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► **To cite this version:**

David Coeurjolly, Jérôme Hulin, Isabelle Sivignon. Finding a Minimum Medial Axis of a Discrete Shape is NP-hard. Theoretical Computer Science, 2008, 206 (1-2), pp.72-79. hal-00332406

**HAL Id: hal-00332406**

**<https://hal.science/hal-00332406>**

Submitted on 20 Oct 2008

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# Finding a Minimum Medial Axis of a Discrete Shape is NP-hard

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## Abstract

The medial axis is a classical representation of digital objects widely used in many applications. However, such a set of balls may not be optimal: subsets of the medial axis may exist without changing the reversibility of the input shape representation. In this article, we first prove that finding a minimum medial axis is an NP-hard problem for the Euclidean distance. Then, we compare two algorithms which compute an approximation of the minimum medial axis, one of them providing bounded approximation results.

*Key words:* Minimum Medial Axis, NP-completeness, bounded approximation algorithm.

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## 1 Introduction

2 In binary images, the *Medial Axis* (MA) of a shape  $\mathcal{S}$  is a classic tool for shape  
3 analysis. It was first proposed by Blum [2] in the continuous plane; then it  
4 was defined by Pfaltz and Rosenfeld in [14] to be the set of centers of all  
5 maximal disks in  $\mathcal{S}$ , a disk being maximal in  $\mathcal{S}$  if it is not included in any  
6 other disk in  $\mathcal{S}$ . This definition allows the medial axis to be computed in a  
7 discrete framework, i.e., if the working space is the rectilinear grid  $\mathbb{Z}^n$ . The  
8 medial axis has the property of being a *reversible* coding: the union of the  
9 disks of  $\text{MA}(\mathcal{S})$  is exactly  $\mathcal{S}$ .

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<sup>1</sup> Supported in part by ANR grant BLAN06-1-138894 (projet OPTICOMB)

10 In order to compute the medial axis of a given discrete shape  $\mathcal{S}$ , we first pro-  
 11 ceed by computing the *Distance Transform* (DT) of  $\mathcal{S}$ . The distance transform  
 12 is a bitmap image in which each point is labelled with the distance to the clos-  
 13 est background point. For either  $d_4$  or  $d_8$  (the discrete counterparts of the  $l_1$   
 14 and  $l_\infty$  norms), any given chamfer distance or the Euclidean distance  $d_E$ , the  
 15 distance transform can be computed in linear time with respect to the number  
 16 of grid points [18,4,7,11]. For the simple distances  $d_4$  and  $d_8$ , MA is extracted  
 17 from DT by picking the local maxima in DT [18,4,16].

18 Polynomial time algorithms exist to extract MA from DT in the case of the  
 19 chamfer norms or the Euclidean distance [16,17]. A Reduced Medial Axis  
 20 (RMA) is presented in [8]: it is a reversible subset of the medial axis, that  
 21 can be computed in linear time. Despite the fact that the medial axis exactly  
 22 describes the shape  $\mathcal{S}$ , it may not be a set with minimum cardinality of balls  
 23 covering  $\mathcal{S}$ : indeed, a maximal disk of the medial axis covered by a union of  
 24 maximal disks is not necessary for the reconstruction of  $\mathcal{S}$ .

25 In this article, we investigate the minimum medial axis problem that aims at  
 26 defining a set of maximal balls with minimum cardinality which cover  $\mathcal{S}$ . This  
 27 problem has already been addressed with algorithms that experimentally filter  
 28 the medial axis [5,15,6,13].

29 In section 2 we first detail some preliminaries and the fundamental defini-  
 30 tions used in the remainder of the paper. Section 3 presents the proof that  
 31 the minimum medial axis problem is NP-hard. Finally, we compare a greedy  
 32 approximation algorithm with the approximation algorithm proposed in [15]  
 33 (Section 4). The greedy approximation algorithm is a first bounded heuristic.

## 34 2 Preliminaries and Related Results

35 First of all, we recall definitions related to the discrete medial axis. Given a  
 36 metric  $d$ , a (open) ball  $B$  of radius  $r$  and center  $p$  is the set of grid points  
 37  $q$  such that  $d(p, q) < r$ . In the following, we consider the Euclidean metric,  
 38 while the extension of the results to other metrics (such as Chamfer norms for  
 39 example) will be discussed in section 5.

40 **Definition 1 (Maximal ball)** *A ball  $B$  is maximal in a discrete shape  $\mathcal{S} \subseteq$*   
 41  *$\mathbb{Z}^n$  if  $B \subseteq \mathcal{S}$  and if  $B$  is not entirely covered by another ball contained in  $\mathcal{S}$ .*

42 Based on this definition, the medial axis is given by:

43 **Definition 2 (Medial axis)** *The medial axis of a shape  $\mathcal{S} \subseteq \mathbb{Z}^n$  is the set*  
 44 *of all maximal balls in  $\mathcal{S}$ .*

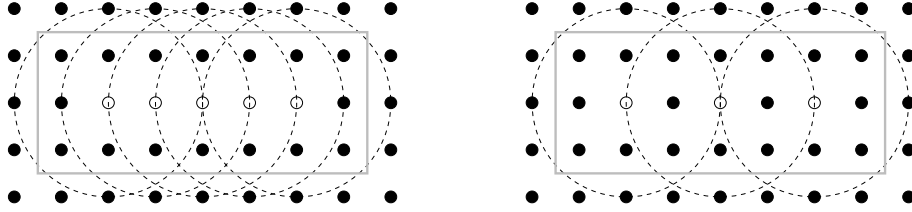


Fig. 1. (*Left*) Unfilled points correspond to the centers of the medial axis balls for the Euclidean metric. In this figure, we represent the discrete maximal balls with the help of their continuous counterpart (open continuous balls) in order to make them distinguishable. (*Right*) A subset of the medial axis the balls of which still cover the entire shape.

