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# EXCITED BROWNIAN MOTIONS

OLIVIER RAIMOND AND BRUNO SCHAPIRA

ABSTRACT. We introduce and study a natural continuous time version of excited random walks. In the case of nonnegative drift, we obtain a necessary and sufficient condition for recurrence. This result is analogous to Zerner's result [Zer1] for excited (or cookie) random walks. We use similar arguments.

## 1. INTRODUCTION

Random processes that interact with their past trajectory have been studied a lot these past years. Reinforced random walks were introduced by Coppersmith and Diaconis, and then studied by Pemantle, Davis and many other authors (see the recent survey [Pem]). Some examples of time and space-continuous processes defined by a stochastic differential equation have also been studied. For example the self-interacting (or attracting) diffusions studied by Benaïm, Ledoux and Raimond (see [BLR, BR2, BR3]) and also by Cranston, Le Jan, Herrmann, Kurtzmann and Roynette (see [CLJ, HR, Ku]), are processes defined by a stochastic differential equation for which the drift term is a function of the present position and of the occupation measure of the past process. Another example which is not solution of a stochastic differential equation was studied by Tóth and Werner [TW].

Carmona, Petit and Yor [CPY], Davis [D2], and Perman and Werner [PW] (see also other references therein) studied what they called a perturbed Brownian motion, which is the real valued process  $X$  defined by

$$X_t = B_t + \alpha \sup_{s \leq t} X_s + \beta \inf_{s \leq t} X_s,$$

where  $B$  is a Brownian motion. This process can be viewed as a weak limit of once edge-reinforced random walks on  $\mathbb{Z}$  (see in particular [D1, D2, W]).

More recently, excited (or cookie) random walks were introduced by Benjamini and Wilson [BW], and then further studied first on  $\mathbb{Z}^d$  [BaS1, BaS2, KZer, Zer1, Zer2], but also on trees [BaS3, V]. In this class of walks, the transition probabilities depend on the number of times the walk has visited the present site. In particular Kosygina and Zerner [KZer] showed that on  $\mathbb{Z}$  and if  $p_i$  is the probability to go from  $x$  to  $x + 1$  after the  $i$ -th visit of  $x$ , then the walk is a.s. recurrent if, and only if,

$$\sum_i (2p_i - 1) \in [-1, 1]$$

and it is a.s. transient otherwise.

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We present what could be a continuous time version of this. Excited Brownian motions considered here are defined by the stochastic differential equation

$$dX_t = dB_t + \varphi(X_s, L_s^{X_s}) ds,$$

for some bounded measurable  $\varphi$ , and where  $B$  is a Brownian motion and  $L_s^x$  is the local time in  $x$  at time  $s$  of  $X$ . We prove here a similar result to the one of Zerner [Zer1] for excited random walks: when  $\varphi(x, l)$  is a nonnegative function constant in  $x$ , recurrence is equivalent to

$$\int_0^\infty \varphi(0, l) dl \leq 1.$$

In particular when  $\varphi(x, l) = b1_{\{l \leq L\}}$ , recurrence is equivalent to  $bL \in [-1, 1]$ . This last case corresponds to the case the drift is  $b$  when the local time is less than  $L$  and 0 after. This is similar with the excited random walk with  $M$  cookies per sites: when a cookie (or more) is present at site  $x$  there is a probability  $p$  to go from  $x$  to  $x + 1$  (and after this visit the cookie is eaten), and when there is no cookie this probability is  $1/2$ , in which case recurrence is equivalent to  $(2p - 1)M \in [-1, 1]$ .

The paper is organized as follows. In the next section we define excited Brownian motions and give some elementary properties. In section 3 we describe the law of the excursions of our process above or below some level. In section 4 we study the property of recurrence and prove a general 0-1 law. In section 5 we study the particular case of nonnegative  $\varphi$  and obtain a necessary and sufficient criterion for recurrence or transience. In the last section, we add some remarks for general  $\varphi$  and prove in particular a necessary condition for recurrence. It is natural to believe that this condition is also sufficient, as it is the case for excited random walks [KZer]. But a proof of this is out of reach with the techniques of the present paper. Our arguments are similar to the ones of [Zer1].

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## 2. DEFINITIONS AND FIRST PROPERTIES

Denote by  $(\Omega, \mathcal{F}, \mathbb{Q}_x)$  the Wiener space, where  $\mathbb{Q}_x$  is the law of a real Brownian motion started at  $x$ . Define  $X_t(\omega) = \omega(t)$  for all  $t \geq 0$  and  $(\mathcal{F}_t, t \geq 0)$  the filtration associated to  $X$ . In the following,  $L_t^y$  denotes the local time process of  $X$  at level  $y$  and at time  $t$ .

Let  $\Lambda$  be the set of measurable bounded functions  $\varphi : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ . The subset of  $\Lambda$  of nonnegative functions will be denoted by  $\Lambda^+$ . We will denote by  $\Lambda_c$  and  $\Lambda_c^+$  the sets of functions  $\varphi$  in  $\Lambda$  (resp. in  $\Lambda^+$ ) such that  $\varphi(x, l)$  is a constant function of  $x$ .

For  $\varphi \in \Lambda$ , set

$$M_t^\varphi := \exp \left( \int_0^t \varphi(X_s, L_s^{X_s}) dX_s - \frac{1}{2} \int_0^t \varphi^2(X_s, L_s^{X_s}) ds \right).$$

Then  $(M_t^\varphi, t \geq 0)$  is an  $(\mathcal{F}_t, t \geq 0)$  martingale. Define  $\mathbb{P}_{x,t}^\varphi$  as the probability measure on  $(\Omega, \mathcal{F}_t)$  having a density  $M_t^\varphi$  with respect to  $\mathbb{Q}_x$  restricted to  $\mathcal{F}_t$ . By consistency, it is possible to construct a (unique) probability measure  $\mathbb{P}_x^\varphi$  on  $\Omega$ , such that  $\mathbb{P}_x^\varphi$  restricted to  $\mathcal{F}_t$  is  $\mathbb{P}_{x,t}^\varphi$ . By the transformation of drift formula (Girsanov

Theorem), one proves that under  $\mathbb{P}_x^\varphi$ ,

$$B_t = X_t - x - \int_0^t \varphi(X_s, L_s^{X_s}) ds$$

is a Brownian motion started at 0.

