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EXCITED BROWNIAN MOTIONS

OLIVIER RAIMOND AND BRUNO SCHAPIRA

ABSTRACT. We study a natural continuous time version of excited random walks, introduced by Norris, Rogers and Williams about twenty years ago. We obtain a necessary and sufficient condition for recurrence and for positive speed. This is analogous to results for excited (or cookie) random walks.

1. INTRODUCTION

Random processes that interact with their past trajectory have been studied a lot these past years. Reinforced random walks were introduced by Coppersmith and Diaconis, and then studied by Pemantle, Davis and many other authors (see the recent survey [Pem]). Some examples of time and space-continuous processes defined by a stochastic differential equation have also been studied. For example the self-interacting (or attracting) diffusions studied by Benaïm, Ledoux and Raimond (see [BeLR, BeR1, BeR2]) and also by Cranston, Le Jan, Herrmann, Kurtzmann and Roynette (see [CLJ, HR, Ku]), are processes defined by a stochastic differential equation for which the drift term is a function of the present position and of the occupation measure of the past process. Another example which is not solution of a stochastic differential equation was studied by Tóth and Werner [TW]. This process is a continuous version of some self-interacting random walks studied by Tóth (see for instance the survey [T]).

Carmona, Petit and Yor [CPY], Davis [D2], and Perman and Werner [PW] (see also other references therein) studied what they called a perturbed Brownian motion, which is the real valued process X defined by

$$X_t = B_t + \alpha \sup_{s \leq t} X_s + \beta \inf_{s \leq t} X_s,$$

where B is a Brownian motion. This process can be viewed has a weak limit of once edge-reinforced random walks on \mathbb{Z} (see in particular [D1, D2, W]).

More recently, excited (or cookie) random walks were introduced by Benjamini and Wilson [BW], and then further studied first on \mathbb{Z}^d [ABK, BaS1, BaS2, BerRa, KZer, Ko1, Ko2, MPiVa, Zer1, Zer2], but also on trees [BaS3, V]. In this class of walks, the transition probabilities depend on the number of times the walk has visited the present site. In particular Zerner [Zer1] and later Kosygina and Zerner [KZer] showed that on \mathbb{Z} , if p_i is the probability to go from x to $x + 1$ after the i -th visit of x and if either $p_i \geq 1/2$ for all i or $p_i = 1/2$ for i large enough, then the walk is a.s. recurrent if, and only if,

$$\sum_i (2p_i - 1) \in [-1, 1],$$

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and it is a.s. transient otherwise. Moreover Basdevant and Singh [BaS1] proved that the random walk has positive speed if and only if

$$\sum_i (2p_i - 1) \notin [-2, 2].$$

We study what could be a continuous time version of this. Excited Brownian motions considered here are defined by the stochastic differential equation

$$dX_t = dB_t + \varphi(X_s, L_s^{X_s}) ds,$$

for some bounded measurable φ , and where B is a Brownian motion and L_s^x is the local time in x at time s of X . These processes were already studied by Norris, Rogers and Williams [NRW], who considered also the case when φ is only supposed to be locally bounded. They obtained Ray–Knight type theorems and derived in some case transience and existence of a positive speed, for instance if φ is continuous, constant in x , nonnegative and increasing in l (in particular when $\varphi(x, l) = l$). Here we mainly concentrate on the case when φ is bounded, but we also discuss the case φ unbounded at the end of the paper. We prove a general criterion for recurrence and existence of a positive speed when φ is constant in x . In the particular case when φ is nonnegative or compactly supported our criterion is similar to the results for excited random walks: recurrence is equivalent to

$$\int_0^\infty \varphi(0, l) dl \in [-1, 1],$$

and nonzero speed is equivalent to

$$\int_0^\infty \varphi(0, l) dl \notin [-2, 2].$$

The paper is organized as follows. In the next section we define excited Brownian motions and give some elementary properties. In section 3 we describe the law of the excursions of these processes above or below some level. In section 4 we study the property of recurrence and prove a general 0-1 law. In section 5 we study the particular case of nonnegative φ and obtain a necessary and sufficient criterion for recurrence or transience. The tools used there are elementary martingale techniques, very similar to the techniques in [Zer1], and apply partly for non homogeneous φ (non necessarily constant in the first variable), for which we obtain a sufficient condition for recurrence (see Corollary 5.6). However they do not apply well for general φ . In section 6 we give a criterion for recurrence and transience without any condition on the sign of φ , but assuming that it is constant in the first variable. For this we use the tools developed in sections 3 and 4 and arguments similar to those in [NRW], in particular a Ray–Knight theorem. We also obtain a law of large number with an explicit expression for the speed. Here again, the proof follows the arguments given in [NRW], but we partly extend and simplify them.

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2. DEFINITIONS AND FIRST PROPERTIES

Denote by $(\Omega, \mathcal{F}, \mathbf{Q}_x)$ the Wiener space, where \mathbf{Q}_x is the law of a real Brownian motion started at x . Define $X_t(\omega) = \omega(t)$ for all $t \geq 0$ and $(\mathcal{F}_t, t \geq 0)$ the filtration

associated to X . In the following, L_t^y denotes the local time process of X at level y and at time t .

Let Λ be the set of measurable bounded functions $\varphi : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$. The subset of Λ of nonnegative functions will be denoted by Λ^+ . We will denote by Λ_c and Λ_c^+ the sets of functions φ in Λ (resp. in Λ^+) such that $\varphi(x, l)$ is a constant function of x .

For $\varphi \in \Lambda$, set

$$M_t^\varphi := \exp \left(\int_0^t \varphi(X_s, L_s^{X_s}) dX_s - \frac{1}{2} \int_0^t \varphi^2(X_s, L_s^{X_s}) ds \right).$$

Then $(M_t^\varphi, t \geq 0)$ is an $(\mathcal{F}_t, t \geq 0)$ martingale. Define $\mathbb{P}_{x,t}^\varphi$ as the probability measure on (Ω, \mathcal{F}_t) having a density M_t^φ with respect to \mathbb{Q}_x restricted to \mathcal{F}_t . By consistency, it is possible to construct a (unique) probability measure \mathbb{P}_x^φ on Ω , such that \mathbb{P}_x^φ restricted to \mathcal{F}_t is $\mathbb{P}_{x,t}^\varphi$. By the transformation of drift formula (Girsanov Theorem), one proves that under \mathbb{P}_x^φ ,

$$B_t = X_t - x - \int_0^t \varphi(X_s, L_s^{X_s}) ds,$$

is a Brownian motion started at 0.

This proves (the uniqueness follows by a similar argument: start with \mathbb{P}_x^φ the law of a solution and then construct $\mathbb{P}_{x,t}^\varphi$) the following

Proposition 2.1. *Let $(x, \varphi) \in \mathbb{R} \times \Lambda$. Then there is a unique solution (X, B) to the equation*

$$(1) \quad X_t = x + B_t + \int_0^t \varphi(X_s, L_s^{X_s}) ds,$$

with L_t^y the local time of X at level y and at time t , and such that B is a Brownian motion started at 0.

For clarity, we will sometimes write \mathbb{Q} for \mathbb{Q}_0 and \mathbb{P} for \mathbb{P}_0^φ , if there is no ambiguity on φ . We will use the notation \mathbb{E} and \mathbb{E}_x^φ for the expectations with respect to \mathbb{P} and \mathbb{P}_x^φ . For other probability measures μ , the expectation of a random variable Z will simply be denoted by $\mu(Z)$.

Let $(x, \varphi) \in \mathbb{R} \times \Lambda$. Under \mathbb{P}_x^φ , for all stopping times T , on the event $\{T < \infty\}$,

$$(2) \quad \text{the law of } (X_{t+T}, t \geq 0) \text{ given } \mathcal{F}_T \text{ is } \mathbb{P}_{X_T}^{\varphi_T},$$

where $\varphi_T \in \Lambda$ is defined by

$$\varphi_T(y, l) = \varphi(y, L_T^y + l).$$

Note that (X_t, φ_t) is a Markov process and that (2) is just the strong Markov property for this process.

