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On Peres approach to Fradkin-Bacry-Ruegg-Souriau's perihelion vector

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We solve explicitly the differential system obtained by Peres for the construction of a conserved vector associated to any central potential. We then obtain a very direct access to the discontinuous behavior of this Fradkin-Bacry-Ruegg-Souriau perihelion vector.

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I. INTRODUCTION

Since the pioneer works of Ermanno, Bernoulli and Laplace during eighteenth century^{1,2,4} we know that the Kepler system presents a specific vector conserved quantity, the so-called Laplace-Runge-Lenz vector, which provides in particular a very simple access to the orbit equation. Today, the existence of generalized Laplace-Runge-Lenz vectors for various extensions of the Kepler problem have been put in evidence¹.

This supplementary constant of motion has a dynamical origin which is directly linked to the existence, for these systems, of a symmetry group larger than the space-time's one^{5,6,7,8}. As established by Jauch and Hill⁹ in the 2D case and by Fradkin¹⁰ in the 3D case, the isotropic harmonic oscillator shares the same feature and admits also a supplementary dynamical conserved quantity but of tensorial type, the Fradkin-Jauch-Hill tensor^{1,6,11}.

Bacry, Ruegg, Souriau¹² and Fradkin¹³ made an important additional step when they showed that all three dimensional dynamical problems involving central potentials possess the extended symmetry algebra O_4 and SU_3 , a result soon generalized by Mukunda¹⁴ for any hamiltonian system with n degrees of freedom. As noted by Bacry, Ruegg and Souriau in their seminal paper¹², the general ability to construct such a local Lie algebra of dynamical symmetry is not surprising since all the $2n$ -dimensional symplectic manifolds are locally isomorphic¹⁵. However the following step to ensure a global dynamical symmetry is to verify that the finite canonical transformations generated by this algebra form a group, a feature of the Kepler problem. As established by Stehle and Han¹⁶, the existence of such a "global" higher symmetry group, admitting finitely multivalued realizations on the phase space of the system, is a necessary condition for the existence of degeneracies.

Bacry, Ruegg, Souriau¹², as much as Fradkin¹³, gave an explicit construction scheme for a vector with constancy properties associated to every central potential, Fradkin's derivation being very direct and explicit. Bacry, Ruegg and Souriau already noted that such a vector is exceptionally one-valued, as in the Kepler case. The first explicit demonstration of the multivalued behavior of these Fradkin-Bacry-Ruegg-Souriau (FBRs) vectors have been established by Serebrennikov, Shabad¹⁷, Buch and Denman¹⁸ when these authors pointed out that in specific cases Fradkin vectors are in fact only piecewise conserved.

In 1979, Peres¹⁹, using a distinct approach, rediscovered the FBRs generalized Laplace-Runge-Lenz vector. He obtained a differential system for the coefficients of the vectors which presents singularities at the apsidal positions. Without solving explicitly the differential system, Peres inferred from this singular behavior the existence of discontinuous changes of direction of the vector which is then only piecewise constant. The correspondence between Fradkin's and Peres approaches have been studied by Yoshida²⁰.

A complete answer to the question concerning the status of conserved quantity which could be attributed to the FBRs vector has been given by Holas and March²¹. They showed explicitly that at each pericenter (or apocenter, depending on the chosen initial conditions), the FBRs vector changes its direction abruptly from an angle double of the apsidal one. A last attempt to obtain a "true" vector constant of motion in every conservative central force field has been made by Yan²² but, as shown a short time after by Holas and March²³, Yan's construction coincides in fact with the FBRs perihelion vector and then presents the same discontinuities. The generalized Laplace-Runge-Lenz vector can also be deduce in a direct and elegant manner from the equation of motion¹.

Note finally that, in a very recent contribution, Ballesteros and al.²⁴ use FBRs type vectors to establish an optimal extension of Bertrand's theorem^{25,26} to curved spaces.

In this paper, we give an explicit solution to the differential system obtained by Peres¹⁹. Using then a complex formulation, particularly convenient for the treatment of planar problems^{27,28}, we recover in a very simple manner the discontinuous behavior of the FBRs vector studied by Holas and March²¹.