This proves (the uniqueness follows by a similar argument: start with  $\mathbb{P}_x^\varphi$  the law of a solution and then construct  $\mathbb{P}_{x,t}^\varphi$ ) the following

**Proposition 2.1.** *Let  $(x, \varphi) \in \mathbb{R} \times \Lambda$ . Then there is a unique solution  $(X, B)$  to the equation*

$$(1) \quad X_t = x + B_t + \int_0^t \varphi(X_s, L_s^{X_s}) ds$$

with  $L_t^y$  the local time of  $X$  at level  $y$  and at time  $t$ , and such that  $B$  is a Brownian motion started at 0.

For clarity, we will sometimes write  $\mathbb{Q}$  for  $\mathbb{Q}_0$  and  $\mathbb{P}$  for  $\mathbb{P}_0^\varphi$ , if there is no ambiguity on  $\varphi$ . We will use the notation  $\mathbb{E}$  and  $\mathbb{E}_x^\varphi$  for the expectations with respect to  $\mathbb{P}$  and  $\mathbb{P}_x^\varphi$ . For other probability measures  $\mu$ , the expectation of a random variable  $Z$  will simply be denoted by  $\mu(Z)$ .

Let  $(x, \varphi) \in \mathbb{R} \times \Lambda$ . Under  $\mathbb{P}_x^\varphi$ , for all stopping times  $T$ , on the event  $\{T < \infty\}$ ,

$$(2) \quad \text{the law of } (X_{t+T}, t \geq 0) \text{ given } \mathcal{F}_T \text{ is } \mathbb{P}_{X_T}^{\varphi_T},$$

where  $\varphi_T \in \Lambda$  is defined by

$$\varphi_T(y, l) = \varphi(y, L_T^y + l).$$

Note that  $(X_t, \varphi_t)$  is a Markov process and that (2) is just the strong Markov property for this process.

We denote by  $D_t$  the drift accumulated at time  $t$  :

$$D_t = \int_0^t \varphi(X_s, L_s^{X_s}) ds.$$

**Lemma 2.2.** *Set  $h(x, l) = \int_0^l \varphi(x, u) du$ . The drift term  $D_t$  is also equal to*

$$\int_{\mathbb{R}} h(x, L_t^x) dx.$$

*Proof.* This follows from the occupation times formula given in exercise (1.15) in the chapter VI of Revuz–Yor [RY].  $\square$

In the following, we set for any Borel set  $A$  of  $\mathbb{R}$ ,

$$D_t^A = \int_A h(x, L_t^x) dx.$$

We will use also the notation  $D_t^+$ ,  $D_t^-$  and  $D_t^k$ ,  $k \in \mathbb{Z}$ , respectively for  $D_t^{\mathbb{R}^+}$ ,  $D_t^{\mathbb{R}^-}$  and  $D_t^{(k, k+1)}$ . Note that (this is still a consequence of Exercise (1.15) Chapter VI in [RY])

$$D_t^A = \int_0^t \varphi(X_s, L_s^{X_s}) 1_A(X_s) ds.$$

**Lemma 2.3.** *Let  $(\varphi_n)_{n \geq 0} \in \Lambda$  be a sequence of functions, such that  $\varphi_n(x, l)$  converges toward  $\varphi(x, l)$ , when  $n \rightarrow +\infty$ , for a.e.  $x$  and  $l$ , and such that*

$$\sup_{x, n, u} |\varphi_n(x, u)| < +\infty.$$

*Then  $\mathbb{P}_x^{\varphi_n}$  converges weakly on  $\Omega$  toward  $\mathbb{P}_x^\varphi$  for all  $x$ .*

*Proof.* Let  $Z$  be a bounded continuous  $\mathcal{F}_t$ -measurable random variable. We have to prove that  $\mathbb{E}_x^{\varphi_n}(Z)$  converges towards  $\mathbb{E}_x^\varphi(Z)$ . Since  $\mathbb{E}_x^{\varphi_n}(Z)$  and  $\mathbb{E}_x^\varphi(Z)$  are respectively equal to  $\mathbb{Q}_x(ZM_t^{\varphi_n})$  and to  $\mathbb{Q}_x(ZM_t^\varphi)$  it suffices to prove that  $M_t^{\varphi_n}$  converges in  $L^2$  towards  $M_t^\varphi$ . Using Itô calculus, we prove that

$$\frac{d}{dt} \mathbb{Q}_x[(M_t^{\varphi_n} - M_t^\varphi)^2] \leq 2\mathbb{Q}_x[(\varphi_n - \varphi)^2(X_t, L_t^{X_t})(M_t^\varphi)^2] + C\mathbb{Q}_x[(M_t^{\varphi_n} - M_t^\varphi)^2],$$

for some constant  $C > 0$ . By dominated convergence, the first term of the right hand term converges to 0. We conclude by using Gronwall's lemma.  $\square$

### 3. CONSTRUCTION WITH EXCURSIONS

Define the processes  $A^+$  and  $A^-$  as follows:

$$A_t^+ = \int_0^t 1_{\{X_s > 0\}} ds \quad \text{and} \quad A_t^- = \int_0^t 1_{\{X_s < 0\}} ds.$$

Define the right-continuous inverses of  $A^+$  and  $A^-$  as

$$\kappa^+(t) = \inf\{u > 0 \mid A_u^+ > t\} \quad \text{and} \quad \kappa^-(t) = \inf\{u > 0 \mid A_u^- > t\}.$$

Define the two processes  $X^+$  and  $X^-$  by

$$X_t^+ = X_{\kappa^+(t)} \quad \text{and} \quad X_t^- = X_{\kappa^-(t)}.$$

Denote by  $\mathbb{Q}^+$  and  $\mathbb{Q}^-$  the laws respectively of  $X^+$  and  $X^-$  under  $\mathbb{Q}$ , and let  $\mathbb{Q}_t^+$  and  $\mathbb{Q}_t^-$  respectively be their restrictions to  $\mathcal{F}_t$  (then  $\mathbb{Q}_t^\pm$  is the law of  $(X_s^\pm; s \leq t)$ ). It is known that  $\mathbb{Q}^+$  (resp.  $\mathbb{Q}^-$ ) is the law of a Brownian motion reflected above 0 (resp. below 0) and started at 0. The process  $\beta$ , defined by

$$\beta_t := X_t - L_t^0 \quad (\text{resp. } \beta_t := X_t + L_t^0)$$

(recall that  $L^0$  is the local time process in 0 of  $X$ ) is a Brownian motion under  $\mathbb{Q}^+$  (resp. under  $\mathbb{Q}^-$ ). Denote by  $N_t^\varphi$  the martingale on  $(\Omega, \mathcal{F}, \mathbb{Q}^\pm)$  defined by

$$N_t^\varphi := \exp\left(\int_0^t \varphi(X_s, L_s^{X_s}) d\beta_s - \frac{1}{2} \int_0^t \varphi^2(X_s, L_s^{X_s}) ds\right).$$

Let  $\mathbb{P}^{\varphi, \pm}$  be the measures whose restrictions  $\mathbb{P}_t^{\varphi, \pm}$  to  $\mathcal{F}_t$  are defined by

$$\mathbb{P}_t^{\varphi, \pm} := N_t^\varphi \cdot \mathbb{Q}_t^\pm \quad \forall t \geq 0.$$

Note that, by using Girsanov Theorem, on the space  $(\Omega, \mathcal{F}, \mathbb{P}^{\varphi, \pm})$ ,

$$(3) \quad X_t = \beta_t^\pm \pm L_t + \int_0^t \varphi(X_s, L_s^{X_s}) ds,$$

with  $\beta_t^\pm$  a Brownian motion.

