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Weak error for stable driven SDEs : expansion of the densities.

Valentin Konakov · Stéphane Menozzi

June 9, 2009

Abstract Consider a multidimensional SDE of the form $X_t = x + \int_0^t b(X_{s-})ds + \int_0^t f(X_{s-})dZ_s$ where $(Z_s)_{s \geq 0}$ is a symmetric stable process. Under suitable assumptions on the coefficients the unique strong solution of the above equation admits a density w.r.t. the Lebesgue measure and so does its Euler scheme. Using a parametrix approach, we derive an error expansion w.r.t. the time step for the difference of these densities.

Keywords Symmetric stable processes · parametrix · Euler scheme

Mathematics Subject Classification (2000) 60H30 · 65C30 · 60G52

1 Introduction

Consider the following \mathbb{R}^d -valued Stochastic Differential Equation (SDE in short)

$$X_t = x + \int_0^t b(X_{s-})ds + \int_0^t f(X_{s-})dZ_s, \quad (1.1)$$

where b, f are respectively Lipschitz continuous mappings from \mathbb{R}^d to \mathbb{R}^d and \mathbb{R}^d to $\mathbb{R}^d \otimes \mathbb{R}^d$ and $(Z_s)_{s \geq 0}$ is a general Lévy process. The previous assumptions guarantee the existence of a unique strong solution to (1.1). Also, this solution satisfies the strong Markov property, see e.g. Theorem 7 and 32 Chapter 5 in Protter [Pro04]. Let $T > 0$ be a fixed time horizon and $(X_t^N)_{t \in \Lambda}$ a given approximation scheme of $(X_t)_{t \in [0, T]}$ associated to the time step $h := T/N$, $N \in \mathbb{N}^*$ on the grid $\Lambda := \{t_i := ih, i \in \llbracket 0, N \rrbracket\}$. When speaking about weak approximation of (1.1) two kind of quantities are of interest. The first one writes

$$\mathcal{E}_1(x, T, N) := \mathbb{E}_x[g(X_T)] - \mathbb{E}_x[g(X_T^N)]$$

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for a suitable class of test functions g . The second one concerns, when it exists, the approximation of the density p of the original SDE (1.1). If the approximation scheme $(X_t^N)_{t \in A}$ admits as well a density p^N , the quantity under study becomes

$$\mathcal{E}_2(x, y, T, N) := (p - p^N)(T, x, y).$$

In both cases, the goal is to give a bound or an error expansion of these quantities in terms of h . If p and p^N exist, the control \mathcal{E}_2 gives more information than \mathcal{E}_1 . Anyhow, the typical assumptions and techniques associated to the study of \mathcal{E}_1 and \mathcal{E}_2 are quite of different nature.

In the continuous case, i.e. $Z_s = bs + \sigma W_s$ where $(W_s)_{s \geq 0}$ is a standard d -dimensional Brownian motion, provided the test function g and the coefficients b, f are sufficiently smooth, without any additional assumption on the generator Talay and Tubaro [TT90] derive an error expansion at order 1 for $\mathcal{E}_1(x, T, N)$ when $(X_t^N)_{t \in A}$ is the Euler approximation. Their proof is based on standard stochastic analysis tools: Itô's expansions and stochastic flows. To obtain the same kind of result for bounded Borel functions g some non degeneracy has to be assumed, namely hypoellipticity of the underlying diffusion, and the proof relies on Malliavin calculus techniques, see Bally and Talay [BT96a]. The authors also manage to extend their results to $\mathcal{E}_2(x, y, T, N)$ for a slightly modified Euler scheme [BT96b].

Anyhow, in the uniformly elliptic case, the most natural approach to handle the estimation of the quantity $\mathcal{E}_2(x, y, T, N)$ consists in using the so called "parametrix" technique introduced to obtain existence and controls on the fundamental solutions of PDEs, see e.g. Mc Kean and Singer [MS67] or Friedman [Fri64]. Roughly speaking it consists in expressing the density of X_T in terms of an infinite sum of suitable iterated kernels applied to the density of an SDE with constant coefficients. This has been done successfully by Konakov and Mammen [KM02]. The main advantage of this approach is that the density of the solution X_T and the Euler approximation X_T^N can be expressed in the same form and therefore quite directly compared. Furthermore this technique turns out to be quite robust and can be applied as soon as good controls on the densities p, p^N and their derivatives are available, see e.g. [KMM08] for an extension to a slightly degenerate framework.

Consider now the general case, i.e. for all $u \in \mathbb{R}^d$, $\mathbb{E}[\exp(i\langle u, Z_t \rangle)] = \exp(t\psi(u))$ with

$$\psi(u) = \left\{ i\langle \gamma, u \rangle - \sigma^2 \frac{|u|^2}{2} + \int \left(\exp(i\langle u, z \rangle) - 1 - i\langle u, z \rangle \mathbb{I}_{|z| \leq 1} \right) \nu(dz) \right\},$$

where $\gamma \in \mathbb{R}^d$, $\sigma \in \mathbb{R}$ and ν stands for the Lévy measure of the process. For smooth functions f, g and under additional assumptions on the behavior at infinity of ν , that is integrability conditions of the large jumps, Protter and Talay [PT97], manage to get a control at order one or even an error expansion for $\mathcal{E}_1(x, T, N)$ with the same approach as in [TT90]. In this work the approximation is the Euler scheme which for a general Lévy measure ν cannot always be simulated on a computer.

The quantity $\mathcal{E}_1(x, T, N)$ for a given simulatable approximation of the Euler scheme has also been studied by Jacod *et al.* [JKMP05] who derived bounds at order 1. Moment conditions are also assumed.

In this work, we consider the case where $(Z_t)_{t \geq 0}$ is an α -stable symmetric process, $\alpha \in (0, 2)$. Under suitable non degeneracy assumptions on its coefficients specified below

(see **(A-1)**-**(A-3)**), equation (1.1) is known to have a density p w.r.t. the Lebesgue measure. This can be proved via a Malliavin calculus-Bismut integration by parts approach, see e.g. Bichteller *et al.* [BGJ87]. Also, a direct construction of this density using a parametrix expansion has been obtained by Kolokoltsov [Kol00] who derived as well "Aronson's like" bounds with time singularity depending on the index α of the stable process $(Z_t)_{t \geq 0}$.

Analogously to the "diffusion case" the first step of the parametrix is to consider that the density $p(T, x, y)$ of (1.1) can be approximated by the density of the process $\tilde{X}_t^y = x + b(y)t + f(y)Z_t$ at time T . Namely, we freeze the coefficients in (1.1) at the final spatial point. The next crucial point is to obtain sharp estimates of the stable density $\tilde{p}^y(T, x, \cdot)$ of \tilde{X}_T^y and its derivatives in order to solve the parametrix integral equations.

Stable driven SDEs appear in various applicative fields, from mathematical physics to electrical engineering or financial mathematics, see [IP06], [SK74] or [JMW96], therefore their approximation becomes of interest. To approximate equation (1.1), setting $\phi(t) := \inf\{t_i : t_i \leq t < t_{i+1}\}$, we introduce the Euler scheme

$$X_t^N = x + \int_0^t b(X_{\phi(s)}^N) ds + \int_0^t f(X_{\phi(s)}^N) dZ_s. \quad (1.2)$$

The computation of the above scheme only requires to be able to simulate exactly the increments of $(Z_t)_{t \geq 0}$, which up to a self similarity argument only amounts to simulate a stable law. This aspect is for instance discussed in Samorodnitsky and Taqqu [ST94], Weron and Weron [WR95] or Section 3 of [PT97]. Under the same assumptions **(A-1)**-**(A-3)**, the Euler scheme defined in (1.2) also has a density p^N .

Observe that the results of [PT97], [JKMP05] cannot be directly applied, even for the study of $\mathcal{E}_1(x, T, N)$, since stable laws have heavy tails. Comparing the parametrix developments of p and p^N we obtain an expansion with leading term of order 1 in h for $\mathcal{E}_2(x, y, T, N)$. The parametrix expansion of p is discussed in [Kol00], see also Section 3 and Appendix, whereas the parametrix expansion of p^N can be related to the ideas developed in [KM00, KM02] for the diffusive case corresponding to an index of stability equal to 2.

This result also emphasizes the robustness of the method that naturally extends to a broad class of processes. Let us mention that, using a Malliavin calculus approach, Hausenblas [Hau02], derived an upper bound of order one w.r.t. h for the quantity $\mathcal{E}_1(x, T, N)$, $g \in L^\infty$ in the scalar case. Concerning functional limit theorems for the approximation of stable driven SDEs we refer to the work of Jacod [Jac04].

The paper is organized as follows. In Section 2 we state our standing assumptions and main results. In Section 3 we prove the existence of the densities for both the stable driven equation and its Euler scheme and also give a parametrix representation of these densities. Section 4 is dedicated to the proof of the main results. Eventually, we state in Section 5 weaker assumptions under which our main result holds and we also briefly discuss how to extend it to the case of a stable process perturbed by a compound Poisson process.

2 Assumptions and Main results

2.1 Assumptions

Since our arguments rely on techniques that involve the generator of (1.1), it is convenient to compute it before stating our assumptions. Let $(Z_t)_{t \geq 0}$ be the process associated to the \mathbb{R}^d valued α -stable law, $\alpha \in (0, 2)$, with characteristic function defined for all $u \in \mathbb{R}^d$ by

$$\begin{aligned} \widehat{\mu}(u) &= \mathbb{E}[\exp(i\langle u, Z_1 \rangle)] \\ &= \exp \left[i\langle \gamma, u \rangle + \int_{S^{d-1}} \int_0^\infty \left(e^{i\langle u, \rho s \rangle} - 1 - \frac{i\langle u, \rho s \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \widetilde{\lambda}(ds) \right], \end{aligned} \quad (2.1)$$

where $\gamma \in \mathbb{R}^d$ and $\widetilde{\lambda}$ is a (finite Borelian) measure on the unit sphere S^{d-1} . In the following we consider *symmetric* stable processes, that is for every A in the Borel σ -field $\mathcal{B}(S^{d-1})$, $\widetilde{\lambda}(A) = \widetilde{\lambda}(-A)$. Thus, for all $t \geq 0$, $u \in \mathbb{R}^d$,

$$\mathbb{E}[\exp(i\langle u, Z_t \rangle)] = \exp(it\langle \gamma, u \rangle - t \int_{S^{d-1}} |\langle u, s \rangle|^\alpha \lambda(ds)), \quad (2.2)$$

with $\lambda := C_\alpha \widetilde{\lambda}$, $C_\alpha := \alpha^{-1} \left(\Gamma(1-\alpha) \mathbb{I}_{\alpha \in (0,1)} + \Gamma(2-\alpha)(\alpha-1)^{-1} \mathbb{I}_{\alpha \in (1,2)} \right) \times \cos\left(\frac{\pi\alpha}{2}\right) + \frac{\pi\alpha}{2} \mathbb{I}_{\alpha=1}$, see e.g. Theorem 2.3.1 in [ST94] and Lemma 2, Chapter XVII.4 in [Fel66] for the expression of C_α .