II. BASICS OF 2D MOTION IN A CENTRAL POTENTIAL

1. Complex formulation

We consider a planar motion $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}_{(O, \vec{u}_x, \vec{u}_y)}$ for a particle of mass $m = 1$ submitted to a potential $U(\vec{r})$, eventually singular at the origin. We choose to adopt a complex formulation where we represent the position by its corresponding affix $z(t) = x(t) + iy(t)$, the potential being then viewed as a real valued function of z , $U(z, \bar{z})$, defined on \mathbb{C} or \mathbb{C}^* . The gradient of any real valued function of z and \bar{z} is then given by :

$$\vec{\nabla}U(\vec{r}) \rightarrow 2 \frac{\partial U(z, \bar{z})}{\partial \bar{z}} \quad (1)$$

where $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$.

The equation of motion for our system takes the form:

$$\ddot{z} + 2 \frac{\partial U(z, \bar{z})}{\partial \bar{z}} = 0 \quad (2)$$

where the dot represents the time derivative.

For two vectors \vec{A} and \vec{B} in the $(O, \vec{u}_x, \vec{u}_y)$ plane, $\vec{A} \times \vec{B}$ is represented in complex form by the real quantity $\text{Im}(\bar{A}B)$, A and B be being the complex affixes of \vec{A} and \vec{B} respectively.

The angular momentum (which necessarily conserves its direction orthogonal to the plane) $\vec{L}(t) = \vec{r}(t) \times \dot{\vec{r}}(t) = L(t) \vec{k}$, admits then the following correspondent :

$$L = \text{Im}(\bar{z}(t)\dot{z}(t)) = \frac{1}{2i} (\bar{z}\dot{z} - \dot{\bar{z}}z) \quad (3)$$

In the case of a central potential $U(z, \bar{z}) = U(|z|) = U(r)$, we have:

$$\vec{\nabla}U(r) \rightarrow \frac{z}{r} U'(r) \quad (4)$$

where $f'(x) = \frac{df(x)}{dx}$ is the usual derivative of a function f of one variable x . Eq.(2) becomes simply:

$$\ddot{z} + \frac{z}{r} U'(r) = 0 \quad (5)$$

Using this result, the angular momentum's conservation is immediate (see Eq.(3)):

$$\dot{L} = \frac{1}{2i} (\bar{z}\ddot{z} - \ddot{\bar{z}}z) = 0 \quad (6)$$

2. Radial equation of motion

If we use a polar representation for z , $z = re^{i\theta}$, with $\theta \in [0, 2\pi[$, we have:

$$\widehat{(r^2)} = 2\dot{r}r = \dot{z}\bar{z} + z\dot{\bar{z}} \quad (7)$$

which gives:

$$\dot{r} = \frac{1}{2r} (\dot{z}\bar{z} + z\dot{\bar{z}}) = \frac{1}{r} (z\dot{\bar{z}} + iL) \quad (8)$$

and (since $e^{2i\theta} = \frac{z}{\bar{z}}$) :

$$\dot{\theta}(t) = \frac{\bar{z}\dot{z} - \dot{\bar{z}}z}{2ir^2} = \frac{L}{r^2(t)} \quad (9)$$

Then, we can write:

$$\dot{z} = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} = \left(\dot{r} + i\frac{L}{r}\right) \frac{z}{r} \quad (10)$$

and:

$$\ddot{z} = \left(\ddot{r} - \frac{L^2}{r^4}\right) \frac{z}{r} \quad (11)$$

Then using Eq.(5), we deduce an equation for the radial motion:

$$\ddot{r}(t) = \frac{L^2}{r^3(t)} - U'(r(t)) \quad (12)$$

which is readily integrated as :

$$(\dot{r})^2 = 2 \left(E - \frac{L^2}{2r^2} - U(r) \right) \quad (13)$$

E is a constant of integration, which can be identified with the energy of the system, since from Eq.(10) we have:

$$\frac{1}{2} |\dot{z}|^2 = \frac{1}{2} \left(\frac{\dot{r}}{r} + i\frac{L}{r^2} \right) z \left(\frac{\dot{r}}{r} - i\frac{L}{r^2} \right) \bar{z} = \frac{1}{2} \left((\dot{r})^2 + \frac{L^2}{r^2} \right) = E - U(r) \quad (14)$$

If we note $T(r) = E - U(r)$, Eq.(12) takes the form:

$$(\dot{r}(t))^2 = \frac{L^2}{r^2} f(r(t)), \quad (15)$$

where :

$$f(r) = \frac{2r^2}{L^2} T(r) - 1 = \frac{2r^2}{L^2} (E - V_L(r)), \quad (16)$$

$V_L(r) = U(r) + \frac{L^2}{2r^2}$ being the radial effective potential.