Set  $\tilde{\mathbb{P}}^\varphi := \mathbb{P}^{\varphi, +} \otimes \mathbb{P}^{\varphi, -}$  and let  $(X^1, X^2)$  be the canonical process of law  $\tilde{\mathbb{P}}^\varphi$ . Then  $X^1$  and  $X^2$  are independent and respectively distributed like  $\mathbb{P}^{\varphi, +}$  and  $\mathbb{P}^{\varphi, -}$ .

Denote by  $L^{(1)}$  and  $L^{(2)}$  the local time processes in 0 of  $X^1$  and  $X^2$ , and define their right continuous inverses  $\tau^1$  and  $\tau^2$  by

$$\tau_s^i = \inf\{t \mid L_t^{(i)} > s\} \quad \text{for } i \in \{1, 2\}.$$

Denote by  $e^1$  and  $e^2$  their excursion processes out of 0: for  $s \leq L_\infty^{(i)}$ ,

$$e_s^i(u) = X_{\tau_{s-}^i + u}^i \quad \text{for all } u \in (0, \tau_s^i - \tau_{s-}^i),$$

if  $\tau_s^i - \tau_{s-}^i > 0$ , and  $e_s^i = 0$  otherwise. Let now  $e$  be the excursion process obtained by adding  $e^1$  and  $e^2$ .

Denote by  $\Xi$  the measurable transformation that reconstructs a process out of its excursion process (see [RY] Proposition (2.5) p.482). Note that  $\Xi$  is not a one to one map, it is only surjective (think of processes having an infinite excursion out of 0, in which case  $L_\infty^0 < \infty$ ).

**Proposition 3.1.** *The following hold*

- (i) *The law of the process  $\Xi e$  is  $\mathbb{P}$ .*
- (ii)  $((\Xi e)_t^+, t \leq L_\infty^{(2)}) = (X_t^1, t \leq L_\infty^{(2)})$ .
- (iii)  $((\Xi e)_t^-, t \leq L_\infty^{(1)}) = (X_t^2, t \leq L_\infty^{(1)})$ .
- (iv) *Denote by  $L$  the local time in 0 of  $\Xi e$ . Then for all  $t \in \mathbb{R}^+ \cup \{\infty\}$ ,  $L_t = L_t^{(1)} \wedge L_t^{(2)}$ .*

*Proof.* Assume first that  $\varphi(x, l) = 0$  if  $x \in (-c, c)$ , for some constant  $c > 0$ . For any  $\epsilon \in (-c, c)$ , define a process  $X^\epsilon$  as follows: set  $T_0^\epsilon = 0$  and for  $n \geq 1$ ,

$$S_n^\epsilon = \inf\{t \geq T_{n-1}^\epsilon \mid X_t \in \{-\epsilon, \epsilon\}\},$$

$$T_n^\epsilon = \inf\{t \geq S_n^\epsilon \mid X_t = 0\}.$$

Define also

$$A_\epsilon(t) = \sum_{n \geq 1} (T_n^\epsilon \wedge t - S_n^\epsilon \wedge t),$$

and let  $\kappa_\epsilon(t)$  be the right-continuous inverse of  $A_\epsilon^+$ . Then set

$$X_t^\epsilon := X_{\kappa_\epsilon(t)}.$$

Now observe that during each time-interval  $(S_n^\epsilon, T_n^\epsilon)$ , the local time in 0 of  $X$  cannot increase. And for  $t \in (A_\epsilon(S_n^\epsilon), A_\epsilon(T_n^\epsilon))$ ,  $\kappa_\epsilon(t) = S_n^\epsilon + (t - A_\epsilon(S_n^\epsilon))$ . So by (3), during the intervals  $(A_\epsilon(S_n^\epsilon), A_\epsilon(T_n^\epsilon))$ , if  $X$  follows the law of  $\Xi e$ , then  $X^\epsilon$  is solution of the SDE

$$dX_t^\epsilon = dR_t^\epsilon + \varphi\left(X_t^\epsilon, L_{\kappa_\epsilon(t)}^{X_t^\epsilon}\right) dt,$$

where  $L$  is the local time of  $X$ , and  $R^\epsilon$  is the Brownian motion defined by

$$R_t^\epsilon = \sum_{k=1}^{n-1} (B_{T_k^\epsilon} - B_{S_k^\epsilon}) + (B_{\kappa_\epsilon(t)} - B_{S_n^\epsilon})$$

for  $t \in (A_\epsilon(S_n^\epsilon), A_\epsilon(T_n^\epsilon))$ .

Denote by  $L^{\epsilon, \cdot}$  the local time process of  $X^\epsilon$ . Then

$$L_{\kappa_\epsilon(t)}^x = L_t^{\epsilon, x} \quad \forall t \geq 0 \quad \forall x \notin (-c, c).$$

Since  $\varphi(x, l) = 0$  when  $x \in (-c, c)$ ,

$$\varphi(x, L_{\kappa_\epsilon(t)}^x) = \varphi(x, L_t^{\epsilon, x}) \quad \forall t \geq 0 \quad \forall x \in \mathbb{R}.$$

Thus  $X^\epsilon$  satisfies in fact the SDE:

$$(4) \quad dX_t^\epsilon = dR_t^\epsilon + \varphi\left(X_t^\epsilon, L_t^{\epsilon, X_t^\epsilon}\right) dt,$$

up to the first time it hits 0. And when it hits 0, it jumps instantaneously to  $\epsilon$  or to  $-\epsilon$  with probability 1/2, independently of its past trajectory. Note that this determines the law of  $X^\epsilon$ .

But it follows also from (1), that if  $X$  has law  $\mathbb{P}$ , then  $X^\epsilon$  solves as well the SDE (4) up to the first time it hits 0, and then jump to  $\epsilon$  or  $-\epsilon$  with probability 1/2 (since there is no drift in  $(-c, c)$ ). Since moreover  $X^\epsilon$  converges to  $X$ , when  $\epsilon$  goes to 0, we conclude that the law of  $\Xi e$  is  $\mathbb{P}$ .