45 In the remainder of the paper, we focus on dimension 2. By definition, the  
 46 medial axis of a shape  $\mathcal{S}$  is a reversible encoding of  $\mathcal{S}$ . Indeed given the cen-  
 47 ters and the radii associated to the medial axis balls, the input shape  $\mathcal{S}$  can  
 48 be reconstructed entirely (this process is called the Reverse Distance Trans-  
 49 formation [18,3,4,19,8]).

50 However, this representation is not minimum in the number of balls as illus-  
 51 trated in Figure 1: the set of balls with highlighted centers in the left shape is  
 52 the medial axis given by Definition 2. However, if we consider the subset of the  
 53 medial axis depicted in the right figure, we still have a reversible description  
 54 of the shape with fewer balls. In the following, we define the  $k$ -medial axis of  
 55 a shape as follows:

56 **Definition 3 ( $k$ -Medial axis ( $k$ -MA))** A  $k$ -medial axis of a shape  $\mathcal{S} \subseteq$   
 57  $\mathbb{Z}^n$  is a subset of the medial axis of  $\mathcal{S}$  with  $k$  balls which entirely covers  $\mathcal{S}$ .

58 In this paper, we address the problem of finding a subset of the medial axis  
 59 that still covers all points of  $\mathcal{S}$ . In the remainder of the paper, we illustrate  
 60 the proofs with discrete ball coverings of several complex discrete objects. In  
 61 order to help the reader, we choose to represent each discrete ball with the  
 62 polygon defined by the convex hull of the grid points inside this ball.

63 In computational geometry, covering a polygon with a minimum number of a  
 64 specific shape (*e.g.* convex polygons, squares, rectangles, ...) usually leads to  
 65 NP-complete or NP-hard problems [10]. From the literature, a related result  
 66 proposed in [1] concerns the minimum decomposition of an orthogonal poly-  
 67 gon into squares. At first sight, this result seems to be closely related to the  
 68  $k$ -MAP for the  $d_8$  metric. However, in the discrete case,  $d_8$  balls are centered  
 69 on grid points and thus have odd widths. Due to this specificity, results of  
 70 [1] cannot be used neither for the  $d_8$  nor the Euclidean metrics. However, the  
 71 proof given in the following sections is inspired by this related work.

### 72 3 NP-completeness of the $k$ -Medial Axis Problem

73 **Definition 4 ( $k$ -Medial Axis Problem ( $k$ -MAP))** *Given a discrete shape*  
74  $\mathcal{S} \subseteq \mathbb{Z}^2$  *of finite cardinality and a positive integer  $k$ , does  $\mathcal{S}$  admit a  $k$ -MA ?*

75 In order to prove the NP-hardness of  $k$ -MAP, we use a polynomial reduction  
76 of the Planar-4 3-SAT problem. Let  $\phi(V, C)$  be the boolean formula in Con-  
77 junctive Normal Form (CNF) consisting of a list  $C$  of clauses over a set  $V$  of  
78 variables. The *formula-graph*  $G(\phi(V, C))$  of a CNF formula  $\phi(V, C)$  is the bi-  
79 partite graph in which each vertex is either a variable  $v \in V$  or a clause  $c \in C$ ,  
80 and there is an edge between a variable  $v \in V$  and a clause  $c \in C$  if  $v$  occurs in  
81  $c$ . A *Planar 3-SAT* formula  $\phi$  is a CNF formula for which the formula-graph  
82  $G(\phi)$  is planar and each clause is a 3-clause (i.e., a clause having exactly 3  
83 literals).

84 In the following, we prefer a reduction based on the Planar-4 3-SAT problem:  
85 an instance of this problem is an instance of Planar 3-SAT such that the  
86 degree of each vertex associated to a variable in the formula-graph is bounded  
87 by 4. In other words, a variable may appear at most four times in the boolean  
88 formula.

89 **Definition 5 (Planar-4 3-SAT Problem)** *Given a Planar-4 3-SAT formula*  
90  $\phi(V, C)$ , *does there exist a truth assignement of the variables in  $V$  which sat-*  
91 *isfies all the clauses in  $C$  ?*

92 Planar-4 3-SAT was shown to be NP-complete in [12].

93 The reduction from any given Planar-4 3-SAT formula  $\phi$  to an instance of  
94  $k$ -MAP consists in constructing a discrete shape  $\mathcal{S}(\phi)$  and finding an integer  
95  $k(\phi)$  in polynomial time such that  $\phi$  is satisfiable if and only if  $\mathcal{S}(\phi)$  can be  
96 covered by  $k(\phi)$  balls.