We denote by D_t the drift accumulated at time t :

$$D_t = \int_0^t \varphi(X_s, L_s^{X_s}) ds.$$

Lemma 2.2. *Set $h(x, l) = \int_0^l \varphi(x, u) du$. The drift term D_t is also equal to*

$$\int_{\mathbb{R}} h(x, L_t^x) dx.$$

Proof. This follows from the occupation times formula given in exercise (1.15) in the chapter VI of Revuz–Yor [RY]. \square

In the following, we set for any Borel set A of \mathbb{R} ,

$$D_t^A = \int_A h(x, L_t^x) dx.$$

We will use also the notation D_t^+ , D_t^- and D_t^k , $k \in \mathbb{Z}$, respectively for $D_t^{\mathbb{R}^+}$, $D_t^{\mathbb{R}^-}$ and $D_t^{(k, k+1)}$. Note that (this is still a consequence of Exercise (1.15) Chapter VI in [RY])

$$D_t^A = \int_0^t \varphi(X_s, L_s^{X_s}) 1_A(X_s) ds.$$

Lemma 2.3. *Let $(\varphi_n)_{n \geq 0} \in \Lambda$ be a sequence of functions. Assume that for a.e. x and l , $\varphi_n(x, l)$ converges toward $\varphi(x, l)$ when $n \rightarrow +\infty$. Assume also that*

$$\sup_{x, n, u} |\varphi_n(x, u)| < +\infty.$$

Then $\mathbb{P}_x^{\varphi_n}$ converges weakly on Ω toward \mathbb{P}_x^φ for all x .

Proof. Let Z be a bounded continuous \mathcal{F}_t -measurable random variable. We have to prove that $\mathbb{E}_x^{\varphi_n}(Z)$ converges towards $\mathbb{E}_x^\varphi(Z)$. Since $\mathbb{E}_x^{\varphi_n}(Z)$ and $\mathbb{E}_x^\varphi(Z)$ are respectively equal to $\mathbb{Q}_x(ZM_t^{\varphi_n})$ and to $\mathbb{Q}_x(ZM_t^\varphi)$ it suffices to prove that $M_t^{\varphi_n}$ converges in L^2 towards M_t^φ . Using Itô calculus, we prove that

$$\frac{d}{dt} \mathbb{Q}_x[(M_t^{\varphi_n} - M_t^\varphi)^2] \leq 2\mathbb{Q}_x[(\varphi_n - \varphi)^2(X_t, L_t^{X_t})(M_t^\varphi)^2] + C\mathbb{Q}_x[(M_t^{\varphi_n} - M_t^\varphi)^2],$$

for some constant $C > 0$. By dominated convergence, the first term of the right hand term converges to 0. We conclude by using Gronwall's lemma. \square

3. CONSTRUCTION WITH EXCURSIONS

Define the processes A^+ and A^- as follows:

$$A_t^+ = \int_0^t 1_{\{X_s > 0\}} ds \quad \text{and} \quad A_t^- = \int_0^t 1_{\{X_s < 0\}} ds.$$

Define the right-continuous inverses of A^+ and A^- as

$$\kappa^+(t) = \inf\{u > 0 \mid A_u^+ > t\} \quad \text{and} \quad \kappa^-(t) = \inf\{u > 0 \mid A_u^- > t\}.$$

Define the two processes X^+ and X^- by

$$X_t^+ = X_{\kappa^+(t)} \quad \text{and} \quad X_t^- = X_{\kappa^-(t)}.$$

Denote by \mathbb{Q}^+ and \mathbb{Q}^- the laws respectively of X^+ and X^- under \mathbb{Q} , and let \mathbb{Q}_t^+ and \mathbb{Q}_t^- respectively be their restrictions to \mathcal{F}_t (then \mathbb{Q}_t^\pm is the law of $(X_s^\pm; s \leq t)$). It is known that \mathbb{Q}^+ (resp. \mathbb{Q}^-) is the law of a Brownian motion reflected above 0 (resp. below 0) and started at 0. The process β , defined by

$$\beta_t := X_t - L_t^0 \quad (\text{resp. } \beta_t := X_t + L_t^0),$$

(recall that L^0 is the local time process in 0 of X) is a Brownian motion under \mathbb{Q}^+ (resp. under \mathbb{Q}^-). Denote by N_t^φ the martingale on $(\Omega, \mathcal{F}, \mathbb{Q}^\pm)$ defined by

$$N_t^\varphi := \exp\left(\int_0^t \varphi(X_s, L_s^{X_s}) d\beta_s - \frac{1}{2} \int_0^t \varphi^2(X_s, L_s^{X_s}) ds\right).$$

Let $\mathbb{P}^{\varphi, \pm}$ be the measures whose restrictions $\mathbb{P}_t^{\varphi, \pm}$ to \mathcal{F}_t are defined by

$$\mathbb{P}_t^{\varphi, \pm} := N_t^\varphi \cdot \mathbb{Q}_t^\pm \quad \forall t \geq 0.$$

Note that, by using Girsanov Theorem, on the space $(\Omega, \mathcal{F}, \mathbb{P}^{\varphi, \pm})$,

$$(3) \quad X_t = \beta_t^\pm \pm L_t + \int_0^t \varphi(X_s, L_s^{X_s}) ds,$$

with β_t^\pm a Brownian motion.

Set $\tilde{\mathbb{P}}^\varphi := \mathbb{P}^{\varphi, +} \otimes \mathbb{P}^{\varphi, -}$ and let (X^1, X^2) be the canonical process of law $\tilde{\mathbb{P}}^\varphi$. Then X^1 and X^2 are independent and respectively distributed like $\mathbb{P}^{\varphi, +}$ and $\mathbb{P}^{\varphi, -}$. Denote by $L^{(1)}$ and $L^{(2)}$ the local time processes in 0 of X^1 and X^2 , and define their right continuous inverses τ^1 and τ^2 by

$$\tau_s^i = \inf\{t \mid L_t^{(i)} > s\} \quad \text{for } i \in \{1, 2\}.$$

Denote by e^1 and e^2 their excursion processes out of 0: for $s \leq L_\infty^{(i)}$,

$$e_s^i(u) = X_{\tau_{s-}^i + u}^i \quad \text{for all } u \in (0, \tau_s^i - \tau_{s-}^i),$$

if $\tau_s^i - \tau_{s-}^i > 0$, and $e_s^i = 0$ otherwise. Let now e be the excursion process obtained by adding e^1 and e^2 .

Denote by Ξ the measurable transformation that reconstructs a process out of its excursion process (see [RY] Proposition (2.5) p.482). Note that Ξ is not a one to one map, it is only surjective (think of processes having an infinite excursion out of 0, in which case $L_\infty^0 < \infty$).

Proposition 3.1. *The following hold*

- (i) *The law of the process Ξe is \mathbb{P} .*
- (ii) $((\Xi e)_t^+, t \leq L_\infty^{(2)}) = (X_t^1, t \leq L_\infty^{(2)})$.
- (iii) $((\Xi e)_t^-, t \leq L_\infty^{(1)}) = (X_t^2, t \leq L_\infty^{(1)})$.
- (iv) *Denote by L the local time in 0 of Ξe . Then for all $t \in \mathbb{R}^+ \cup \{\infty\}$, $L_t = L_t^{(1)} \wedge L_t^{(2)}$.*

Proof. Assume first that $\varphi(x, l) = 0$ if $x \in (-c, c)$, for some constant $c > 0$. For any $\epsilon \in (-c, c)$, define a process X^ϵ as follows: set $T_0^\epsilon = 0$ and for $n \geq 1$,

$$S_n^\epsilon = \inf\{t \geq T_{n-1}^\epsilon \mid X_t \in \{-\epsilon, \epsilon\}\},$$

$$T_n^\epsilon = \inf\{t \geq S_n^\epsilon \mid X_t = 0\}.$$

Define also

$$A_\epsilon(t) = \sum_{n \geq 1} (T_n^\epsilon \wedge t - S_n^\epsilon \wedge t),$$

and let $\kappa_\epsilon(t)$ be the right-continuous inverse of A_ϵ^+ . Then set

$$X_t^\epsilon := X_{\kappa_\epsilon(t)}.$$

Now observe that during each time-interval $(S_n^\epsilon, T_n^\epsilon)$, the local time in 0 of X cannot increase. And for $t \in (A_\epsilon(S_n^\epsilon), A_\epsilon(T_n^\epsilon))$, $\kappa_\epsilon(t) = S_n^\epsilon + (t - A_\epsilon(S_n^\epsilon))$. So by (3), during the intervals $(A_\epsilon(S_n^\epsilon), A_\epsilon(T_n^\epsilon))$, if X follows the law of Ξe , then X^ϵ is solution of the SDE

$$dX_t^\epsilon = dR_t^\epsilon + \varphi\left(X_t^\epsilon, L_{\kappa_\epsilon(t)}^{X_t^\epsilon}\right) dt,$$

where L is the local time of X , and R^ϵ is the Brownian motion defined by

$$R_t^\epsilon = \sum_{k=1}^{n-1} (B_{T_k^\epsilon} - B_{S_k^\epsilon}) + (B_{\kappa_\epsilon(t)} - B_{S_n^\epsilon}),$$

for $t \in (A_\epsilon(S_n^\epsilon), A_\epsilon(T_n^\epsilon))$.

