}, x > 0, \quad (4.62)$$

$$= 0, \text{ for } \lim_{V \uparrow \infty} (X_1(V)/V^{\delta/2}) = 0. \quad (4.63)$$

*Proof:*

To ensure that the correlation function be monotonously decreasing for periodic boundary conditions, we suppose that the arguments  $0 < X_\nu \leq \frac{1}{2}V^{\alpha_\nu}$ ,  $\nu = 1, 2, 3$ , see Proposition 4.1.

The first step of the proof is to study the case  $\rho < \rho_c(\beta)$ :

Since  $\sigma(\beta, \rho) = 0$  (Theorem 1.1), by Lemma 4.1 we get  $\sigma_X(\beta, \rho) = 0$  for any vector  $X(V) \in \Lambda$ .

The second step is to study the case  $\rho > \rho_c(\beta)$ :

For  $\alpha_1 < 1/2$  by Definition 4.1 we have:

$$\begin{aligned} \sigma_X(\beta, \rho) &= \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_{0,I}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right) + \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_I^* \setminus \Lambda_{0,I}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right) \\ &= \lim_{V \uparrow \infty} \left( \frac{1}{V} \sum_{j=1}^{\infty} e^{-Aj/V} \right) + \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_I^* \setminus \Lambda_{0,I}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right) \end{aligned} \quad (4.64)$$

where  $\Lambda_I^*$  is the dual vector space given by equation (3.29) and  $\Lambda_{0,I}^* = \{k = 0\}$  is the subset corresponding to the condensate defined by (3.30). The first term of (4.64) is equal to  $\rho_0(\beta, \rho)$  by virtue of Theorem 3.1, so given that  $\sigma(\beta, \rho) = \rho_0(\beta, \rho)$  (Theorem 1.1) and by Lemma 4.1 the second term of (4.64) is null and we get the result.

For  $\alpha_1 = 1/2$  by Definition 4.1 we obtain:

$$\begin{aligned} \sigma_X(\beta, \rho) &= \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_{0,II}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right) + \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_{II}^* \setminus \Lambda_{0,II}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right) \\ &= \lim_{V \uparrow \infty} \left( \frac{1}{V} \sum_{j=1}^{\infty} e^{-Bj/V} \sum_{n_1 \in \mathbb{Z}^1} e^{-\pi \lambda_\beta^2 n_1^2 (j/V)} e^{iX_1(V)(n_1/V)} \right) + \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_{II}^* \setminus \Lambda_{0,II}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right) \end{aligned} \quad (4.65)$$

where  $\Lambda_{II}^*$  is the dual vector space given by equation (3.38) and  $\Lambda_{0,II}^*$  is the subset corresponding to the condensate defined by (3.39).

If  $\lim_{V \uparrow \infty} (X_1(V)/V^{\alpha_1}) = 0$ , the first term in the right-hand side of (4.65) is equal to  $\rho_0(\beta, \rho)$  by virtue of Theorem 3.2. So given that  $\sigma(\beta, \rho) = \rho_0(\beta, \rho)$  (Theorem 1.1) and by Lemma 4.1 the second term in (4.65) is null and we get the result.

Let  $\lim_{V \uparrow \infty} (X_1(V)/V^{\alpha_1}) = x$ ,  $0 < x \leq 1/2$ . Since the sum inside the limit in the first term of the right-hand side of (4.65) is a Darboux-Riemann sum, one get:

$$\int_{\mathbb{R}^+} d\chi e^{-B\chi} \sum_{n_1 \in \mathbb{Z}^1} e^{-\pi \lambda_\beta^2 n_1^2 \chi} e^{ixn_1} = \sum_{n_1 \in \mathbb{Z}^1} \frac{\cos \pi n_1 x}{\lambda_\beta^2 n_1^2 + B}$$

One can verify that the precedent expression is a decreasing function of  $x$  for  $0 < x \leq 1/2$  and so that the first part of (4.65) is a decreasing function of  $X_\nu$ ,  $\nu = 1, 2, 3$  on  $[0, L_\nu/2]$ . Since the s-two-point correlation function is a decreasing function of  $X_\nu$ ,  $\nu = 1, 2, 3$  the second part of (4.65) is too. The

first term of (4.65) is decreasing and the s-two point correlation function is a decreasing function, so the second term of (4.65) is decreasing too. Moreover, we have shown that for  $\lim_{V \uparrow \infty} (X_1(V)/V^{\alpha_1}) = 0$  the second part of (4.65) is null consequently by the precedent arguments the second part of (4.65) is null.

