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WEAK ERROR FOR STABLE DRIVEN SDES : EXPANSION OF THE DENSITIES.

BY VALENTIN KONAKOV, STÉPHANE MENOZZI

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Abstract Consider a multidimensional SDE of the form $X_t = x + \int_0^t b(X_{s-})ds + \int_0^t f(X_{s-})dZ_s$ where $(Z_s)_{s \geq 0}$ is a symmetric stable process. Under suitable assumptions on the coefficients the unique strong solution of the above equation admits a density w.r.t. the Lebesgue measure and so does its Euler scheme. Using a parametrix approach, we derive an error expansion at order 1 w.r.t. the time step for the difference of these densities.

1. Introduction. Consider the following \mathbb{R}^d -valued Stochastic Differential Equation (SDE in short)

$$(1.1) \quad X_t = x + \int_0^t b(X_{s-})ds + \int_0^t f(X_{s-})dZ_s,$$

where b, f are respectively Lipschitz continuous mappings from \mathbb{R}^d to \mathbb{R}^d and \mathbb{R}^d to $\mathbb{R}^d \otimes \mathbb{R}^d$ and $(Z_s)_{s \geq 0}$ is a general Lévy process. The previous assumptions guarantee the existence of a unique strong solution to (1.1). Also, this solution satisfies the strong Markov property, see e.g. Theorem 7 and 32 Chapter 5 in Protter [Pro04]. Let $T > 0$ be a fixed time horizon $T > 0$ and $(X_t^N)_{t \in \Lambda}$ a given approximation scheme of $(X_t)_{t \in [0, T]}$ associated to the time step $h := T/N$, $N \in \mathbb{N}^*$ on the grid $\Lambda := \{t_i := ih, i \in \llbracket 0, N \rrbracket\}$. When speaking about weak approximation of (1.1) two kind of quantities are of interest. The first one writes

$$\mathcal{E}_1(x, T, N) := \mathbb{E}_x[g(X_T)] - E_x[g(X_T^N)]$$

for a suitable class of test functions g . The second one concerns, when it exists, the approximation of the density p of the original SDE (1.1). If the approximation scheme $(X_t^N)_{t \in \Lambda}$ admits as well a density p^N , the quantity under study becomes

$$\mathcal{E}_2(x, y, T, N) := (p - p^N)(T, x, y).$$

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In both cases, the goal is to give a bound or an error expansion of these quantities in terms of h . If p and p^N exist, the control \mathcal{E}_2 gives more information than \mathcal{E}_1 . Anyhow, the typical assumptions and techniques associated to the study of \mathcal{E}_1 and \mathcal{E}_2 are quite of different nature.

In the continuous case, i.e. $Z_s = bs + \sigma W_s$ where $(W_s)_{s \geq 0}$ is a standard d -dimensional Brownian motion, provided the test function g and the coefficients b, f are sufficiently smooth, without any additional assumption on the generator Talay and Tubaro [TT90] derive an error expansion at order 1 for $\mathcal{E}_1(x, T, N)$ when $(X_t^N)_{t \in \Lambda}$ is the Euler approximation. Their proof is based on standard stochastic analysis tools: Itô's expansions and stochastic flows. To obtain the same kind of result for bounded Borel functions g some non degeneracy has to be assumed, namely hypoellipticity of the underlying diffusion, and the proof relies on Malliavin calculus techniques, see Bally and Talay [BT96a]. The authors also manage to extend their results to $\mathcal{E}_2(x, y, T, N)$ for a slightly modified Euler scheme [BT96b].

Anyhow, the most natural approach to handle the estimation of the quantity $\mathcal{E}_2(T, x, y, N)$ consists in using the so called "parametrix" technique introduced to obtain existence and controls on the fundamental solutions of PDEs, see e.g. Mc Kean and Singer [MS67] or Friedman [Fri64]. Roughly speaking it consists in expressing the density of X_T in terms of a infinite sum of suitable iterated kernels applied to the density of an SDE with constant coefficients. This has been done successfully by Konakov and Mammen [KM02] in a uniformly elliptic setting. The main advantage of this approach is that the density of the solution X_T and the Euler approximation X_T^N can be expressed in the same form and therefore quite directly compared. Furthermore this technique turns out to be quite robust and can be applied as soon as good controls on the densities p, p^N and their derivatives are available, see e.g. [KMM08] for an extension to a slight degenerate framework.

Consider now the general case, i.e. $\mathbb{E}[\exp(i\langle u, Z_t \rangle)] = \exp(t\psi(u))$ where

$$\psi(u) = \left\{ i\langle u, \gamma \rangle - \sigma^2 \frac{|u|^2}{2} + \int \left(\exp(i\langle u, z \rangle) - 1 - i\langle u, z \rangle \mathbb{I}_{|z| \leq 1} \right) \nu(dz) \right\}.$$

For smooth functions f, g and under additional assumptions on the behavior at infinity of ν , that is integrability conditions of the large jumps, Protter and Talay [PT97], manage to get a control at order one or even an error expansion for $\mathcal{E}_1(x, T, N)$ with the same approach as in [TT90]. In this work the approximation is the Euler scheme which for a general Lévy measure ν cannot always be simulated on a computer.

The quantity $\mathcal{E}_1(T, x, N)$ for a given simulatable approximation of the Euler scheme has also been studied by Jacod *et al.* [JKMP05] who derived

bounds at order 1. Moment conditions are also assumed.

In this article we manage to exploit the parametrix approach of [KM02] to derive a precise error expansion, with leading term of order 1, for $\mathcal{E}_2(T, x, y, N)$ when $(Z_t)_{t \geq 0}$ is an α -stable process, $\alpha \in (0, 2)$. The previous results cannot be directly applied, even for the study of $\mathcal{E}_1(T, x, N)$, since stable laws have large tails.

Under suitable non degeneracy assumptions on its coefficients specified below (see **(A-1)**-**(A-3)**), equation (1.1) is known to have a density p w.r.t. the Lebesgue measure. This can be proved via a Malliavin calculus-Bismut integration by parts approach, see e.g. Bichteller *et al.* [BGJ87]. Also, a direct construction of this density using a parametrix expansion has been obtained by Kolokoltsov [Kol00] who derived as well "Aronson's like" bounds with time singularity depending on the index α of the stable process Z .

Stable driven SDEs appear in various applicative fields, from mathematical physics to electrical engineering or financial mathematics, see [IP06], [SK74] or [JMW96], therefore their approximation becomes of interest. To approximate equation (1.1), setting $\phi(t) := \inf\{t_i : t_i \leq t < t_{i+1}\}$, we introduce the Euler scheme

$$(1.2) \quad X_t^N = x + \int_0^t b(X_{\phi(s)}^N) ds + \int_0^t f(X_{\phi(s)}^N) dZ_s.$$

The computation of the above scheme only requires to be able to simulate exactly the increments of Z , which up to a self similarity argument only amounts to simulate a stable law. This aspect is for instance discussed in Samorodnitsky and Taqqu [ST94], Weron and Weron [WR95] or Section 3 of [PT97]. Under the same assumptions **(A-1)**-**(A-3)**, the Euler scheme defined in (1.2) also has a density p^N .