From Eq.(15) and Eq.(16), we see that the radial oscillation motion presents, for bounded orbits, two turning point r_m and r_M , corresponding to the pericenters et apocenters (apsidal positions) of the orbital motion. They are roots of the equation:

$$f(r) = 0 \Leftrightarrow V_L(r) = E \quad (17)$$

In the sequel, we will suppose that r_m and r_M , are simple roots, that is $V_L'(r_m) < 0$ and $V_L'(r_M) < 0$. At the apsidal positions, $\dot{r}_m = \dot{r}_M = 0$ and:

$$\dot{z}_j = i\dot{\theta}_j z_j = i\frac{L}{r_j^2} z_j, \quad j = m, M \quad (18)$$

The i factor in the right hand side of Eq.(18) implies that, for apsidal positions, the instantaneous speed vector is orthogonal to the position vector.

Note that if $z(t)$ and $\dot{z}(t)$ are continuous and differentiable for every values of t , the differentiability domain of $r(t)$ is strictly limited to the real interval $]r_m, r_M[$.

To extract from Eq.(15) the radial celerity $\dot{r}(t)$ as a real-valued function, the square root has to be constructed with a pre-defined sign. To take account of the turn back in the radial motion, this sign changes when r reaches the extremal values r_m and r_M , which corresponds to change the square root determination.

We obtain:

$$\dot{r}(t) = \pm \frac{L}{r(t)} \sqrt{f(r(t))} \quad (19)$$

or:

$$dr(t) = \pm \frac{L}{r} \sqrt{f(r)} dt \quad (20)$$

The $+$ sign is chosen when r moves from r_m to r_M (increasing phase of the radial oscillation, $dr(t) > 0$) and the $-$ sign during the reverse motion (decreasing phase of the radial oscillation, $dr(t) < 0$). To be more precise, we will index each phase of the motion by an integer number $k \in \mathbb{N}$, the even value $k = 2n$ corresponding to the decreasing phase of the n -th period (except for $n = 0$) and the odd value $k = 2n + 1$ corresponding to the increasing phase of the same period. With this convention, we then have:

$$dr(t) = (-1)^{k+1} \frac{L}{r(t)} \sqrt{f(r(t))} dt \quad (21)$$

and:

$$\dot{r}(t) = (-1)^{k+1} \frac{L}{r(t)} \sqrt{f(r(t))} \quad (22)$$

Eq.(21) gives implicitly the solution of the radial equation of motion Eq.(12).

3. Orbital equation and r -parametrization of the motion

In the k -phase of the motion, in every point except the apsidal ones, $\dot{r}(t)$ is a monotonic function (see Eq.(22)) and we can choose to parametrize the motion by r rather than by t .

For instance:

$$\frac{d\theta}{dr(t)} = \frac{\dot{\theta}}{\dot{r}} = (-1)^{k+1} \frac{1}{r \sqrt{f(r)}} \quad (23)$$

where we have used Eq.(22) and Eq.(9).

Choosing a reference time t_0 during the same phase, we can write:

$$\theta(r(t)) = \theta(r_0) + (-1)^{k+1} \int_{r_0}^{r(t)} \frac{d\rho}{\rho \sqrt{f(\rho)}} \quad (24)$$

with $r(t_0) = r_0$ and the orbital equation for this phase writes:

$$\theta(r) = \theta(r_0) + (-1)^{k+1} \int_{r_0}^r \frac{d\rho}{\rho \sqrt{f(\rho)}} \quad (25)$$

We have to be more careful if we want to use r as a global parameter for the motion. Indeed, since $r(t)$ is a periodic function of t , $t(r)$ and every monotonic function of t , as θ (see Eq.(9)), will be multivalued. Therefore, at every k phase of the motion will be attached a different branch of the function.

For simplicity, we choose the initial condition:

$$\begin{cases} r(t=0) = r_m \\ \theta(t=0) = 0 \end{cases} \quad (26)$$

Then, the different determinations of $\theta(r)$ corresponding to each phase of the motion are given by:

$$\text{In the first phase of the motion : } \theta(r(t)) = g(r(t)) \quad (27)$$

$$\text{In the second phase of the motion : } \theta(r(t)) = \Phi - \int_{r_M}^{r(t)} \frac{d\rho}{\rho\sqrt{f(\rho)}} = 2\Phi - g(r(t))$$

$$\begin{aligned} \text{In the third phase of the motion : } \theta(r(t)) &= 2\Phi + g(r(t)) \\ &\dots \end{aligned} \quad (28)$$

where:

$$g(r(t)) = \int_{r_m}^{r(t)} \frac{d\rho}{\rho\sqrt{f(\rho)}}, \quad (29)$$

$\Phi = g(r_M) = \int_{r_m}^{r_M} \frac{dr'}{r'\sqrt{f(r')}}$ being the apsidal angle.