To finish the proof of (i), for  $c > 0$ , define  $\varphi_c$  by

$$\varphi_c(x, l) = \varphi(x, l)1(x \notin [-c, c]).$$

By Lemma 2.3 we know that  $\mathbb{P}_0^{\varphi_c}$  converges toward  $\mathbb{P}_0^\varphi$ , when  $c \rightarrow 0$ . It can be also seen that  $\mathbb{P}^{\varphi_c, +}$  and  $\mathbb{P}^{\varphi_c, -}$  converge respectively towards  $\mathbb{P}^{\varphi, +}$  and  $\mathbb{P}^{\varphi, -}$ . Since  $\Xi e$  is a measurable transformation of  $(X^1, X^2)$ , we can conclude.

Assertions (ii), (iii) and (iv) are immediate: in the construction of  $\Xi e$ , one needs only to know  $e_s$  for  $s \leq L_\infty = L_\infty^{(1)} \wedge L_\infty^{(2)}$ . So  $(\Xi e)^+$  and  $(\Xi e)^-$  can be respectively reconstructed with the positive and negative excursions of  $(e_s, s \leq L_\infty^{(1)} \wedge L_\infty^{(2)})$ .  $\square$

#### 4. RECURRENCE, TRANSCIENCE AND A 0 – 1 LAW

Let  $\varphi \in \Lambda$ . Consider the events  $R_a := \{L_\infty^a = +\infty\}$ ,  $a \in \mathbb{R}$ . Using conditional Borel-Cantelli lemma, one can prove that for all  $x, y, z$ , and all  $\varphi \in \Lambda$ ,  $\mathbb{P}_x^\varphi$ -a.s.,  $R_y = R_z$ . In the following, we will denote by  $R$  the event of recurrence ( $= R_a$  for all  $a$ ).

We will first study the question of recurrence and transience for the processes  $X^1$  and  $X^2$  separately, where  $X^1$  and  $X^2$  are independent respectively of law  $\mathbb{P}^{\varphi, +}$  and  $\mathbb{P}^{\varphi, -}$ . Note that we still have for all  $x \geq 0$  (resp.  $x \leq 0$ ), and all  $\varphi \in \Lambda$ ,  $\mathbb{P}^{\varphi, +}$ -a.s. (resp.  $\mathbb{P}^{\varphi, -}$ -a.s.),  $R_x = R_0 (= R)$ . So in all cases,  $R$  is the event  $\{L_\infty^0 = \infty\}$ .

Fix  $x \in \mathbb{R}$  and denote by  $\varphi_x$  the function in  $\Lambda$  such that for all  $(y, l) \in \mathbb{R} \times \mathbb{R}^+$ ,

$$\varphi_x(y, l) = \varphi(x + y, l).$$

In the following,  $\mathbb{P}_x^{\varphi, \pm}$  will denote  $\mathbb{P}^{\varphi_x, \pm}$ .

**Proposition 4.1.** *For all  $x \in \mathbb{R}$  and all  $\varphi \in \Lambda$ , the following holds*

$$(5) \quad \mathbb{P}_x^\varphi(R) = \mathbb{P}_x^{\varphi, +}(R) \times \mathbb{P}_x^{\varphi, -}(R).$$

*Proof.* This is a straightforward application of Proposition 3.1 since

$$\begin{aligned} \mathbb{P}_x^\varphi(R) &= \tilde{\mathbb{P}}^{\varphi_x}(L_\infty = \infty) \\ &= \tilde{\mathbb{P}}^{\varphi_x}(L_\infty^{(1)} = L_\infty^{(2)} = \infty). \end{aligned}$$

and we conclude since  $L_\infty^{(1)}$  and  $L_\infty^{(2)}$  are independent.  $\square$

For  $t > 0$  and  $a \in \mathbb{R}$ , set  $\sigma_t^a = \inf\{s > 0 \mid \int_0^s 1_{\{X_u \geq a\}} du > t\}$ . Then we have

**Lemma 4.2.** *Let  $a \in \mathbb{R}$ ,  $x, y \leq a$  and  $\varphi, \psi \in \Lambda$  be such that  $\varphi(z, l) = \psi(z, l)$  for all  $z \geq a$  and all  $l \geq 0$ . Then  $(X_{\sigma_t^a} - a, t \geq 0)$  has the same distribution under  $\mathbb{P}_x^{\varphi, +}$  and under  $\mathbb{P}_y^{\psi, +}$ . In particular,*

$$\mathbb{P}_x^{\varphi, +}(L_\infty^a = \infty) = \mathbb{P}_y^{\psi, +}(L_\infty^a = \infty).$$

*Proof.* The lemma follows immediately from Proposition 3.1.  $\square$

Note that a similar proposition holds for  $\mathbb{P}^{\varphi,-}$ . A direct consequence of this lemma is that

**Proposition 4.3.** *For all  $x \in \mathbb{R}$ ,  $\mathbb{P}_x^{\varphi,\pm}(R) = \mathbb{P}^{\varphi,\pm}(R)$ .*

With this in hands, we can prove the

**Proposition 4.4** (Zero-one law). *Let  $\varphi \in \Lambda$ . Then  $\mathbb{P}^{\varphi,+}(R)$  and  $\mathbb{P}^{\varphi,-}(R)$  both belong to  $\{0, 1\}$ .*

*Proof.* Assume that  $\mathbb{P}^{\varphi,+}(R) > 0$ . By martingale convergence theorem,  $\mathbb{P}^{\varphi,+}$ -a.s.

$$\begin{aligned} 1_R &= \lim_{z \rightarrow +\infty} \mathbb{P}_x^{\varphi,+}(R \mid \mathcal{F}_{T_z}) \\ &= \lim_{z \rightarrow +\infty} \mathbb{P}_z^{\varphi T_z,+}(R) \\ &= \mathbb{P}^{\varphi,+}(R), \end{aligned}$$

by application of Lemma 4.2 and Proposition 4.3. Thus  $\mathbb{P}^{\varphi,+}(R) = 1$ , which proves the proposition.  $\square$

Denote by  $T$  the event of transience. Then  $T = R^c = \{L_\infty^0 < \infty\}$ . Note that  $\mathbb{P}^{\varphi,\pm}$ -a.s.,  $T = \{X \rightarrow \pm\infty\}$ , and that  $\mathbb{P}_x^\varphi$ -a.s.,  $T = \{X \rightarrow +\infty\} \cup \{X \rightarrow -\infty\}$  and that  $\tilde{\mathbb{P}}^\varphi$ -a.s.,  $\{\exists e \text{ is transient}\} = \{X^1 \rightarrow \infty\} \cup \{X^2 \rightarrow -\infty\}$ . Then we have the criterion

**Proposition 4.5.** *Let  $\varphi \in \Lambda$ .*

- (i)  $\mathbb{P}_x^\varphi(R) = 1$  for all  $x$  if, and only if,  $\mathbb{P}^{\varphi,+}(R) = \mathbb{P}^{\varphi,-}(R) = 1$ .
- (ii)  $\mathbb{P}_x^\varphi(R) = 0$  for all  $x$  if, and only if,  $\mathbb{P}^{\varphi,+}(R) = 0$  or  $\mathbb{P}^{\varphi,-}(R) = 0$ .