#### 97 3.1 Variables

98 Let us first consider a geometrical interpretation of variables. Figure 2 presents  
99 a 4-connected discrete object, so called *variable gadget* in the following, defined  
100 by the set of grid points below the horizontal dashed line. The eight vertical  
101 parts of width 3 of the gadget (numbered on Figure 2) are called the *extremities*  
102 of the variable gadget. These extremities are used to plug the “wires” that  
103 represent the edges of a formula-graph.

104 Any minimum covering of this object has 72 balls. This comes first from the  
105 fact that all the balls depicted with a thick border belong to any minimum

106 covering; hence 40 balls are required. Moreover, on the remaining part, any two  
 107 of the 32 circled points (on Figure 2) cannot be covered by a single ball, which  
 108 proves that at least 72 balls are required to cover a variable gadget. Finally,  
 109 coverings with exactly 72 balls can be exhibited (see Figure 2), which proves  
 110 that a minimum covering has 72 balls. Then, if we consider the point  $p$  depicted  
 111 in Figure 2,  $p$  can be covered by two different balls, which in turn implies  
 112 two minimum different coverings. None of these minimum coverings allow  
 113 protrusions from both one odd extremity and one even extremity. However,  
 114 one minimum covering allows balls to protrude out at all odd extremities  
 115 by one row of grid points (Figure 2 top); while another minimum covering  
 116 allows balls to protrude out at all even extremities also by one row of grid  
 117 points (Figure 2 bottom). These two coverings mimic the two possible truth  
 118 assignments of a variable. Without loss of generality, the first covering will  
 119 correspond to a True assignment, and the other one to a False assignment of  
 120 the variable.

121 If the gadget represents the variable  $x$ , then each odd extremity carries the  
 122 literal  $x$ , while each even extremity carries the literal  $\bar{x}$ . A protrusion from a  
 123 variable extremity can be viewed as a signal 'True' sent from the variable to  
 124 the clauses. Thus, wires which are used to connect variables and clauses are  
 125 plugged on odd extremities for positive literals and on even extremities for  
 126 negative literals.

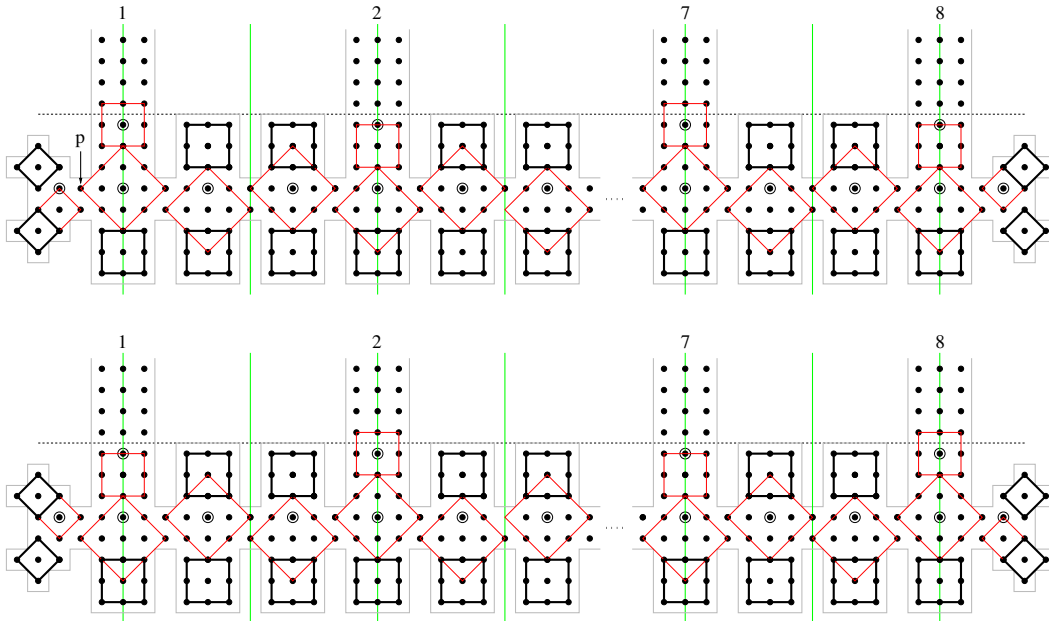


Fig. 2. Two minimum coverings of a variable gadget, corresponding to a True assignment of the variable (top), and False assignment (bottom). Balls with a thick border belong to any minimum covering; any two circled points cannot be covered by a single ball.

127 Note that this object and its decomposition are invariant under rotation of

128 angle  $\frac{\pi}{2}$ . Furthermore, the extremities are centered on abscissas with equal  
 129 values modulo 6 (represented by vertical lines of Figure 2). This property will  
 130 be used to align the objects and to connect them to each other.

### 131 3.2 Wires

132 In order to connect variables to clauses, we need wires that correspond to edges  
 133 in the embedding of the formula-graph. A wire must be designed such that it  
 134 carries either a 'True' signal (protrusion), or a 'False' signal (no protrusion)  
 135 from variable extremities to clauses without altering the signal (see Fig. 3).  
 136 We can define a straight wire of width 3 and whose length is equivalent to  
 137  $0 \pmod 3$ , so that the signal sent at the left extremity of the wire will be  
 138 propagated to the right extremity. Furthermore a wire can be bent at angle  
 139  $\frac{\pi}{2}$  (see Fig. 3). In this case, two minimum decompositions still exist such that  
 140 if a ball protrudes from one extremity of the wire, then another ball also  
 141 protrudes out at the other extremity. Furthermore, straight wires and bends  
 142 can be designed such that the alignment of the abscissa and ordinates of the  
 143 shape is preserved (*i.e.* is constant modulo 3).