Denote by $L^{\epsilon, \cdot}$ the local time process of X^ϵ . Then

$$L_{\kappa_\epsilon(t)}^x = L_t^{\epsilon, x} \quad \forall t \geq 0 \quad \forall x \notin (-c, c).$$

Since $\varphi(x, l) = 0$ when $x \in (-c, c)$,

$$\varphi(x, L_{\kappa_\epsilon(t)}^x) = \varphi(x, L_t^{\epsilon, x}) \quad \forall t \geq 0 \quad \forall x \in \mathbb{R}.$$

Thus X^ϵ satisfies in fact the SDE:

$$(4) \quad dX_t^\epsilon = dR_t^\epsilon + \varphi(X_t^\epsilon, L_t^{\epsilon, X_t^\epsilon}) dt,$$

up to the first time it hits 0. And when it hits 0, it jumps instantaneously to ϵ or to $-\epsilon$ with probability 1/2, independently of its past trajectory. Note that this determines the law of X^ϵ .

But it follows also from (1), that if X has law \mathbb{P} , then X^ϵ solves as well the SDE (4) up to the first time it hits 0, and then jump to ϵ or $-\epsilon$ with probability 1/2 (since there is no drift in $(-c, c)$). Since moreover X^ϵ converges to X , when ϵ goes to 0, we conclude that the law of Ξe is \mathbb{P} .

To finish the proof of (i), for $c > 0$, define φ_c by

$$\varphi_c(x, l) = \varphi(x, l) 1(x \notin [-c, c]).$$

By Lemma 2.3 we know that $\mathbb{P}_0^{\varphi_c}$ converges toward \mathbb{P}_0^φ , when $c \rightarrow 0$. It can be also seen that $\mathbb{P}^{\varphi_c, +}$ and $\mathbb{P}^{\varphi_c, -}$ converge respectively towards $\mathbb{P}^{\varphi, +}$ and $\mathbb{P}^{\varphi, -}$. Since Ξe is a measurable transformation of (X^1, X^2) , we can conclude.

Assertions (ii), (iii) and (iv) are immediate: in the construction of Ξe , one needs only to know e_s for $s \leq L_\infty = L_\infty^{(1)} \wedge L_\infty^{(2)}$. So $(\Xi e)^+$ and $(\Xi e)^-$ can be respectively reconstructed with the positive and negative excursions of $(e_s, s \leq L_\infty^{(1)} \wedge L_\infty^{(2)})$. \square

4. A 0 – 1 LAW FOR RECURRENCE AND TRANSIENCE

Let $\varphi \in \Lambda$. Consider the events $R_a := \{L_\infty^a = +\infty\}$, $a \in \mathbb{R}$. Using conditional Borel-Cantelli lemma, one can prove that for all x, y, z , and all $\varphi \in \Lambda$, \mathbb{P}_x^φ -a.s., $R_y = R_z$. In the following, we will denote by R the event of recurrence (= R_a for all a).

We will first study the question of recurrence and transience for the processes X^1 and X^2 separately, where X^1 and X^2 are independent respectively of law $\mathbb{P}^{\varphi, +}$ and $\mathbb{P}^{\varphi, -}$. Note that we still have for all $x \geq 0$ (resp. $x \leq 0$), and all $\varphi \in \Lambda$, $\mathbb{P}^{\varphi, +}$ -a.s. (resp. $\mathbb{P}^{\varphi, -}$ -a.s.), $R_x = R_0 (= R)$. So in all cases, R is the event $\{L_\infty^0 = \infty\}$.

Fix $x \in \mathbb{R}$ and denote by φ_x the function in Λ such that for all $(y, l) \in \mathbb{R} \times \mathbb{R}^+$,

$$\varphi_x(y, l) = \varphi(x + y, l).$$

In the following, $\mathbb{P}_x^{\varphi, \pm}$ will denote $\mathbb{P}^{\varphi_x, \pm}$.

Proposition 4.1. *For all $x \in \mathbb{R}$ and all $\varphi \in \Lambda$, the following holds*

$$(5) \quad \mathbb{P}_x^\varphi(R) = \mathbb{P}_x^{\varphi, +}(R) \times \mathbb{P}_x^{\varphi, -}(R).$$

Proof. This is a straightforward application of Proposition 3.1 since

$$\begin{aligned} \mathbb{P}_x^\varphi(R) &= \tilde{\mathbb{P}}^{\varphi_x}(L_\infty = \infty) \\ &= \tilde{\mathbb{P}}^{\varphi_x}(L_\infty^{(1)} = L_\infty^{(2)} = \infty), \end{aligned}$$

and we conclude since $L_\infty^{(1)}$ and $L_\infty^{(2)}$ are independent. \square

For $t > 0$ and $a \in \mathbb{R}$, set $\sigma_t^a = \inf\{s > 0 \mid \int_0^s 1_{\{X_u \geq a\}} du > t\}$. Then we have

Lemma 4.2. *Let $a \in \mathbb{R}$, $x, y \leq a$ and $\varphi, \psi \in \Lambda$ be such that $\varphi(z, l) = \psi(z, l)$ for all $z \geq a$ and all $l \geq 0$. Then $(X_{\sigma_t^a} - a, t \geq 0)$ has the same distribution under $\mathbb{P}_x^{\varphi,+}$ and under $\mathbb{P}_y^{\psi,+}$. In particular,*

$$\mathbb{P}_x^{\varphi,+}(L_\infty^a = \infty) = \mathbb{P}_y^{\psi,+}(L_\infty^a = \infty).$$

Proof. The lemma follows immediately from Proposition 3.1. \square

Note that a similar proposition holds for $\mathbb{P}^{\varphi,-}$. A direct consequence of this lemma is that

Proposition 4.3. *For all $x \in \mathbb{R}$, $\mathbb{P}_x^{\varphi,\pm}(R) = \mathbb{P}^{\varphi,\pm}(R)$.*

With this in hands, we can prove the

Proposition 4.4 (Zero-one law). *Let $\varphi \in \Lambda$. Then $\mathbb{P}^{\varphi,+}(R)$ and $\mathbb{P}^{\varphi,-}(R)$ both belong to $\{0, 1\}$.*

Proof. Assume that $\mathbb{P}^{\varphi,+}(R) > 0$. By martingale convergence theorem, $\mathbb{P}^{\varphi,+}$ -a.s.

$$\begin{aligned} 1_R &= \lim_{z \rightarrow +\infty} \mathbb{P}^{\varphi,+}(R \mid \mathcal{F}_{T_z}) \\ &= \lim_{z \rightarrow +\infty} \mathbb{P}_z^{\varphi_{T_z},+}(R) \\ &= \mathbb{P}^{\varphi,+}(R), \end{aligned}$$

by application of Lemma 4.2 and Proposition 4.3. Thus $\mathbb{P}^{\varphi,+}(R) = 1$, which proves the proposition. \square

Denote by T the event of transience. Then $T = R^c = \{L_\infty^0 < \infty\}$. Note that $\mathbb{P}^{\varphi,\pm}$ -a.s., $T = \{X \rightarrow \pm\infty\}$, and that \mathbb{P}_x^φ -a.s., $T = \{X \rightarrow +\infty\} \cup \{X \rightarrow -\infty\}$ and that $\tilde{\mathbb{P}}^\varphi$ -a.s., $\{\exists e \text{ is transient}\} = \{X^1 \rightarrow \infty\} \cup \{X^2 \rightarrow -\infty\}$. Then we have the criterion

Proposition 4.5. *Let $\varphi \in \Lambda$.*

- (i) $\mathbb{P}_x^\varphi(R) = 1$ for all x if, and only if, $\mathbb{P}^{\varphi,+}(R) = \mathbb{P}^{\varphi,-}(R) = 1$.
- (ii) $\mathbb{P}_x^\varphi(R) = 0$ for all x if, and only if, $\mathbb{P}^{\varphi,+}(R) = 0$ or $\mathbb{P}^{\varphi,-}(R) = 0$.