*Proof.* This is a straightforward consequence of Propositions 4.1, 4.3 and 4.4.  $\square$

## 5. CASE $\varphi \geq 0$

In this section, we will consider functions belonging to  $\Lambda^+$ . In this case, it is obvious that  $\mathbb{P}^{\varphi,-}(R) = 1$ . Thus the recurrence property only depends on  $\mathbb{P}^{\varphi,+}(R)$ .

Set  $T_a = \inf\{t > 0 \mid X_t = a\}$ .

**Lemma 5.1.** *Let  $\varphi \in \Lambda^+$ , for which there exists  $x_0$  such that for all  $x \leq x_0$  and all  $l \geq 0$ ,  $\varphi(x, l) = \varphi(x_0, l)$ , and where  $\varphi(x_0, l)$  is non zero function of  $l$ . Then, for all  $a \geq x$ ,*

$$\mathbb{E}_x^\varphi(D_{T_a}) = a - x.$$

*Proof.* Without losing generality, we prove this result for  $x = 0$ . Take  $a > 0$ . We have for all  $n$ ,

$$0 = \mathbb{E}_0^\varphi(B_{T_a \wedge n}) = \mathbb{E}_0^\varphi(X_{T_a \wedge n}) - \mathbb{E}_0^\varphi(D_{T_a \wedge n}).$$

By monotone convergence,  $\lim_{n \rightarrow \infty} \mathbb{E}_0^\varphi(D_{T_a \wedge n}) = \mathbb{E}_0^\varphi(D_{T_a})$ . Thus, if we can prove that  $\lim_{n \rightarrow \infty} \mathbb{E}_0^\varphi(X_{T_a \wedge n}) = a$ , the lemma will follow. Since  $\mathbb{P}_0^\varphi$ -a.s.,  $X_{T_a} = a$  and  $T_a < \infty$ , this is equivalent to prove that

$$\lim_{n \rightarrow \infty} \mathbb{E}_0^\varphi[X_n 1_{\{n < T_a\}}] = 0.$$

For all  $n$ ,

$$\min_{t < T_a} X_t \leq X_n 1_{\{n < T_a\}} \leq a \quad \mathbb{P}_0^\varphi - a.s.$$



If one proves that  $\min_{t < T_a} X_t$  is integrable, we can conclude by dominated convergence. One has for all  $c > 0$ ,

$$\begin{aligned} \mathbb{E}_0^\varphi \left[ -\min_{t < T_a} X_t \right] &\leq \mathbb{E}_0^\varphi [D_{T_a}/c] \\ &\quad + \sum_{i \geq 1} i \times \mathbb{P}_0^\varphi \left[ -\min_{t < T_a} X_t \in [i-1, i], D_{T_a} < ci \right]. \end{aligned}$$

Clearly  $\mathbb{E}_0^\varphi [D_{T_a}/c] \leq a/c < +\infty$ . Moreover for all  $i \geq 1$ ,

$$(6) \quad \mathbb{P}_0^\varphi \left[ -\min_{t < T_a} X_t \in [i-1, i], D_{T_a} < ci \right] \leq \mathbb{P}_0^\varphi [T_{-i} < T_a, D_{T_a} < ci].$$

Now on the event  $\{T_{-i} < T_a\}$ ,

$$D_{T_a} \geq \sum_{k=1}^i \left( \int_{-k}^{-k+1} h(y, L_{T_{-k}}^y) dy \right) \mathbf{1}_{\{T_{-k} < \infty\}}.$$

Set

$$\alpha_k := \left( \int_{-k}^{-k+1} h(y, L_{T_{-k}}^y) dy \right) \mathbf{1}_{\{T_{-k} < \infty\}}.$$

We will prove that there exists a constant  $\alpha > 0$ , such that for all integer  $k$  greater than  $-x_0$ ,

$$(7) \quad \mathbb{E}_0^\varphi [\alpha_{k+1} \mid \mathcal{F}_{T_{-k}}] \geq \alpha \mathbf{1}_{\{T_{-k} < +\infty\}}.$$

By (2) we have

$$\begin{aligned} \mathbb{E}_0^\varphi [\alpha_{k+1} \mid \mathcal{F}_{T_{-k}}] &= \mathbb{E}_{-k}^{\varphi^{T_{-k}}} [\alpha_{k+1}] \mathbf{1}_{\{T_{-k} < \infty\}} \\ &= \mathbb{E}_0^{\psi_k} [\alpha_1] \mathbf{1}_{\{T_{-k} < \infty\}} \end{aligned}$$

with  $\psi_k(x, l) = \varphi(x_0, l)$  for  $x < 0$  and  $\psi_k(x, l) = \varphi(x - k, L_{T_{-k}}^{x-k})$  for  $x \geq 0$ . Note that by Proposition 3.1,  $\Xi e$  will reach level  $-1$  in finite time if, and only if,

$$L_\infty^{(1)} \geq L_{T_{-1}}^{(2)},$$

where  $T_{-1}$  denotes also the hitting time of  $-1$  for  $X^2$ . Thus

$$\mathbb{E}_0^{\psi_k} [\alpha_1] = \tilde{\mathbb{E}}^{\psi_k} \left[ F(X_t^2, t \leq T_{-1}) \mathbf{1}_{\{L_\infty^{(1)} \geq L_{T_{-1}}^{(2)}\}} \right],$$

with

$$F(X_t^2, t \leq T_{-1}) = \int_{-1}^0 h(y, L_{T_{-1}}^{(2),y}) dy,$$

where  $L^{(2),y}$  is the local time in  $y$  of  $X^2$ . It is possible to couple  $X^1$  with a Brownian motion  $Y^+$  reflected at 0, started at 0, with drift  $\|\varphi\|_\infty$ , and such that  $X_t^1 \leq Y_t^+$  for all  $t \geq 0$ . Note that the law of  $Y^+$  is  $\mathbb{P}^{\psi,+}$ , where  $\psi(x, l) = \|\varphi\|_\infty$  if  $x \geq 0$  and  $\psi(x, l) = \varphi(x_0, l)$  if  $x < 0$ . Moreover,

$$L_\infty^{(1)} \geq L_\infty^+,$$

with  $L^+$  the local time in 0 of  $Y^+$ . These properties imply that

$$\mathbb{E}_0^{\psi_k} [\alpha_1] \geq \alpha := \mathbb{E}_0^\psi [\alpha_1].$$