We obtain, as one could expect an expansion with leading term of order 1 in h for $\mathcal{E}_2(T, x, y, N)$. The main idea of the proof consists in comparing the parametrix developments of p and p^N . The parametrix expansion of p is discussed in [Kol00], see also Section 3 and Appendix, whereas the parametrix expansion of p^N can be related to the ideas developed in [KM00, KM02] for the diffusive case corresponding to an index of stability equal to 2.

This result also emphasizes the robustness of the method that naturally extends to a broad class of processes. Let us mention that, using a Malliavin calculus approach, Hausenblas [Hau02], derived an upper bound of order one w.r.t. h for the quantity $\mathcal{E}_1(T, x, N)$, $g \in L^\infty$ in the scalar case. Hence, our result generalizes hers. Concerning functional limit theorems for the approximation of stable driven SDEs we refer to the work of Jacod [Jac04].

The paper is organized as follows. In Section 2 we state our standing assumptions and main results. In Section 3 we prove the existence of the densities for both the stable driven equation and its Euler scheme and also give a parametrix representation of these densities. Section 4 is dedicated to the proof of the main results. Eventually we give some extensions in Section 5, namely we state weaker assumptions under which our main result holds and we also extend it to the case of a stable process perturbed by a compound Poisson process.

2. Assumptions and Main results.

2.1. *Assumptions.* Since our arguments rely on techniques that involve the generator of (1.1), it is convenient to compute it before stating our assumptions. Let $(Z_t)_{t \geq 0}$ be the process associated to the \mathbb{R}^d valued α -stable law, $\alpha \in (0, 2)$, with characteristic function

$$(2.1) \quad \widehat{\mu}(z) = \exp \left[i \langle \gamma, z \rangle + \int_{S^{d-1}} \int_0^\infty \left(e^{i \langle z, \rho s \rangle} - 1 - \frac{i \langle z, \rho s \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \widetilde{\lambda}(ds) \right],$$

where $\gamma \in \mathbb{R}^d$ and $\widetilde{\lambda}$ is a (finite Borelian) measure on the unit sphere S^{d-1} . In the following we consider *symmetric* stable processes, that is for every A in the Borel σ -field $\mathcal{B}(S^{d-1})$, $\widetilde{\lambda}(A) = \widetilde{\lambda}(-A)$. Thus, for all $t \geq 0$,

$$(2.2) \quad \mathbb{E}[\exp(i \langle z, Z_t \rangle)] = \exp(it \langle \gamma, z \rangle - t \int_{S^{d-1}} |\langle z, s \rangle|^\alpha \lambda(ds)),$$

with $\lambda := C_\alpha \widetilde{\lambda}$, $C_\alpha := \alpha^{-1} \left(\Gamma(1 - \alpha) \mathbb{I}_{\alpha \in (0,1)} + \Gamma(2 - \alpha)(\alpha - 1)^{-1} \mathbb{I}_{\alpha \in (1,2)} \right) \times \cos\left(\frac{\pi\alpha}{2}\right) + \frac{\pi\alpha}{2} \mathbb{I}_{\alpha=1}$, see e.g. Theorem 2.3.1 in [ST94] and Lemma 2, Chapter XVII.4 in [Fel66] for the expression of C_α .

Recalling that Z is a pure jump semi-martingale, we deduce from Itô's formula for semi-martingales that for every $\psi \in C_0^2(\mathbb{R}^d)$

$$\begin{aligned} \psi(X_t) - \psi(x) &= \int_{0+}^t \langle \nabla \psi(X_{s-}), (b(X_{s-}) ds + f(X_{s-}) dZ_s) \rangle + \\ &\sum_{0 < s \leq t} [\psi(X_{s-} + f(X_{s-}) \Delta Z_s) - \psi(X_{s-}) - \langle \nabla \psi(X_{s-}), f(X_{s-}) \Delta Z_s \rangle]. \end{aligned}$$

In addition we have the componentwise canonical decomposition for Z_t with the truncation function $h(z) = z 1_{|z| \leq 1}$ (see Theorem 2.43 Chapter II in Jacod and Shiryaev [JS87])

$$(2.3) \quad Z_t = \int_0^t \int_{|z| \leq 1} z [\mu(\omega, dz, ds) - \nu(dz) ds] + \int_0^t \int_{|z| > 1} z \mu(\omega, dz, ds) + \gamma t$$

where $\mu(\omega, dt, dz) := \sum_{s \geq 0} \mathbb{I}_{\Delta Z_s(\omega) \neq 0} \delta_{s, \Delta Z_s(\omega)}(dt, dz)$ is the jump measure of Z and ν its Lévy measure. Hence,

$$\begin{aligned} \psi(X_t) - \psi(x) &= M_t + \int_0^t \langle \nabla \psi(X_{s-}), f(X_{s-})\gamma + b(X_{s-}) \rangle ds + \\ &\sum_{s \in (0, t]} [\psi(X_{s-} + f(X_{s-})\Delta Z_s) - \psi(X_{s-}) - \langle \nabla \psi(X_{s-}), f(X_{s-})\Delta Z_s \mathbb{I}_{|\Delta Z_s| \leq 1} \rangle] \end{aligned}$$

where $M_t = \int_0^t \int \langle \nabla \psi(X_{s-}), f(X_{s-})z \mathbb{I}_{|z| \leq 1} \rangle [\mu(\omega, dz, ds) - \nu(dz) ds]$. Since f is assumed to be bounded and $\psi \in C_0^2(\mathbb{R}^d)$, we deduce from equations (2.1)-(2.2) and corollary 3 p. 73 in Protter [Pro04] that $(M_t)_{t \geq 0}$ is a square integrable martingale. Thus the generator of the unique strong solution of (1.1) writes:

$$\begin{aligned} A\psi(x) &= \lim_{t \rightarrow 0} \frac{\mathbb{E}_x \psi(X_t) - \psi(x)}{t} = \langle \nabla \psi(x), f(x)\gamma + b(x) \rangle + \\ &\int \left[\psi(x + f(x)z) - \psi(x) - \langle \nabla \psi(x), f(x)z \rangle \mathbb{I}_{|z| \leq 1} \right] \nu(dz) = \\ &\langle \gamma(x), \nabla \psi(x) \rangle + \int_{S^{d-1}} \int_0^\infty \left[\psi(x + \rho s) - \psi(x) - \frac{\langle \rho s, \nabla \psi(x) \rangle}{1 + \rho^2} \right] \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}(x, ds) \end{aligned} \quad (2.4)$$

where $\gamma(x) = b(x) + f(x)\gamma + \int_{\mathbb{R}^d} f(x)z \left(\frac{1}{1 + |f(x)z|^2} - \mathbb{I}_{|z| \leq 1} \right) \nu(dz) = b(x) + f(x)\gamma$, recalling the symmetry. We denote for all $A \in \mathcal{B}(\mathbb{R}^d)$, $\nu(x, A) := \nu(\{z \in \mathbb{R}^d : f(x)z \in A\})$. Then $\tilde{\lambda}(x, \cdot)$ stands for the spherical part of $\nu(x, \cdot)$.

We now introduce our assumptions. Fix now an integer $q \geq 2$, we assume that

(A-1) The spherical measure $\tilde{\lambda}$ has a C^q surface density and $\exists(C_1, C_2) \in (\mathbb{R}^+)^2$, $\forall \bar{p} \in S^{d-1}$, $0 < C_1 \leq \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda(ds) \leq C_2$.