144 Now, if we consider a complex wire with straight parts and bends, the signals  
 145 are propagated during the construction of the minimum covering from one  
 146 extremity to the other one (by induction on the number of bends and straight  
 147 parts).

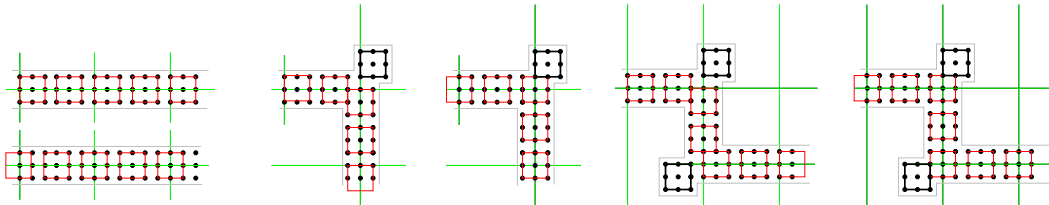


Fig. 3. Wires carrying 'True' or 'False' signals - from left to right: a straight wire, a simple bend, a shift.

### 148 3.3 Clauses

149 Finally, we introduce the *clause gadget*, a component that geometrically sim-  
 150 ulates a clause. This gadget is the set of grid points to the right of the vertical  
 151 dashed line in Fig.4. Note that this gadget is not symmetrical because we shall  
 152 not allow an open ball of radius  $\sqrt{8}$  to be placed in its center.

153 Again, the 5 balls depicted with a thick border belong to any minimum cov-  
 154 ering. Furthermore, any two of the 5 circled points (on Fig.4, left) cannot be  
 155 covered by a single ball. Thus, independently covering this gadget requires at

156 least  $5+5=10$  balls. However, if one open ball of radius 2 is protruding from  
 157 some wire by one column, carrying a 'True' signal (e.g. the upper one in Fig.4,  
 158 middle), then minimally covering the remainder of the gadget can be done  
 159 with only 9 balls. Similarly, if two or three wires are carrying a protrusion, a  
 160 minimum covering of the remainder of the clause gadget also has cardinality  
 161 9. The case of three protrusions appears on the right in Fig.4, showing that  
 162 even here 9 balls are still necessary to finish covering the gadget (similarly,  
 163 any two of the 4 circled points cannot be covered by a single ball). Note that  
 164 in general there may be several possible minimum coverings of the gadget,  
 165 although only one is drawn here in each case.

166 According to these observations, it follows that the clause gadget can be min-  
 167 imally covered by 10 balls if and only if no input protrusion is observed, in  
 168 other words if and only if the corresponding clause is not satisfied. Otherwise,  
 169 if at least one literal of the clause is set to 'True' (protrusion), implying that  
 170 the clause is satisfied, then only 9 balls are necessary to cover the remainder  
 171 of the gadget.

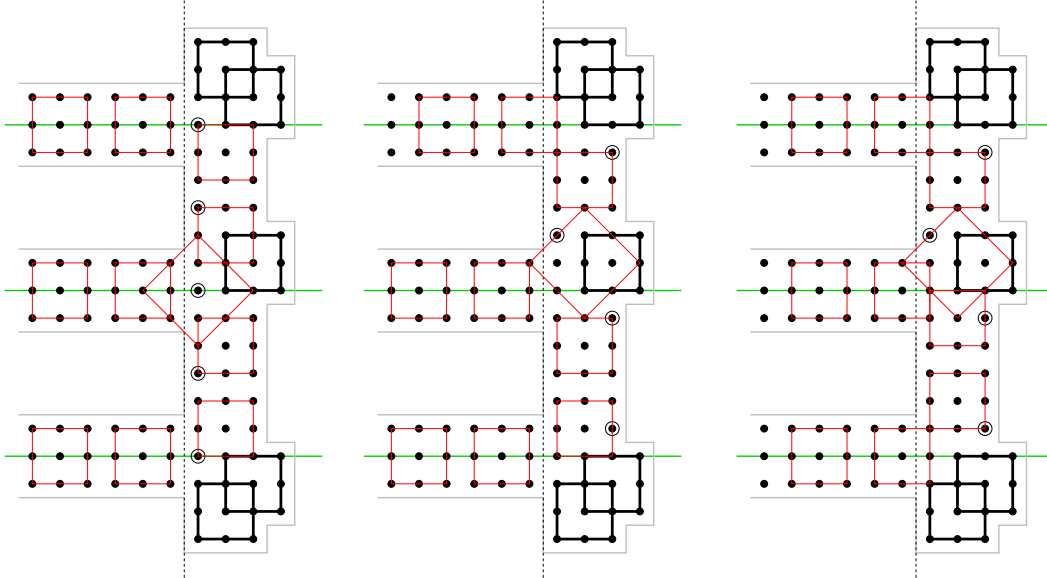


Fig. 4. Three minimum coverings of a clause gadget, depending on the following in-  
 put signals (from left to right): False-False-False, True-False-False, True-True-True.  
 Balls with a thick border belong to any minimum covering; any two circled points  
 cannot be covered by a single ball.