It remains to see that  $\alpha$  is positive. For any  $t > 0$ , it is larger than

$$\mathbb{E}_0^\psi [\alpha_1 \mathbf{1}_{\{T_{-1} \leq t\}}].$$

By using Girsanov's transform it suffices to prove that for some  $t > 0$ ,

$$\mathbb{Q} [\alpha_1 1_{\{T_{-1} \leq t\}}] > 0.$$

But

$$\mathbb{Q} [\alpha_1 1_{\{T_{-1} \leq t\}}] = \int_{-1}^0 \mathbb{Q} \left[ h(y, L_{T_{-1}}^y) 1_{\{T_{-1} \leq t\}} \right] dy.$$

By continuity of  $h$  and  $L$  it suffices to prove that for some  $y \in (-1, 0)$  and  $t > 0$ ,

$$\mathbb{Q} \left[ h(y, L_{T_{-1}}^y) 1_{\{T_{-1} \leq t\}} \right] > 0.$$

But this is clear for  $y = 1/2$  for instance, since for any  $M > 0$ ,  $\mathbb{Q}[L_{T_{-1}}^{1/2} > M] > 0$ .

Next (7) shows that the process  $(M_i, i \geq 0)$  defined by

$$M_i = \sum_{k=1}^i (\alpha_k - \alpha 1_{\{T_{-k} < +\infty\}}),$$

is a sub-martingale with respect to the filtration  $(\mathcal{F}_{T_{-i}})_{i \geq 1}$ . Moreover on the event  $\{T_{-i} < T_a\}$ , one has

$$M_i = \sum_{k=1}^i (\alpha_k - \alpha).$$

So by taking  $c = \alpha/2$  in (6) one gets

$$\mathbb{P} [T_{-i} < T_a, D_{T_a} < c(i+1)] \leq \mathbb{P}[M_i \leq -\alpha i/2],$$

for all  $i \geq 0$ . We conclude now that  $\mathbb{E}[-\min_{t < T_a} X_t]$  is finite by using standard results on martingales.  $\square$

**Remark 5.2.** As noticed also by Zerner in [Zer1], the condition  $\varphi(x_0, \cdot) \neq 0$  is necessary (the result being obviously false if  $\varphi = 0$ ).

**Remark 5.3.** This lemma can be extended to the case where  $\varphi$  is random and stationary in the sense that for all  $z$ ,  $\varphi^z$  and  $\varphi$  have the same law, where  $\varphi^z$  is defined by  $\varphi^z(x, l) = \varphi(x+z, l)$ .

**Lemma 5.4.** *Let  $\varphi \in \Lambda_c^+$ . Then*

$$\mathbb{E}_0^\varphi(D_\infty^k) \leq 1,$$

for all  $k \geq 0$ .

*Proof.* It is the same proof than for Lemma 11 in Zerner [Zer1]. We reproduce it here for completeness. Note first that  $\mathbb{E}_0^\varphi[D_\infty^k] = \mathbb{P}_k^{\varphi,+}(D_\infty^0)$ , which does not depend on  $k$  since  $\varphi \in \Lambda_c$ . For  $K \geq 1$  and  $i \leq K-1$ ,

$$D_{T_K} \geq D_{T_K}^+ = \sum_{j=0}^{K-1} D_{T_K}^j \geq \sum_{j=0}^{K-1-i} D_{T_K}^j \geq \sum_{j=0}^{K-1-i} D_{T_{j+i}}^j.$$

Using Lemma 5.1,

$$K = \mathbb{E}_0^\varphi[D_{T_K}] \geq \sum_{j=0}^{K-1-i} \mathbb{E}_0^\varphi[D_{T_{j+i}}^j] \geq (K-i) \mathbb{E}_0^\varphi[D_{T_i}^0].$$

Letting  $i \rightarrow \infty$ , we conclude.  $\square$

**Lemma 5.5.** *Let  $\varphi \in \Lambda^+$  be such that  $\mathbb{P}^{\varphi,+}(R) = 0$ . Then*

$$\lim_{z \rightarrow \infty} \mathbb{E}_0^\varphi(D_{T_z}^+)/z = 1.$$

*If moreover  $\varphi \in \Lambda_c^+$ , then*

$$\mathbb{E}_0^\varphi(D_\infty^0) = 1.$$

*Proof.* The proof of the first part follows the proof of Lemma 6 in Zerner [Zer1]. We write it here for completeness. First, since  $\mathbb{E}^{0,\varphi}(D_{T_z}^+) = \mathbb{P}^{\varphi,+}(D_{T_z})$  and  $\mathbb{P}^{\varphi,+}(R)$  is a function of  $(\varphi(x, \cdot))_{x \geq 0}$ , we can assume that  $\varphi(x, l) = 1$  for all  $x \leq 0$  and all  $l \geq 0$ . Thus Lemma 5.1 can be applied:  $\mathbb{E}_0^\varphi(D_{T_z}) = z$ . So, it suffices to prove that  $\lim_{z \rightarrow \infty} \mathbb{E}_0^\varphi(D_{T_z}^-)/z = 0$ .