Recalling that $(Z_t)_{t \geq 0}$ is a quadratic pure jump semimartingale, we deduce from Itô's formula for semimartingales that for every $\psi \in C_0^2(\mathbb{R}^d)$

$$\begin{aligned} \psi(X_t) - \psi(x) &= \int_{0+}^t \langle \nabla \psi(X_{s-}), (b(X_{s-}) ds + f(X_{s-}) dZ_s) \rangle + \\ &\sum_{0 < s \leq t} [\psi(X_{s-} + f(X_{s-}) \Delta Z_s) - \psi(X_{s-}) - \langle \nabla \psi(X_{s-}), f(X_{s-}) \Delta Z_s \rangle]. \end{aligned}$$

In addition we have the componentwise canonical decomposition for $(Z_t)_{t \geq 0}$ with the truncation function $h(z) = z \mathbb{I}_{|z| \leq 1}$ (see Theorem 2.43 Chapter II in Jacod and Shiryaev [JS87])

$$Z_t = \int_0^t \int_{|z| \leq 1} z [\mu(\omega, dz, ds) - \nu(dz) ds] + \int_0^t \int_{|z| > 1} z \mu(\omega, dz, ds) + \gamma t \quad (2.3)$$

where $\mu(\omega, dz, dt) := \sum_{s \geq 0} \mathbb{I}_{\Delta Z_s(\omega) \neq 0} \delta_{\Delta Z_s(\omega), s}(dz, dt)$ is the jump measure of Z and ν its Lévy measure. Hence,

$$\begin{aligned} \psi(X_t) - \psi(x) &= M_t + \int_{0+}^t \langle \nabla \psi(X_{s-}), f(X_{s-}) \gamma + b(X_{s-}) \rangle ds + \\ &\int_{0+}^t \int_{\mathbb{R}^d} [\psi(X_{s-} + f(X_{s-})z) - \psi(X_{s-}) - \langle \nabla \psi(X_{s-}), f(X_{s-})z \mathbb{I}_{|z| \leq 1} \rangle] \nu(dz) ds \end{aligned}$$

where $M_t = \int_{0+}^t \int_{\mathbb{R}^d} (\psi(X_{s-} + f(X_{s-})z) - \psi(X_{s-})) [\mu(\omega, dz, ds) - \nu(dz) ds]$. Since we assume in the following f to be bounded (see assumption **(A-2)** below) and $\psi \in C_0^2(\mathbb{R}^d)$, we deduce from equations (2.1)-(2.2) and corollary 3 p. 73 in Protter [Pro04]

that $(M_t)_{t \geq 0}$ is a square integrable martingale. Thus if Φ denotes the generator of the unique strong solution of (1.1) then:

$$\begin{aligned} \Phi\psi(x) &= \lim_{t \rightarrow 0} \frac{\mathbb{E}_x \psi(X_t) - \psi(x)}{t} = \langle \nabla \psi(x), f(x)\gamma + b(x) \rangle + \\ &\int_{\mathbb{R}^d} \left[\psi(x + f(x)z) - \psi(x) - \langle \nabla \psi(x), f(x)z \rangle \mathbb{I}_{|z| \leq 1} \right] \nu(dz) = \\ &\langle \gamma(x), \nabla \psi(x) \rangle + \int_{S^{d-1}} \int_0^\infty \left[\psi(x + \rho s) - \psi(x) - \frac{\langle \rho s, \nabla \psi(x) \rangle}{1 + \rho^2} \right] \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}_{f(x)}(ds) \end{aligned} \quad (2.4)$$

where $\gamma(x) := b(x) + f(x)\gamma + \int_{\mathbb{R}^d} f(x)z \left(\frac{1}{1+|f(x)z|^2} - \mathbb{I}_{|z| \leq 1} \right) \nu(dz) = b(x) + f(x)\gamma$, recalling the symmetry. We denote for all $A \in \mathcal{B}(\mathbb{R}^d)$, $\nu_{f(x)}(A) := \nu(\{z \in \mathbb{R}^d : f(x)z \in A\})$. Then $\tilde{\lambda}_{f(x)}(\cdot)$ stands for the spherical part of $\nu_{f(x)}(\cdot)$, see e.g. Theorem 14.3 p.78 in Sato [Sat99] for details. Note that for $f(x) \equiv I_d$, $\nu_{f(x)} = \nu$ and therefore $\tilde{\lambda}_{f(x)} = \tilde{\lambda}$ appearing in (2.1).

We now introduce our assumptions. Fix an integer $q \geq 2$. We assume that

(A-1) For $d \geq 2$, the spherical measure $\tilde{\lambda}$ appearing in (2.1) has a $C^q(S^{d-1})$ surface density and there exist constants $0 < C_1 \leq C_2 < +\infty$, $\forall p \in \mathbb{R}^d$,

$$C_1 |p|^\alpha \leq \int_{S^{d-1}} |\langle p, s \rangle|^\alpha \lambda(ds) \leq C_2 |p|^\alpha,$$

with λ defined after (2.2).

Note that for $d = 1$, with the convention $S^0 = \{-1, 1\}$, the symmetry gives $\lambda(A) = C_\lambda(\delta_{-1} + \delta_1)(A)$, $\forall A \in \mathcal{B}(\mathbb{R})$ with $C_\lambda > 0$. Equation (2.2) yields

$$\forall u \in \mathbb{R}, \hat{\mu}(u) = \mathbb{E}[\exp(iuZ_1)] = \exp(iu\gamma - 2C_\lambda |u|^\alpha).$$

The second point of the above assumption thus holds with $C_1 = C_2 = 2C_\lambda$.

(A-2) The coefficients b and f and their derivatives up to order q are uniformly bounded in x . Thus, for $1 < \alpha < 2$, $\gamma(x) := b(x) + f(x)\gamma$ is uniformly bounded. We impose for $0 < \alpha \leq 1$, $\gamma(x) = 0$ for all $x \in \mathbb{R}^d$.

Remark 21 *The zero drift condition comes from the fact that for $\alpha \in (0, 1]$ the addition of a drift of order t does not correspond to a negligible term in small time with respect to the natural scale $t^{1/\alpha}$, see Appendix A for details.*

(A-3) There exist constants $0 < \underline{c} \leq \bar{c} < +\infty$ s.t. for all $x \in \mathbb{R}^d, \xi \in \mathbb{R}^d$,

$$\underline{c} |\xi|^2 \leq \langle f(x)\xi, \xi \rangle \leq \bar{c} |\xi|^2.$$

Remark 22 *The uniform ellipticity condition (A-3) allows to have good controls on the measure $\tilde{\lambda}_{f(x)}(\cdot)$ introduced in (2.4). In particular, as a consequence of (A-1), (A-3) one gets that there exist constants $0 < \underline{C}_1 = \underline{C}_1(\underline{c}, d, \alpha) \leq \bar{C}_2 = \bar{C}_2(\bar{c}, d, \alpha) < +\infty$ s.t. $\forall p \in \mathbb{R}^d, x \in \mathbb{R}^d$,*

$$\underline{C}_1 |p|^\alpha \leq \int_{S^{d-1}} |\langle p, s \rangle|^\alpha \lambda_{f(x)}(ds) \leq \bar{C}_2 |p|^\alpha, \quad (2.5)$$

where analogously to (2.2) $\lambda_{f(x)} = C_\alpha \tilde{\lambda}_{f(x)}$.

From now on we assume that Assumptions **(A-1)**-**(A-3)** are in force.

In the following we denote by C a positive generic constant that can depend on α, d , the bounds appearing in the previous assumptions but neither on N nor on the spatial points involved. Its value may change from line to line. Other possible dependencies, especially w.r.t. the final time T are explicitly specified. Concerning functional spaces, we denote by $C_b(\mathbb{R}^d)$ the Banach space of continuous bounded functions on \mathbb{R}^d endowed with the sup-norm, by $C_{b,0}(\mathbb{R}^d)$ its closed subspace of functions vanishing at infinity and by $C_b^k(\mathbb{R}^d)$, $k \in \mathbb{N}^*$, the Banach space of continuous functions having bounded derivatives up to and including the order k with the norm $\|f\| := \max_{l \leq k} \sup_{x \in \mathbb{R}^d} |f^{(l)}(x)|$.

2.2 Main results

Proposition 21 *For every $t > 0$ the solution X_t (resp. X_t^N) of (1.1) (resp. (1.2)) has a density $p(t, x, \cdot)$ (resp. $p^N(t, x, \cdot)$) w.r.t. the Lebesgue measure. Additionally, as a function of the space variables this density is in $C_b^q(\mathbb{R}^d \times \mathbb{R}^d)$ if $\alpha > 1$ and in $C_b^{q-1}(\mathbb{R}^d \times \mathbb{R}^d)$ if $\alpha \leq 1$.*

To state the theorem we first need some notation. Introduce for a smooth function $\varphi(t, x, y)$ the integro-differential operators:

$$\begin{aligned} \Phi\varphi(t, x, y) &= \langle \gamma(x), \nabla_x \varphi(t, x, y) \rangle + \\ &\int_{S^{d-1}} \int_0^\infty \left(\varphi(t, x + \rho s, y) - \varphi(t, x, y) - \frac{\langle \rho s, \nabla_x \varphi(t, x, y) \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}_{f(x)}(ds), \end{aligned} \quad (2.6)$$

$\tilde{\Phi}^* \varphi(t, x, y) = \tilde{\Phi}_y \varphi(t, x, y)$ where for $\xi \in \mathbb{R}^d$,

$$\begin{aligned} \tilde{\Phi}_\xi \varphi(t, x, y) &= \langle \gamma(\xi), \nabla_x \varphi(t, x, y) \rangle + \\ &\int_{S^{d-1}} \int_0^\infty \left(\varphi(t, x + \rho s, y) - \varphi(t, x, y) - \frac{\langle \rho s, \nabla_x \varphi(t, x, y) \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}_{f(\xi)}(ds). \end{aligned} \quad (2.7)$$

Define also, $(\tilde{\Phi}_*)^m \varphi(t, x, y) = (\tilde{\Phi}_\xi)^m \varphi(t, x, y) |_{\xi=x}, \forall m \in \mathbb{N}^*$. Note that we have $\tilde{\Phi}_* \varphi(t, x, y) = \Phi \varphi(t, x, y)$ but in general $(\tilde{\Phi}_*)^m \varphi(t, x, y) \neq (\Phi)^m \varphi(t, x, y)$ for $m \geq 2$.

Define now, for $t > 0$, the kernel $H(t, x, y) := (\Phi - \tilde{\Phi}_y) \tilde{p}^y(t, x, y)$ where $\tilde{p}^y(t, x, y)$ denotes the density at point y of $\tilde{X}_t = x + b(y)t + f(y)Z_t$. Note that the variable y acts here twice: as the argument of the density and as a defining quantity of the process $\tilde{X}_t (\equiv \tilde{X}_{t,x,y})$, i.e. the coefficients are frozen in y . Eventually we introduce the continuous and discrete convolution operators

$$\begin{aligned} \varphi \otimes \psi(t, x, y) &= \int_0^t du \int dz \varphi(u, x, z) \psi(t - u, z, y), \forall t \in [0, T], \\ \varphi \otimes_N \psi(t, x, y) &= \int_0^t du \int dz \varphi(\phi(u), x, z) \psi(t - \phi(u), z, y), \forall t \in \{(t_i)_{i \in [1, N]}\}, \end{aligned}$$

with $\phi(u)$ is defined just before (1.2) and denotes the largest discretization time lower or equal to u . Also $\varphi \otimes H^{(0)} = \varphi$ and $\varphi \otimes H^{(r)} = (\varphi \otimes H^{(r-1)}) \otimes H$ stands for the r -fold convolution.

Theorem 21 Suppose $q > d + 4$. Take $0 < M \leq q - (d + 4)$. There exists a function $R_M(T, x, y)$ with $|R_M(T, x, y)| \leq C_M(T) \left(\frac{1}{1+|y-x|^{d+\alpha}} \right) := \rho_{\alpha, M}(T, y - x)$ for some positive constant $C_M(T)$ such that

$$(p - p^N)(T, x, y) = \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} \left[p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d \right] (T, x, y) - \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} p^N \right] (T, x, y) + h^M R_M(T, x, y)$$

where we use the convention that $\sum_{l=1}^0 \dots = 0$ and $\forall t \in \{(t_i)_{i \in [1, N]}\}$, $p^d(t, x, y) := \sum_{r=0}^{\infty} (\tilde{p} \otimes_N H^{(r)})(t, x, y)$. It holds that

$$\sum_{l=1}^{M-1} \left| \left(p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d \right) (T, x, y) \right| \leq \rho_{\alpha, M}(T, y - x),$$

$$\sum_{k=1}^{M-1} \left| \left(p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} p^N \right) (T, x, y) \right| \leq \rho_{\alpha, M}(T, y - x).$$

Remark 23 In the above expression, one writes for all $l \in [1, M - 1]$,

$$(\Phi - \tilde{\Phi}^*)^{l+1} \varphi(t, x, y) = \sum_{k=1}^{l+1} C_{l+1}^k \Phi^k (-\tilde{\Phi}^*)^{l+1-k} \varphi(t, x, y),$$

whereas, $\forall k \in [1, M - 1]$,

$$\begin{aligned} (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} \varphi(t, x, y) &= \underbrace{[(\tilde{\Phi}_\xi - \tilde{\Phi}_y) \cdots (\tilde{\Phi}_\xi - \tilde{\Phi}_y)]}_{(k+1) \text{ times}} \varphi(t, x, y) \Big|_{\xi=x} \\ &= (\tilde{\Phi}_\xi - \tilde{\Phi}_y)^{k+1} \varphi(t, x, y) \Big|_{\xi=x}. \end{aligned}$$

Remark 24 The terms in the previous expansion depend on N . Anyhow using iteratively the Theorem and controls on $\otimes_N - \otimes$ (see also Lemma 41) it is possible to obtain an expansion with terms independent of N . For small M explicit formulas are thus easily derived but in all generality the terms become less transparent. For $M = 2$ one gets

$$\begin{aligned} (p - p^N)(T, x, y) &= \frac{h}{2} \left(p \otimes_N (\Phi - \tilde{\Phi}^*)^2 p^d - p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^2 p^N \right) (T, x, y) \\ &\quad + h^2 R_2(T, x, y) \\ &= \frac{h}{2} \left(p \otimes (\Phi - \tilde{\Phi}^*)^2 p - p \otimes (\tilde{\Phi}_* - \tilde{\Phi}^*)^2 p \right) (T, x, y) \\ &\quad + h^2 \tilde{R}_2(T, x, y) = \frac{h}{2} (p \otimes (\Phi^2 - \tilde{\Phi}_*^2) p) (T, x, y) + h^2 \tilde{R}_2(T, x, y), \end{aligned}$$

where $\tilde{R}_2(T, x, y) \leq C(T) \rho_{\alpha, 2}(T, y - x)$ for some positive constant $C(T)$.