### 172 3.4 Overall Construction and Proof

173 Given a Planar-4 3-SAT formula  $\phi(V, C)$ , we are now ready to construct  $\mathcal{S}(\phi)$   
 174 by drawing a variable gadget for each variable vertex in  $G(\phi)$ , a clause gadget  
 175 for each clause vertex in  $G(\phi)$ , and drawing wires corresponding to the edges



176 in  $G(\phi)$ , thus linking each literal (the extremity of a variable gadget) to every  
 177 clause where it occurs.

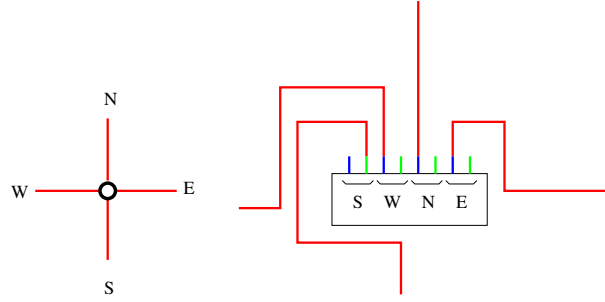


Fig. 5. Illustration of the transformation of a vertex of the planar orthogonal embedding into a variable gadget. In this case, the associated variable appears four times in  $\phi$ , three times as a positive literal, and once as a negative literal.

178 **Lemma 1** *The shape  $\mathcal{S}(\phi)$  can be computed in polynomial time in the size of*  
 179  *$\phi$ .*

180 **PROOF.** We know from [20] that every planar graph with  $n$  vertices (with  
 181 degree  $\leq 4$ ) can be embedded in a rectilinear grid in polynomial time and  
 182 space. This algorithm produces an orthogonal drawing such that edges are  
 183 intersection free 4-connected discrete curves. Since our variable gadgets and  
 184 clause gadgets have a constant size and our wires have constant width, and  
 185 since  $\phi$  is an instance of Planar-4 3-SAT, it is clear that the construction of  
 186  $\mathcal{S}(\phi)$  can also be done in polynomial time and space. For example, Figure 5  
 187 illustrates how to bend the orthogonal drawing edges in order to connect them  
 188 to our variable gadget extremities.  $\square$

189 In the following, let  $w(\phi)$  denote the minimum number of balls necessary to  
 190 cover the wires of  $\mathcal{S}(\phi)$ , and let  $k(\phi(V, C)) = 72 \cdot |V| + w(\phi) + 9 \cdot |C|$ .

191 **Lemma 2** *If the formula  $\phi$  is satisfiable, then there exists a covering of  $\mathcal{S}(\phi)$*   
 192 *with  $k(\phi)$  maximal balls.*

193 **PROOF.** Given a truth assignment  $T$  of the variables  $V$  of  $\phi$  such that  
 194  $\phi$  is satisfiable, the following algorithm builds a covering of  $\mathcal{S}(\phi)$  with  $k(\phi)$   
 195 maximal balls:

- 196 • cover the variable gadgets according to the truth assignment  $T$  ('True' or  
 197 'False' value for each variable): each one requires 72 balls allowing protrusions in each extremity carrying a 'True' assignment (Section 3.1);  
 198

- 199 • cover the wires: since the grid embedding of  $G(\phi)$  is computed in polynomial  
200 time, so is  $w(\phi)$ ; the protrusions from the extremities of the variables are  
201 transmitted to the clause gadgets;
- 202 • cover the clause gadgets: since  $\phi$  is satisfiable, at least one input wire of  
203 each clause gadget carries a protrusion which implies that 9 maximal balls  
204 are enough to cover each clause gadgets (Section 3.3).

205 Altogether,  $72 \cdot |V| + w(\phi) + 9 \cdot |C| = k(\phi)$  maximal balls are used in this cov-  
206 ering.  $\square$

207 **Lemma 3** *If there exists a covering of  $\mathcal{S}(\phi)$  with  $k(\phi)$  maximal balls, then the*  
208 *formula  $\phi$  is satisfiable.*

209 **PROOF.** Suppose that there exists a covering of  $\mathcal{S}(\phi)$  with  $k(\phi)$  maximal  
210 balls. By construction,  $72 \cdot |V|$  plus  $w(\phi)$  maximal balls are required to cover  
211 the  $|V|$  variable gadgets and the wires of  $\mathcal{S}(\phi)$ . This leaves us with  $k(\phi) -$   
212  $72 \cdot |V| - w(\phi) = 9 \cdot |C|$  maximal balls to cover the clause gadgets. Since there  
213 are  $|C|$  clause gadgets, each one is totally covered with 9 maximal balls in  
214 the covering, which is possible only if at least one input wire of each clause  
215 gadget carries a protrusion (Section 3.3). By construction, this means that the  
216 clauses are all satisfied, and in turn that  $\phi$  is satisfiable.  $\square$