For  $i \geq 1$ , let  $\sigma_i = \inf\{j \geq T_i \mid X_j = 0\}$ . We have, for  $z$  an integer,

$$\mathbb{E}_0^\varphi[D_{T_z}^-] = \sum_{i=0}^{z-1} \mathbb{E}_0^\varphi[D_{T_{i+1}}^- - D_{T_i}^-].$$

Note that

$$\begin{aligned} \mathbb{E}_0^\varphi[D_{T_{i+1}}^- - D_{T_i}^-] &= \mathbb{E}_0^\varphi \left[ 1_{\{\sigma_i < T_{i+1}\}} (D_{T_{i+1}}^- - D_{T_i}^-) \right] \\ &= \mathbb{E}_0^\varphi \left[ 1_{\{\sigma_i < T_{i+1}\}} \mathbb{E}_0^{\varphi_{\sigma_i}}(D_{T_{i+1}}^-) \right] \\ &\leq (i+1) \mathbb{P}_0^\varphi(\sigma_i < T_{i+1}) \end{aligned}$$

using again Lemma 5.1. Thus it remains to prove that

$$\lim_{z \rightarrow \infty} \frac{1}{z} \sum_{i=1}^{z+1} i \mathbb{P}_0^\varphi(\sigma_i < T_{i+1}) = 0.$$

Let  $Y_i = \mathbb{P}_0^\varphi[\sigma_i < T_{i+1} \mid \mathcal{F}_{T_i}]$ . Since  $\mathbb{P}_0^\varphi(R) = 0$ , the conditional Borel-Cantelli lemma implies that  $\mathbb{P}_0^\varphi$ -a.s.,  $\sum_i Y_i < +\infty$ . Since  $Y_i \leq 1/i$  ( $X$  being greater than a Brownian motion), for all positive  $\epsilon$ ,

$$i \times \mathbb{P}_0^\varphi(\sigma_i < T_{i+1}) \leq \epsilon + \mathbb{P}_0^\varphi(Y_i \geq \epsilon/i).$$

This implies that

$$\frac{1}{z} \sum_{i=1}^{z+1} i \mathbb{P}_0^\varphi(\sigma_i < T_{i+1}) \leq \epsilon + \frac{1}{z} \mathbb{E}_0^\varphi \left[ \sum_{i=1}^{z+1} 1_{\{Y_i \geq \epsilon/i\}} \right].$$

But since  $\sum_i Y_i < \infty$  a.s., the density of the  $i \leq z$  such that  $Y_i \leq \epsilon/i$  tends to 0 when  $z$  tends to  $\infty$ . Thus the preceding sum converges to 0. This concludes the proof of the first part.

The second part is immediate (see Zerner [Zer1] Theorem 12). Since  $\varphi \in \Lambda_c^+$ , for all  $K \geq 0$ ,

$$\mathbb{E}_0^\varphi[D_\infty^0] = \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}_0^\varphi[D_\infty^k] \geq \frac{1}{K} \mathbb{E}_0^\varphi[D_{T_K}^+].$$

We conclude using the first part of the lemma.  $\square$

The next result gives a sufficient criterion for recurrence, when we only know that  $\varphi \in \Lambda^+$ . For  $\varphi \in \Lambda_c^+$ , we will obtain a necessary and sufficient condition (see Theorem 5.10 below).

**Corollary 5.6.** Let  $\varphi \in \Lambda^+$ . For  $x \in \mathbb{R}$ , set  $\delta^x(\varphi) = \int_0^\infty \varphi(x, u) du$ . If

$$\liminf_{z \rightarrow +\infty} \frac{1}{z} \int_0^z \delta^x(\varphi) dx < 1,$$

then  $\mathbf{P}^{\varphi,+}(R) = 1$ .

*Proof.* Since P-a.s.  $D_{T_z}^+ \leq \int_0^z \delta^x(\varphi) dx$ , if  $\liminf \frac{1}{z} \int_0^z \delta^x(\varphi) dx < 1$ , then

$$\liminf \mathbf{E}_0^\varphi(D_{T_z}^+)/z < 1.$$

We conclude by using Lemma 5.5.  $\square$

**Lemma 5.7.** Let  $\varphi \in \Lambda^+$  be such that  $\mathbf{P}^{\varphi,+}(R) = 1$ . Then

$$\mathbf{P}_0^\varphi(T_{-1} = +\infty) > 0.$$

*Proof.* By using Proposition 3.1,

$$\mathbf{P}_0^\varphi(T_{-1} = +\infty) = \tilde{\mathbf{P}}^\varphi(L_\infty^{(1)} < L_{T_{-1}}^{(2)}),$$

where  $T_{-1}$  denotes also the hitting time of  $-1$  for  $X^2$ . Since  $X^1$  and  $X^2$  are independent and since  $\mathbf{P}^{\varphi,+}(R) = 0$  implies that  $\tilde{\mathbf{P}}^\varphi$ -a.s.,  $L_\infty^{(1)} < +\infty$ , it suffices to prove that for any  $l > 0$ ,

$$\mathbf{P}^{\varphi,-}(L_{T_{-1}}^0 > l) > 0.$$

Equivalently it suffices to prove that for any  $l > 0$ , there exists  $t > 0$  such that

$$\mathbf{P}^{\varphi,-}(L_{T_{-1}}^0 > l \text{ and } T_{-1} \leq t) > 0.$$

By absolute continuity of  $\mathbf{P}^{\varphi,-}_{|\mathcal{F}_t}$  and  $\mathbf{Q}^-_{|\mathcal{F}_t}$  this is equivalent to

$$\mathbf{Q}^-(L_{T_{-1}} > l \text{ and } T_{-1} \leq t) > 0.$$

But this is well known. Thus the lemma is proved.  $\square$

**Lemma 5.8.** Let  $\varphi \in \Lambda^+$  be such that  $\mathbf{P}^{\varphi,+}(R) = 0$ . Then for any  $M > 0$ ,

$$\mathbf{P}_0^\varphi[L_\infty^0 < M] > 0.$$

*Proof.* We have

$$\begin{aligned} \mathbf{P}_0^\varphi[L_\infty^0 < M] &\geq \mathbf{P}_0^\varphi[L_{T_1}^0 < M \text{ and } X_t > 0 \quad \forall t > T_1] \\ &\geq \mathbf{E}_0^\varphi \left[ 1_{\{L_{T_1}^0 < M\}} \mathbf{P}_1^{\varphi T_1}(T_0 = \infty) \right]. \end{aligned}$$