For  $\alpha_1 > 1/2$  by Definition 4.1 and by virtue of (3.50) we have:

$$\begin{aligned} \sigma_X(\beta, \rho) &= \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_{0,III}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right) + \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_{III}^* \setminus \Lambda_{0,III}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right) \\ &= \lim_{V \uparrow \infty} \left( \frac{1}{V^\delta} \frac{1}{V^{2\alpha_1-1}} \sum_{j=1}^{\infty} e^{-Cj/V^\delta} \sum_{n_1 = -[c'V^{2\alpha_1-1+\epsilon}]}^{[c'V^{2\alpha_1-1+\epsilon}]} e^{-\pi\lambda_\beta^2(j/V^\delta)(n_1/V^{2\alpha_1-1})^2} e^{iX_1(V)n_1/V^{\alpha_1}} \right) \\ &\quad + \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_{III}^* \setminus \Lambda_{0,III}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right) \end{aligned} \quad (4.66)$$

where  $\Lambda_{III}^*$  is the dual vector space given by equation (3.45) and  $\Lambda_{0,III}^*$  is the subset corresponding to the condensate defined by (3.46).

If  $\lim_{V \uparrow \infty} (X_1(V)/V^{\delta/2}) = 0$ , the right-hand side of (4.66) is equal to  $\rho_0(\beta, \rho)$  by virtue of Theorem 3.3. Since  $\sigma(\beta, \rho) = \rho_0(\beta, \rho)$  (Theorem 1.1), by Lemma 4.1 the second term of (4.66) is null and we get the result.

Let  $\lim_{V \uparrow \infty} (X_1(V)/V^{\delta/2}) = x$ ,  $x > 0$ ,  $\delta = 2(1 - \alpha_1)$ . Then the sum inside the limit in the first term of (4.66) is a double Darboux-Riemann sum, which implies:

$$\int_{\mathbb{R}^+} d\xi e^{-B\xi} \int_{\chi \in \mathbb{R}} d\chi e^{-\pi\lambda_\beta^2 \xi \chi^2} e^{ix\chi} = \frac{\lambda_\beta \sqrt{\pi}}{C} e^{-x\sqrt{C}/\sqrt{\pi}\lambda_\beta}.$$

By the same argument as in the case  $\alpha_1 = 1/2$  the second part of (4.66) is null (the precedent expression is a decreasing function of  $x$  for  $x > 0$ ) and thus we obtain the result.

Let  $\lim_{V \uparrow \infty} (X_1(V)/V^{\delta/2}) = \infty$ . since the correlation function is decreasing function for  $0 < X_\nu \leq V^{\alpha_\nu}/2$  (see Proposition 4.1), it is uniformly bounded by the precedent estimate with  $X(V) = xV^\delta$ ,  $x > 0$ :

$$\int_{\mathbb{R}^+} d\xi e^{-B\xi} \int_{\chi \in \mathbb{R}} d\chi e^{-\pi\lambda_\beta^2 \xi \chi^2} e^{ix\chi} + \lim_{V \uparrow \infty} \left( \sum_{k \in \Lambda_{III}^* \setminus \Lambda_{0,III}^*} \rho_\Lambda(k) e^{ik \cdot X(V)} \right).$$

When  $x$  tends to infinity, the first part goes to zero (By Riemann-Lesbegue theorem) then and the precedent arguments show that the second part is also null. This conclude the proof.  $\square$

We give here the classification of the s-ODLRO free three cases of Casimir boxes:

**Theorem 4.2** *If one takes the Casimir boxes with  $\alpha \leq 1/2$  then for a fixed density  $\rho > \rho_c(\beta)$  the PBG manifests macroscopic-ODLRO in the three directions. If  $\alpha > 1/2$  then for a fixed density  $\rho > \rho_c(\beta)$  the PBG manifests microscopic-ODLRO in direction  $x_\nu$  and macroscopic-ODLRO in the other directions.*

*Proof:*

By Definitions 4.2, 4.3, and Theorem 4.1, we obtain the proof of the theorem.  $\square$

It is remarkable that for type I and II of g-BEC in Casimir boxes corresponding to cases  $\alpha_1 \leq 1/2$  the condensate is spatially macroscopic whereas for the case  $\alpha_1 > 1/2$  the condensate is spatially macroscopic in two directions but microscopic in the most anisotropic direction  $x_1$ . It is naturally to guess that there is a link between the size of s-long cycles and the size of the spatial delocalization of the condensate. We can see this explicitly in [U] where the competition between the delocalization  $X$  and the size of cycle  $j$  indicate that the maximal size of the delocalizations is of the order of the square root of the size of the s-long cycles (e.g.  $V^{\delta/2}$ ,  $\delta = 2(1 - \alpha_1)$  in the case of the PBG in Casimir boxes with  $\alpha_1 > 1/2$ ).