(A-2) The coefficients b and f as well as their derivatives up to order q are uniformly bounded in x . Thus, for $1 < \alpha < 2$, $\gamma(x) := b(x) + f(x)\gamma$ is uniformly bounded. We impose for $0 < \alpha \leq 1$, $\gamma(x) = 0$ for all $x \in \mathbb{R}^d$.

Remark 2.1 *The zero drift condition comes from the fact that for $\alpha \in (0, 1]$ the addition of a drift of order t does not correspond to a negligible term in small time with respect to the natural scale $t^{1/\alpha}$, see Appendix A for details.*

(A-3) There exist constants $0 < \underline{c} \leq \bar{c}$ s.t. for all $x \in \mathbb{R}^d, \xi \in \mathbb{R}^d$,

$$\underline{c}|\xi|^2 \leq \langle f(x)\xi, \xi \rangle \leq \bar{c}|\xi|^2.$$

Remark 2.2 *The uniform ellipticity condition (A-3) allows to have good controls on the measure $\tilde{\lambda}(x, \cdot)$. In particular, a consequence of (A-1), (A-3) one gets that there exist constants $\underline{C}_1 = \underline{C}_1(\underline{c}, d, \alpha) \leq \overline{C}_1 = \overline{C}_1(\overline{c}, d, \alpha)$ s.t. $\forall \bar{p} \in S^{d-1}, x \in \mathbb{R}^d$,*

$$(2.5) \quad 0 < \underline{C}_1 \leq \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda(x, ds) \leq \overline{C}_2.$$

From now on we assume that Assumptions (A-1)-(A-3) are in force.

In the following we denote by C a positive generic constant that can depend on α, d , the bounds appearing in the previous assumptions but neither on N nor on the spatial points involved. Other possible dependencies, especially w.r.t. the final time T are explicitly specified.

2.2. Main results.

Proposition 2.1 *For every $t > 0$ the solution X_t (resp. X_t^N) of (1.1) (resp. (1.2)) has a density $p(t, x, \cdot)$ (resp. $p^N(t, x, \cdot)$) w.r.t. the Lebesgue measure.*

To state the theorem we first need some notation. Introduce for a smooth function $\varphi(t, x, z)$ the integro-differential operators:

$$(2.6) \quad \Phi \varphi(t, x, z) = \langle \gamma(x), \nabla_x \varphi(t, x, z) \rangle + \int_{S^{d-1}} \int_0^\infty \left(\varphi(t, x + \rho \tilde{s}, z) - \varphi(t, x, z) - \frac{\langle \rho \tilde{s}, \nabla_x \varphi(t, x, z) \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}(x, d\tilde{s}),$$

$$(2.7)$$

$\tilde{\Phi}^* \varphi(t, x, y) = \tilde{\Phi}_y \varphi(t, x, y)$ where for $\xi \in \mathbb{R}^d$,

$$(2.8) \quad \tilde{\Phi}_\xi \varphi(t, x, z) = \langle \gamma(\xi), \nabla_x \varphi(t, x, z) \rangle + \int_{S^{d-1}} \int_0^\infty \left(\varphi(t, x + \rho \tilde{s}, z) - \varphi(t, x, z) - \frac{\langle \rho \tilde{s}, \nabla_x \varphi(t, x, z) \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}(\xi, d\tilde{s}).$$

Define also, $(\tilde{\Phi}_*)^m \varphi(t, x, y) = (\tilde{\Phi}_\xi)^m \varphi(t, x, y) |_{\xi=x}, \forall m \in \mathbb{N}^*$. Note that we have $\tilde{\Phi}_* \varphi(t, x, y) = \Phi \varphi(t, x, y)$ but in general $(\tilde{\Phi}_*)^m \varphi(t, x, y) \neq (\Phi)^m \varphi(t, x, y)$ for $m \geq 2$.

Define now, for $t > 0$, the kernel $H(t, x, y) := (\Phi - \tilde{\Phi}_y) \tilde{p}(t, x, y)$ where $\tilde{p}(t, x, y)$ denotes the density at point y of $\tilde{X}_t = x + b(y)t + f(y)Z_t$. Eventually we introduce the discrete and continuous convolution operators

$$\varphi \otimes \psi(t, x, y) = \int_0^t ds \int dz \varphi(s, x, z) \psi(t-s, z, y), \forall t \in [0, T],$$

$$\varphi \otimes_N \psi(t, x, y) = \int_0^t ds \int dz \varphi(\phi(s), x, z) \psi(t - \phi(s), z, y), \forall t \in \{(t_i)_{i \in [1, N]}\}.$$

Also $\varphi \otimes H^{(r)} = (\varphi \otimes H^{(r-1)}) \otimes H$, $\varphi \otimes H^{(0)} = \varphi$ stands for the r -fold convolution.

Theorem 2.1 *Suppose $q > d - 4$. Take $M < (q - d - 4)$. There exists a function $R(x, y)$ with $|R(x, y)| \leq C \left(\frac{1}{1+|y-x|^{d+\alpha}} \right) := \rho_\alpha(y-x)$ for some positive constant $C := C(T)$ such that*

$$(p - p^N)(T, x, y) = \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} \left[p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d \right] (T, x, y) - \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} p^N \right] (T, x, y) + h^M R(x, y)$$

where $p^d(T, x, y) = \sum_{r=0}^{\infty} (\tilde{p} \otimes_N H^{(r)})(T, x, y)$. It holds that

$$\sum_{l=1}^{M-1} \left| \left(p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d \right) (T, x, y) \right| \leq C \rho_\alpha(y-x),$$

$$\sum_{k=1}^{M-1} \left| \left(p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} p^N \right) (T, x, y) \right| \leq C \rho_\alpha(y-x).$$

Remark 2.3 *In the above expression, one writes for all $l \in \llbracket 1, M \rrbracket$,*

$$(\Phi - \tilde{\Phi}^*)^{l+1} \varphi(t, x, y) = \sum_{k=1}^{l+1} C_{l+1}^k \Phi^k (-\tilde{\Phi}^*)^{l+1-k} \varphi(t, x, y),$$

whereas, $\forall k \in \llbracket 1, m \rrbracket$, $(\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} \varphi(t, x, y) = (\tilde{\Phi}_\xi - \tilde{\Phi}_y)^{k+1} \varphi(t, x, y) \Big|_{\xi=x}$.

Remark 2.4 *The terms in the previous expansion depend on N . Anyhow using iteratively the Theorem and controls on $\otimes_N - \otimes$ (see also Lemma 4.1) it is possible to obtain an expansion with terms independent of N . For small M explicit formulas are thus easily derived but in all generality the terms become less transparent. For $M = 2$ one gets*

$$(p - p^N)(T, x, y) = \frac{h}{2} \left(p \otimes_N (\Phi - \tilde{\Phi}^*)^2 p^d - p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^2 p^N \right) (T, x, y) + h^2 R(x, y)$$

$$= \frac{h}{2} \left(p \otimes (\Phi - \tilde{\Phi}^*)^2 p - p \otimes (\tilde{\Phi}_* - \tilde{\Phi}^*)^2 p \right) (T, x, y)$$

$$+ h^2 \tilde{R}(x, y) = \frac{h}{2} (p \otimes (\Phi^2 - \tilde{\Phi}_*^2) p) (T, x, y) + h^2 \tilde{R}(x, y),$$

where $\tilde{R}(x, y) \leq \tilde{C} \rho_\alpha(y-x)$.