217 According to lemmas 2 and 3, there exists a truth assignment of the variables  
218 in  $V$  which satisfies all the clauses in  $\phi$  if and only if there exists a covering  
219 of  $\mathcal{S}(\phi)$  with cardinality  $k(\phi) = 72 \cdot |V| + w(\phi) + 9 \cdot |C|$ . Thus, if any instance  
220 of the  $k$ -Medial Axis Problem could be solved in polynomial time, then we  
221 would have a polynomial time algorithm to solve the Planar-4 3-SAT Problem.  
222 Therefore, the  $k$ -MAP Problem is NP-hard. It is also clear that the  $k$ -MAP  
223 problem is in NP, since we can easily verify in polynomial time whether a set of  
224  $k$  balls covers a discrete shape  $\mathcal{S}$ . Consequently, we have the following theorem:

225 **Theorem 4**  *$k$ -MAP is an NP-complete problem.*

226 As a consequence, finding a  $k$ -MA with minimum  $k$  of a shape  $\mathcal{S}$  is NP-hard.

## 227 4 Approximation Algorithms and Heuristics

228 Even if the theoretical problem is NP-hard, approximation algorithms can  
229 be designed to find the  $k$ -MA with the smallest possible  $k$ . In the literature,  
230 several authors have discussed simplification techniques to extract an approxi-  
231 mation of the  $k$ -MA with minimum cardinality [5,15,6,13]. When dealing with

232 NP-hard problems, we usually want to have bounded heuristics in the sense  
233 that the results given by the approximation algorithm will always be at most  
234 at a given distance from the optimal solution.

235 In the following, we first detail the simplification algorithm proposed by Rag-  
236 nemalm and Borgefors [15] and extended to 3-D by Borgefors and Nyström  
237 [6]. Then, we compare their result with a simple but bounded heuristic de-  
238 rived from the MINSETCOVER problem. These algorithms are presented in a  
239 generic way, for any dimension. The experiments are conducted in dimension  
240 3, which is the highest standard dimension for digital objects. Even if the  
241 NP-completeness proof is established in dimension 2 in the previous sections,  
242 a similar result in dimension 3 can be conjectured.

#### 243 4.1 Ragnemalm and Borgefors Simplification Algorithm

244 The algorithm is quite simple but provides interesting results: we first con-  
245 struct a covering map  $CM(p) : \mathcal{S} \rightarrow \mathbb{Z}$  where we count for each discrete  
246 point  $p \in \mathcal{S}$ , the number of discrete maximal balls containing  $p$ . Basically, if  
247 a ball  $B$  contains a grid point  $p$  for which  $CM(p) = 1$ , then  $B$  is necessary to  
248 maintain the reconstruction and  $B$  belongs to any  $k$ -MA. Based on this idea,  
249 the approximation algorithm can be sketched as follows: let  $\mathcal{F} = \text{MA}(\mathcal{S})$ , we  
250 consider each ball  $B$  of  $\mathcal{F}$  by increasing radii. If for all points  $p \in B$  we have  
251  $CM(p) > 1$ , then we decide to remove  $B$  from  $\mathcal{F}$  and we decrease by one the  
252 value of  $CM(p)$  for each  $p \in B$ . Then, we process the next ball.

253 The resulting set  $\hat{\mathcal{F}}$  may be such that  $|\hat{\mathcal{F}}| < |\mathcal{F}|$ . In [15], the author illustrates  
254 the reduction rates with several shapes in dimension 2 but no simplification  
255 rate is formally given in the general case. In our experiments, instead of con-  
256 sidering the medial axis of  $\mathcal{S}$ , we set  $\mathcal{F} = \text{RMA}(\mathcal{S})$  [8].

257 If  $\mathcal{F} = \{B_i, i = 1 \dots k\}$ , the overall computational cost of this algorithm is  
258  $O(\sum_{i=1}^k |B_i| + k \log k)$ .

#### 259 4.2 Greedy Algorithm: a Bounded Heuristic

260 To have a bounded heuristic, let us consider another problem called the MIN-  
261 SETCOVER problem [9]: an instance  $(\mathcal{S}, \mathcal{F})$  of the MINSETCOVER consists of  
262 a finite set  $\mathcal{S}$  and a family  $\mathcal{F}$  of subsets of  $\mathcal{S}$ , such that every element of  $\mathcal{S}$   
263 belongs to at least one subset of  $\mathcal{F}$ . The problem is to find a family of sub-  
264 sets  $\mathcal{F}^* \subseteq \mathcal{F}$  with minimum cardinality such that  $\mathcal{F}^*$  still covers  $\mathcal{S}$ . From the  
265 optimization MINSETCOVER problem, we can define the following decision  
266 problem: can we cover  $\mathcal{S}$  with a family  $\mathcal{F}^*$  such that  $|\mathcal{F}^*| \leq k$  for a given

267  $k$ ? This decision problem is known to be NP-complete [9]. Replacing  $\mathcal{S}$  by a  
 268 discrete object and  $\mathcal{F}$  by the medial axis, we have a specific instance of the  
 269 MINSETCOVER problem.