Proof. This is a straightforward consequence of Propositions 4.1, 4.3 and 4.4. \square

Let us remark that when X is transient it is possible to have $\mathbb{P}(X \rightarrow +\infty) = 1 - \mathbb{P}(X \rightarrow -\infty) \in (0, 1)$. For instance take $\varphi \in \Lambda$ such that $\mathbb{P}^{\varphi,+}(R) = 0$, i.e. one has $\mathbb{P}^{\varphi,+}$ -a.s. $X \rightarrow +\infty$. Then define $\psi \in \Lambda$ by $\psi(x, u) = \varphi(x, u)$ if $x \geq 0$ and $\psi(x, u) = -\varphi(x, u)$ if $x < 0$. Then by symmetry we have

$$\mathbb{P}_0^\psi(X \rightarrow +\infty) = \mathbb{P}_0^\psi(X \rightarrow -\infty) = 1/2.$$

More generally,

$$\mathbb{P}_0^\varphi(X \rightarrow +\infty) = \tilde{\mathbb{P}}^\varphi(L_\infty^{(1)} < L_\infty^{(2)}),$$

and

$$\mathbf{P}_0^\varphi(X \rightarrow -\infty) = \tilde{\mathbf{P}}^\varphi(L_\infty^{(1)} > L_\infty^{(2)}).$$

So, if $L_\infty^{(1)}$ and $L_\infty^{(2)}$ are finite random variables (i.e. if X^1 and X^2 are transient), since they are independent, these two probabilities are positive (see Lemma 5.8 below).

5. CASE WHEN $\varphi \geq 0$

In this section, we will consider functions belonging to Λ^+ . In this case, it is obvious that $\mathbf{P}^{\varphi,-}(R) = 1$. Thus the recurrence property only depends on $\mathbf{P}^{\varphi,+}(R)$.

Set $T_a = \inf\{t > 0 \mid X_t = a\}$.

Lemma 5.1. *Let $\varphi \in \Lambda^+$, for which there exists x_0 such that for all $x \leq x_0$ and all $l \geq 0$, $\varphi(x, l) = \varphi(x_0, l)$, and where $\varphi(x_0, l)$ is non zero function of l . Then, for all $a \geq x$,*

$$\mathbf{E}_x^\varphi(D_{T_a}) = a - x.$$

Proof. Without losing generality, we prove this result for $x = 0$. Take $a > 0$. We have for all n ,

$$0 = \mathbf{E}_0^\varphi(B_{T_a \wedge n}) = \mathbf{E}_0^\varphi(X_{T_a \wedge n}) - \mathbf{E}_0^\varphi(D_{T_a \wedge n}).$$

By monotone convergence, $\lim_{n \rightarrow \infty} \mathbf{E}_0^\varphi(D_{T_a \wedge n}) = \mathbf{E}_0^\varphi(D_{T_a})$. Thus, if we can prove that $\lim_{n \rightarrow \infty} \mathbf{E}_0^\varphi(X_{T_a \wedge n}) = a$, the lemma will follow. Since \mathbf{P}_0^φ -a.s., $X_{T_a} = a$ and $T_a < \infty$, this is equivalent to prove that

$$\lim_{n \rightarrow \infty} \mathbf{E}_0^\varphi[X_n 1_{\{n < T_a\}}] = 0.$$

For all n ,

$$\min_{t < T_a} X_t \leq X_n 1_{\{n < T_a\}} \leq a \quad \mathbf{P}_0^\varphi - a.s.$$

If one proves that $\min_{t < T_a} X_t$ is integrable, we can conclude by dominated convergence. One has for all $c > 0$,

$$\begin{aligned} \mathbf{E}_0^\varphi \left[-\min_{t < T_a} X_t \right] &\leq \mathbf{E}_0^\varphi [D_{T_a}/c] \\ &\quad + \sum_{i \geq 1} i \times \mathbf{P}_0^\varphi \left[-\min_{t < T_a} X_t \in [i-1, i], D_{T_a} < ci \right]. \end{aligned}$$

Clearly $\mathbf{E}_0^\varphi[D_{T_a}/c] \leq a/c < +\infty$. Moreover for all $i \geq 1$,

$$(6) \quad \mathbf{P}_0^\varphi \left[-\min_{t < T_a} X_t \in [i-1, i], D_{T_a} < ci \right] \leq \mathbf{P}_0^\varphi [T_{-i} < T_a, D_{T_a} < ci].$$

Now on the event $\{T_{-i} < T_a\}$,

$$D_{T_a} \geq \sum_{k=1}^i \left(\int_{-k}^{-k+1} h(y, L_{T_{-k}}^y) dy \right) 1_{\{T_{-k} < \infty\}}.$$

Set

$$\alpha_k := \left(\int_{-k}^{-k+1} h(y, L_{T_{-k}}^y) dy \right) 1_{\{T_{-k} < \infty\}}.$$

We will prove that there exists a constant $\alpha > 0$, such that for all integer k greater than $-x_0$,

$$(7) \quad \mathbf{E}_0^\varphi[\alpha_{k+1} \mid \mathcal{F}_{T_{-k}}] \geq \alpha 1_{\{T_{-k} < +\infty\}}.$$

By (2) we have

$$\begin{aligned} \mathbf{E}_0^\varphi[\alpha_{k+1} \mid \mathcal{F}_{T_{-k}}] &= \mathbf{E}_{-k}^{\varphi_{T_{-k}}}[\alpha_{k+1}] 1_{\{T_{-k} < \infty\}} \\ &= \mathbf{E}_0^{\psi_k}[\alpha_1] 1_{\{T_{-k} < \infty\}}, \end{aligned}$$

with $\psi_k(x, l) = \varphi(x_0, l)$ for $x < 0$ and $\psi_k(x, l) = \varphi(x - k, L_{T_{-k}}^{x-k})$ for $x \geq 0$. Note that by Proposition 3.1, Ξe will reach level -1 in finite time if, and only if,

$$L_\infty^{(1)} \geq L_{T_{-1}}^{(2)},$$

where T_{-1} denotes also the hitting time of -1 for X^2 . Thus

$$\mathbf{E}_0^{\psi_k}[\alpha_1] = \tilde{\mathbf{E}}^{\psi_k} \left[F(X_t^2, t \leq T_{-1}) 1_{\{L_\infty^{(1)} \geq L_{T_{-1}}^{(2)}\}} \right],$$

with

$$F(X_t^2, t \leq T_{-1}) = \int_{-1}^0 h(y, L_{T_{-1}}^{(2),y}) dy,$$

where $L^{(2),y}$ is the local time in y of X^2 . It is possible to couple X^1 with a Brownian motion Y^+ reflected at 0, started at 0, with drift $\|\varphi\|_\infty$, and such that $X_t^1 \leq Y_t^+$ for all $t \geq 0$. Note that the law of Y^+ is $\mathbf{P}^{\psi,+}$, where $\psi(x, l) = \|\varphi\|_\infty$ if $x \geq 0$ and $\psi(x, l) = \varphi(x_0, l)$ if $x < 0$. Moreover,

$$L_\infty^{(1)} \geq L_\infty^+,$$

with L^+ the local time in 0 of Y^+ . These properties imply that

$$\mathbf{E}_0^{\psi_k}[\alpha_1] \geq \alpha := \mathbf{E}_0^\psi[\alpha_1].$$

It remains to see that α is positive. For any $t > 0$, it is larger than

$$\mathbf{E}_0^\psi[\alpha_1 1_{\{T_{-1} \leq t\}}].$$

By using Girsanov's transform it suffices to prove that for some $t > 0$,

$$\mathbf{Q}[\alpha_1 1_{\{T_{-1} \leq t\}}] > 0.$$

But

$$\mathbf{Q}[\alpha_1 1_{\{T_{-1} \leq t\}}] = \int_{-1}^0 \mathbf{Q}[h(y, L_{T_{-1}}^y) 1_{\{T_{-1} \leq t\}}] dy.$$

By continuity of h and L it suffices to prove that for some $y \in (-1, 0)$ and $t > 0$,

$$\mathbf{Q}[h(y, L_{T_{-1}}^y) 1_{\{T_{-1} \leq t\}}] > 0.$$

But this is clear for $y = 1/2$ for instance, since for any $M > 0$, $\mathbf{Q}[L_{T_{-1}}^{1/2} > M] > 0$.