3. Stable driven equations and their Euler scheme: existence of the density and associated parametrix expansion.

3.1. Stable driven equation.

3.1.1. *Existence of the density.* For X_t , Proposition 2.1 derives from the following result due to Kolokoltsov [Kol00]. In our current framework, using a parametrix approach he constructs a function $p(t, x, y)$ which is everywhere non-negative, belongs to $C_0(\mathbb{R}^d)$ (continuous functions of \mathbb{R}^d vanishing at infinity) for each $t > 0$, satisfies the semigroup identity (or the Chapman-Kolmogorov equation) and $\int p(t, x, \eta) d\eta = 1$. Set now for an arbitrary $\psi \in C(\mathbb{R}^d)$ and $t > 0$, $(R_t\psi)(x) = \int_{\mathbb{R}^d} p(t, x, \xi)\psi(\xi)d\xi$. Then the following result holds true (Proposition 3.4 in [Kol00]).

Theorem 3.1 (i) $(R_t\psi)(x)$ tends to $\psi(x)$ as $t \rightarrow 0$ for each x and any $\psi \in C(\mathbb{R}^d)$.

(ii) If $\alpha > 1$, or $\alpha \leq 1$, then R_t is a continuous operator from $C(\mathbb{R}^d)$ into $C^l(\mathbb{R}^d)$ with norm of order $O(t^{-l/\alpha})$ for all $l \leq q$, or $l \leq q - 1$ respectively.

(iii) If $\psi \in C(\mathbb{R}^d)$ and $t > 0$, the function $(R_t\psi)(x)$ satisfies equation

$$(3.1) \quad \begin{aligned} \partial_t u(t, x) &= \langle \gamma(x), \nabla_x u(t, x) \rangle + \\ &\int_{S^{d-1}} \int_0^\infty \left(u(t, x + \rho s) - u(t, x) - \frac{\langle \rho s, \nabla_x u(t, x) \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}(x, ds). \end{aligned}$$

(iv) The Cauchy problem for (3.1) can have at most one solution in the class of continuous functions belonging to $C_0(\mathbb{R}^d)$ for each t . This solution is necessary non-negative whenever the initial function is non-negative.

In other words p is a *fundamental solution* of the Cauchy problem associated to (3.1). From (ii) it is also a regularizing kernel.

It follows from (3.6) and (3.15) in [Kol00] that the transition function $P(t, x, \Gamma) = \int_\Gamma p(t, x, y) dy$ is stochastically continuous and, hence, is uniquely determined by its weak infinitesimal operator (see Dynkin [Dyn63], Theorem 2.3, p. 56). Then p can be identified with the transition probability density of the solution $X_t = (X_t^1, \dots, X_t^d)$ of the equation (1.1) by comparison of the r.h.s. of (3.1) and (2.4).

Remark 3.1 *The existence of the density is discussed in Bichteler et al. [BGJ87], where it is proved thanks to a Bismut-Malliavin approach. This technique requires the computation of a tangent equation associated to the gradient flow that involves the derivatives of the coefficients of equation (1.1) implying some additional smoothness of those coefficients, see e.g. Theorem 6.48 of the above reference.*

3.1.2. *Parametrix expansion of the density.* For the sake of completeness we now specify how to get through a "parametrix" approach a series expansion for the density $p(t, x, y)$.

Introduce, for all $x, y \in \mathbb{R}^d$ the following stochastic "frozen" stable driven equation $\tilde{X} = \tilde{X}_{s,x,y}$ defined for $s \leq t$ by

$$(3.2) \quad \tilde{X}_t = x + \int_s^t b(y) du + \int_s^t f(y) dZ_u.$$

By computation of the Fourier transform of Z and Fourier inversion the transition densities \tilde{p}^y of \tilde{X}_t explicitly write

$$\begin{aligned} \tilde{p}^y(t-s, x, z) &= \frac{1}{(2\pi)^d} \int e^{-i\langle z-x-(t-s)\gamma(y), p \rangle} \\ &\times \exp \left\{ -(t-s) \int_{S^{d-1}} |\langle p, \tilde{s} \rangle|^\alpha \lambda(y, d\tilde{s}) \right\} dp = \tilde{p}(t-s, x, z). \end{aligned}$$

One can directly check that this transition density satisfies the equation

$$(3.3) \quad \begin{aligned} \frac{\partial u}{\partial t}(t, x, z) &= \langle \gamma(y), \nabla_x u(t, x, z) \rangle + \\ &\int_{S^{d-1}} \int_0^\infty \left(u(t, x + \rho \tilde{s}, z) - u(t, x, z) - \frac{\langle \rho \tilde{s}, \nabla_x u(t, x, z) \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}(y, d\tilde{s}). \end{aligned}$$

Now, from equations (3.3) and (3.1) and with the notations of Section 2, the densities of the solutions of (3.2) and (1.1) satisfy respectively

$$(3.4) \quad \begin{aligned} \frac{\partial \tilde{p}}{\partial t} &= \tilde{\Phi}_y \tilde{p}, \text{ for } t \geq 0, \tilde{p}(0, x, z) = \delta(z - x), \\ \frac{\partial p}{\partial t} &= \Phi p, \text{ for } x \in \mathbb{R}^d, t \geq 0, p(0, x, z) = \delta(z - x). \end{aligned}$$

We will speak about the operators appearing in (3.4) as the "frozen" and "unfrozen" ones.

From (3.4), we derive a parametrix expansion for the density $p(t, x, \cdot)$ of the solution of (1.1).

Proposition 3.1 (Parametrix expansion of the density) *With the notations of Section 2, the following representation holds*

$$(3.5) \quad p(t, x, y) = \sum_{r=0}^{\infty} (\tilde{p} \otimes H^{(r)})(t, x, y).$$

Proof. The function \tilde{p} , fundamental solution of (3.4), satisfies the equation

$$(3.6) \quad \frac{\partial \tilde{p}}{\partial t}(t, x, y) = \tilde{\Phi}_x \tilde{p}(t, x, y) - H(t, x, y).$$

We have now from (3.4) and (3.6)

$$\begin{aligned} (p - \tilde{p})(t, x, y) &= \int_0^t ds \frac{\partial}{\partial s} \left[\int p(s, x, z) \tilde{p}(t-s, z, y) dz \right] = \\ &= \int_0^t ds \int \left[\left({}^t \tilde{\Phi}_z p \right)(s, x, z) \tilde{p}(t-s, z, y) - p(s, x, z) \tilde{\Phi}_y \tilde{p}(t-s, z, y) \right] dz \\ &= p \otimes H(t, x, y), \end{aligned}$$

where ${}^t \tilde{\Phi}_z$ stands for the adjoint of $\tilde{\Phi}_z$. Note that we considered the solution $\tilde{p}(t, x, y)$ at the same point y where we "froze" the right hand side of the equation. The representation (3.5) then follows by simple iteration. \square