The greedy approximation algorithm is presented in 1. Even if this algorithm is simple, it provides a bounded approximation: if we denote  $H(d) = \sum_{i=1}^d \frac{1}{i}$ ,  $H_{\mathcal{F}} = H(\max |B|, B \in \mathcal{F})$  and  $\mathcal{F}^*$  the  $k$ -MA, the greedy algorithm produces a set  $\hat{\mathcal{F}}$  such that:

$$|\hat{\mathcal{F}}| \leq H_{\mathcal{F}} \cdot |\mathcal{F}^*|$$

---

**Algorithm 1:** Greedy algorithm for MINSETCOVER.

---

**Data:**  $\mathcal{S}$  and  $\mathcal{F}$

**Result:** the approximated solution  $\hat{\mathcal{F}}$

$U = \mathcal{S};$

$\hat{\mathcal{F}} = \emptyset;$

**while**  $U \neq \emptyset$  **do**

Select  $B \in \mathcal{F}$  that maximizes  $|B \cap U|;$   
 $U = U - B;$   
 $\hat{\mathcal{F}} = \hat{\mathcal{F}} \cup \{B\};$

**return**  $\hat{\mathcal{F}}$

---

270  
271

272 If we consider  $\mathcal{S}$  as a discrete object and  $\mathcal{F}$  the medial axis of  $\mathcal{S}$ , the medial  
 273 axis simplification problem is a sub-problem of MINSETCOVER. Hence, Algo-  
 274 rithm 1 provides a bounded heuristic for the medial axis reduction and this  
 275 is, at the time of writing, the only known approximation algorithm for the  
 276 minimum  $k$ -MA for which we have an approximation factor. Despite the fact  
 277 that Algorithm 1 has a computational cost in  $O(|\mathcal{S}||\mathcal{F}| \min(|\mathcal{S}|, |\mathcal{F}|))$ , a linear  
 278 in time algorithm can be designed, for instance in  $O(\sum_{i=1}^k |B_i|)$  [9, Section  
 279 37.3].

### 280 4.3 Experiments

281 In Figure 6, we present some experiments of both approximation algorithms.  
 282 Two observations can be addressed: first, the reduction rate is very interest-  
 283 ing since almost half of the medial axis balls can be removed. Secondly, the  
 284 computational time of both algorithms are similar.

285 Despite the fact that Ragnemalm and Borgefors's algorithm gives slightly  
 286 better results, the theoretical bound provided by the greedy algorithm makes  
 287 this approach a bit more satisfactory.

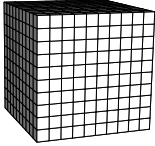
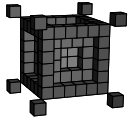
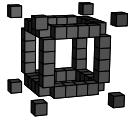
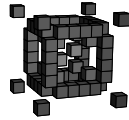
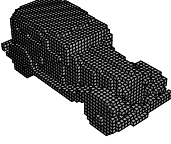
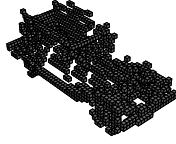
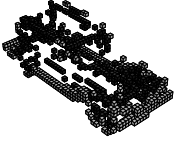
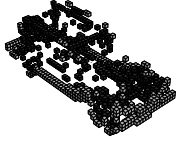




Objet	$\mathcal{F} = \text{MA}(\mathcal{S})$	$\hat{\mathcal{F}}$ RAGNEMALM ET AL.	$\hat{\mathcal{F}}$ Greedy
	 104	 56 (-46%) [ $< 0.01\text{s}$ ]	 66 (-36%) [ $< 0.01\text{s}$ ]
	 1292	 795 (-38%) [0.1s]	 820 (-36%) [0.19s]
	 17238	 6177 (-64%) [48.53s]	 6553 (-62%) [57.79s]

Fig. 6. Experimental analysis of simplification algorithms: *(from left to right)* Discrete 3-D objects, the discrete medial axis (ball centers), simplification obtained by [15] (ball centers), simplification obtained by the proposed greedy algorithm (ball centers). The cardinality of the sets are given below the figure with the reduction ratio (in percent) and the computational time.

## 288 5 Discussion and Conclusion

289 In this paper, we prove that finding a  $k$ -medial axis with minimum cardinality  
290  $k$  of a discrete shape is an NP-hard problem. To do so, we provide a poly-  
291 nomial reduction from the Planar-4 3-SAT problem to the  $k$ -MAP. We also  
292 experimentally compare the greedy approximation algorithm which provides  
293 a bounded approximation, with existing simplification algorithms.

294 In the proof, we have considered the Euclidean distance based medial axis. In  
295 order to derive a proof for the other metrics, new gadgets must be defined.  
296 Some cases are trivial, such as the  $d_8$  case for which only the variable gadget  
297 must be redefined (see Figure 7). Concerning other metrics, even if the gadgets  
298 may be difficult to design, we conjecture that theoretical results may be the  
299 same.

300 Future works concern both the complexity of specific restrictions of the

301  $k$ -MAP, and the approximation algorithms. Concerning the theoretical part,  
 302 the result we give induces the construction of very specific discrete shapes,  
 303 whose genus depends on the number of cycles in the Planar-4 3-SAT instance.  
 304 Thus, an important question is whether  $k$ -MA is still NP-complete in the  
 305 case of connected discrete shapes without holes. Concerning approximation  
 306 algorithms, experiments show that the results of the greedy approximation  
 307 algorithm are slightly worse than other existing algorithms. An important fu-  
 308 ture work is to merge the two approaches to improve the results while keeping  
 309 the bounded approximation.