Next (7) shows that the process $(M_i, i \geq 0)$ defined by

$$M_i = \sum_{k=1}^i (\alpha_k - \alpha 1_{\{T_{-k} < +\infty\}}),$$

is a sub-martingale with respect to the filtration $(\mathcal{F}_{T_{-i}})_{i \geq 1}$. Moreover on the event $\{T_{-i} < T_a\}$, one has

$$M_i = \sum_{k=1}^i (\alpha_k - \alpha).$$

So by taking $c = \alpha/2$ in (6) one gets

$$\mathbf{P}[T_{-i} < T_a, D_{T_a} < c(i+1)] \leq \mathbf{P}[M_i \leq -\alpha i/2],$$

for all $i \geq 0$. We conclude now that $\mathbb{E}[-\min_{t < T_a} X_t]$ is finite by using standard results on martingales. \square

Remark 5.2. As noticed also by Zerner in [Zer1], the condition $\varphi(x_0, \cdot) \neq 0$ is necessary (the result being obviously false if $\varphi = 0$).

Remark 5.3. This lemma can be extended to the case when φ is random and stationary in the sense that for all z , φ^z and φ have the same law, where φ^z is defined by $\varphi^z(x, l) = \varphi(x + z, l)$.

Lemma 5.4. *Let $\varphi \in \Lambda_c^+$. Then*

$$\mathbb{E}_0^\varphi(D_\infty^k) \leq 1,$$

for all $k \geq 0$.

Proof. It is the same proof than for Lemma 11 in Zerner [Zer1]. We reproduce it here for completeness. Note first that $\mathbb{E}_0^\varphi[D_\infty^k] = \mathbb{P}_k^{\varphi, +}(D_\infty^0)$, which does not depend on k since $\varphi \in \Lambda_c$. For $K \geq 1$ and $i \leq K - 1$,

$$D_{T_K} \geq D_{T_K}^+ = \sum_{j=0}^{K-1} D_{T_K}^j \geq \sum_{j=0}^{K-1-i} D_{T_K}^j \geq \sum_{j=0}^{K-1-i} D_{T_{j+i}}^j.$$

Using Lemma 5.1,

$$K = \mathbb{E}_0^\varphi[D_{T_K}] \geq \sum_{j=0}^{K-1-i} \mathbb{E}_0^\varphi[D_{T_{j+i}}^j] \geq (K - i) \mathbb{E}_0^\varphi[D_{T_i}^0].$$

Letting $i \rightarrow \infty$, we conclude. \square

Lemma 5.5. *Let $\varphi \in \Lambda^+$ be such that $\mathbb{P}^{\varphi, +}(R) = 0$. Then*

$$\lim_{z \rightarrow \infty} \mathbb{E}_0^\varphi(D_{T_z}^+)/z = 1.$$

If moreover $\varphi \in \Lambda_c^+$, then

$$\mathbb{E}_0^\varphi(D_\infty^0) = 1.$$

Proof. The proof of the first part follows the proof of Lemma 6 in Zerner [Zer1]. We write it here for completeness. First, since $\mathbb{E}_0^\varphi(D_{T_z}^+) = \mathbb{P}^{\varphi, +}(D_{T_z})$ and $\mathbb{P}^{\varphi, +}(R)$ is a function of $(\varphi(x, \cdot))_{x \geq 0}$, we can assume that $\varphi(x, l) = 1$ for all $x \leq 0$ and all $l \geq 0$. Thus Lemma 5.1 can be applied: $\mathbb{E}_0^\varphi(D_{T_z}) = z$. So, it suffices to prove that $\lim_{z \rightarrow \infty} \mathbb{E}_0^\varphi(D_{T_z}^-)/z = 0$.

For $i \geq 1$, let $\sigma_i = \inf\{j \geq T_i \mid X_j = 0\}$. We have, for z an integer,

$$\mathbb{E}_0^\varphi[D_{T_z}^-] = \sum_{i=0}^{z-1} \mathbb{E}_0^\varphi[D_{T_{i+1}}^- - D_{T_i}^-].$$

Note that

$$\begin{aligned} \mathbb{E}_0^\varphi[D_{T_{i+1}}^- - D_{T_i}^-] &= \mathbb{E}_0^\varphi \left[1_{\{\sigma_i < T_{i+1}\}} (D_{T_{i+1}}^- - D_{T_i}^-) \right] \\ &= \mathbb{E}_0^\varphi \left[1_{\{\sigma_i < T_{i+1}\}} \mathbb{E}_0^{\varphi^{\sigma_i}}(D_{T_{i+1}}^-) \right] \\ &\leq (i + 1) \mathbb{P}_0^\varphi(\sigma_i < T_{i+1}), \end{aligned}$$

using again Lemma 5.1. Thus it remains to prove that

$$\lim_{z \rightarrow \infty} \frac{1}{z} \sum_{i=1}^{z+1} i \mathbb{P}_0^\varphi(\sigma_i < T_{i+1}) = 0.$$

Let $Y_i = \mathbb{P}_0^\varphi[\sigma_i < T_{i+1} \mid \mathcal{F}_{T_i}]$. Since $\mathbb{P}_0^\varphi(R) = 0$, the conditional Borel-Cantelli lemma implies that \mathbb{P}_0^φ -a.s., $\sum_i Y_i < +\infty$. Since $Y_i \leq 1/i$ (X being greater than a Brownian motion), for all positive ϵ ,

$$i \times \mathbb{P}_0^\varphi(\sigma_i < T_{i+1}) \leq \epsilon + \mathbb{P}_0^\varphi(Y_i \geq \epsilon/i).$$

This implies that

$$\frac{1}{z} \sum_{i=1}^{z+1} i \mathbb{P}_0^\varphi(\sigma_i < T_{i+1}) \leq \epsilon + \frac{1}{z} \mathbb{E}_0^\varphi \left[\sum_{i=1}^{z+1} 1_{\{Y_i \geq \epsilon/i\}} \right].$$

But since $\sum_i Y_i < \infty$ a.s., the density of the $i \leq z$ such that $Y_i \leq \epsilon/i$ tends to 0 when z tends to ∞ . Thus the preceding sum converges to 0. This concludes the proof of the first part.

The second part is immediate (see Zerner [Zer1] Theorem 12). Since $\varphi \in \Lambda_c^+$, for all $K \geq 0$,

$$\mathbb{E}_0^\varphi[D_\infty^0] = \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}_0^\varphi[D_\infty^k] \geq \frac{1}{K} \mathbb{E}_0^\varphi[D_{T_K}^+].$$

We conclude using the first part of the lemma. \square

The next result gives a sufficient condition for recurrence, when we only know that $\varphi \in \Lambda^+$. For $\varphi \in \Lambda_c^+$, we will obtain a necessary and sufficient condition in Theorem 5.9 below.

Corollary 5.6. *Let $\varphi \in \Lambda^+$. For $x \in \mathbb{R}$, set $\delta^x(\varphi) = \int_0^\infty \varphi(x, u) du$. If*

$$\liminf_{z \rightarrow +\infty} \frac{1}{z} \int_0^z \delta^x(\varphi) dx < 1,$$

then $\mathbb{P}^{\varphi,+}(R) = 1$.

Proof. Since P-a.s. $D_{T_z}^+ \leq \int_0^z \delta^x(\varphi) dx$, if $\liminf \frac{1}{z} \int_0^z \delta^x(\varphi) dx < 1$, then

$$\liminf \mathbb{E}_0^\varphi(D_{T_z}^+)/z < 1.$$

We conclude by using Lemma 5.5. \square

Lemma 5.7. *Let $\varphi \in \Lambda$ be such that $\mathbb{P}^{\varphi,+}(R) = 0$. Then*

$$\mathbb{P}_0^\varphi(T_{-1} = +\infty) > 0.$$

Proof. By using Proposition 3.1,

$$\mathbb{P}_0^\varphi(T_{-1} = +\infty) = \tilde{\mathbb{P}}^\varphi(L_\infty^{(1)} < L_{T_{-1}}^{(2)}),$$

where T_{-1} denotes also the hitting time of -1 for X^2 . Since X^1 and X^2 are independent and since $\mathbb{P}^{\varphi,+}(R) = 0$ implies that $\tilde{\mathbb{P}}^\varphi$ -a.s., $L_\infty^{(1)} < +\infty$, it suffices to prove that for any $l > 0$,

$$\mathbb{P}^{\varphi,-}(L_{T_{-1}}^0 > l) > 0.$$

Equivalently it suffices to prove that for any $l > 0$, there exists $t > 0$ such that