More generally, we will write:

$$\text{In the } k\text{-th phase of the motion : } \theta_k(r(t)) = 2n\Phi + (-1)^{k+1}g(r(t)) \quad (30)$$

where n is the integral part of $\frac{k}{2}$.

From Eq.(22) and Eq.(23), we deduce the following expression for the instantaneous velocity in the k -th phase:

$$\dot{z}(t) = \left((-1)^{k+1} \sqrt{f(r(t))} + i \right) \frac{L}{r^2(t)} z(t) = \frac{L}{r(t)} \left((-1)^{k+1} \sqrt{f(r(t))} + i \right) e^{i\theta_k(r(t))} \quad (31)$$

$\theta_k(r)$ being given by Eq.(30).

Note that, since $\dot{z} = \lambda z$ where λ has a nonzero imaginary part, Eq.(31) ensures the non-colinearity of \vec{r} and $\dot{\vec{r}}$.

In the sequel, we will have to use the vector $\dot{\vec{r}} \times \vec{L}$ which has the following complex correspondent :

$$\dot{\vec{r}}(t) \times \vec{L} \rightarrow iL\dot{z}(t) = - \left(1 + (-1)^k i \sqrt{f(r(t))} \right) \frac{L^2}{r^2(t)} z(t) \quad (32)$$

We see immediately that $\vec{L} \times \dot{\vec{r}}$ and \vec{r} are always linearly independent, except at the apsidal positions.

III. PERES APPROACH TO THE FBRS VECTOR

In the special case of the Kepler system $U(r) = -r^{-1}$, the system admits the Laplace-Runge-Lenz vector, $\vec{\mathcal{A}}_{LRL} = \dot{\vec{r}} \times \vec{L} - \frac{1}{r}\vec{r}$, as a supplementary conserved quantity¹. In the complex formulation used above, this gives :

$$\vec{\mathcal{A}}_{LRL} \rightarrow \mathcal{A}_{LRL} = iL\dot{z} - \frac{1}{r}z \quad (33)$$

Following Peres¹⁹, for the general central potential $U(r)$, we will then look for a supplementary conserved vector of the form:

$$\vec{\mathcal{A}} = \dot{\vec{r}} \times \vec{L} \left(\frac{r^2}{L^2} a(r) \right) + b(r) \vec{r} \quad (34)$$

whose associated complex correspondent is:

$$\mathcal{A} = \left(\frac{r^2}{L^2} a(r) \right) iL\dot{z} + b(r)z, \quad (35)$$

$a(r)$ and $b(r)$ being two real valued functions that we will have to determine (the factor $\frac{r^2}{L^2}$ in the first term has been introduced for future convenience).

Using Eq.(32), we can rewrite \mathcal{A} in the k -th phase, as:

$$\mathcal{A}_k = \left(b(r) - a(r) + (-1)^{k+1} i\sqrt{f(r)}a(r) \right) z \quad (36)$$

4. Differential system for the coefficients of the FBR vector

On every time interval I on which \mathcal{A} is constant we must have:

$$\dot{\mathcal{A}} = 0 \quad (37)$$

that is, with Eq.(31):

$$0 = \frac{d}{dt} \left(b(r) - a(r) + (-1)^{k+1} i\sqrt{f(r)}a(r) \right) z + \left(b(r) - a(r) + (-1)^{k+1} i\sqrt{f(r)}a(r) \right) \dot{z} \quad (38)$$

$$\begin{aligned} & i \left(b'(r) - a'(r) + (-1)^{k+1} i \left(\sqrt{f(r)}a'(r) + \frac{a(r)f'(r)}{2\sqrt{f(r)}} \right) \right) z \\ & + \left(b(r) - a(r) + (-1)^{k+1} i\sqrt{f(r)}a(r) \right) \left((-1)^{k+1} \sqrt{f(r)} + i \right) \frac{L}{r^2} z = 0 \end{aligned} \quad (39)$$