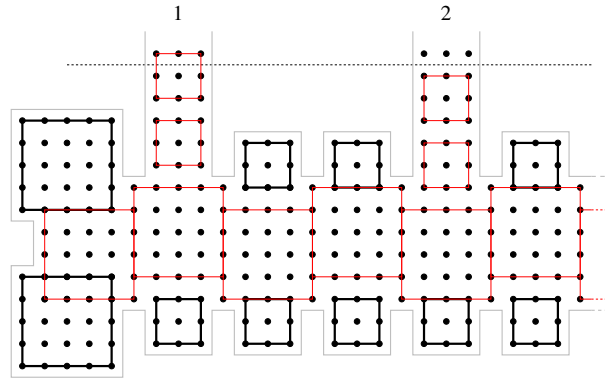


Fig. 7. Outline of a variable gadget for  $d_8$

## 310 References

- 311 [1] L. J. Aupperle, H. E. Conn, J. M. Keil, and J. O'Rourke. Covering orthogonal  
 312 polygons with squares. In *Proceedings of the 26th Annual Allerton Conf. Comm.*  
 313 *Control Comput.*, pages 97–106, 1988.
- 314 [2] H. Blum. A transformation for extracting new descriptors of shape. In  
 315 W. Wathen-Dunn, editor, *Models for the Perception of Speech and Visual Form*,  
 316 pages 362–380, Cambridge, 1967. MIT Press.
- 317 [3] G. Borgefors. Distance transformations in arbitrary dimensions. *Computer*  
 318 *Vision, Graphics and Image Processing*, 27:321–345, 1984.
- 319 [4] G. Borgefors. Distance transformations in digital images. *Computer Vision,*  
 320 *Graphics, and Image Processing*, 34(3):344–371, June 1986.
- 321 [5] G. Borgefors and I. Nyström. Quantitative shape analysis of volume images –  
 322 reducing the set of centres of maximal spheres. In *Proc. SSAB Symposium on*  
 323 *Image Analysis*, pages 5–8, Linköping, Sweden, March 1995.
- 324 [6] G. Borgefors and I. Nyström. Efficient shape representation by minimizing the  
 325 set of centers of maximal discs/spheres. *Pattern Recognition Letters*, 18:465–  
 326 472, 1997.

- 327 [7] H. Breu, J. Gil, D. Kirkpatrick, and M. Werman. Linear time euclidean distance  
328 transform algorithms. *IEEE Transactions on Pattern Analysis and Machine*  
329 *Intelligence*, 17(5):529–533, 1995.
- 330 [8] D. Coeurjolly and A. Montanvert. Optimal separable algorithms to compute the  
331 reverse euclidean distance transformation and discrete medial axis in arbitrary  
332 dimension. *IEEE Transactions on Pattern Analysis and Machine Intelligence*,  
333 29(3):437–448, mar 2007.
- 334 [9] T. Cormen, C. Leiserson, and R. Rivest. *Introduction to Algorithms*. MIT Press,  
335 1990.
- 336 [10] J. E. Goodman and J. O’Rourke, editors. *Handbook of Discrete and*  
337 *Computational Geometry*. CRC Press, 1997.
- 338 [11] T. Hirata. A unified linear-time algorithm for computing distance maps.  
339 *Information Processing Letters*, 58(3):129–133, May 1996.
- 340 [12] K. Jansen and H. Müller. The minimum broadcast time problem for several  
341 processor networks. *Theoretical Computer Science*, 147(1–2):69–85, 7 August  
342 1995.
- 343 [13] F. Nilsson and P.-E. Danielsson. Finding the minimal set of maximum disks for  
344 binary objects. *Graphical models and image processing*, 59(1):55–60, January  
345 1997.
- 346 [14] J.L. Pfaltz and A. Rosenfeld. Computer representation of planar regions by  
347 their skeletons. *Comm. of ACM*, 10:119–125, feb 1967.
- 348 [15] I. Ragnemalm and G. Borgefors. *The Euclidean Distance Transform*,, chapter  
349 Towards a minimal shape representation using maximal discs, pages 245–260.  
350 Linköping Studies in Science and Technology. Dissertations No. 304., Linköping  
351 University, apr 1993.
- 352 [16] E. Remy and E. Thiel. Medial axis for chamfer distances: computing look-up  
353 tables and neighbourhoods in 2D or 3D. *Pattern Recognition Letters*, 23(6):649–  
354 661, 2002.
- 355 [17] E. Remy and E. Thiel. Exact Medial Axis with Euclidean Distance. *Image and*  
356 *Vision Computing*, 23(2):167–175, 2005.
- 357 [18] A. Rosenfeld and J. L. Pfaltz. Sequential operations in digital picture  
358 processing. *Journal of the ACM*, 13(4):471–494, October 1966.
- 359 [19] T. Saito and J.-I. Toriwaki. Reverse distance transformation and skeletons  
360 based upon the euclidean metric for  $n$ -dimensional digital pictures. *IECE*  
361 *Trans. Inf. & Syst.*, E77-D(9):1005–1016, September 1994.
- 362 [20] R. Tamassia. On embedding a graph in the grid with the minimum number of  
363 bends. *SIAM J. Comput.*, pages 421–444, 1987.