$$\mathbb{P}^{\varphi,-}(L_{T_{-1}}^0 > l \text{ and } T_{-1} \leq t) > 0.$$

By absolute continuity of $\mathbb{P}_{|\mathcal{F}_t}^{\varphi, -}$ and $\mathbb{Q}_{|\mathcal{F}_t}^-$ this is equivalent to

$$\mathbb{Q}^-(L_{T_{-1}} > l \text{ and } T_{-1} \leq t) > 0.$$

But this is well known. Thus the lemma is proved. \square

Lemma 5.8. *Let $\varphi \in \Lambda$ be such that $\mathbb{P}_0^\varphi(R) = 0$. Then for any $M > 0$,*

$$\mathbb{P}_0^\varphi[L_\infty^0 < M] > 0.$$

Proof. Since $\mathbb{P}_0^\varphi(R) = 0$, Proposition 4.1 shows that $\mathbb{P}^{\varphi, +}(R) = 0$ or $\mathbb{P}^{p, -}(R) = 0$. Assume for instance that $\mathbb{P}^{\varphi, +}(R) = 0$, the other case being similar. We have

$$\begin{aligned} \mathbb{P}_0^\varphi[L_\infty^0 < M] &\geq \mathbb{P}_0^\varphi[T_1 < +\infty, L_{T_1}^0 < M \text{ and } X_t > 0 \quad \forall t > T_1] \\ &\geq \mathbb{E}_0^\varphi \left[1_{\{T_1 < +\infty, L_{T_1}^0 < M\}} \mathbb{P}_1^{\varphi T_1}(T_0 = \infty) \right]. \end{aligned}$$

But Lemma 5.7 implies that a.s., $\mathbb{P}_1^{\varphi T_1}(T_0 = \infty) > 0$. So it remains to prove that $\mathbb{P}_0^\varphi(T_1 < +\infty, L_{T_1}^0 < M) > 0$. Like in the previous lemma, by absolute continuity, it suffices to prove that

$$\mathbb{Q}(L_{T_1}^0 < M) > 0.$$

But again this is well known. Thus the lemma is proved. \square

Finally we obtain the

Theorem 5.9. *Let $\varphi \in \Lambda_c^+$. Then*

$$\mathbb{P}^{\varphi, +}(R) = 1 \iff \int_0^\infty \varphi(0, u) \, du \leq 1.$$

Proof. We prove this for φ such that $\varphi(x, l) = \varphi(0, l)$ for all x . By Lemma 5.4, $\mathbb{E}_0^\varphi(D_\infty^0) \leq 1$. But if $\mathbb{P}^{\varphi, +}(R) = 1$, then \mathbb{P}_0^φ -a.s. $L_\infty^x = +\infty$, for all x . So, by using the occupation time formula (see Lemma 2.2) we have $\mathbb{E}_0^\varphi(D_\infty^0) = \int_0^\infty \varphi(0, u) \, du$. This gives the necessary condition. Reciprocally, if $\mathbb{P}^{\varphi, +}(R) = 0$, we saw in Lemma 5.5 that $\mathbb{E}_0^\varphi(D_\infty^0) = 1$. But by Lemma 5.8, we have $\mathbb{E}_0^\varphi(D_\infty^0) = \mathbb{E}_0^\varphi(h(0, L_\infty^0)) < \int_0^\infty \varphi(0, u) \, du$, which gives the sufficient condition and concludes the proof of the theorem. \square

Note that if $\varphi \in \Lambda$ is such that for some $a \in \mathbb{R}$, $\varphi(x, l) = \varphi(a, l) \geq 0$ for all $x \geq a$ and all $l \geq 0$, then

$$\mathbb{P}^{\varphi, +}(R) = 1 \iff \int_0^\infty \varphi(a, u) \, du \leq 1.$$

This can be proved using the fact that $\mathbb{P}^{\varphi, +}(R) = \mathbb{P}^{\varphi a, +}(R)$ which does not depend on $\varphi(x, \cdot)$, for $x < a$.

Remark 5.10. With the technique used in this section one could prove as well that for any $\varphi \in \Lambda_c$, such that $\int_0^{+\infty} |\varphi(0, u)| \, du < +\infty$, recurrence implies $\int_0^\infty \varphi(0, u) \, du \in [-1, 1]$. But as we will see, this last condition is not sufficient for recurrence. In fact the necessary and sufficient condition we will obtain in the next section requires more sophisticated tools. In particular we will use a Ray–Knight theorem.

6. GENERAL CRITERION FOR RECURRENCE AND LAW OF LARGE NUMBERS

6.1. A Ray–Knight theorem. By following the proof of the usual Ray–Knight theorem for Brownian motion given for instance in [RY] Theorem (2.2) p.455, a Ray–Knight theorem can be obtained. Such a theorem is proved in [NRW] (Theorem 2). For $a \geq 0$ and $0 \leq x \leq a$, let

$$Z_x^{(a)} := L_{T_a}^{(1), a-x},$$

where we recall that $L^{(1), \cdot}$ is the local time process of X^1 and T_a is the first time X^1 hits a . Recall also that the law of X^1 is $\mathbb{P}^{\varphi, +}$.

Theorem 6.1 (Ray–Knight). *The process $(Z_x^{(a)}, x \in [0, a])$ is a non-homogeneous Markov process started at 0 solution of the following SDE:*

$$(8) \quad dZ_x^{(a)} = 2\sqrt{Z_x^{(a)}} d\beta_x + 2(1 - h(a - x, Z_x^{(a)})) dx,$$

with β a Brownian motion.

Proof. We just sketch the proof. It is actually proved in [NRW]. Applying Tanaka's formula at time T_a :

$$(X_{T_a} - (a - x))^+ = \int_0^{T_a} 1_{\{X_s > a-x\}} dB_s + \int_0^{T_a} 1_{\{X_s > a-x\}} \varphi(X_s, L_s^{X_s}) ds + \frac{1}{2} Z_x^{(a)}.$$

Now one has $(X_{T_a} - (a - x))^+ = x$, $M_x = \int_0^{T_a} 1_{\{X_s > a-x\}} dB_s$ is a martingale (in the spatial variable x) and the extended occupation formula gives

$$\int_0^{T_a} 1_{\{X_s > a-x\}} \varphi(X_s, L_s^{X_s}) ds = \int_0^x h(a - z, Z_z^{(a)}) dz.$$

Thus $Z_x^{(a)}$ is a semimartingale, whose quadratic variation is $4 \int_0^x Z_z^{(a)} dz$ (see Theorem (2.2) p.455 in [RY]). This proves the proposition. \square

6.2. Criterion for recurrence. In the rest of this section, we assume that $\varphi \in \Lambda_c$. We will note $h(x, l) \equiv h(l)$ for all x and l . Note that the Ray–Knight theorem 6.1 implies that for $a > 0$, we have

Proposition 6.2 (Ray–Knight). *The process $(Z_x^{(a)}, x \in [0, a])$ is a diffusion started at 0 with generator $L = 2z d^2/dz^2 + 2(1 - h(z))d/dz$. Thus $Z^{(a)}$ solves the following SDE:*

$$(9) \quad dZ_x^{(a)} = 2\sqrt{Z_x^{(a)}} d\beta_x + 2(1 - h(Z_x^{(a)})) dx,$$

with β a Brownian motion.

In the following, Z will denote a diffusion started at 0 with generator L . Then $(Z_x^{(a)}, 0 \leq x \leq a)$ and $(Z_x, 0 \leq x \leq a)$ have the same law. As noticed in [NRW] Theorem 3, the diffusion Z admits an invariant measure π given by

$$\pi(x) = c \cdot \exp \left[- \int_0^x h(l) \frac{dl}{l} \right],$$

with c some positive constant. When π is a finite measure, c is chosen such that π is a probability measure. This implies the following generalization of Theorem 5.9 :

Theorem 6.3. *Let $\varphi \in \Lambda_c$. Then*

$$\mathbb{P}^{\varphi,+}(R) = 1 \iff \int_0^\infty \exp\left[-\int_0^x h(l) \frac{dl}{l}\right] dx = +\infty.$$

Proof. Assume that $\mathbb{P}^{\varphi,+}(R) = 1$. Then $Z_a^{(a)} = L_{T_a}^{(1),0}$ converges a.s. towards ∞ as $a \rightarrow \infty$. Thus Z_a (which is equal in law to $Z_a^{(a)}$) converges in probability towards ∞ . Thus Z is not positive recurrent, i.e. π is not a finite measure.