## 5 Concluding remarks

In this paper we introduce a new concept of BEC which is called the scaled BEC (s-BEC) to adapt the London scaling approach to the problem of g-BEC, which introduce the van den Berg-Lewis-Pulé classification of BEC in three types (I,II,III) illustrated in the particular case of the PBG in Casimir boxes. This is a first formal step necessarily to study with more finely the different case of g-BEC for the PBG in Casimir boxes, we can see this by virtue of Propositions 2.1 and 2.2.

The fundamental question that we put in this paper is: what is the relation of different types of g-BEC (I,II,III) with the long cycles and with the ODLRO ?

We give the following answers concerning the PBG in Casimir boxes:

- we introduced new concepts of short/long cycles called scaled short/long cycles (s-short/long cycles, see Definition 2.3) to distinguish the different types of g-BEC, see Theorems 3.1, 3.2 3.3 and Remark 2.2 and Proposition 2.4. This paper is based on the estimation of the size of s-long cycles in the condensate, see Definitions 2.4 at 2.8. If the size of s-long cycles is macroscopic, then the g-BEC is of type I or II and if the s-long cycles are microscopic, then the g-BEC is of type III.

- we introduced new concept of ODLRO called scaled ODLRO (s-ODLRO, see Definition 4.1), to distinguish the different types of g-BEC, see Theorems 4.1, 4.2. Our arguments are based on the estimate of the delocalization of the condensate, see Definitions 4.2, 4.3. If the delocalization is macroscopic in the three directions, then the g-BEC is of type I or II, and if in one of the three dimension the delocalization is microscopic, then the g-BEC is of type III.

Some technical remarks. For simplicity we consider here the PBG with periodic boundary conditions for simplicity but one can adapt present paper to Dirichlet or Newman boundary conditions. It does not change the concept and the results. However if one take attractive boundary conditions, see [V-V-Z], we can suppose that the result should be different, since in this case the condensate is localized on two modes and it is not homogeneous. One can suppose that the s-long cycles are macroscopic but the most interesting is the delocalization of the condensate.

We see that the s-long cycles are more fine and more adapted to BEC classification than the s-ODLRO and s-BEC since by virtue of Proposition 2.2, Proposition 2.4 and Theorem 4.1, only s-long cycles can identify the condensate of type I on the zero mode because it can decompose the sum over cycles with a function well selected. However only the s-BEC can discriminate evidently type I and II since type II provide by an infinite number of modes. One can see that the proof of the Theorem I, II and III result of an analysis of geometric series easily done via s-long cycles, for this reason we can say that s-long cycles is a well adapt technic to study classification of BEC. Another reason is that the concept of cycles is independant of the representation of the gas (Feynman-Kac versus spectral representation). Besides we studied s-ODLRO with spectral approach but it is important to remark that we can do this analysis in the Feynman-Kac representation of partition function. In this framework it is the relation between s-long cycles and the delocalization of the condensate which appeared, see formula (A10) in [U], we have a competition between long cycles and spatial delocalization  $(K_\Lambda(j\beta; x, x') = \frac{1}{\lambda_\beta^3 j^{3/2}} \sum_{z \in \mathbb{Z}^3} e^{-(x-x'+Lz)^2/2\lambda_\beta^2 j})$ ,

where  $K_\Lambda(\beta)$  is the kernel of the Gibbs semigroup  $e^{-j\beta T_\Lambda^{N=1}}$ , for the PBG with periodic boundary conditions in the cubic case  $\Lambda = L^3$ ).

Why this approach is interesting ? One can see that the concept of g-BEC is well formulated for the PBG but not for the interacting Bose gas. Then how to study the classification of BEC for the interacting Bose gas if we have no more indications about the spectral properties of the gas? (Take e.g. an interaction  $U = \sum_{k \in \Lambda^*} g N_k^2 / 2V$ ,  $g > 0$ , see [Br-Z]). This problem can be solved using the approach developed in this paper. Thus the next step will be the application of these new methods for some models of interacting Bose gas.

We took here Casimir boxes to illustrate our concepts. But we can use *van den Berg* boxes which is a generalization of Casimir boxes ( $\Lambda_L = L_1(L) \times L_2(L) \times L_3(L)$  with  $V_L := |\Lambda_L| = L_1(L)L_2(L)L_3(L)$  where  $L_i(L)$  are functions of a parameter  $L$  such that  $\lim_{L \rightarrow \infty} L_i(L) = \infty$ ). These boxes are very interesting because with particular choice of the functions  $L_i(L)$ , e.g. (see[vdB])  $L_1(L) = L_2(L) = e^L, L_3(L) = L$

with Diriclet boundary conditions, for  $\rho_c(\beta) < \rho < \rho_m(\beta)$  the g-BEC is of type III and for  $\rho > \rho_m(\beta)$  it seems to have the *coexistence* of g-BEC of type I and type III, where  $\rho_c(\beta)$  is the critical density defined by (1.5) and  $\rho_m(\beta)$  is a critical density defined in this particular case by  $\rho_m(\beta) = \rho_c(\beta) + 1/\beta\pi$ . In fact  $\rho_c(\beta)$  divided regimes condensate - non condensate and  $\rho_m(\beta)$  divided the different types of generalized BEC. It is natural now to study this curious phenomena using our approach, which will be a subject of another paper.