3 Stable driven equations and their Euler scheme: existence of the density and associated parametrix expansion

3.1 Stable driven equation

3.1.1 Existence of the density

In this section, we freely use the notations introduced at the end of Section 2.1 for functional spaces.

For $(X_t)_{t \geq 0}$, Proposition 21 derives from the following result due to Kolokoltsov [Kol00]. Using a parametrix approach he constructs a function $p(t, x, y)$ which is everywhere non-negative, belongs to $C_{b,0}(\mathbb{R}^d)$ for each $t > 0$, satisfies the semigroup identity (or the Chapman-Kolmogorov equation) and $\int p(t, x, \eta) d\eta = 1$. Set now for an arbitrary $\psi \in C_b(\mathbb{R}^d)$ and $t > 0$, $(R_t \psi)(x) = \int_{\mathbb{R}^d} p(t, x, \xi) \psi(\xi) d\xi$. Then the following result holds true (Proposition 3.4 in [Kol00]).

Theorem 31 (i) $(R_t \psi)(x)$ tends to $\psi(x)$ as $t \rightarrow 0$ for each x and any $\psi \in C_b(\mathbb{R}^d)$.
(ii) If $\alpha > 1$, or $\alpha \leq 1$, then R_t is a continuous operator from $C_b(\mathbb{R}^d)$ into $C_b^l(\mathbb{R}^d)$ with operator norm of order $O(t^{-l/\alpha})$ for all $l \leq q$, or $l \leq q - 1$ respectively.
(iii) If $\psi \in C_b(\mathbb{R}^d)$ and $t > 0$, the function $v(t, x) := (R_t \psi)(x)$ satisfies equation

$$\partial_t v(t, x) = \langle \gamma(x), \nabla_x v(t, x) \rangle + \int_{S^{d-1}} \int_0^\infty \left(v(t, x + \rho s) - v(t, x) - \frac{\langle \rho s, \nabla_x v(t, x) \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}_{f(x)}(ds). \quad (3.1)$$

(iv) The Cauchy problem for (3.1) can have at most one solution in the class of continuous functions belonging to $C_{b,0}(\mathbb{R}^d)$ for each $t > 0$. This solution is necessary non-negative whenever the initial function is non-negative.

In other words p is a fundamental solution of the Cauchy problem associated to (3.1). From (ii) it is also a regularizing kernel.

It follows from (3.6) and (3.15) in [Kol00] that the transition function $P(t, x, \Gamma) = \int_{\Gamma} p(t, x, y) dy$ is stochastically continuous and, hence, is uniquely determined by its weak infinitesimal operator (see Dynkin [Dyn63], Theorem 2.3, p. 56). Then p can be identified with the transition probability density of the solution $X_t = (X_t^1, \dots, X_t^d)$ of the equation (1.1) by comparison of the r.h.s. of (3.1) and (2.4). The smoothness stated in Proposition 21 can be derived from point (ii).

Remark 31 The existence of the density is discussed in Bichteler et al. [BGJ87], where it is proved thanks to a Bismut-Malliavin approach. This technique requires the computation of a tangent equation associated to the gradient flow that involves the derivatives of the coefficients of equation (1.1). Thus, some additional smoothness of the coefficients is needed, see e.g. Theorem 6.48 of the above reference.

3.1.2 Parametrix expansion of the density

For the sake of completeness we now specify how to get through a "parametrix" approach a series expansion for the density $p(t, x, y)$.

Introduce, for all $x, y \in \mathbb{R}^d$ the following stochastic "frozen" stable driven equation $\tilde{X}_t \equiv \tilde{X}_{t,x,y}$ defined for $t \geq 0$ by

$$\tilde{X}_t = x + \int_0^t b(y) du + \int_0^t f(y) dZ_u. \quad (3.2)$$

By computation of the Fourier transform of Z_t and Fourier inversion the transition density $\tilde{p}^y(t, x, z)$ of \tilde{X}_t at point $z \in \mathbb{R}^d$ explicitly writes

$$\begin{aligned} \tilde{p}^y(t, x, z) &= \frac{1}{(2\pi)^d} \int e^{-i\langle z-x-t\gamma(y), p \rangle} \\ &\times \exp \left\{ -t \int_{S^{d-1}} |\langle p, s \rangle|^\alpha \lambda_{f(y)}(ds) \right\} dp. \end{aligned} \quad (3.3)$$

One can directly check that this transition density satisfies the equation

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x, z) &= \langle \gamma(y), \nabla_x v(t, x, z) \rangle + \\ \int_{S^{d-1}} \int_0^\infty \left(v(t, x + \rho s, z) - v(t, x, z) - \frac{\langle \rho s, \nabla_x v(t, x, z) \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}_{f(y)}(ds) \\ &= \tilde{\Phi}_y v(t, x, z). \end{aligned} \quad (3.4)$$

Now, from equations (3.4) and (3.1) and with the notations of Section 2, the densities of the solutions of (3.2) and (1.1) satisfy respectively

$$\begin{aligned} \frac{\partial \tilde{p}^y}{\partial t}(t, x, z) &= \tilde{\Phi}_y \tilde{p}^y(t, x, z), \text{ for } t > 0, (x, z) \in (\mathbb{R}^d)^2, \tilde{p}^y(0, x, z) = \delta(z - x), \\ \frac{\partial p}{\partial t}(t, x, z) &= \Phi p(t, x, z), \text{ for } t > 0, (x, z) \in (\mathbb{R}^d)^2, p(0, x, z) = \delta(z - x). \end{aligned} \quad (3.5)$$

Note carefully that the derivatives in $\tilde{\Phi}_y$ are taken w.r.t. the x variable.

We will speak about the operators appearing in (3.5) as the "frozen" and "unfrozen" ones.

From (3.5), we derive a parametrix expansion for the density $p(t, x, \cdot)$ of the solution of (1.1). In the following $\forall (t, x, z) \in \mathbb{R}^{+*} \times (\mathbb{R}^d)^2$, $\tilde{p}(t, x, z) := \tilde{p}^z(t, x, z)$. Hence, $\forall (t, z, y) \in \mathbb{R}^{+*} \times (\mathbb{R}^d)^2$, $H(t, z, y) = (\Phi - \tilde{\Phi}_y) \tilde{p}(t, z, y) = (\tilde{\Phi}_z - \tilde{\Phi}_y) \tilde{p}(t, z, y)$.

Proposition 31 (Parametrix expansion of the density) *With the notations of Section 2, the following representation holds*

$$p(t, x, y) = \sum_{r=0}^{\infty} (\tilde{p} \otimes H^{(r)})(t, x, y). \quad (3.6)$$

Proof. Equations (3.5) correspond to the forward Kolmogorov equations. Consider now the backward equation for p , namely, $\partial_s p(s, x, z) = {}^t \tilde{\Phi}_z p(s, x, z)$ where ${}^t \tilde{\Phi}_z$ stands for the adjoint operator of $\tilde{\Phi}_z$ and the derivatives are taken w.r.t. z . Differentiating under the integral we have from (3.5)

$$\begin{aligned} (p - \tilde{p})(t, x, y) &= \int_0^t ds \frac{\partial}{\partial s} \left[\int p(s, x, z) \tilde{p}(t-s, z, y) dz \right] = \\ &\int_0^t ds \int [({}^t \tilde{\Phi}_z p)(s, x, z) \tilde{p}(t-s, z, y) - p(s, x, z) \tilde{\Phi}_y \tilde{p}(t-s, z, y)] dz \\ &= p \otimes H(t, x, y). \end{aligned}$$

The representation (3.6) then follows by simple iteration. \square

Remark 32 Note that the previous expansion is "formal". The convergence of the r.h.s. in (3.6) is investigated in the proof of Theorem 3.1 in [Kol00] and can also be derived with the controls of Lemmas 44 and 45. For the sake of completeness, a short proof of this convergence is also given in Appendix A.

3.2 Euler scheme

We consider now, for given $N \in \mathbb{N}^*$, the Euler scheme for equation (1.1) at the discretization times:

$$X_0^N = x, X_{t_{i+1}}^N = X_{t_i}^N + b(X_{t_i}^N)h + f(X_{t_i}^N)(Z_{t_{i+1}} - Z_{t_i})$$

recalling $h = T/N$. For each $N \in \mathbb{N}^*$, $(X_{t_i}^N)_{i \in \llbracket 0, N \rrbracket}$ is a Markov chain. Given the past $\{X_{t_l}^N = x_l, l \in \llbracket 0, i \rrbracket\}$, the conditional distribution of the innovations $b(X_{t_i}^N)h + f(X_{t_i}^N)(Z_{t_{i+1}} - Z_{t_i})$ has conditional density $\tilde{p}^{x_i}(h, 0, \cdot)$ (with the notation of (3.2), (3.3)).

Now, in order to give an expansion of the density of the Euler scheme similar to equation (3.6), that will also be the starting point for our error expansion, we need to define, for fixed j, k , $0 \leq j < k \leq N$ and $x, y \in \mathbb{R}^d$ additional "frozen" Markov chains $(\tilde{X}_{t_i}^N)_{i \in \llbracket j, k \rrbracket} = (\tilde{X}_{t_i, x, y}^N)_{i \in \llbracket j, k \rrbracket}$. Their dynamics is described by

$$\tilde{X}_{t_j}^N = x, \tilde{X}_{t_{i+1}}^N = \tilde{X}_{t_i}^N + b(y)h + f(y)(Z_{t_{i+1}} - Z_{t_i}), \quad i \in \llbracket j, k-1 \rrbracket.$$

Given the past $\{\tilde{X}_{t_l}^N = x_l, l \in \llbracket j, i \rrbracket\}$, the conditional distribution of the innovations $b(y)h + f(y)(Z_{t_{i+1}} - Z_{t_i})$ has conditional density $\tilde{p}^y(h, 0, \cdot)$ and, hence, does not depend on the past. Note that for the grid points $(t_i)_{i \in \llbracket 0, N \rrbracket}$ the transition densities of the solution $\tilde{X}_{s, x, y}^N$ of (3.2) coincide with the transition densities of the chain $\tilde{X}_{t_j, x, y}^N$ for $N \in \mathbb{N}^*$, $x, y \in \mathbb{R}^d$ and $s = t_j$.

For all $0 \leq j < k \leq N$, $(x, y) \in (\mathbb{R}^d)^2$, we denote by $p^N(t_k - t_j, x, y)$ and $\tilde{p}^N(t_k - t_j, x, y)$ the transition probability densities between times t_j and t_k from point x to y of the chains X^N and \tilde{X}^N respectively. In particular,

$$\tilde{p}^N(t_k - t_j, x, y) = \tilde{p}^y(t_k - t_j, x, y) = \tilde{p}(t_k - t_j, x, y). \quad (3.7)$$

Before stating the parametrix expansion of p^N in terms of \tilde{p}^N , we need to introduce a kernel H_N that is the "discrete" analogue of H :

$$H_N(t_k - t_j, x, y) = \left\{ L_N - \tilde{L}_N^y \right\} \tilde{p}^N(t_k - t_j, x, y), \quad (3.8)$$

with

$$\begin{aligned} L_N \varphi(t_k - t_j, x, y) &= h^{-1} \left\{ \int p^N(h, x, z) \varphi(t_k - t_{j+1}, z, y) dz - \varphi(t_k - t_{j+1}, x, y) \right\}, \\ \tilde{L}_N^y \varphi(t_k - t_j, x, y) &= h^{-1} \left\{ \int \tilde{p}^y(h, x, z) \varphi(t_k - t_{j+1}, z, y) dz - \varphi(t_k - t_{j+1}, x, y) \right\}. \end{aligned}$$

Note that the previous definitions yield $p^N(h, x, z) = \tilde{p}^x(h, x, z)$. We also mention that, because of the discretisation, there is a slight "shift" in time in the definition of H_N . Namely we have $t_k - t_{j+1}$ instead of the somehow expected $t_k - t_j$.