Using Eq.(22), this becomes:

$$(-1)^{k+1} \sqrt{f(r)} \left(b'(r) - a'(r) + \frac{b(r) - 2a(r)}{r} \right) + i \left(\sqrt{f(r)}a'(r) + \left(\frac{f'(r)}{2\sqrt{f(r)}} + \frac{f(r) - 1}{r} \right) a(r) + b(r) \right) = 0 \quad (40)$$

Consequently, in the k -phase of the motion, on the interval of radial values $]r_m, r_M[$, we must have:

$$\begin{cases} a'(r) - b'(r) + \frac{2a(r) - b(r)}{r} = 0 \\ r f(r) a'(r) + a(r) \left(r \frac{f'(r)}{2} + f(r) - 1 \right) + b(r) = 0 \end{cases} \quad (41)$$

Extracting $a'(r)$ from the second equation above and substituting in the first one, we obtain, in matrix form:

$$\begin{pmatrix} a(r) \\ b(r) \end{pmatrix}' = \frac{-1}{rf(r)} \begin{pmatrix} r \frac{f'(r)}{2} + f(r) - 1 & 1 \\ r \frac{f'(r)}{2} - f(r) - 1 & f(r) + 1 \end{pmatrix} \begin{pmatrix} a(r) \\ b(r) \end{pmatrix} \quad (42)$$

5. Exact solution

Peres¹⁹, starting from a first order differential system equivalent to Eq.(41) transforms it in a second order differential equation for the coefficient $a(r)$. Nevertheless, the structure of this last equation is, at first sight, rather complicated and the author restricts its analysis to a characterization of the singularities.

Things are much more transparent if we choose to work with a slightly different unknown function.

Indeed, it is readily seen that the first equation in Eq.(41) can be rewritten:

$$ra'(r) + 2a(r) = (rb(r))' \quad (43)$$

Introducing an auxiliary function $u(r)$ defined by:

$$a(r) = u'(r), \quad (44)$$

the above equation Eq.(41) becomes:

$$ru''(r) + 2u'(r) = (ru(r))'' = (rb(r))' \quad (45)$$

Then:

$$b(r) = \frac{1}{r} (ru(r))' + b_0 = u'(r) + \frac{1}{r}u(r) \quad (46)$$

where the integration constant b_0 has been chosen equal to 0.

We can report the expressions Eq.(44) and Eq.(46) of $a(r)$ and $b(r)$ in terms of $u(r)$ in the second equation of Eq.(41). This gives:

$$\left(u'(r) + \frac{1}{r}u(r)\right)' = \frac{-1}{rf(r)} \left(u'(r) \left(r \frac{f'(r)}{2} - f(r) - 1\right) + (u'(r) + \frac{1}{r}u(r)) (f(r) + 1)\right) \quad (47)$$

or :

$$u''(r) + \left(\frac{1}{r} + \frac{f'(r)}{2f(r)}\right) u'(r) + \frac{1}{r^2 f(r)} u(r) = 0 \quad (48)$$

In a more compact form, we obtain the following second order differential equation for $u(r)$:

$$u''(r) - \left(\log \left(\frac{1}{r\sqrt{f(r)}}\right)\right)' u'(r) + \left(\frac{1}{r\sqrt{f(r)}}\right)^2 u(r) = 0 \quad (49)$$

or (see Eq.(29)):

$$u''(r) - \frac{g''(r)}{g'(r)} u'(r) + (g'(r))^2 u(r) = 0 \quad (50)$$

The coefficients in Eq.(50) depending only on $g(r)$, if we define:

$$u(r) = v(g(r)) \quad (51)$$

Eq.(50) becomes then:

$$(g'(r))^2 v^{(2)}(g(r)) + g''(r) v^{(1)}(g(r)) - \frac{g''(r)}{g'(r)} (g'(r) v^{(1)}(g(r))) + (g'(r))^2 v(g(r)) = 0 \quad (52)$$

that is, simply:

$$v^{(2)}(g(r)) + v(g(r)) = 0 \quad (53)$$

where $v^{(n)}(x) = \frac{d^n v}{dx^n}$.