An interesting perspective is the study of the finite-size scaling effect in the Bose gas via this scaling approach [B-D-T]. Indeed, we can estimate the impact of the s-long cycles to the finite-size corrections to the grand canonical pressure. An application is the calculation of the impact of the s-long cycles (s-BEC) for the Casimir effect in the perfect Bose gas between two slabs [M-Z]. An other application is the study of the scaling formation of the condensate around the critical point where the quantity of particles in s-long cycles are not visible in thermodynamical limit but the number is infinite (e.g. of the order  $O(V^{2/3})$  which is microscopic but large).

The concept of s-ODLRO can be interesting to study of the correlation function of the condensate for the Bose gas in weak harmonic trap (which corresponding to the experimental situation for cold atoms confined in magnetic trap). In fact the scaling laws (size of the spatial delocalization  $X(V)$  and energetic delocalization  $\eta(V)$ ) could give us a scaling characterization of the geometry of the condensate cloud in this system. In another paper we will adapt the scaling approach to this system.

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### References

- [A-S] M.Abramowitz, I.A.Stegun, eds (1972), Handbook of Mathematical functions with formulas, graphs and mathematical tables, New York: Dover Publications.
- [B-D-T] J.G.Brankov, D.M.Danchev, N.S.Tonchev, (World Scientific Publishing Co. Pte. Ltd, Theory of critical phenomena in finite-size systems: scaling and quantum effects, Series on modern condensed matter physics; vol. 9, 2000).
- [Br-Z] J.B.Bru, V.A.Zagrebnov, A model of Imperfect Bose gas with a non-conventionnal BEC of type III, *J.Math.Phys.* **45**, 1606-1622, (2004).
- [D-M-P] T.C.Dorlas, Ph.A.Martin, J.V.Pulè, Long Cycles in a Perturbed Mean Field Model of a Boson Gas, *Journal of Statistical Physics* **121**, 433-461 (2005).
- [F] R.P.Feynman, Atomic Theory of the  $\lambda$  Transition in Helium, *The Physical Review* **91**, (1953).
- [G] J.Ginibre, "Some applications of fonctionnal integration in statistical mechanics" Statistical Mechanics and Quantum Field Theory, C. DeWitt, R.Stora, eds, Les Houches, Gordon and Breach, 1971, p. 327.
- [L] F.London On the Bose-Einstein condensation, *Phys. Rev.* **54**, 947-954 (1938).
- [M] Ph.A.Martin, Quantum mayer graph: application to Bose and Coulomb gases, *Acta Physica Polonica B* vol. 34 (2003).
- [M-B] D.C.Brydges, Ph.A.Martin, Coulomb system at low density, *Journal of Statistical Physics* **96**, 1163-1330 (1999).
- [M-Z] Ph.A.Martin, V.A.Zagrebnov, The Casimir effect for the Bose-gas in slabs, *Europhysics Letters* **73**(1), 15-20 (2006).



- [P-O] O.Penrose, L.Onsager Bose-Einstein Condensation and Liquid Helium, *Physical Review* **104**, (1956).
- [S] A. Sütö, Percolation transition in the Bose gas: II, *J.Phys.A:Math.Gen* **35**, 6995-7002 (2002).
- [U] D. Ueltschi, Feynman cycles in the Bose gas, *Journal of Mathematical Physics* **47**, (2006).
- [vdB] M. van den Berg On Condensation in the Free-Boson Gas and the Spectrum of the Laplacian, *Journal of Statistical Physics* Vol. 31, No. 3, (1983).
- [vdB-L] M. van den Berg, J.T.Lewis, On generalized condensation in the free boson gas, *Physica 110A*, 550-564 (1982).
- [vdB-L-P] M. van den Berg, J.T.Lewis and J.V.Pulè A general theory of Bose-Einstein condensation, *Helv.Phys.Acta* **59**, 1271-1288 (1986).
- [V-V-Z] L.Vandedevenne, A.Verbeure and V.A.Zagrebnov, Equilibrium states for the Boson gas, *J.Math.Phys.* **45**, 1606–1622, (2004).