Lemma 31 For $0 \leq j < k \leq N$ the following formula holds:

$$p^N(t_k - t_j, x, y) = \sum_{r=0}^{k-j} (\tilde{p}^N \otimes_N H_N^{(r)})(t_k - t_j, x, y) \quad (3.9)$$

where in the calculation of $\tilde{p}^N \otimes_N H_N^{(r)}$ (r -fold convolution) we define

$$p^N(0, x, y) = \tilde{p}^N(0, x, y) = \delta(x - y).$$

The proof of this lemma is given in [KM00], Lemma 3.6 and does not rely on the specific distribution of the innovations.

Remark 33 With the convention that $H_N^{(r)} = 0$ for $r > k - j$, equation (33) also writes

$$p^N(t_k - t_j, x, y) = \sum_{r=0}^{\infty} (\tilde{p}^N \otimes_N H_N^{(r)})(t_k - t_j, x, y).$$

This expression will often be used in the sequel.

4 Proof of the main results

In this section, we state in Subsection 4.1 the various points needed to prove Theorem 21. The proofs are postponed to Subsection 4.2. As mentioned earlier, the key idea consists in comparing the parametrix expansions of the densities p and p^N respectively given by (3.6) and (33). In the whole section we suppose that the assumptions of Theorem 21 hold.

4.1 Proof of Theorem 21

For the previously mentioned comparison to be possible we first need to estimate a difference between the transition density $p(T, x, y)$ and $p^d(T, x, y) := \sum_{r \geq 0} \tilde{p}^N \otimes_N$

$H^{(r)}(T, x, y)$ which is the analogous of (3.6) up to the discrete time convolution (i.e. \otimes replaced by \otimes_N).

Lemma 41 (Time discretization) One has:

$$(p - p^d)(T, x, y) = \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} \left(p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d \right) (T, x, y) + h^M R_{M,1}(T, x, y)$$

with

$$\sum_{l=1}^{M-1} \left| \left(p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d \right) (T, x, y) \right| + |R_{M,1}(T, x, y)| \leq \rho_{\alpha, M}(T, y - x).$$

Then the comparison between p^d and p^N is controlled with the following

Lemma 42 (Comparison of the discrete densities) *The following expansion holds:*

$$(p^d - p^N)(T, x, y) = - \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} p^N \right] (T, x, y) \\ + h^M R_{M,2}(T, x, y)$$

where

$$R_{M,2}(T, x, y) = - \frac{1}{M!} \int_0^1 (1-\tau)^M \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{M+1} \tilde{p}_\tau^\Delta \right] (T, x, y) d\tau,$$

$$\forall t \in \{(t_i)_{i \in \llbracket 1, N \rrbracket}\}, \tilde{p}_\tau^\Delta(t_i, x, y) = \sum_{r=0}^{\infty} \tilde{p}_\tau \otimes_N H_N^{(r)}(t_i, x, y), \tilde{p}_0^\Delta = p^N, \\ \text{and } \forall \tau \in [0, 1], \tilde{p}_\tau(t, x, y) = \int_{\mathbb{R}^d} \tilde{p}^x(\tau h, x, z) \tilde{p}^y(t - \tau h, z, y) dz.$$

In particular $\tilde{p}_0(t, x, y) = \tilde{p}^y(t, x, y)$. Also,

$$\sum_{k=1}^{M-1} \left| \left(p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} p^N \right) (T, x, y) \right| + |R_{M,2}(T, x, y)| \leq \rho_{\alpha, M}(T, y - x).$$

Theorem 21 is then a direct consequence of Lemmas 41 and 42.

4.2 Proofs of the technical Lemmas

Proof of Lemma 41. We start from the recurrence relation for $r \in \mathbb{N}^*$

$$\tilde{p} \otimes H^{(r)} - \tilde{p} \otimes_N H^{(r)} = \left[(\tilde{p} \otimes H^{(r-1)}) \otimes H - (\tilde{p} \otimes H^{(r-1)}) \otimes_N H \right] \\ + \left[(\tilde{p} \otimes H^{(r-1)}) - (\tilde{p} \otimes_N H^{(r-1)}) \right] \otimes_N H.$$

Summing up these terms over $r \in \mathbb{N}^*$ and using the linearity of \otimes and \otimes_N we get $p - p^d = p \otimes H - p \otimes_N H + (p - p^d) \otimes_N H$. An iterative application of this identity yields

$$(p - p^d)(T, x, y) = \sum_{r=0}^{\infty} [p \otimes H - p \otimes_N H] \otimes_N H^{(r)}(T, x, y). \quad (4.1)$$

By definition, for all $k \in \llbracket 1, N \rrbracket$,

$$[p \otimes H - p \otimes_N H](t_k, x, y) = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} ds \int [p(s, x, z) H(t_k - s, z, y) \\ - p(t_j, x, z) H(t_k - t_j, z, y)] dz. \quad (4.2)$$

A Taylor expansion of the function $\theta(s, z) := p(s, x, z)H(t_k - s, z, y)$ in the interval $[t_j, s] \subseteq [t_j, t_{j+1}]$ gives

$$\begin{aligned} \int [\theta(s, z) - \theta(t_j, z)] dz &= \sum_{l=1}^{M-1} \frac{(s-t_j)^l}{l!} \int \partial_\tau^l \theta(\tau, z) \Big|_{\tau=t_j} dz + \\ &\frac{(s-t_j)^M}{(M-1)!} \int_0^1 (1-\delta)^{M-1} \int \partial_\tau^M \theta(\tau, z) \Big|_{\tau=\tau_j(s,\delta)} dz d\delta, \end{aligned} \quad (4.3)$$

where $\tau_j(s, \delta) = t_j + \delta(s - t_j)$. Note now that $-\partial_s p(t-s, x, z) = \Phi p(t-s, x, z)$, $\partial_t p(t-s, x, z) = {}^t\Phi p(t-s, x, z)$. Here ${}^t\Phi = {}^t\tilde{\Phi}_z$ is the adjoint operator of Φ where the derivatives have to be taken w.r.t. z . Hence, $\Phi p(t-s, x, z) = {}^t\Phi p(t-s, x, z)$. The same identity also holds for \tilde{p} with $\Phi, {}^t\Phi$ respectively replaced by $\tilde{\Phi}, {}^t\tilde{\Phi}^*$. We therefore derive

$$\begin{aligned} \int \partial_\tau \theta(\tau, z) \Big|_{\tau=t_j} dz &= \int \partial_\tau [p(\tau, x, z)] \Big|_{\tau=t_j} H(t_k - t_j, z, y) dz \\ &+ \int p(t_j, x, z) \partial_\tau [H(t_k - \tau, z, y)] \Big|_{\tau=t_j} dz \\ &= \int {}^t\tilde{\Phi}_z p(t_j, x, z) (\Phi - \tilde{\Phi}^*) \tilde{p}(t_k - t_j, z, y) dz \\ &- \int p(t_j, x, z) (\Phi - \tilde{\Phi}^*) \tilde{\Phi}^* \tilde{p}(t_k - t_j, z, y) dz \\ &= \int p(t_j, x, z) (\Phi - \tilde{\Phi}^*)^2 \tilde{p}(t_k - t_j, z, y) dz. \end{aligned}$$

Iterating the differentiation we get

$$\int \partial_\tau^l \theta(\tau, z) \Big|_{\tau=t_j} dz = \int p(t_j, x, z) (\Phi - \tilde{\Phi}^*)^{l+1} \tilde{p}(t_k - t_j, z, y) dz, \quad (4.4)$$

where we recall that for two operators A and B we denote by $(A - B)^k$ the following sum $(A - B)^k = \sum_{j=0}^k C_k^j A^{k-j} (-B)^j$.

Plugging (4.3) and (4.4) into (4.2) we get

$$\begin{aligned} [p \otimes H - p \otimes_N H](t_k, x, y) &= \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} \tilde{p}(t_k, x, y) \\ &+ h^M \tilde{R}_{M,1}(t_k, x, y) \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} \tilde{R}_{M,1}(t_k, x, y) &= \frac{1}{(M-1)!} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} [h^{-1}(s-t_j)]^M \int_0^1 (1-\delta)^{M-1} \times \\ &\int \partial_\tau^M [p(\tau, x, z) H(t_k - \tau, z, y)] \Big|_{\tau=\tau_j(s,\delta)} ds dz d\delta. \end{aligned} \quad (4.6)$$

Plugging (4.5) and (4.6) into (4.1) we get

$$\begin{aligned} (p - p^d)(T, x, y) &= \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} \times \sum_{r=0}^{\infty} p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} \tilde{p} \otimes_N H^{(r)}(T, x, y) \\ &+ h^M R_{M,1}(T, x, y) \end{aligned} \quad (4.7)$$

with $R_{M,1}(T, x, y) = \sum_{r=0}^{\infty} (\tilde{R}_{M,1} \otimes_N H^{(r)})(T, x, y)$.

Now we apply that for a linear operator S and its adjoint ${}^t S$ we have $p \otimes_N S \tilde{p} = {}^t S p \otimes_N \tilde{p}$. This gives

$$\begin{aligned} & \sum_{r=0}^{\infty} p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} \tilde{p} \otimes_N H^{(r)}(T, x, y) = \\ & {}^t \left[(\Phi - \tilde{\Phi}^*)^{l+1} \right] p \otimes_N \sum_{r=0}^{\infty} (\tilde{p} \otimes_N H^{(r)})(T, x, y) = p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d(T, x, y), \end{aligned}$$

which plugged into (4.7) gives the desired expansion. The stated bound follows by application of the estimates given in Lemma 43 below. We only give the proof for the first summand, the other terms of the sum over l and the remainder $R_{M,1}(T, x, y)$ can be handled in a similar way. Write

$$\begin{aligned} p \otimes_N (\Phi - \tilde{\Phi}^*)^2 p^d(T, x, y) &= \sum_{j=0}^{N-1} h \int p(t_j, x, z) (\Phi - \tilde{\Phi}^*)^2 p^d(T - t_j, z, y) dz \\ &:= S_1 + S_2, \end{aligned}$$

where in S_1 (resp. S_2) the sum is taken over $I_1 := \{j \in \llbracket 0, \lfloor \frac{N-1}{2} \rrbracket\}$ (resp. $I_2 := \{j \in \llbracket \lfloor \frac{N-1}{2} \rrbracket + 1, N-1 \rrbracket\}$). For S_1 (resp. S_2), $p^d(T - t_j, z, y)$ (resp. $p(t_j, x, z)$) is non singular. From Lemma 43 below equation (4.12), there exists $C := C(T)$ s.t. for all $(x, y, z) \in (\mathbb{R}^d)^3$,

$$\begin{aligned} p(s, x, z) &\leq C \tilde{p}^y(s, x, z), \quad p^d(s, z, y) \leq C \tilde{p}^y(s, z, y), \quad \forall s \in]0, T], \\ |(\Phi - \tilde{\Phi}^*)^2 p^d(T - t_j, z, y)| &\leq C \tilde{p}^y(T - t_j, z, y), \quad \forall j \in I_1, \\ \left| {}^t \left[(\Phi - \tilde{\Phi}^*)^2 \right] p(t_j, x, z) \right| &\leq C \tilde{p}^y(t_j, x, z), \quad \forall j \in I_2. \end{aligned} \quad (4.8)$$

The semigroup property for \tilde{p}^y yields $|S_1| + |S_2| \leq C \tilde{p}(T, x, y)$. One eventually checks from Proposition A1 that $\tilde{p}(T, x, y) := \tilde{p}^y(T, x, y) \leq \rho_{\alpha, M}(T, y - x)$. \square

Proof of Lemma 42. Let us denote by $\mathcal{F}[\psi](z) = \int \exp(i\langle z, p \rangle) \psi(p) dp$ the Fourier transform of a function ψ . Introduce now for all u, t , $u < t$, $u, t \in \{(t_i)_{i \in \llbracket 0, N \rrbracket}\}$, $p \in \mathbb{R}^d$,

$$\begin{aligned} \psi(p) &= h(L_N - \tilde{L}_N^y) \tilde{p}^y(t - u, x, p) \\ &= \int p^N(h, x, w) \tilde{p}^y(t - (u + h), w, p) dw - \tilde{p}^y(t - u, x, p). \end{aligned}$$

Note that in particular according to (3.8), $\psi(y) = h H_N(t - u, x, y)$. Taking the characteristic functions of the densities involved in the above equation, we obtain from (3.3) and (3.8) that

$$\mathcal{F}[\psi](z) := G_z(1) - G_z(0)$$

with

$$\begin{aligned} G_z(\tau) &= \exp \left[i \langle x, z \rangle + i(t - u) \langle \gamma(y), z \rangle + i\tau h \langle \Delta \gamma^{x, y}, z \rangle \right. \\ &\quad \left. - \int_{S^{d-1}} |\langle z, s \rangle|^\alpha \left[(t - u) \lambda_{f(y)}(ds) + \tau h \Delta \lambda^{x, y}(ds) \right] \right], \end{aligned}$$

where $\Delta\gamma^{x,y} = \gamma(x) - \gamma(y)$, $\Delta\lambda^{x,y}(ds) = \lambda_{f(x)}(ds) - \lambda_{f(y)}(ds)$. Note in particular that $\forall \tau \in [0, 1]$,

$$G_z(\tau) = G_z(0) \times \exp\left(\tau h \left[i \langle \Delta\gamma^{x,y}, z \rangle - \int_{S^{d-1}} | \langle z, s \rangle |^\alpha \Delta\lambda^{x,y}(ds) \right]\right). \quad (4.9)$$