The resolution of Eq.(53) is immediate and gives:

$$u(r) = \alpha \cos g(r) + \beta \sin g(r), \quad \alpha, \beta \in \mathbb{R} \quad (54)$$

Reporting this result in Eq.(44) and Eq.(46), we finally have:

$$\begin{cases} a(r) = \frac{1}{r\sqrt{f(r)}} (\beta \cos g(r) - \alpha \sin g(r)) \\ b(r) = a(r) + \frac{1}{r} (\alpha \cos g(r) + \beta \sin g(r)) \end{cases}, \quad \alpha, \beta \in \mathbb{R} \quad (55)$$

that is:

$$\begin{cases} a(r) = \frac{\gamma}{r\sqrt{f(r)}} \cos(g(r) + \phi) \\ b(r) = a(r) + \frac{\gamma}{r} \sin(g(r) + \phi) \end{cases}, \quad \gamma \in \mathbb{R}, \phi \in [0, 2\pi[\quad (56)$$

6. Discontinuous behavior of \mathcal{A}

The preceding result Eq.(56) determines completely the form of \mathcal{A} on $]r_m, r_M[$ during a given phase of the motion. Indeed, inserting Eq.(56) in Eq.(36), we obtain:

$$\mathcal{A}_k = \frac{\gamma}{r} (\sin(g(r) + \phi) + (-1)^{k+1} i \cos(g(r) + \phi)) z(r) = (-1)^{k+1} i \gamma e^{(-1)^k i(g(r) + \phi)} e^{i\theta_k(r)} \quad (57)$$

Using Eq.(30), this becomes:

$$\mathcal{A}_k = (-1)^{k+1} i \gamma e^{(-1)^k i(g(r) + \phi)} e^{i(2n\Phi + (-1)^{k+1} g(r))} = (-1)^{k+1} \gamma i e^{(-1)^k i\phi + 2ni\Phi} \quad (58)$$

where n is the integer part of $\frac{k}{2}$.

Under a more detailed form, we have:

$$\begin{cases} \mathcal{A}_{2n} = -\gamma e^{i\frac{\pi}{2} + i\phi + 2ni\Phi} \\ \mathcal{A}_{2n+1} = \gamma e^{i\frac{\pi}{2} - i\phi + 2ni\Phi} \end{cases} \quad (59)$$

The integration constants γ and ϕ for and determine then respectively the modulus and argument of the $\vec{\mathcal{A}}$ vector affix. The freedom in the choice of the parameters γ and ϕ induces that $\vec{\mathcal{A}}$ can be identified with any vector of the plane. We have the same type of result for the specific Kepler problem. Indeed, in this case, we can add to the Laplace-Runge-Lenz vector $\vec{\mathcal{A}}_K$ a second conserved vector, $\vec{\mathcal{S}}_K = \vec{L} \times \vec{\mathcal{A}}_K$, the so-called Hamilton vector^{1,3,11}, whose constancy is a direct consequence of these ones of \vec{L} and $\vec{\mathcal{A}}_K$. Any linear combination of this two orthogonal vectors being conserved, we can build a conserved vector corresponding to any vector of the plane. $\vec{\mathcal{S}}_K$ and $\vec{\mathcal{A}}_K$ are particularly interesting choices because they give the directions of the minor and major axes of the elliptical trajectory.

If we want that the functional forms of $a(r)$ and $b(r)$ being globally defined during all the motion, we have to keep the same values for γ and ϕ in every phase. With the choice $\phi = \frac{\pi}{2}$, Eq.(59) becomes:

$$\mathcal{A}_{2n} = \mathcal{A}_{2n+1} = \gamma e^{2ni\Phi} \quad (60)$$

$\vec{\mathcal{A}}$ is then constant when we pass from a even phase to the following odd one, that is when we cross the apocenter. But after a complete period of oscillation, when we reach the pericenter again, we pass from $k = 2n$ to $k = 2n + 1$. At this moment the value of \mathcal{A}_{2n+1} change to \mathcal{A}_{2n+1} with:

$$\mathcal{A}_{2n+2} = \mathcal{A}_{2n+1} e^{2i\Phi} \quad (61)$$

This corresponds to a 2Φ rotation of the associated vector. We recover here the discontinuity jumps (observed by Serebrennikov, Shabad, Buch and Denman^{17,18} and Peres¹⁹ and first studied in a detailed by Holas and March²¹) which make FBR vector only a piecewise conserved quantity: the perihelion vector \mathcal{A} presents discontinuities at each pericenter (with the choice made here and at each apocenter if we choose the initial condition $r(t = 0) = r_M$) corresponding to a rotation of two times the apsidal angle.