But Lemma 5.7 implies that a.s.,  $\mathbf{P}_1^{\varphi T_1}(T_0 = \infty) > 0$ . So it remains to prove that  $\mathbf{P}_0^\varphi(L_{T_1}^0 < M) > 0$ . Like in the previous lemma, by absolute continuity, it suffices to prove that

$$\mathbf{Q}(L_{T_1}^0 < M) > 0.$$

But again this is well known. Thus the lemma is proved.  $\square$

**Remark 5.9.** This lemma holds as well for any  $\varphi \in \Lambda$ , such that  $\mathbf{P}_0^\varphi(R) = 0$ .

Finally we obtain the

**Theorem 5.10.** Let  $\varphi \in \Lambda_c^+$ . Then

$$\mathbf{P}^{\varphi,+}(R) = 1 \iff \int_0^\infty \varphi(0, u) du \leq 1.$$

*Proof.* We prove this for  $\varphi$  such that  $\varphi(x, l) = \varphi(0, l)$  for all  $x$ . By Lemma 5.4,  $\mathbf{E}_0^\varphi(D_\infty^0) \leq 1$ . But if  $\mathbf{P}^{\varphi,+}(R) = 1$ , then  $\mathbf{P}_0^\varphi$ -a.s.  $L_\infty^x = +\infty$ , for all  $x$ . So, by using the occupation time formula (see Lemma 2.2) we have  $\mathbf{E}_0^\varphi(D_\infty^0) = \int_0^\infty \varphi(0, u) du$ . This gives the necessary condition. Reciprocally, if  $\mathbf{P}^{\varphi,+}(R) = 0$ , we saw in Lemma 5.5 that  $\mathbf{E}_0^\varphi(D_\infty^0) = 1$ . But by Lemma 5.8, we have  $\mathbf{E}_0^\varphi(D_\infty^0) = \mathbf{E}_0^\varphi(h(0, L_\infty^0)) < \int_0^\infty \varphi(0, u) du$ , which gives the sufficient condition and concludes the proof of the theorem.  $\square$

Note that if  $\varphi \in \Lambda$  is such that for some  $a \in \mathbb{R}$ ,  $\varphi(x, l) = \varphi(a, l) \geq 0$  for all  $x \geq a$  and all  $l \geq 0$ , then

$$\mathbf{P}^{\varphi,+}(R) = 1 \iff \int_0^\infty \varphi(a, u) du \leq 1.$$

This can be proved using the fact that  $\mathbf{P}^{\varphi,+}(R) = \mathbf{P}^{\varphi_a,+}(R)$  which does not depend on  $\varphi(x, \cdot)$ , for  $x < a$ .

## 6. ADDITIONAL REMARKS FOR GENERAL $\varphi$

Our first remark concerns the question of transience. More precisely we could ask if it would be possible to have  $\mathbf{P}(X \rightarrow +\infty) = 1 - \mathbf{P}(X \rightarrow -\infty) \in (0, 1)$ . The answer is yes. For instance take  $\varphi \in \Lambda$  such that  $\mathbf{P}^{\varphi,+}(R) = 0$ , i.e. one has  $\mathbf{P}^{\varphi,+}$ -a.s.  $X \rightarrow +\infty$ . Then define  $\psi \in \Lambda$  by  $\psi(x, u) = \varphi(x, u)$  if  $x \geq 0$  and  $\psi(x, u) = -\varphi(x, u)$  if  $x < 0$ . Then by symmetry we have

$$\mathbf{P}_0^\psi(X \rightarrow +\infty) = \mathbf{P}_0^\psi(X \rightarrow -\infty) = 1/2.$$

More generally,

$$\mathbf{P}_0^\varphi(X \rightarrow +\infty) = \tilde{\mathbf{P}}^\varphi(L_\infty^1 < L_\infty^2),$$

and

$$\mathbf{P}_0^\varphi(X \rightarrow -\infty) = \tilde{\mathbf{P}}^\varphi(L_\infty^1 > L_\infty^2).$$

So, if  $L_\infty^1$  and  $L_\infty^2$  are finite random variables, since they are independent with support  $\mathbb{R}^+$ , these two probabilities are positive. Note that the previous section gives a criterion for these random variables to be finite when  $\varphi$  is nonnegative or nonpositive at infinity.

To summarize:  $\mathbf{P}_0^\varphi(R) = 1$  if there exists  $a > 0$  such that

- $\varphi(x, l) \leq 0$  for  $x \geq a$  and  $\varphi(x, l) \geq 0$  for  $x \leq -a$  or that;
- $\varphi(x, l) = \varphi(a, l) \geq 0$  for  $x \geq a$  and  $\varphi(x, l) \geq 0$  for  $x \leq -a$ , and  $\int_0^\infty \varphi(a, u) du \leq 1$ , or that;
- $\varphi(x, l) = \varphi(a, l) \geq 0$  for  $x \geq a$ ,  $\varphi(x, l) = \varphi(-a, l) \leq 0$  for  $x \leq -a$ ,  $\int_0^\infty \varphi(a, u) du \leq 1$  and  $\int_0^\infty \varphi(-a, u) du \geq -1$  or that;
- $\varphi(x, l) \leq 0$  for  $x \leq -a$  and  $\varphi(x, l) = \varphi(-a, l) \leq 0$  for  $x \leq -a$  and  $\int_0^\infty \varphi(0, u) du \geq -1$ .

Similarly  $\mathbf{P}_0^\varphi(R) = 0$  if there exists  $a > 0$  such that

- $\varphi(x, l) = \varphi(a, l) \geq 0$  for  $x \geq a$  with  $\int_0^\infty \varphi(a, u) du > 1$  or that;
- $\varphi(x, l) = \varphi(-a, l) \leq 0$  for  $x \leq -a$  with  $\int_0^\infty \varphi(-a, u) du < -1$ .

Our second remark gives a general necessary condition for recurrence.

**Proposition 6.1.** *Let  $\varphi \in \Lambda_c$ . If*

$$(8) \quad \int_0^{+\infty} |\varphi(0, u)| du < +\infty,$$

then

$$P_0^\varphi(R) = 1 \implies \int_0^\infty \varphi(0, u) \, du \in [-1, 1].$$

*Proof.* Let

$$D_t^{(1)} := \int_0^t \varphi \left( X_s^1, L_s^{(1), X_s^{(1)}} \right) \, ds = \int_0^\infty h(x, L_t^{(1), x}) \, dx,$$

be the total drift accumulated by  $X^{(1)}$  up to time  $t$ . By dominated convergence theorem, that we can apply by using (8), one get for all  $z \geq 0$ ,

$$z = \tilde{E}^\varphi \left[ X_{T_z}^1 \right] = \tilde{E}^\varphi \left[ L_{T_z}^{(1)} \right] + \tilde{E}^\varphi \left[ D_{T_z}^{(1)} \right].$$

So for all  $z \geq 0$ ,

$$\tilde{E}^\varphi \left[ D_{T_z}^{(1)} \right] \leq z.$$

Since  $\varphi \in \Lambda_c$ , this implies

$$\tilde{E}^\varphi \left[ D_\infty^{(1), 0} \right] \leq 1,$$

with evident notation. But if  $X$  is recurrent, then  $X^1$  is also recurrent and

$$D_\infty^{(1), 0} = \int_0^\infty \varphi(0, u) \, du,$$

which proves that

$$\int_0^\infty \varphi(0, u) \, du \leq 1.$$

The other inequality is obtained similarly by using the recurrence of  $X^2$ . This concludes the proof of the proposition.  $\square$

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