A Taylor expansion yields $\mathcal{F}[\psi](z) = \sum_{k=1}^M \frac{1}{k!} G_z^{(k)}(0) + \frac{1}{M!} \int_0^1 (1-\tau)^M G_z^{(M+1)}(\tau) d\tau$. From (4.9), one derives that for $k \in \mathbb{N}^*$:

$$\frac{1}{k!} G_z^{(k)}(0) = \frac{h^k}{k!} G_z(0) \times \left[i \langle \Delta\gamma^{x,y}, z \rangle - \int_{S^{d-1}} | \langle z, s \rangle |^\alpha \Delta\lambda^{x,y}(ds) \right]^k.$$

Observe now that $G_z(0) = \mathcal{F}[\theta](z)$, $\theta(p) := \tilde{p}^y(t-u, x, p)$. Using the well-known properties of the Fourier transform one gets for all $k \in \llbracket 1, M \rrbracket$

$$G_z^{(k)}(0) = h^k \mathcal{F} \left[\left(\tilde{\Phi}_\xi - \tilde{\Phi}_y \right)^k \theta \right] \Big|_{\xi=x} (z),$$

where the operators $\tilde{\Phi}$ are applied w.r.t. the x component and the Fourier transform is applied w.r.t. the p component of $\tilde{p}^y(t-u, x, p)$. Also, in the above writing, we compute the Fourier transform for an arbitrary fixed $\xi \in \mathbb{R}^d$ and we then put $\xi = x$.

Hence,

$$\begin{aligned} \mathcal{F}[\psi](z) &= \sum_{k=1}^M \frac{1}{k!} G_z^{(k)}(0) + \frac{1}{M!} \int_0^1 (1-\tau)^M G_z^{(M+1)}(\tau) d\tau = \\ &= \sum_{k=1}^M \frac{h^k}{k!} \mathcal{F} \left[\left(\tilde{\Phi}_\xi - \tilde{\Phi}_y \right)^k \theta \right] \Big|_{\xi=x} (z) + \\ &= \frac{h^{M+1}}{M!} \mathcal{F} \left[\int_0^1 (1-\tau)^M \left[\left(\tilde{\Phi}_\xi - \tilde{\Phi}_y \right)^{M+1} \theta_\tau \right] \Big|_{\xi=x} d\tau \right] (z), \end{aligned}$$

where $\forall \tau \in [0, 1]$, $\theta_\tau(p) := \int_{\mathbb{R}^d} \tilde{p}^x(\tau h, x, z) \tilde{p}^y(t-u-\tau h, z, p) dz$. Taking the inverse Fourier transform and putting $p = y$ in the above equation, observing that $H(t-u, x, y) = (\tilde{\Phi}_* - \tilde{\Phi}^*) \tilde{p}^y(t-u, x, y)$, we obtain

$$\begin{aligned} (H_N - H)(t-u, x, y) &= \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} \left(\tilde{\Phi}_* - \tilde{\Phi}^* \right)^{k+1} \tilde{p}(t-u, x, y) + \\ &= \frac{h^M}{M!} \int_0^1 (1-\tau)^M \left(\tilde{\Phi}_* - \tilde{\Phi}^* \right)^{M+1} \tilde{p}_\tau(t-u, x, y) d\tau. \quad (4.10) \end{aligned}$$

Recall now that

$$(p^d - p^N)(T, x, y) = \sum_{r=0}^{\infty} [(\tilde{p} \otimes_N H^{(r)}) - (\tilde{p} \otimes_N H_N^{(r)})](T, x, y)$$

where we put $(\tilde{p} \otimes_N H_N^{(r)})(T, x, y) = 0$ for $hr > T$. Summing over $r \in \mathbb{N}$ in the identity

$$\begin{aligned} & (\tilde{p} \otimes_N H^{(r)} - \tilde{p} \otimes_N H_N^{(r)})(T, x, y) = \\ & \left((\tilde{p} \otimes_N H^{(r-1)}) \otimes_N (H - H_N) \right)(T, x, y) + \\ & \left((\tilde{p} \otimes_N H^{(r-1)} - \tilde{p} \otimes_N H_N^{(r-1)}) \otimes_N H_N \right)(T, x, y) \end{aligned}$$

one gets

$$(p^d - p^N)(T, x, y) = [p^d \otimes_N (H - H_N) + (p^d - p^N) \otimes_N H_N](T, x, y).$$

By iterative application of the last identity we obtain

$$(p^d - p^N)(T, x, y) = \sum_{r=0}^{\infty} [p^d \otimes_N (H - H_N)] \otimes_N H_N^{(r)}(T, x, y).$$

We get from (4.10) that for all $t \in \{(t_i)_{i \in \llbracket 1, N \rrbracket}\}$:

$$\begin{aligned} (p^d \otimes_N (H - H_N))(T, x, y) &= - \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} \tilde{p} \right](T, x, y) \\ &\quad - \frac{h^M}{M!} \int_0^1 (1-\tau)^M \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{M+1} \tilde{p}_\tau \right](T, x, y) d\tau. \end{aligned}$$

Eventually,

$$\begin{aligned} (p^d - p^N)(T, x, y) &= - \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} p^N \right](T, x, y) \\ &\quad + h^M R_{M,2}(T, x, y), \end{aligned}$$

$$R_{M,2}(T, x, y) = - \frac{1}{M!} \int_0^1 (1-\tau)^M \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{M+1} \tilde{p}_\tau^\Delta \right](T, x, y) d\tau$$

$$\forall t \in \{(t_i)_{i \in \llbracket 1, N \rrbracket}\}, \tilde{p}_\tau^\Delta(t, x, y) = \sum_{r=0}^{\infty} \tilde{p}_\tau \otimes_N H_N^{(r)}(t, x, y), \tilde{p}_0^\Delta = p^N.$$

This proves the expansion part of the Lemma. The bound follows as in the previous proof from Lemma 43. \square

We now state Lemma 43 that allows to control the rests appearing in the expansions of Lemmas 41 and 42.

Lemma 43 *Let $q > d + 4$. For all multi-indices a, b s.t. $|a| + |b| < q - (d + 4)$, the following inequalities hold:*

$$\left| D_y^a D_x^b p^d(t_k, x, y) \right| + \left| D_y^a D_x^b p^N(t_k, x, y) \right| \leq C t_k^{-\frac{|a|+|b|}{\alpha}} \tilde{p}(t_k, x, y), k \in \llbracket 1, n \rrbracket, \quad (4.11)$$

$$\left| D_y^a D_x^b p(t, x, y) \right| \leq C t^{-\frac{|a|+|b|}{\alpha}} \tilde{p}(t, x, y), 0 < t \leq T.$$

Also, $\exists C := C(T)$ s.t. for all $(x, y, z) \in (\mathbb{R}^d)^3$, $s \in]0, T]$,

$$\begin{aligned} p(s, x, z) &\leq C \tilde{p}^y(s, x, z), \quad p^d(s, z, y) \leq C \tilde{p}^y(s, z, y), \\ |(\Phi - \tilde{\Phi}^*)^k p^d(s, z, y)| &\leq C s^{-\frac{|k|}{\alpha}} \tilde{p}^y(s, z, y), \\ \left| t \left[(\Phi - \tilde{\Phi}^*)^k \right] p(s, x, z) \right| &\leq C s^{-\frac{|k|}{\alpha}} \tilde{p}^y(s, x, z). \end{aligned} \quad (4.12)$$

To conclude the proof it remains to prove Lemma 43 The first step is to get bounds on partial derivatives of the transition densities \tilde{p} and p . The following estimates generalize the ones obtained in [Kol00], Propositions 2.1-2.3.

Lemma 44 *Let $q > d + 4$. There exists a constant $C > 1$ such that the following estimates hold uniformly for α in any compact subset of the interval $(0, 2)$ and for all $0 < t \leq T$, $x, y, z \in \mathbb{R}^d$ and $|a| < q - (d + 4)$*

$$\left| D_z^a \tilde{p}^y(t, x, z) \right| \leq \frac{C}{t^{|a|/\alpha}} \tilde{p}^y(t, x, z), \quad (4.13)$$

$$\left| D_z^a \tilde{p}^y(t, x, z) \right| \leq \frac{C}{|z - \gamma(y)t - x|^{|a|}} \tilde{p}^y(t, x, z). \quad (4.14)$$

Remark 41 *Equation (4.13) extends to the stable case what is widely known in the Gaussian framework. Namely, each derivation of the density in space remains homogeneous to a stable density up to a multiplicative additional singularity of order $t^{-1/\alpha}$.*

Proof. From now on we assume w.l.o.g. that $d \geq 3$, the cases $d \in \{1, 2\}$ can be addressed more directly. To proceed with the computations, we need to specify a useful change of coordinates. Namely, for a given direction $\zeta \in \mathbb{R}^d \setminus \{0\}$ introduce for $p \in \mathbb{R}^d$ the spherical coordinates $(\rho, \vartheta, \varphi_2, \dots, \varphi_{d-1})$, $\rho = |p|$ with first coordinate or main axis directed along ζ , that is

$$\begin{aligned} p_1 &= \rho \cos \vartheta, \quad p_2 = \rho \sin \vartheta \cos \varphi_2, \quad p_3 = \rho \sin \vartheta \sin \varphi_2 \cos \varphi_3, \dots \\ p_{d-1} &= \rho \sin \vartheta \sin \varphi_2 \dots \sin \varphi_{d-2} \cos \varphi_{d-1}, \\ p_d &= \rho \sin \vartheta \sin \varphi_2 \dots \sin \varphi_{d-2} \sin \varphi_{d-1}, \end{aligned} \quad (4.15)$$

$\vartheta \in [0, \pi]$, $\varphi_i \in [0, \pi]$, $i \in [2, d-2]$, $\varphi_{d-1} \in [0, 2\pi]$. Consider then the coordinates (v, τ, ϕ) where $\tau = \cos \vartheta$ and $v = \rho |\zeta|$, with $v \in \mathbb{R}^+$, $\tau \in [-1, 1]$, $\phi = (\varphi_2, \dots, \varphi_{d-1}) \in [0, \pi]^{d-3} \times [0, 2\pi]$. In the following we write $p = p(v, \tau, \phi)$ for the previous r.h.s. in (4.15) written in these new coordinates that is

$$\begin{aligned} p_1 &= |\zeta|^{-1} v \tau, \quad p_2 = |\zeta|^{-1} v (1 - \tau^2)^{1/2} \cos \varphi_2, \\ p_3 &= |\zeta|^{-1} v (1 - \tau^2)^{1/2} \sin \varphi_2 \cos \varphi_3, \dots \\ p_{d-1} &= |\zeta|^{-1} v (-\tau^2)^{1/2} \sin \varphi_2 \dots \sin \varphi_{d-2} \cos \varphi_{d-1}, \\ p_d &= |\zeta|^{-1} v (-\tau^2)^{1/2} \sin \varphi_2 \dots \sin \varphi_{d-2} \sin \varphi_{d-1}, \end{aligned} \quad (4.16)$$

and $\bar{p}(\tau, \phi) = p(|\zeta|, \tau, \phi)$.