7. Harmonic oscillator, Kepler problem and Bertrand's theorem

For $\Phi = \pi$, which is the case of the Kepler problem $U(r) = -r^{-1}$ for every values of the characteristic parameters of the motion L and E , $\vec{\mathcal{A}}$ is a true vector conserved quantity and is identical to the usual Laplace-Runge-Lenz vector $\vec{\mathcal{A}}_K$. Indeed, in this case, with $E < 0$, introducing Clairaut's variable⁴ $u = \frac{1}{r}$, we have (see Eq.(29) and (see Eq.(16)):

$$g(r) = -\frac{L}{\sqrt{2}} \int_{u_m = \frac{1}{r_m}}^u \frac{dv}{\sqrt{-|E| + v - \frac{L^2}{2}v^2}} \quad (62)$$

which gives²⁹ for $1 - 2L^2|E| = e^2 > 0$:

$$g(r) = \left[\arg \sin \left(\frac{1 - L^2 v}{e} \right) \right]_{u_m}^{u = \frac{1}{r}} \quad (63)$$

Since r_m is the smallest root of $f(r) = \frac{2r^2}{L^2} \left(-|E| + \frac{1}{r} - \frac{L^2}{2r^2} \right) = 0$, that is, $u_m = \frac{1}{r_m}$ is the greatest root of $-|E| + u - \frac{L^2}{2}u^2 = 0$, we obtain:

$$u_m = \frac{1 + e}{L^2} \quad (64)$$

and:

$$g(r) = \arg \sin \left(\frac{1 - \frac{L^2}{r}}{e} \right) + \frac{\pi}{2} \quad (65)$$

Then (see Eq.(56) with $\phi = \frac{\pi}{2}$):

$$\begin{cases} a(r) = \frac{\gamma}{r\sqrt{f(r)}} \cos \left(g(r) + \frac{\pi}{2} \right) = -\frac{\gamma}{r\sqrt{f(r)}} \sqrt{1 - \left(\frac{r-L^2}{er} \right)^2} = -\gamma \frac{L^2}{er^2} \\ b(r) = a(r) + \frac{\gamma}{r} \sin \left(g(r) + \frac{\pi}{2} \right) = -\gamma \frac{L^2}{er^2} - \frac{\gamma}{r} \frac{r-L^2}{er} = -\gamma \frac{1}{er} \end{cases} \quad (66)$$

Taking $\gamma = -e$, we recover the coefficients of the usual Laplace-Runge-Lenz vector (see Eq.(33)):

$$\begin{cases} a(r) = \frac{L^2}{r^2} \\ b(r) = \frac{1}{r} \end{cases} \quad (67)$$

In the isotropic harmonic oscillator problem (Hooke's problem) $\Phi = \frac{\pi}{2}$, for every L and E . The global direction is then conserved but the sense of $\vec{\mathcal{A}}$ is alterned at each pericenter crossing : $\vec{\mathcal{A}} \rightarrow -\vec{\mathcal{A}}$ ^{17,18}. The generalized Hamilton vector $\vec{\mathcal{S}} = \vec{L} \times \vec{\mathcal{A}}$ is subject to the same phenomenon : $\vec{\mathcal{S}} \rightarrow -\vec{\mathcal{S}}$. As for the Fradkin tensor $\mathcal{T} = \frac{1}{2} \dot{\vec{r}} \otimes \dot{\vec{r}} + \frac{\omega^2}{2} \vec{r} \otimes \vec{r}$, recalling that it can be written as¹¹ $\mathcal{T} = \frac{\omega^2}{2} \vec{\mathcal{A}} \otimes \vec{\mathcal{A}} + \frac{1}{2|\vec{\mathcal{A}}|^2} \vec{\mathcal{S}} \otimes \vec{\mathcal{S}}$, we see immediately that it's a global invariant of the motion.

As established by Bertrand^{25,26} more than one century ago, Kepler and Hooke potentials are the only central potentials for which the apsidal angle is commensurable with π for every values of the initial parameters of the motion. For all the other central potentials, this condition, which is necessary for the closure of the orbit, is obtained only for specific values of E and L . In these cases, as established by Holas and March²¹, it is still possible for an orbit of multiplicity n to build global geometrical invariants in form of n -arm stars by using the n distinct FBRS vectors associated to the system. Nevertheless, as we have seen before, the existence of a general (that is for every initial condition) true invariant vector or tensor is a specificity of Kepler and Hooke problems respectively.

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