Assume now that $\mathbb{P}^{\varphi,+}(R) = 0$. Then $Z_a^{(a)} = L_{T_a}^{(1),0}$ converges a.s. towards $L_\infty^{(1),0} < \infty$ as $a \rightarrow \infty$. Thus Z_a converges in law towards $L_\infty^{(1),0}$. This implies that Z is positive recurrent, i.e. π is a finite measure. \square

Of course an analogue result holds as well for X^2 . The criterion becomes:

$$\mathbb{P}^{\varphi,-}(R) = 1 \iff \int_0^\infty \exp\left[\int_0^x h(l) \frac{dl}{l}\right] dx = +\infty.$$

In particular, X^1 and X^2 cannot be both transient.

Observe now that if $\lim_{l \rightarrow +\infty} h(l)$ exists, and equals let say h_∞ , then

$$\mathbb{P}_0^\varphi(R) = 1 \implies h_\infty \in [-1, 1],$$

and

$$\mathbb{P}_0^\varphi(R) = 0 \implies h_\infty \notin (-1, 1).$$

But in the critical case $|h_\infty| = 1$ both recurrence and transience regimes may hold. For instance if $h(l) - 1 \sim \alpha / \ln l$ in $+\infty$, then $\mathbb{P}_0^\varphi(R) = 1$ if $\alpha < 1$, whereas $\mathbb{P}_0^\varphi(R) = 0$ if $\alpha > 1$.

Let us finish this subsection with this last remark: Let $\zeta_x := L_\infty^{(1),x}$, with $x \geq 0$. Assume Z is positive recurrent (then X^1 is transient). Using the fact that Z is reversible, it can be seen that ζ is a diffusion of generator L , with initial distribution π (see Theorem 3 in [NRW]).

6.3. Law of large numbers. Our next result is a strong law of large numbers with an explicit expression for the speed.

Theorem 6.4. *Let $\varphi \in \Lambda_c$.*

- (i) *Assume π is a probability measure and denote by v the mean of π (i.e. $v = \int_0^\infty x\pi(x)dx \in [0, \infty]$). Then $\tilde{\mathbb{P}}^\varphi$ -a.s.*

$$\lim_{t \rightarrow \infty} \frac{X_t^1}{t} = \frac{1}{v}.$$

- (ii) *Assume π is an infinite measure. Then, $\tilde{\mathbb{P}}^\varphi$ -a.s.*

$$\lim_{t \rightarrow \infty} \frac{X_t^1}{t} = 0.$$

Proof. We take $v = \infty$ when π is an infinite measure. We first prove that (in all cases) $\tilde{\mathbb{P}}^\varphi$ -a.s.

$$(10) \quad \frac{T_K}{K} \rightarrow v,$$

when $K \rightarrow +\infty$. Fix $N \geq 1$. Since $\varphi \in \Lambda_c$,

$$(11) \quad \frac{1}{K} T_K \geq \frac{1}{K} \sum_{i=1}^{[K/N]} (T_{N(i+1)} - T_{Ni}) \geq \frac{1}{K} \sum_{i=1}^{[K/N]} U_i,$$

where the U_i are i.i.d. random variables distributed like T_N (U_i is the time spent in $[Ni, N(i+1)]$ between times T_{Ni} and $T_{N(i+1)}$). Now (in the following \mathbf{E} denotes the expectation with respect to $\tilde{\mathbf{P}}^\varphi$)

$$\mathbf{E}[Z_a] = \mathbf{E}[Z_a^{(a)}] = \mathbf{E}[L_{T_a}^{(1),0}]$$

is an increasing function of a . By monotone convergence $\mathbf{E}[Z_a]$ converges towards $\mathbf{E}[Z_\infty] \equiv v$. It is standard to see by using (9) that $\mathbf{E}[Z_a] < \infty$ for all a (φ being bounded, $2(1-h(z))$ is dominated by $c(1+z)$ for some positive c). Thus for all $N \geq 1$,

$$\mathbf{E}[T_N] = \int_0^N \mathbf{E}[L_{T_N}^{(1),a}] da = \int_0^N \mathbf{E}[Z_a] da \leq N \mathbf{E}[Z_N] < +\infty.$$

Therefore (11) shows that

$$\begin{aligned} \liminf_{K \rightarrow \infty} \frac{1}{K} T_K &\geq \frac{1}{N} \mathbf{E}[T_N] \\ &\geq \frac{1}{N} \int_0^N \mathbf{E}[Z_a] da. \end{aligned}$$

By taking the limit as $N \rightarrow \infty$, we get

$$(12) \quad \liminf_{K \rightarrow \infty} \frac{1}{K} T_K \geq v.$$

So we can conclude when $v = \infty$. Assume now that $v < \infty$. Then $\zeta_a = L_\infty^{(1),a}$ is a diffusion of generator L with initial distribution π . Now

$$T_K = \int_0^K L_{T_K}^{(1),a} da \leq \int_0^K \zeta_a da,$$

so that

$$\limsup_{K \rightarrow \infty} \frac{1}{K} T_K \leq v,$$

by the well known ergodic theorem for positive recurrent diffusions (see, for instance, [IMcK] §6.8). Together with (12) this proves (10).

Note that for all $t \in (T_N, T_{N+1})$.

$$(13) \quad \frac{X_t^1}{t} \leq \frac{N+1}{T_N}.$$

The law of large numbers follows immediately when $v = \infty$. Assume now that $v < +\infty$. Since φ is bounded, for all N , $(X_t^1, t \geq T_N)$ dominates a Brownian motion with drift $-\|\varphi\|_\infty$ started at N and absorbed at 0. So for all $\epsilon > 0$, there exist constants $c > 0$ and $C > 0$ such that

$$\mathbf{P}^{\varphi,+} \left[\inf_{t \in (T_N, T_N + cN)} X_t < (1 - \epsilon)N \right] \leq C \exp(-cN),$$

for all $N \geq 1$. So $\tilde{\mathbf{P}}^\varphi$ -a.s. we have

$$\inf_{t \in (T_N, T_N + cN)} X_t^1 \geq (1 - \epsilon)N,$$

for all N large enough. Moreover since a.s. $\lim_{N \rightarrow \infty} T_N/N = v$, we have $T_{N+1} - T_N = o(N)$. So a.s. for N large enough,

$$\inf_{t \in (T_N, T_{N+1})} X_t^1 \geq (1 - \epsilon)N.$$

Then a.s. for N large enough and $t \in (T_N, T_{N+1})$,

$$\frac{X_t^1}{t} \geq (1 - \epsilon) \frac{N}{T_{N+1}}.$$

By (10), a.s. $T_N \sim vN$. So this together with (13) finishes the proof of the theorem. \square

Here also an analogous result holds for X^2 . One has $\tilde{\mathbb{P}}^\varphi$ -a.s.

$$\frac{X_t^2}{t} \rightarrow -\frac{1}{v'},$$

where

$$v' = \int_0^\infty x \exp \left[\int_0^x h(l) \frac{dl}{l} \right] dx.$$

Now if $\varphi \in \Lambda_c$, as we already observed, X^1 and X^2 cannot be both transient. But assume for instance that, let say, X^1 is transient and X^2 is recurrent. Then for t large enough $X_t^1 = X_{t+U}$, where U is the finite random variable equal to the total time spent by X in the negative part. So X satisfies the same law of large numbers than X^1 , namely a.s.

$$\frac{X_t}{t} \rightarrow \frac{1}{v}.$$

Remark 6.5. Let us comment now on the case when φ is only supposed to be locally bounded. Assume that φ is constant in the first variable and that $\liminf_{l \rightarrow \infty} h(l)/l > -\infty$ (this includes the case $\inf_{l \geq 0} \varphi(0, l) > -\infty$). Then, as we will see in the appendix below, the process X^1 of law $\mathbb{P}^{\varphi, +}$ is still well defined. Moreover, as the reader may check, Proposition 4.4, Theorem 6.3 and Theorem 6.4 hold as well for X^1 . Using the same method, like in [NRW], it is also possible to construct \mathbb{P}^φ , assuming that $\liminf_{l \rightarrow \infty} h(l) > 0$. Then the study of \mathbb{P}^φ can be reduced to the study of $\mathbb{P}^{\varphi, +}$. But this method does not look to apply to construct $\mathbb{P}^{\varphi, -}$. However the following basic question remains:

Question: Is it possible to find φ for which $\mathbb{P}^{\varphi, +}$ can be defined and such that $\mathbb{P}^{\varphi, +}(\sup_{t \geq 0} X_t < +\infty) > 0$?