Without loss of generality we suppose $\gamma(y) = 0$. The first step consists in differentiating w.r.t z the inverse Fourier transform for $\tilde{p}^y(t, x, z)$

$$\tilde{p}^y(t, x, z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp \left\{ -t \int_{S^{d-1}} |\langle p, s \rangle|^\alpha \lambda_{f(y)}(ds) \right\} \exp(-i \langle p, z - x \rangle) dp. \quad (4.17)$$

For $z = x$, (2.5) and standard computations directly give estimate (4.13). Thus, in the following we also assume $z \neq x$ and use the previous spherical coordinates (v, τ, ϕ) derived from (4.15) setting $\zeta = z - x$ as the main axis. We obtain:

$$D_z^a \tilde{p}^y(t, x, z) = \frac{1}{(2\pi)^d |z-x|^{|a|+d}} \int_0^\infty dv v^{|a|+d-1} \times \\ \int_{-1}^1 d\tau \int_{[0, \pi]^{d-3} \times [0, 2\pi]} d\phi \Psi(v, \tau, |a|) \exp \left\{ -t \frac{v^\alpha}{|z-x|^\alpha} \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda_{f(y)}(ds) \right\} \times \\ \tau^{a_1} (1-\tau^2)^{\frac{|a|-a_1+d-3}{2}} h(\phi, a), \quad (4.18)$$

where $\bar{p} = p/|p|$, $a = (a_1, \dots, a_d) \in \mathbb{N}^d$ and $\Psi(v, \tau, |a|) = (-1)^{|a|/2} \cos(v\tau) \mathbb{I}_{|a| \text{ even}} + (-1)^{(|a|+1)/2} \sin(v\tau) \mathbb{I}_{|a| \text{ odd}}$,

$$h(\phi, a) = \{ (\cos \varphi_2)^{a_2} (\sin \varphi_2 \cos \varphi_3)^{a_3} \times \dots \times (\sin \varphi_2 \dots \sin \varphi_{d-2} \cos \varphi_{d-1})^{a_{d-1}} \\ \times (\sin \varphi_2 \dots \sin \varphi_{d-2} \sin \varphi_{d-1})^{a_d} \} \times V(\phi) \\ V(\phi) = (\sin \varphi_2)^{d-3} (\sin \varphi_3)^{d-4} \times \dots \times (\sin \varphi_{d-3})^2 \sin \varphi_{d-2}.$$

We consider, first the case $|z-x|/t^{1/\alpha} \leq \bar{C}$, for a sufficiently small positive constant \bar{C} . In this case we expand the trigonometric function $\Psi(v, \tau, |a|)$ in (4.18) in power series and change the variable of integration $\frac{t^{1/\alpha} v}{|z-x|}$ to w in each term. This gives for all $k \in \mathbb{N}$,

$$D_z^a \tilde{p}^y(t, x, z) = \frac{C_{|a|}}{t^{\frac{|a|+d}{\alpha}}} \left\{ \sum_{m=0}^k \frac{(-1)^m}{(2m + \mathbb{I}_{|a| \text{ odd}})!} e_m^{|a|} \left(\frac{|z-x|}{t^{1/\alpha}} \right)^{2m + \mathbb{I}_{|a| \text{ odd}}} \right. \\ \left. + R_{k+1}^{|a|} \right\}, \quad C_{|a|} = \frac{(-1)^{(|a| + \mathbb{I}_{|a| \text{ odd}})/2}}{(2\pi)^d}, \quad (4.19)$$

where $\forall m \in [1, k]$,

$$e_m^{|a|} = \int_0^\infty dw \int_{-1}^1 d\tau \int_{[0, \pi]^{d-3} \times [0, 2\pi]} d\phi \exp \left\{ -w^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda_{f(y)}(ds) \right\} \\ w^{|a|+2m+d-\mathbb{I}_{|a| \text{ even}}} \times \tau^{a_1+2m+\mathbb{I}_{|a| \text{ odd}}} (1-\tau^2)^{\frac{|a|-a_1+d-3}{2}} h(\phi, a), \\ |R_{k+1}^{|a|}| \leq \frac{|e_{k+1}^{|a|}|}{(2(k+1) + \mathbb{I}_{|a| \text{ odd}})!} \left(\frac{|z-x|}{t^{1/\alpha}} \right)^{2(k+1) + \mathbb{I}_{|a| \text{ odd}}}. \quad (4.20)$$

To simplify the notations we omit the dependence of the coefficients of our expansions on the direction $\zeta = z - x$. From **(A-1)**, **(A-2)** and (2.5) one then derives the following bound:

$$\left| e_m^{|a|} \right| \leq \frac{A_{d-2}}{\alpha C_1^{\frac{|a|+2m+d+\mathbb{I}_{|a| \text{ odd}}}{\alpha}}} \Gamma \left(\frac{|a| + 2m + d + \mathbb{I}_{|a| \text{ odd}}}{\alpha} \right) \\ B \left(m + \frac{a_1 + 1 + \mathbb{I}_{|a| \text{ odd}}}{2}, \frac{|a| - a_1 + d - 1}{2} \right). \quad (4.21)$$

Here A_{d-2} denotes the area of the unit sphere S^{d-2} and B is the β -function. Note that the modulus of each term in the expansion (4.19) serves as an estimate of the remainder in a finite Taylor expansion. From (4.19) we have

$$D_z^a \tilde{p}^y(t, x, z) = \frac{C_{|a|}}{t^{\frac{|a|+d}{\alpha}}} \left(e_0^{|a|} \left(\frac{|z-x|}{t^{1/\alpha}} \right)^{\mathbb{I}_{|a|\text{ odd}}} + R_1^{|a|} \right). \quad (4.22)$$

By Proposition 3.1 (i) from [Kol00] for some \tilde{C} depending on \bar{C} , $\tilde{C}^{-1} t^{-d/\alpha} \leq \tilde{p}^y(t, x, z) \leq \tilde{C} t^{-d/\alpha}$. Hence, recalling that we are considering the case $\frac{|z-x|}{t^{1/\alpha}} \leq \bar{C}$ equations (4.22), (4.21), (4.20) yield

$$|D_z^a \tilde{p}^y(t, x, z)| \leq \frac{C}{t^{\frac{|a|}{\alpha}}} \tilde{p}^y(t, x, z) \leq \frac{C \bar{C}^{|a|}}{|z-x|^{|a|}} \tilde{p}^y(t, x, z). \quad (4.23)$$

To estimate $D_z^a \tilde{p}^y(t, x, z)$ for $|z-x|/t^{1/\alpha} \geq (\bar{C})^{-1}$ we proceed as in Proposition 2.3 of [Kol00]. This gives the following representation $D_z^a \tilde{p}^y(t, x, z) = [D_z^a \tilde{p}^y(t, x, z)]_1 + [D_z^a \tilde{p}^y(t, x, z)]_2$ with

$$\begin{aligned} [D_z^a \tilde{p}^y(t, x, z)]_j &= \frac{1}{(2\pi)^d} \int_0^\infty d\rho \rho^{|a|+d-1} \int_{-1}^1 d\tau \Psi(\rho|z-x|, \tau, |a|) \times \\ & f_j(\tau) \int_{[0, \pi]^{d-3} \times [0, 2\pi]} d\phi \exp\left\{-t\rho^\alpha g_{\lambda_f}(\tau, \phi, y)\right\} h(\phi, a), \quad j = 1, 2, \quad (4.24) \\ g_{\lambda_f}(\tau, \phi, y) &:= \int_{S^{d-1}} |\langle \bar{p}(\tau, \phi), s \rangle|^\alpha \lambda_{f(y)}(ds), \end{aligned}$$

using the notations introduced after (4.15). Here

$$f_1(\tau) = \tau^{a_1} \left(1 - \tau^2\right)^{\frac{|a|-a_1+d-3}{2}} \chi(\tau), \quad f_2(\tau) = \tau^{a_1} \left(1 - \tau^2\right)^{\frac{|a|-a_1+d-3}{2}} (1 - \chi(\tau))$$

where $\chi(\tau)$ is a C^∞ even truncation function $\mathbb{R} \rightarrow [0, 1]$ that equals 1 for $|\tau| \leq 1 - 2\varepsilon$, and 0 for $|\tau| \geq 1 - \varepsilon$ for some $\varepsilon \in (0, \frac{1}{2})$. Because of the symmetry in τ , it is easy to see that the integral in (4.24) is non-zero only if a_1 and $|a|$ are both even or odd. Expanding the exponential at order 2 in (4.24) and making the change of variables $\rho|z-x| = v$ we get

$$[D_z^a \tilde{p}^y(t, x, z)]_1 = \frac{C_{|a|}}{|z-x|^{|a|+d}} \sum_{m=0}^2 \frac{1}{m!} b_m^{|a|} \left(\frac{t}{|z-x|^\alpha} \right)^m, \quad (4.25)$$

where $C_{|a|}$ is defined in (4.19) and for $m \in [0, 1]$,

$$\begin{aligned} b_m^{|a|} &= (-1)^m \int_0^\infty F_m^{|a|}(v) v^{|a|+m\alpha+d-1} dv, \\ F_m^{|a|}(v) &= [\mathbb{I}_{|a| \text{ even}} \text{Re} - \mathbb{I}_{|a| \text{ odd}} \text{Im}] \left[\int_{-\infty}^\infty \exp(-iv\tau) \varphi_m(\tau) d\tau \right], \\ \varphi_m(\tau) &= f_1(\tau) \int_{[0, \pi]^{d-3} \times [0, 2\pi]} g_{\lambda_f}^m(\tau, \phi, y) h(\phi, a) d\phi. \end{aligned}$$

and

$$b_2^{|a|} = 2 \int_0^1 (1-\delta) \int_0^\infty F_{2,\delta}^{|a|}(v) v^{|a|+2\alpha+d-1} dv d\delta,$$

$$F_{2,\delta}^{|a|}(v) = [\mathbb{I}_{|a| \text{ even}} \text{Re} - \mathbb{I}_{|a| \text{ odd}} \text{Im}] \left[\int_{-\infty}^\infty \exp(-iv\tau) \varphi_{2,\delta}(\tau) d\tau \right],$$

$$\varphi_{2,\delta}(\tau) = f_1(\tau) \int_{[0,\pi]^{d-3} \times [0,2\pi]} g_{\lambda_f}^2(\tau, \phi, y) \exp \left\{ -\delta t \left(\frac{v}{|z-x|} \right)^\alpha g_{\lambda_f}(\tau, \phi, y) \right\} \\ \times h(\phi, a) d\phi.$$

To extend the integration to \mathbb{R} in the definition of $(F_m^{|a|}(v))_{m \in \llbracket 0,1 \rrbracket}$, $F_{2,\delta}^{|a|}(v)$, we simply use that the functions $(\varphi_m)_{m \in \llbracket 0,1 \rrbracket}$, $\varphi_{2,\delta}$ have compact support in τ . However, to check that the coefficients $(b_m^{|a|})_{m \in \llbracket 0,2 \rrbracket}$ are well defined, we have to equilibrate at infinity the term in $(v^{|a|+m\alpha+d-1})_{m \in \llbracket 0,2 \rrbracket}$. This can be done computing iterated integration by parts in τ in the definition of $(F_m^{|a|}(v))_{m \in \llbracket 0,1 \rrbracket}$, $F_{2,\delta}^{|a|}(v)$. Namely, if $q > |a| + 4 + d > |a| + 2\alpha + d$, performing q integrations by parts w.r.t. τ one derives that the coefficients $(b_m^{|a|})_{m \in \llbracket 0,2 \rrbracket}$ are well defined. Let us now check that assumption **(A-1)** implies that $\varphi_m(\tau)$, $m = 0, 1$, and $\varphi_{2,\delta}(\tau)$ are C^q functions of τ with compact support. Indeed, for the unit vectors $\bar{p}(\tau + \Delta\tau, \phi)$ and $\bar{p}(\tau, \phi)$, from elementary algebra there exists an orthogonal matrix $A := A(\Delta\tau)$ s.t. $\bar{p}(\tau + \Delta\tau, \phi) = A\bar{p}(\tau, \phi)$. Hence, if $\lambda_{f(x)}(ds) = \Theta(x, s) ds$ where Θ has the previous smoothness one can show

$$\lim_{\Delta\tau \rightarrow 0} \frac{g_{\lambda_f}(\tau + \Delta\tau, \phi, x) - g_{\lambda_f}(\tau, \phi, x)}{\Delta\tau}$$

$$= \lim_{\Delta\tau \rightarrow 0} \frac{\int_{S^{d-1}} \{ |\langle \bar{p}(\tau, \phi), A^* s \rangle|^\alpha - |\langle \bar{p}(\tau, \phi), s \rangle|^\alpha \} \lambda_{f(x)}(ds)}{\Delta\tau}$$

$$= \int_{S^{d-1}} |\langle \bar{p}(\tau, \phi), s \rangle|^\alpha \lim_{\Delta\tau \rightarrow 0} \frac{[\Theta(x, As) - \Theta(x, s)]}{\Delta\tau} ds$$

$$= \int_{S^{d-1}} |\langle \bar{p}(\tau, \phi), s \rangle|^\alpha \Theta'_s(x, s) \beta(\tau, \phi, s) ds,$$

where $\beta(\tau, \phi, s)$ is C^∞ function in τ uniformly bounded in (τ, ϕ, s) in our region. The process can then be iterated other $q - 1$ times.

Thus all coefficients $(b_m^{|a|})_{m \in \llbracket 0,2 \rrbracket}$ are well defined.