Remark 3.2 *Note that the previous expansion is "formal". The convergence of the r.h.s. in (3.5) is investigated in the proof of Theorem 3.1 in [Kol00] and can also be derived with the controls of Lemmas 4.4 and 4.5. For the sake of completeness, a short proof of this convergence is also given in Appendix A.*

3.2. Euler scheme. We consider now, for given $N \in \mathbb{N}^*$, the Euler scheme for equation (1.1) at the discretization times:

$$X_0^N = x, X_{t_{i+1}}^N = X_{t_i}^N + b(X_{t_i}^N)h + f(X_{t_i}^N)(Z_{t_{i+1}} - Z_{t_i})$$

recalling $h = T/N$. For each $N \in \mathbb{N}^*$, $(X_{t_i}^N)_{i \in \llbracket 0, N \rrbracket}$ is a Markov chain. Given the past $X_{t_l}^N = x_l$, $l \in \llbracket 0, i \rrbracket$, the conditional distribution of the innovations $b(X_{t_i}^N)h + f(X_{t_i}^N)(Z_{t_{i+1}} - Z_{t_i})$ has conditional density $\tilde{p}(h, \cdot; \lambda(x_i, \cdot), \gamma(x_i))$.

Now, in order to give an expansion of the density of the Euler scheme similar to equation (3.5), that will also be the starting point for our error expansion, we need to define, for fixed j, k , $0 \leq j < k \leq N$ and $x, y \in \mathbb{R}^d$ additional "frozen" Markov chains $(\tilde{X}_{t_l}^N)_{l \in \llbracket j, k \rrbracket} = (\tilde{X}_{t_l, x, y}^N)_{l \in \llbracket j, k \rrbracket}$. Their dynamics is described by

$$\tilde{X}_{t_j}^N = x, \tilde{X}_{t_{i+1}}^N = \tilde{X}_{t_i}^N + b(y)h + f(y)(Z_{t_{i+1}} - Z_{t_i}), \quad i \in \llbracket j, k-1 \rrbracket.$$

Given the past $\tilde{X}_{t_l}^N = x_l$, $l \in \llbracket j, i \rrbracket$, the conditional distribution of the innovations $b(y)h + f(y)(Z_{t_{i+1}} - Z_{t_i})$ has conditional density $\tilde{p}(h, \cdot; \lambda(y, \cdot), \gamma(y))$ and, hence, does not depend on the past. Note that for the grid points

$(t_i)_{i \in [0, N]}$ the transition densities of the solution $\tilde{X}_{s, x, y}$ of (3.2) coincide with the transition densities of the chain $\tilde{X}_{t_j, x, y}^N$ for $N \in \mathbb{N}^*$, $x, y \in \mathbb{R}^d$ and $s = t_j$.

We denote by $p^N(t_k - t_j, x, y)$ and

$$\tilde{p}^N(t_k - t_j, x, y) = \tilde{p}(t_k - t_j, y - x; \lambda(y, \cdot), \gamma(y))$$

the transition probability densities between times t_j and t_k of the chains X^N and \tilde{X}^N respectively.

Before stating the parametrix expansion of p^N in terms of \tilde{p}^N , we need to introduce a kernel H_N that is the "discrete" analogue of H :

$$(3.7) \quad H_N = \{L_N - \tilde{L}_N\} \tilde{p}^N,$$

with

$$\begin{aligned} L_N \varphi(t_k - t_j, x, y) &= h^{-1} \left\{ \int p^N(h, x, z) \varphi(t_k - t_{j+1}, z, y) dz - \varphi(t_k - t_{j+1}, x, y) \right\}, \\ \tilde{L}_N \varphi(t_k - t_j, x, y) &= h^{-1} \left\{ \int \tilde{p}^{N, y}(h, x, z) \varphi(t_k - t_{j+1}, z, y) dz - \varphi(t_k - t_{j+1}, x, y) \right\}, \end{aligned}$$

$$\tilde{p}^{N, y}(h, x, z) = \tilde{p}(h, z - x; \lambda(y, \cdot), \gamma(y)).$$

Lemma 3.1 *For $0 \leq j < k \leq N$ the following formula holds:*

$$(3.8) \quad p^N(t_k - t_j, x, y) = \sum_{r=0}^{k-j} (\tilde{p}^N \otimes_N H_N^{(r)})(t_k - t_j, x, y)$$

where in the calculation of $\tilde{p}^N \otimes_N H_N^{(r)}$ (r -fold convolution) we define

$$p^N(0, x, y) = \tilde{p}^N(0, x, y) = \delta(x - y).$$

The proof of this lemma is given in [KM00], Lemma 3.6 and does not rely on the specific distribution of the innovations.

4. Proof of the main results. In this section, we state in Subsection 4.1 the various points needed to prove Theorem 2.1. The proofs are postponed to Subsection 4.2. As mentioned earlier, the key idea consists in comparing the parametrix expansions of the densities p and p^N respectively given by (3.5) and (3.8). In the whole section we suppose that the assumptions of Theorem 2.1 hold.

4.1. *Proof of Theorem 2.1.* For the previously mentioned comparison to be possible we first need to estimate a difference between the transition density $p(t, x, y)$ and $p^d(t, x, y)$ which is defined analogously to p but with \otimes replaced by \otimes_N (discrete time convolution).

Lemma 4.1 (Time discretization) *One has:*

$$(p - p^d)(T, x, y) = \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} \left(p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d \right) (T, x, y) + h^M R_1(T, x, y)$$

with

$$\sum_{l=1}^{M-1} \left| \left(p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d \right) (T, x, y) \right| + |R_1(T, x, y)| \leq C \rho_\alpha(y - x)$$

for some constant $C := C(T)$.

Then the comparison between p^d and p^N is controlled with the following

Lemma 4.2 (Comparison of the discrete densities) *The following expansion holds:*

$$(p^d - p^N)(T, x, y) = - \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} p^N \right] (T, x, y) + h^M R_2(T, x, y)$$

where

$$\begin{aligned} R_2(T, x, y) &= - \frac{1}{M!} \int_0^1 (1-\tau)^M \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{M+1} \tilde{p}_\tau^\Delta \right] (T, x, y) d\tau \\ \tilde{p}_\tau^\Delta(T, x, y) &= \sum_{r=0}^{\infty} \tilde{p}_\tau \otimes_N H_N^{(r)}(T, x, y), \tilde{p}_0^\Delta = p^N, \text{ and } \forall \tau \in [0, 1], \\ \tilde{p}_\tau(t, x, y) &= \tilde{p}(y-x; t\lambda(y, \cdot) + \tau h \Delta \lambda^x(y, \cdot), t\gamma(y) + \tau h \Delta \gamma^x(y)) \\ &:= \int_{\mathbb{R}^d} dz \tilde{p}(t, x, z; \lambda(y, \cdot), \gamma(y)) \tilde{p}(z, y; \tau h \Delta \lambda^x(y, \cdot), \tau h \Delta \gamma^x(y)), \\ \Delta \lambda^x(y, \cdot) &= \lambda(x, \cdot) - \lambda(y, \cdot), \Delta \gamma^x(y) = \gamma(x) - \gamma(y). \end{aligned}$$