7. APPENDIX: CONSTRUCTION OF \mathbb{P}^φ AND $\mathbb{P}^{\varphi, +}$ IN CASES φ IS LOCALLY BOUNDED

We assume that φ is constant in the first variable. We note $\varphi(x, l) \equiv \varphi(l)$ for any x and l . We assume also that φ is locally bounded, i.e. $\sup_{l \in [0, L]} \varphi(l) < +\infty$ for any $L > 0$, and $\liminf_{l \rightarrow \infty} h(l)/l > -\infty$. We want to define $\mathbb{P}^{\varphi, +}$. Following [NRW], set $\varphi_n(l) := 1_{\{l \leq n\}} \varphi(l)$. Let $S'_n = \inf\{t \mid L_t^{X_t} > n\}$ and let $S_n = S'_n \wedge n$. According to Theorem B in [NRW], if we can prove that for each $t > 0$,

$$\mathbb{P}^{\varphi_n, +}(S_n < t) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

then by consistency, it will be possible to define $\mathbb{P}^{\varphi, +}$. Observe that it is enough to prove the two statements:

$$\text{for each } a > 0 \quad \mathbb{P}^{\varphi_n, +}(S_n < T_a) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

and for each $t > 0$ and $\epsilon > 0$, there is some $a > 0$ such that

$$\mathbf{P}^{\varphi_{n,+}}(T_a < t) < \epsilon \quad \text{for all } n.$$

The first statement can be proved like in [NRW]: by using the Ray-Knight theorem under $\mathbf{P}^{\varphi_{n,+}}$,

$$\mathbf{P}^{\varphi_{n,+}}(S_n < T_a) = \mathbf{P}^{\varphi_{n,+}}\left(\sup_{0 \leq x \leq a} Z_x^{(a)} > n\right) = \mathbf{P}^Z\left(\sup_{0 \leq x \leq a} Z_x > n\right),$$

where \mathbf{P}^Z denotes the law of the diffusion with generator $L = 2zd^2/dz^2 + 2(1 - h(z))d/dz$. The last term tends to 0 when $n \rightarrow +\infty$, since a.s. Z does not explode in finite time (this follows from the fact that $\liminf_{l \rightarrow \infty} h(l)/l > -\infty$). For the second statement, write:

$$\begin{aligned} \mathbf{P}^{\varphi_{n,+}}(T_a < t) &= \mathbf{P}^{\varphi_{n,+}}\left(\int_0^a Z_x^{(a)} dx < t\right) \\ &\leq \mathbf{P}^{\varphi_{n,+}}\left(\int_0^{a \wedge T_n} Z_x^{(a)} dx < t\right) \\ &\leq \mathbf{P}^Z\left(\int_0^{a \wedge T_n} Z_x dx < t\right), \end{aligned}$$

where T_n denotes the first time $Z^{(a)}$ or Z reaches n . But by using standard scale functions arguments, we can see 0 is an entrance boundary point for the diffusion with generator L . Thus \mathbf{P}^Z -a.s.

$$\int_0^\infty Z_x dx = +\infty.$$

The second statement follows.

We give now some hints to construct \mathbf{P}^φ when $\liminf h(l) > 0$. Using the fact that $\mathbf{P}^{\varphi_{n,-}}(R) = 1$, all the stopping times T_a are finite $\mathbf{P}^{\varphi_{n,-}}$ -a.s. for all positive a . Then the method of [NRW] can be followed.

REFERENCES

- [ABK] **Amir G., Benjamini I, Kozma G.:** *Excited random walk against a wall*, Probab. Theory and Related Fields 140, (2008), 83–102.
- [BaS1] **Basdevant A.-L., Singh A.:** *On the speed of a cookie random walk*, Probab. Theory Related Fields 141, (2008), 625–645.
- [BaS2] **Basdevant A.-L., Singh A.:** *Rate of growth of a transient cookie random walk*, Electron. J. Probab. 13, (2008), 811–851.
- [BaS3] **Basdevant A.-L., Singh A.:** *Recurrence and transience of a multi-excited random walk on a regular tree*, arXiv:0803.3284.
- [BeLR] **Benaïm M., Ledoux, M., Raimond, O.:** *Self-interacting diffusions*, Probab. Theory Relat. Fields 122(1), 1–41 (2002).
- [BeR1] **Benaïm, M., Raimond, O.:** *Self-interacting diffusions II: convergence in law*, Ann. Inst. H. Poincaré Probab. Statist. 39(6), 1043–1055 (2003).
- [BeR2] **Benaïm, M., Raimond, O.:** *Self-interacting diffusions III: the symmetric case*, Ann. Probab. 33(5), 1717–1759 (2005).
- [BW] **Benjamini I., Wilson D. B.:** *Excited random walk*, Electron. Comm. Probab. 8 (electronic), (2003), 86–92.
- [BerRa] **Bérard J., Ramirez A.:** *Central limit theorem for the excited random walk in dimension $d \geq 2$* , Electron. Comm. Probab. 12, (2007), 303–314.
- [CPY] **Carmona P., Petit F., Yor M.:** *Beta variables as times spent in $[0, \infty[$ by certain perturbed Brownian motions*, J. London Math. Soc. (2) 58, (1998), 239–256.

- [CLJ] **Cranston, M., Le Jan, Y.:** *Self-attracting diffusions: two case studies*, Math. Ann. 303, 87–93 (1995).
- [D1] **Davis B.:** *Reinforced random walk*, Probab. Theory Related Fields 84, (1990), 203–229.
- [D2] **Davis B.:** *Weak limits of perturbed Brownian motion and the equation $Y_t = B_t + \alpha \sup\{Y_s : s \leq t\} + \beta \inf\{Y_s : s \leq t\}$* , Ann. Probab. 24, (1996), 2007–2023.
- [HR] **Herrmann, S., Roynette, B.:** *Boundedness and convergence of some self-attracting diffusions*, Math. Ann. 325(1), 81–96 (2003).
- [IMcK] **Itô K., McKean H.P.:** *Diffusion processes and their sample paths*, Berlin Heidelberg New York, Springer, (1965).
- [KZer] **Kosygina E., Zerner M. P. W.:** *Positively and negatively excited random walks on integers, with branching processes*, preprint, arXiv:0801.1924.
- [Ko1] **Kozma G.:** *Excited random walk in three dimensions has positive speed*, arXiv:math/0310305.
- [Ko2] **Kozma G.:** *Excited random walk in two dimensions has linear speed*, arXiv:math/0512535.
- [Ku] **Kurtzmann A.:** *Asymptotic behavior of Self-interacting Diffusions on R^d* , Ph.D. thesis, University of Neuchâtel, (2007).
- [MPiVa] **Mountford T., Pimentel L.P.R., Valle G.:** *On the speed of the one-dimensional excited random walk in the transient regime*, ALEA Lat. Am. J. Probab. Math. Stat. 2, (2006), 279–296 (electronic).
- [NRW] **Norris J.R., Rogers L.C.G., Williams D.:** *Self-avoiding random walk: a Brownian motion model with local time drift*, Probab. Theory Related Fields 74, (1987), 271–287.
- [Pem] **Pemantle R.:** *A survey of random processes with reinforcement*, Probab. Surv. 4 (electronic), (2007), 1–79.
- [PW] **Perman, M., Werner W.:** *Perturbed Brownian motions*, Probab. Theory Related Fields 108, (1997), 357–383.
- [RY] **Revuz D., Yor M.:** *Continuous martingales and Brownian motion*, Springer-Verlag, third ed. (1999).
- [T] **Tóth B.:** *Self-interacting random motions*, Proceedings of the 3rd European Congress of Mathematics, Barcelona 2000, vol. 1, pp. 555–565, Birkhauser, (2001).
- [TW] **Tóth B., Werner W.:** *The true self-repelling motion*, Probab. Theory Related Fields 111, (1998), 375–452.
- [V] **Volkov S.:** *Excited random walk on trees*, Electron. J. Probab. 8, (2003), 15 pp. (electronic).
- [W] **Werner W.:** *Some remarks on perturbed reflecting Brownian motion*, Sémin. Probab. XXIX, LNM 1613, Springer, Berlin, (1995), 37–43.
- [Zer1] **Zerner M. P. W.:** *Multi-excited random walks on integers*, Probab. Theory Relat. Fields 133, (2005), 98–122.
- [Zer2] **Zerner M. P. W.:** *Recurrence and transience of excited random walks on \mathbb{Z}_d and strips*, Electron. Comm. Probab. 11, (2006), 118–128 (electronic).

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