Next, analogously to Proposition 2.3 in [Kol00] (where the case $|a| = 0$ was considered) and with the same rotations of the integration contours for $\alpha \in (0, 1]$, $\alpha \in (1, 2)$, we obtain for all $k \in \mathbb{N}$

$$[D_z^\alpha \tilde{p}^y(t, x, z)]_2 = \frac{C_{|a|}}{|z-x|^{|a|+d}} \left\{ \sum_{m=0}^k \frac{1}{m!} c_m^{|a|} \left(\frac{t}{|z-x|^\alpha} \right)^m + R_{2,k+1}^{|a|} \right\}, \quad (4.26)$$

$$c_m^{|a|} = 2[\mathbb{I}_{|a| \text{ even}} \text{Re} - \mathbb{I}_{|a| \text{ odd}} \text{Im}] \left[\int_{1-2\varepsilon}^1 d\tau \int_{[0,\pi]^{d-3} \times [0,2\pi]} d\phi h(\phi, a) (-g_{\lambda_f}(\tau, \phi))^m \right. \\ \left. \times \exp\left(-\frac{i\pi\alpha m}{2}\right) (-i)^{|a|+d} \tau^{-(\alpha m+d+|a|)} \Gamma(\alpha m + d + |a|) f_2(\tau) \right],$$

and $|R_{2,k+1}^{[a]}| \leq \frac{|c_{k+1}^{[a]}|}{(k+1)!} \left(\frac{t}{|x-z|^\alpha} \right)^{k+1}$. Note that the coefficients $c_m^{[a]}$ are also well defined because τ does not approach zero (recall that $1 - \chi(\tau) \neq 0 \Leftrightarrow |\tau| > 1 - 2\varepsilon$). Precisely $|c_m^{[a]}| \leq 2A_{d-2}C_2^m(1-2\varepsilon)^{-\alpha m+d+|a|}\Gamma(\alpha m+d+|a|)$.

Now the sum of expansions (4.25) and (4.26) gives the expansion for $D_z^a \tilde{p}^y(t, x, z)$. Note that by construction, the first coefficient $b_0^{[a]} + c_0^{[a]}$ does not depend on the spectral measure $\lambda_{f(y)}(\cdot)$ and it vanishes when the spectral measure is uniform (that is $C_1 = C_2 = 1$ in (2.5)). This can be shown by means of representations involving Bessel and Whittaker functions and the same rotations of the integration contours as in Proposition 2.2 of [Kol00], see Appendix B for details. Thus, for all $k \in \mathbb{N}^*$, we get a representation

$$D_z^a \tilde{p}^y(t, x, z) = \frac{C_{|a|}}{|z-x|^{|\alpha|+d}} \left\{ \sum_{m=1}^k \frac{1}{m!} d_m^{[a]} \left(\frac{t}{|z-x|^\alpha} \right)^m + R_{k+1}^{[a]} \right\}, \quad (4.27)$$

where $d_m^{[a]} = b_m^{[a]} + c_m^{[a]}$ with $b_m = 0$ for $m \geq 3$ and $|R_{k+1}^{[a]}| \leq \frac{|d_{k+1}^{[a]}|}{(k+1)!} \left(\frac{t}{|x-z|^\alpha} \right)^{k+1}$. Now, from Proposition 3.1 (ii) in [Kol00] $d_1^0 > 0$. Equation (4.27) yields

$$D_z^a \tilde{p}^y(t, x, z) = \frac{C_{|a|} d_1^0 t}{|z-x|^{|\alpha|+d+\alpha}} \left(\frac{d_1^{[a]}}{d_1^0} + \tilde{R}_2^{[a]} \right), \quad |\tilde{R}_2^{[a]}| \leq \frac{|d_2^{[a]}|}{2d_1^0} \frac{t}{|x-z|^\alpha},$$

$$\tilde{p}^y(t, x, z) = \frac{C_{|a|}}{|z-x|^d} \left(\frac{d_1^0 t}{|z-x|^\alpha} + R_2^0 \right) \geq \frac{C_{|a|} d_1^0 t}{2|z-x|^{d+\alpha}}$$

for sufficiently small \bar{C} . Hence, we have

$$\begin{aligned} |D_z^a \tilde{p}^y(t, x, z)| &\leq \frac{C C_{|a|}}{|z-x|^{|\alpha|}} \frac{d_1^0 t}{|z-x|^{d+\alpha}} \leq \frac{C}{|z-x|^{|\alpha|}} \tilde{p}^y(t, x, z) \\ &\leq \frac{C \bar{C}^{|\alpha|}}{t^{|\alpha|/\alpha}} \tilde{p}^y(t, x, z), \end{aligned} \quad (4.28)$$

recalling that $\frac{t^{1/\alpha}}{|z-x|} \leq \bar{C}$ for the last inequality. W.l.o.g. we can assume $\bar{C} < 1$. It remains to consider the case $|x-z|/t^{1/\alpha} \in]\bar{C}, \bar{C}^{-1}[:= I(\bar{C})$. It follows from (4.18) that $|z-x|^{d+\alpha} \tilde{p}^y(t, x, z)$ and $|z-x|^{d+|a|} D_z^a \tilde{p}^y(t, x, z)$ are continuous functions of $|x-z|/t^{1/\alpha}$. Since the stable density is also strictly positive, we deduce that there exists \tilde{C} s.t. on $I(\bar{C})$, $|D_z^a \tilde{p}^y(t, x, z)| \leq \frac{\tilde{C}}{|z-x|^{|\alpha|+d}} \leq \frac{C}{|z-x|^{|\alpha|}} \tilde{p}^y(t, x, z) \leq \frac{C \bar{C}^{|\alpha|}}{t^{|\alpha|/\alpha}} \tilde{p}^y(t, x, z)$ which concludes the proof. \square

Lemma 45 *Let $q > d + 4$. There exists a constant $C > 1$ s.t. the following estimates hold uniformly for α in any compact subset of the interval $(0, 2)$ and for all $0 < t \leq T$, $x, y, v \in \mathbb{R}^d$ and $|a| + |b| < q - (d + 4)$:*

$$\left| D_y^a D_x^b H(t, x, y) \right| \leq \frac{C}{t^{\frac{|\alpha|+|b|}{\alpha}}} \tilde{p}(t, x, y) \left(1 + \frac{\min(1, |y-x|)}{t} \right), \quad (4.29)$$

$$\left| D_x^b H(t, x, x+v) \right| \leq C \tilde{p}(t, x, x+v) \left(1 + \frac{\min(1, |v|)}{t} \right), \quad (4.30)$$

$$\left| D_y^a D_x^b \tilde{p}(t, x, y) \right| \leq \frac{C}{|y - \gamma(y)t - x|^{|\alpha|+|b|}} \tilde{p}(t, x, y). \quad (4.31)$$

Proof. Inequalities (4.29) and (4.30) follow from the representation

$$\begin{aligned} H(t, x, y) &= \langle \gamma(x) - \gamma(y), \nabla_x \tilde{p}(t, x, y) \rangle + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |p|^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \times \\ & (\lambda_{f(y)}(ds) - \lambda_{f(x)}(ds)) \exp \left\{ -t |p|^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda_{f(y)}(ds) \right\} \times \\ & \exp \{ -i \langle p, y - \gamma(y)t - x \rangle \} dp \end{aligned} \quad (4.32)$$

analogously to the proof of Proposition 2.3 in [Kol00], see also Appendix A where (4.29) is proved for $|a| = |b| = 0$. Inequality (4.30) contains in (3.23') p.748 of that reference. Inequality (4.31) can be derived following the proof of Lemma 44. \square

The proof of Lemma 43 can then be achieved from Lemmas 44 and 45 adapting the arguments in Appendix A concerning the control in terms of the frozen density for the "formal" series appearing in (3.6). See also the proof of Theorem 2.3 in [KM02] or Theorem 3.1 in [Kol00].

5 Extensions and conclusion

A careful examination of the previous proofs shows that the absolute continuity of λ w.r.t. to the Lebesgue measure of S^{d-1} can be removed in **(A-1)** provided the function

$$\begin{aligned} \zeta(t, x, y) &:= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\int_{S^{d-1}} |\langle p, s \rangle|^\alpha \lambda_{f(x)}(ds) \right) \exp \left(-t \int_{S^{d-1}} |\langle p, s \rangle|^\alpha \lambda_{f(y)}(ds) \right) \\ & \times \exp(-i \langle p, x \rangle) dp \end{aligned}$$

has bounded derivatives w.r.t. x up to order q (see Appendix A and the statement of Theorem 3.1 in [Kol00]). Also up to a standard perturbative argument, similar controls on the density can be obtained when we consider (1.1) driven by $(Z_s + P_s)_{s \geq 0}$ where $(P_s)_{s \geq 0}$ is a compound Poisson process with Lévy measure $\nu_P(dz) = f(z)dz$ and $|f(z)| \leq \frac{C}{1+|z|^{d+\beta}}$, $\beta > 0$, see Theorem 4.1 in [Kol00]. In that case our main results remain valid up to a modification of the remainder. Indeed, it is the smallest exponent (or equivalently the largest tail) that leads the asymptotic behavior of $p(t, x, y)$ when $|x - y|$ is large. Thus $\rho_{\alpha, M}(T, y - x)$ has to be replaced by $\rho_{\min(\alpha, \beta), M}(T, y - x)$ in Theorem 21. Eventually, good controls have been obtained on p for stable-like processes, i.e. when the stability index in the generator $A\psi(x)$ in (2.4) can depend on the spatial position x , i.e. α turns to $\alpha(x) \in [\underline{\alpha}, \bar{\alpha}]$ strictly included in $(0, 2]$ (see Section 5 in [Kol00]). Anyhow the processes associated to those generators cannot be approximated by a usual Euler scheme and the previous analysis breaks down. The approximation of such processes will concern further research.

A Control of the parametrix series of the density

For the sake of completeness we provide in this section a complete proof of the control for the r.h.s of (3.6) under our standing Assumptions **(A-1)**-**(A-3)**.

We first sum up in Proposition A1 the various estimates needed to control the convergence of (3.6) following the proof of Theorem 3.1 in [Kol00], namely Proposition 3.1 and its corollary, Lemma 3.1 and Propositions 3.2-3.3 of that reference. These estimates can also be directly derived from the computations of Appendix B.

Proposition A1 *For all K sufficiently large, there exists $C > 0$ s.t. the following estimates hold uniformly for α in any compact subset of $(0, 2)$, for all $x, y, z \in \mathbb{R}^d$ and for all $t \in (0, T]$.*

$$\begin{aligned} C^{-1}t^{-d/\alpha} &\leq \tilde{p}^y(t, x, z) \leq Ct^{-d/\alpha}, \quad |x - z| \leq Kt^{1/\alpha}, \\ \frac{C^{-1}t}{|x - z|^{d+\alpha}} &\leq \tilde{p}^y(t, x, z) \leq \frac{Ct}{|x - z|^{d+\alpha}}, \quad |x - z| \geq Kt^{1/\alpha}, \\ \tilde{p}^z(t, x, z) &\leq C\tilde{p}^y(t, x, z). \end{aligned}$$

Also, there exists $C > 0$ s.t. $\forall (t, x, y) \in [0, T] \times (\mathbb{R}^d)^2$,

$$\int dz \min(1, |z|) \tilde{p}^y(t, 0, z) \leq Ct^\omega, \quad \omega := \min(1, 1/\alpha).$$

For all $s \in (0, t)$

$$\begin{aligned} &\int dz \tilde{p}^z(t - s, x, z) \min(1, |y - z|) s^{-1} \tilde{p}^y(s, z, y) \\ &\leq C(t^{-1} \min(1, |y - x|) + s^{\omega-1}) \tilde{p}^y(t, x, y), \\ &\quad \int dz \min(1, |z - x|) \tilde{p}^z(t - s, x, z) \\ &\times \min(1, |y - z|) s^{-1} \tilde{p}^y(s, z, y) \leq Cs^{\omega-1} \tilde{p}^y(t, x, y), \\ &\quad \int dz \tilde{p}^z(t - s, x, z) \tilde{p}^y(s, z, y) \\ &\leq C\tilde{p}^y(t, x, y). \end{aligned}$$

Introduce now for a given bounded measure η on S^{d-1} the function

$$\begin{aligned} \varphi_\eta(t, z, \lambda_{f(y)}) &:= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dp |p|^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \eta(ds) \\ &\times \exp\left(-t|p|^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda_{f(y)}(ds)\right) \exp(-i\langle p, z \rangle). \end{aligned}$$

With this notation and (4.32) we get

$$H(t, x, y) = \langle \gamma(x) - \gamma(y), \nabla_x \tilde{p}(t, x, y) \rangle + (\varphi_{\lambda_{f(y)}} - \varphi_{\lambda_{f(x)}})(t, y - x - \gamma(y)t, \lambda_{f(y)}).$$

Under our standing assumptions, the mean value theorem yields $|\varphi_{\lambda_{f(y)}} - \varphi_{\lambda_{f(x)}}|(t, y - x - \gamma(y)t, \lambda_{f(y)}) \leq \min(1, |y - x|) \varphi_{\eta_{x,y}}(t, y - x - \gamma(y)t, \lambda_{f(y)})$, where $\eta_{x,y}$ is a bounded measure. Now Proposition 2.5 in [Kol00] states that for a bounded measure η ,

$$\varphi_\eta(t, z, \lambda_{f(y)}) \leq Ct^{-1} \tilde{p}^y(t, 0, z).$$

From Lemmas 44, 45 and the above controls one deduces $|H(t, x, y)| \leq C\tilde{p}^y(t, x, y)(1 + t^{-1} \min(1, |x - y|)) := Cv(t, x, y)$ (which actually gives (4.29) for $a = b = 0$).