Also, there exists $C := C(T) > 0$ s.t.

$$\sum_{l=1}^{M-1} \left| \left(p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{l+1} p^N \right) (T, x, y) \right| + |R_2(T, x, y)| \leq C \rho_\alpha(y - x)$$

4.2. Proofs of the technical Lemmas.

Proof of Lemma 4.1. We start from the recurrence relation for $r \in \mathbb{N}^*$

$$\begin{aligned} \tilde{p} \otimes H^{(r)} - \tilde{p} \otimes_N H^{(r)} &= \left[(\tilde{p} \otimes H^{(r-1)}) \otimes H - (\tilde{p} \otimes H^{(r-1)}) \otimes_N H \right] \\ &\quad + \left[(\tilde{p} \otimes H^{(r-1)}) - (\tilde{p} \otimes_N H^{(r-1)}) \right] \otimes_N H. \end{aligned}$$

Summing up these terms over $r \in \mathbb{N}^*$ and using the linearity of \otimes and \otimes_N we get $p - p^d = p \otimes H - p \otimes_N H + (p - p^d) \otimes_N H$. An iterative application of this identity yields

$$(4.1) \quad p - p^d = \sum_{r=0}^{\infty} [p \otimes H - p \otimes_N H] \otimes_N H^{(r)}.$$

By definition

$$(4.2) \quad \begin{aligned} [p \otimes H - p \otimes_N H](t_k, x, y) &= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} ds \int [p(s, x, z) H(t_k - s, z, y) \\ &\quad - p(t_j, x, z) H(t_k - t_j, z, y)] dz \end{aligned}$$

A Taylor expansion of the function $\theta(s, z) := p(s, x, z)H(t_k - s, z, y)$ in the interval $[t_j, t_{j+1}]$ gives

$$(4.3) \quad \begin{aligned} \int [\theta(s, z) - \theta(t_j, z)] dz &= \sum_{l=1}^{M-1} \frac{(s - t_j)^l}{l!} \int \partial_\tau^l \theta(\tau, z) \Big|_{\tau=t_j} dz + \\ &\quad \frac{(s - t_j)^M}{(M-1)!} \int_0^1 (1 - \delta)^{M-1} \int \partial_\tau^M \theta(\tau, z) \Big|_{\tau=\tau_j(s, \delta)} dz d\delta, \end{aligned}$$

where $\tau_j(s, \delta) = t_j + \delta(s - t_j)$. Note now that $-\partial_s p(t - s, x, z) = \Phi p(t - s, x, z)$, $\partial_t p(t - s, x, z) = {}^t\Phi_z p(t - s, x, z)$, where ${}^t\Phi_z$ is the adjoint operator of Φ_z , so that $\Phi p(t - s, x, z) = {}^t\Phi_z p(t - s, x, z)$. The same identity also holds for \tilde{p} with $\Phi, {}^t\Phi$ respectively replaced by $\tilde{\Phi}^*, {}^t\tilde{\Phi}^*$. We therefore derive

$$\begin{aligned} \int \partial_\tau \theta(\tau, z) \Big|_{\tau=t_j} dz &= \int \partial_\tau [p(\tau, x, z)] \Big|_{\tau=t_j} H(t_k - t_j, z, y) dz \\ &\quad + \int p(t_j, x, z) \partial_\tau [H(t_k - \tau, z, y)] \Big|_{\tau=t_j} dz \\ &= \int {}^t\Phi_z p(t_j, x, z) (\Phi - \tilde{\Phi}^*) \tilde{p}(t_k - t_j, z, y) dz \\ &\quad - \int p(t_j, x, z) (\Phi - \tilde{\Phi}^*) \tilde{\Phi}^* \tilde{p}(t_k - t_j, z, y) dz \\ &= \int p(t_j, x, z) (\Phi - \tilde{\Phi}^*)^2 \tilde{p}(t_k - t_j, z, y) dz. \end{aligned}$$

Iterating the differentiation we get

$$(4.4) \quad \int \partial_\tau^l \theta(\tau, z)|_{\tau=t_j} dz = \int p(t_j, x, z) (\Phi - \tilde{\Phi}^*)^{l+1} \tilde{p}(t_k - t_j, z, y) dz$$

where we recall that for two operators A and B we denote by $(A - B)^k$ the following sum $(A - B)^k = \sum_{j=0}^k C_k^j A^{k-j} (-B)^j$.

Plugging (4.3) and (4.4) into (4.2) we get

$$(4.5) \quad \begin{aligned} [p \otimes H - p \otimes_N H](t_k, x, y) &= \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} \tilde{p}(t_k, x, y) \\ &+ h^M R_M(t_k, x, y) \end{aligned}$$

where

$$(4.6) \quad \begin{aligned} R_M(t_k, x, y) &= \frac{1}{(M-1)!} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} [h^{-1}(s - t_j)]^M \int_0^1 (1 - \delta)^{M-1} \times \\ &\int \partial_\tau^M [p(\tau, x, z) H(t_k - \tau, z, y)]|_{\tau=\tau_j(s, \delta)} ds dz d\delta. \end{aligned}$$

Plugging (4.5) and (4.6) into (4.1) we get

$$(4.7) \quad \begin{aligned} (p - p^d)(T, x, y) &= \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} \times \sum_{r=0}^{\infty} p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} \tilde{p} \otimes_N H^{(r)}(T, x, y) \\ &+ h^M R_1(T, x, y) \end{aligned}$$

with $R_1(T, x, y) = \sum_{r=0}^{\infty} (R_M \otimes_N H^{(r)})(T, x, y)$.

Now we apply that for a linear operator S and its adjoint ${}^t S$ we have $p \otimes_N S \tilde{p} = {}^t S p \otimes_N \tilde{p}$. This gives

$$\begin{aligned} &\sum_{r=0}^{\infty} p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} \tilde{p} \otimes_N H^{(r)}(T, x, y) = \\ &{}^t \left[(\Phi - \tilde{\Phi}^*)^{l+1} \right] p \otimes_N \sum_{r=0}^{\infty} (\tilde{p} \otimes_N H^{(r)})(T, x, y) = p \otimes_N (\Phi - \tilde{\Phi}^*)^{l+1} p^d(T, x, y), \end{aligned}$$

which plugged into (4.7) gives the desired expansion. The stated bound follows by application of the estimates given in Lemma 4.3 below. We only give the proof for the first summand, the other terms can be handled in a similar way. Write

$$\begin{aligned} p \otimes_N (\Phi - \tilde{\Phi}^*)^2 p^d(T, x, y) &= \sum_{i=0}^{N-1} h \int p(t_i, x, z) (\Phi - \tilde{\Phi}^*)^2 p^d(T - t_i, z, y) dz \\ &:= S_1 + S_2, \end{aligned}$$

where in S_1 (resp. S_2) the sum is taken over $I_1 := \{i \in \llbracket 0, \lfloor \frac{N-1}{2} \rrbracket\}$ (resp. $I_2 := \{i \in \llbracket \lfloor N-1/2 \rfloor + 1, N-1 \rrbracket\}$). For S_1 (resp. S_2), $p^d(T-t_i, z, y)$ (resp. $p(t_i, x, z)$) is non singular. One could show from the proof of Lemma 4.3, see also Appendix A, using as well the proof of Proposition 2.5 in [Kol00] that $\exists C := C(T)$,

$$(4.8) \quad \begin{aligned} |(\Phi - \tilde{\Phi}^*)^2 p^d(T-t_i, z, y)| &\leq C \tilde{p}(T-t_i, z, y), \forall i \in I_1, \\ \left| \left((\tilde{\Phi}^*)^t - \Phi^t \right)^2 p(t_i, x, z) \right| &\leq C \tilde{p}(t_i, x, z), \forall i \in I_2. \end{aligned}$$