Introduce now $\beta \circ \psi(t, s, x, y) = \int_{\mathbb{R}^d} \beta(t - s, x, z) \psi(s, z, y) dz$, i.e. \circ is the spatial part of the convolution operator \otimes , and set $\tilde{v}(t, x, y) := tv(t, x, y)$. From Proposition A1 one derives

$$\begin{aligned} \tilde{p} \circ v(t, s, x, y) &\leq C(v(t, x, y) + s^{\omega-1} \tilde{p}(t, x, y)) \\ \tilde{v} \circ v(t, s, x, y) &\leq C(\tilde{v}(t, x, y) + (s^{\omega-1} + (t - s)^\omega) \tilde{p}(t, x, y)). \end{aligned}$$

Recalling $|H(t, x, y)| \leq Cv(t, x, y)$, integrating the above inequalities one gets:

$$|\tilde{p} \otimes H|(t, x, y) \leq C(\tilde{v}(t, x, y) + t^\omega \tilde{p}(t, x, y)), \quad |\tilde{p} \otimes H \otimes H|(t, x, y) \leq C^2 t^\omega (\tilde{p} + \tilde{v})(t, x, y).$$

An induction yields, for all $k \in \mathbb{N}^*$:

$$\begin{aligned} |\tilde{p} \otimes H^{(2k)}|(t, x, y) &\leq \frac{C^{2k} t^{k\omega}}{(k!)^2} (\tilde{p} + \tilde{v})(t, x, y), \\ |\tilde{p} \otimes H^{(2k+1)}|(t, x, y) &\leq \frac{C^{2k+1} t^{k\omega}}{k!(k + \mathbb{I}_{\alpha \in (0,1]})!} (tv(t, x, y) + t^\omega \tilde{p}(t, x, y) \mathbb{I}_{\alpha > 1}), \end{aligned}$$

and the the required control, i.e. $p(t, x, y) \leq C\tilde{p}(t, x, y)$. The controls on the derivatives can be proved in a similar way, up to suitable rearrangements of the variable of integration, see p.747 and 748 in [Kol00]. Also the whole proof can be carried out for p^d, p^N . \square

Remark A1 *To conclude, note that by arguments similar to those used to prove Proposition A1, one gets*

$$|H \otimes H(t, x, y)| \leq Ct^{\omega-1}\tilde{p}(t, x, y),$$

which turns to be a useful estimate to derive (4.8) following the above proof.

B Additional computations concerning the derivatives of the density

In this section we prove that $b_0^{|a|} + c_0^{|a|} = 0$, justifying that the first index in (4.27) is one.

Odd dimensions d

From the definitions in the proof of Lemma 44 , it is enough to show

$$\begin{aligned} [\mathbb{I}_{|a| \text{ even Re}} - \mathbb{I}_{|a| \text{ odd Im}}] \left\{ \int_0^\infty \left[\int_{\mathbb{R}} \exp(-iv\tau) f_1(\tau) d\tau \right] v^{|a|+d-1} dv \right. \\ \left. + 2 \int_{1-2\varepsilon}^1 (i\tau)^{-(d+|a|)} \Gamma(d+|a|) f_2(\tau) d\tau \right\} = 0. \end{aligned} \quad (\text{B.1})$$

Note that since d is odd, if $|a|$ is odd $i^{-(d+|a|)} = (-1)^{\frac{d+|a|}{2}}$ and if $|a|$ is even $i^{-(d+|a|)} = i^{-1}(-1)^{\frac{d-1+|a|}{2}}$. Hence, the contribution of the second term in (B.1) vanishes and the condition writes

$$[\mathbb{I}_{|a| \text{ even Re}} - \mathbb{I}_{|a| \text{ odd Im}}] \left\{ \int_0^\infty \left[\int_{\mathbb{R}} \exp(-iv\tau) f_1(\tau) d\tau \right] v^{|a|+d-1} dv \right\} = 0. \quad (\text{B.2})$$

Denote $G_1(v) = \int_{\mathbb{R}} \exp(-iv\tau) f_1(\tau) d\tau$. Remind that $|a|$ and a_1 have the same parity, see proof of Lemma 44.

a) For even $|a|, a_1$, G_1 is even and belongs to a Schwartz space of functions. Since d is odd, by inverse Fourier transform, Equation (B.2) reduces to

$$\begin{aligned} \int_0^\infty G_1(v) v^{|a|+d-1} dv &= \frac{1}{2} \int_{\mathbb{R}} G_1(v) v^{|a|+d-1} dv \\ &= \frac{(-i)^{|a|+d-1} (2\pi)^d}{2} f_1^{(|a|+d-1)}(0) = 0. \end{aligned}$$

The equality $f_1^{(|a|+d-1)}(0) = 0$ follows from the Leibniz differentiation rule for the product $\tau^{a_1} \times (1-\tau^2)^{\frac{|a|-a_1+d-3}{2}}$ and the identity $|a|+d-1 = a_1 + (|a|-a_1+d-3) + 2$. Thus (B.2) holds in this case.

b) Analogously, for odd $|a|, a_1$, $-\text{Im}(G_1(v))$ is odd and belongs to a Schwartz space of functions. The function $(-\text{Im}G_1(v))v^{|a|+d-1}$ is even. Thus

$$\int_0^\infty (-\text{Im}G_1(v)) v^{|a|+d-1} dv = \frac{(-i)^{|a|+d}}{2} f_1^{(|a|+d-1)}(0) = 0$$

for the same previous reasons and equation (B.2) holds in this case as well.

Even dimensions d

We assume in this section that the spectral measure is uniform with $C_1 = C_2 = 1$ in **(A-1)**. For $|a|$ and $a_1 = 2m$ even, equation (4.18) can be rewritten as

$$D_z^a \tilde{p}^y(t, x, z) = \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \quad (\text{B.3})$$

$$\times \int_0^\infty dv v^{|a|+d-1} \exp\left(-t \frac{v^\alpha}{|z-x|^\alpha}\right) \int_{-1}^1 (1-\tau^2)^{N_j-1/2} \cos(v\tau) d\tau$$

where $A_{d-2}^a = \int_{[0,\pi]^{d-3} \times [0,2\pi]} h(\phi, a) d\phi$ and $N_j = \frac{|a|-a_1+d-2+2j}{2}$, $j \in \llbracket 0, m \rrbracket$. Now recalling the definition of the Bessel function $J_n(z) := \frac{(z/2)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^1 (1-t^2)^{n-1/2} \cos(zt) dt$ which is well defined for $n > 1/2$ on $\mathbb{C} \setminus (-\infty, 0)$, we get

$$D_z^a \tilde{p}^y(t, x, z) = \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j 2^{N_j} \Gamma(N_j + \frac{1}{2}) \sqrt{\pi}$$

$$\times \int_0^\infty v^{\frac{|a|+a_1+d-2j}{2}} \exp\left(-t \frac{v^\alpha}{|z-x|^\alpha}\right) J_{N_j}(v) dv$$

$$= \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \Gamma(N_j + \frac{1}{2}) 2^{N_j+1/2} \quad (\text{B.4})$$

$$\times \text{Re} \int_0^\infty \exp\left(-t \frac{v^\alpha}{|z-x|^\alpha}\right) \exp\left[\left(\frac{1}{2}N_j + \frac{1}{4}\right)\pi i\right] W_{0, N_j}(2iv) v^{N_j'} dv,$$

where $W_{0,n}(z) = \frac{\exp(-z/2)}{\Gamma(n+\frac{1}{2})} \int_0^\infty [t(1+t/z)]^{n-1/2} e^{-t} dt$, $n > 1/2$, $z \in \mathbb{C} \setminus (-\infty, 0)$, is the Whittaker function and for $z > 0$,

$$J_n(z) = 2\text{Re} \left[\frac{1}{\sqrt{2\pi z}} \exp\left(\frac{1}{2}(n + \frac{1}{2})\pi i\right) W_{0,n}(2iz) \right]$$

(relation (2.10) from [Kol00]) and $N_j' = \frac{|a|+a_1+d-2j-1}{2}$, $j \in \llbracket 0, m \rrbracket$. For $\alpha \in (0, 1]$, from Cauchy's theorem we can change the integration path in (B.4) to the negative imaginary half line. Setting then $v = -i\xi$ we obtain

$$D_z^a \tilde{p}^y(t, x, z) = \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \Gamma(N_j + \frac{1}{2}) 2^{N_j+1/2}$$

$$\times (-1)^{j-m} \text{Re} \left[-i \int_0^\infty \exp\left(-t \frac{\xi^\alpha}{|z-x|^\alpha}\right) \exp\left(-\frac{i\pi\alpha}{2}\right) W_{0, N_j}(2\xi) \xi^{N_j'} d\xi \right].$$

Recalling the definition of W_{0, N_j} , we conclude expanding the exponential in power series that the first term is 0.

For $\alpha \in (1, 2)$, using the same arguments we can rotate the initial contour through the angle $-\pi/(2\alpha)$. Setting then $\eta = e^{i\frac{\pi}{2\alpha}} v$ we get

$$D_z^a \tilde{p}^y(t, x, z) = \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \Gamma(N_j + \frac{1}{2}) 2^{N_j+1/2}$$

$$\times \text{Re} \int_0^\infty \exp\left(it \frac{\eta^\alpha}{|z-x|^\alpha} + \left(\frac{1}{2}N_j + \frac{1}{4}\right)\pi i - \frac{i\pi}{2\alpha}(N_j' + 1)\right)$$

$$\times W_{0, N_j}(2\eta \exp\left\{\frac{i\pi(\alpha-1)}{2\alpha}\right\}) \eta^{N_j'} d\eta.$$

Taylor's formula for $\exp(it\frac{\eta^\alpha}{|z-x|^\alpha})$ yields for the first term, $\forall j \in \llbracket 0, m \rrbracket$,

$$I_\alpha^j := \operatorname{Re} \left\{ \exp \left[\left(\frac{1}{2} N_j + \frac{1}{4} \right) \pi i - \frac{i\pi}{2\alpha} (N_j' + 1) \right] \int_0^\infty W_{0, N_j} \left(2\eta \exp \left\{ \frac{i\pi(\alpha-1)}{2\alpha} \right\} \right) \eta^{N_j'} d\eta \right\}.$$

At last, we rotate the contour through the angle $-\frac{\pi(\alpha-1)}{2\alpha}$. Setting $\xi = \eta \exp \left(\frac{i\pi(\alpha-1)}{2\alpha} \right)$ we obtain $I_\alpha^j = \operatorname{Re} \left\{ -i(-1)^{j-m} \int_0^\infty W_{0, N_j} (2\xi) \xi^{N_j'} d\xi \right\} = 0$.

For $|a|$ and $a_1 = 2m + 1$ odd,

$$\begin{aligned} D_z^a \tilde{p}^y(t, x, z) &= \frac{(-1)^{(|a|+1)/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|\alpha|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \\ &\times \int_0^\infty dv v^{|\alpha|+d-1} \exp \left(-t \frac{v^\alpha}{|z-x|^\alpha} \right) \int_{-1}^1 (1-\tau^2)^{N_j-1/2} \tau \sin(v\tau) d\tau \\ &= \frac{(-1)^{(|a|+1)/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|\alpha|+d}} \sum_{j=0}^m (-1)^{2m+1-j} C_m^j \int_0^\infty dv v^{|\alpha|+d-2} \exp \left(-t \frac{v^\alpha}{|z-x|^\alpha} \right) \\ &\times \int_{-1}^1 (1-\tau^2)^{N_j-1/2} \tau d(\cos(v\tau)) \\ &= \frac{(-1)^{(|a|+1)/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|\alpha|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \int_0^\infty dv v^{|\alpha|+d-2} \exp \left(-t \frac{v^\alpha}{|z-x|^\alpha} \right) \\ &\times \int_{-1}^1 \cos(v\tau) \times \left[(1-2N_j)\tau^2(1-\tau^2)^{N_j-3/2} + (1-\tau^2)^{N_j-1/2} \right] d\tau. \end{aligned}$$

The above integrals have the same form as in (B.3) and can be estimated similarly.

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