Thus the semi group property yields $|S_1| + |S_2| \leq C \tilde{p}(T, x, y)$. One eventually checks from Proposition A.1 that $\tilde{p}(T, x, y) \leq C \rho_\alpha(y-x)$. \square

Proof of Lemma 4.2. Let us denote by $\mathcal{F}[\psi](z) = \int \exp(i\langle z, p \rangle) \psi(p) dp$ the Fourier transform of a function ψ . Introduce now for all s, t , $s < t$, $s, t \in \{(t_i)_{i \in \llbracket 0, N \rrbracket}\}$, $p \in \mathbb{R}^d$,

$$\begin{aligned} \psi(p) &= h(L_N^x - \tilde{L}_N^y) \tilde{p}^{N,y}(t-s, x, p) \\ &= \int p^N(h, x, w) \tilde{p}^{N,y}(t-(s+h), w, p) dw - \tilde{p}^{N,y}(t-s, x, p). \end{aligned}$$

Note, that in particular according to (3.7), $\psi(y) = hH_N(t-s, x, y)$. We obtain from (2.1) and (3.7) that

$$\mathcal{F}[\psi](z) := G_z(1) - G_z(0)$$

with

$$G_z(\delta) = \exp[i\langle x, z \rangle + i(t-s)\langle \gamma(y), z \rangle + i\delta h \langle \Delta \gamma^x(y), z \rangle + \int_{S^{d-1}} \int_0^\infty \left(e^{i\langle z, \rho \tilde{s} \rangle} - 1 - \frac{i\langle z, \rho \tilde{s} \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \left[(t-s) \tilde{\lambda}(y, d\tilde{s}) + \delta h \Delta \tilde{\lambda}^x(y, d\tilde{s}) \right],$$

denoting $\Delta \gamma^\xi(y) = \gamma(\xi) - \gamma(y)$, $\Delta \tilde{\lambda}^\xi(y, d\tilde{s}) = \tilde{\lambda}(\xi, d\tilde{s}) - \tilde{\lambda}(y, d\tilde{s})$. A Taylor expansion yields $\mathcal{F}[\psi](z) = \sum_{k=1}^M \frac{1}{k!} G_z^{(k)}(0) + \frac{1}{M!} \int_0^1 (1-\tau)^M G_z^{(M+1)}(\tau) d\tau$. From the previous expression of $G_z(\delta)$ one derives that for $k \in \mathbb{N}^*$:

$$\begin{aligned} \frac{1}{k!} G_z^{(k)}(0) &= \frac{1}{k!} G_z(0) \times \\ &\left[ih \langle \Delta \gamma^x(y), z \rangle + h \int_{S^{d-1}} \int_0^\infty \left(e^{i\langle z, \rho \tilde{s} \rangle} - 1 - \frac{i\langle z, \rho \tilde{s} \rangle}{1 + \rho^2} \right) \frac{d\rho}{\rho^{1+\alpha}} \Delta \tilde{\lambda}^x(y, d\tilde{s}) \right]^k \end{aligned}$$

where $G_z(0) = \mathcal{F}[\theta](z)$, $\theta(p) := \tilde{p}(t-s, x, p; \lambda(y, \cdot), \gamma(y))$. Using the well-known properties of the Fourier transform one gets for all $k \in \llbracket 1, M \rrbracket$

$$G_z^{(k)}(0) = \mathcal{F} \left[h^k \left(\tilde{\Phi}_x - \tilde{\Phi}_y \right)^k \theta \right] (z),$$

where the operators are applied w.r.t. the x component. Hence,

$$\begin{aligned} \mathcal{F}[\psi](z) &= \sum_{k=1}^M \frac{1}{k!} G_z^{(k)}(0) + \frac{1}{M!} \int_0^1 (1-\tau)^M G_z^{(M+1)}(\tau) d\tau = \\ &= \sum_{k=1}^M \frac{h^k}{k!} \mathcal{F} \left[(\tilde{\Phi}_x - \tilde{\Phi}_y)^k \theta \right] (z) + \mathcal{F} \left[\frac{h^{M+1}}{M!} \int_0^1 (1-\tau)^M (\tilde{\Phi}_x - \tilde{\Phi}_y)^{M+1} \theta_\tau d\tau \right] (z), \end{aligned}$$

where $\forall \tau \in [0, 1]$, $\theta_\tau(p) = \tilde{p}(p-x; (t-s)\lambda(y, \cdot) + \tau h \Delta \lambda^x(y, \cdot), (t-s)\gamma(y) + \tau h \Delta \gamma^x(y))$. Taking the inverse Fourier transform at point y in the above equation, observing that $H(t-s, x, y) = (\tilde{\Phi}_* - \tilde{\Phi}^*) \tilde{p}(t-s, x, y; \lambda(y, \cdot), \gamma(y))$, we obtain

$$(4.9) \quad \begin{aligned} (H_N - H)(t-s, x, y) &= \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} \tilde{p}(t-s, x, y) + \\ &= \frac{h^M}{M!} \int_0^1 (1-\tau)^M (\tilde{\Phi}_* - \tilde{\Phi}^*)^{M+1} \tilde{p}_\tau(t-s, y-x) d\tau. \end{aligned}$$

Recall that for $t \in \{(t_i)_{i \in [1, N]}\}$

$$(p^d - p^N)(t, x, y) = \sum_{r=0}^{\infty} [(\tilde{p} \otimes_N H^{(r)})(t, x, y) - (\tilde{p} \otimes_N H_N^{(r)})(t, x, y)]$$

where we put $(\tilde{p} \otimes_N H_N^{(r)})(t, x, y) = 0$ for $hr > t$. Summing over $r \in \mathbb{N}$ in the identity

$$\begin{aligned} (\tilde{p} \otimes_N H^{(r)} - \tilde{p} \otimes_N H_N^{(r)})(t, x, y) &= \\ &= \left((\tilde{p} \otimes_N H^{(r-1)}) \otimes_N (H - H_N) \right)(t, x, y) + \\ &= \left((\tilde{p} \otimes_N H^{(r-1)} - \tilde{p} \otimes_N H_N^{(r-1)}) \otimes_N H_N \right)(t, x, y) \end{aligned}$$

one gets

$$(p^d - p^N)(t, x, y) = \left[p^d \otimes_N (H - H_N) + (p^d - p^N) \otimes_N H_N \right](t, x, y).$$

By iterative application of the last identity we obtain

$$p^d - p^N = \sum_{r=0}^{\infty} \left[p^d \otimes_N (H - H_N) \right] \otimes_N H_N^{(r)}.$$

We get from (4.9)

$$\begin{aligned} (p^d \otimes_N (H - H_N))(t, x, y) &= - \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} \tilde{p} \right](t, x, y) \\ &= - \frac{h^M}{M!} \int_0^1 (1-\tau)^M \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{M+1} \tilde{p}_\tau \right](t, x, y) d\tau. \end{aligned}$$

Eventually,

$$(p^d - p^N)(t, x, y) = - \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!} \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{k+1} p^N \right] (t, x, y) + h^M R_2(t, x, y),$$

$$R_2(t, x, y) = - \frac{1}{M!} \int_0^1 (1-\tau)^M \left[p^d \otimes_N (\tilde{\Phi}_* - \tilde{\Phi}^*)^{M+1} \tilde{p}_\tau^\Delta \right] (t, x, y) d\tau$$

$$\tilde{p}_\tau^\Delta(t, x, y) = \sum_{r=0}^{\infty} \tilde{p}_\tau \otimes_N H_N^{(r)}(t, x, y), \tilde{p}_0^\Delta = p^N.$$

This proves the expansion part of the Lemma. The bound follows as in the previous proof from Lemma 4.3. \square

We now state Lemma 4.3 that allows to control the rests appearing in the expansions of Lemmas 4.1 and 4.2.

Lemma 4.3 *For all multi-indices a, b s.t. $|a| + |b| < q - (d+4)$, the following inequalities hold:*

(4.10)

$$\left| D_y^a D_x^b p^d(t_k, x, y) \right| + \left| D_y^a D_x^b p^N(t_k, x, y) \right| \leq C t_k^{-\frac{|a|+|b|}{\alpha}} \tilde{p}(t_k, x, y), k \in \llbracket 1, n \rrbracket,$$

$$\left| D_y^a D_x^b p(t, x, y) \right| \leq C t^{-\frac{|a|+|b|}{\alpha}} \tilde{p}(t, x, y), 0 < t \leq T.$$

To conclude the proof it remains to prove Lemma 4.3 The first step is to get bounds on partial derivatives of the transition densities \tilde{p} and p . The following estimates generalize the ones obtained in [Kol00], Propositions 2.1-2.3. From now on we assume w.l.o.g. that $d \geq 3$, the cases $d \in \{1, 2\}$ can be addressed more directly. To proceed with the computations, we need to specify a useful change of coordinates. Namely, for a given direction $\zeta \in \mathbb{R}^d \setminus \{0\}$ introduce for $p \in \mathbb{R}^d$ the spherical coordinates $(\rho, \vartheta, \varphi_2, \dots, \varphi_{d-1})$, $\rho = |p|$ with first coordinate or main axis directed along ζ , that is

$$\begin{aligned} p_1 &= \rho \cos \vartheta, \quad p_2 = \rho \sin \vartheta \cos \varphi_2, \quad p_3 = \rho \sin \vartheta \sin \varphi_2 \cos \varphi_3, \dots \\ p_{d-1} &= \rho \sin \vartheta \sin \varphi_2 \dots \sin \varphi_{d-2} \cos \varphi_{d-1}, \\ (4.11) \quad p_d &= \rho \sin \vartheta \sin \varphi_2 \dots \sin \varphi_{d-2} \sin \varphi_{d-1}, \end{aligned}$$

$\vartheta \in [0, \pi], \varphi_i \in [0, \pi], i \in \llbracket 2, d-2 \rrbracket, \varphi_{d-1} \in [0, 2\pi]$. Consider then the coordinates (v, τ, ϕ) where $\tau = \cos \vartheta$ and $v = \rho |\zeta|$, with $v \in \mathbb{R}^+, \tau \in [-1, 1], \phi = (\varphi_2, \dots, \varphi_{d-1}) \in [0, \pi]^{d-3} \times [0, 2\pi]$. In the following we write $p(v, \tau, \phi)$ for the previous r.h.s. in (4.11) written in these new coordinates and $\bar{p}(\tau, \phi) = p(|\zeta|, \tau, \phi)$.

Lemma 4.4 *There exists a constant $C > 1$ such that the following estimates hold uniformly for α in any compact subset of the interval $(0, 2)$ and for all $0 < t \leq T, x, z$ and $|a| < q - (d + 4)$*

$$(4.12) \quad |D_z^\alpha \tilde{p}(t, x, z)| \leq \frac{C}{t^{|\alpha|/\alpha}} \tilde{p}(t, x, z),$$

$$(4.13) \quad |D_z^\alpha \tilde{p}(t, x, z)| \leq \frac{C}{|z - \gamma(y)t - x|^{|\alpha|}} \tilde{p}(t, x, z),$$

recalling $\tilde{p}(t, x, z) := \tilde{p}(t, x, z; \lambda(y, \cdot), \gamma(y))$.

Proof. Without loss of generality we suppose $\gamma(y) = 0$. The first step consists in differentiating w.r.t z the inverse Fourier transform for $\tilde{p}(t, x, z)$

$$(4.14) \quad \tilde{p}(t, x, z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp \left\{ -t \int_{S^{d-1}} |\langle p, s \rangle|^\alpha \lambda(y, ds) \right\} \exp(-i \langle p, z - x \rangle) dp.$$

For $z = x$, (2.5) and standard computations directly give estimate (4.12). Thus, in the following we also assume $z \neq x$ and use the previous spherical coordinates (v, τ, ϕ) derived from (4.11) setting $\zeta = z - x$ as the main axis. We obtain:

$$(4.15) \quad D_z^\alpha \tilde{p}(t, x, z) = \frac{1}{(2\pi)^d |z - x|^{|\alpha|+d}} \int_0^\infty dv v^{|\alpha|+d-1} \times \\ \int_{-1}^1 d\tau \int_{[0, \pi]^{d-3} \times [0, 2\pi]} d\phi \Psi(v, \tau, |a|) \exp \left\{ -t \frac{v^\alpha}{|z - x|^\alpha} \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda(y, ds) \right\} \times \\ \tau^{a_1} (1 - \tau^2)^{\frac{|a| - a_1 + d - 3}{2}} h(\phi, a),$$

where $\bar{p} = p/|p|$, $a = (a_1, \dots, a_d) \in \mathbb{N}^d$ and $\Psi(v, \tau, |a|) = (-1)^{|a|/2} \cos(v\tau) \mathbb{I}_{|a| \text{ even}} + (-1)^{(|a|+1)/2} \sin(v\tau) \mathbb{I}_{|a| \text{ odd}}$,

$$h(\phi, a) = \{ (\cos \varphi_2)^{a_2} (\sin \varphi_2 \cos \varphi_3)^{a_3} \times \dots \times (\sin \varphi_2 \dots \sin \varphi_{d-2} \cos \varphi_{d-1})^{a_{d-1}} \\ \times (\sin \varphi_2 \dots \sin \varphi_{d-2} \sin \varphi_{d-1})^{a_d} \} \times V(\phi) \\ V(\phi) = (\sin \varphi_2)^{d-3} (\sin \varphi_3)^{d-4} \times \dots \times (\sin \varphi_{d-3})^2 \sin \varphi_{d-2}.$$

We consider, first the case $|z - x|/t^{1/\alpha} \leq \overline{C}$, for a sufficiently small positive constant \overline{C} . In this case we expand the trigonometric function $\Psi(v, \tau, |a|)$

in (4.15) in power series and change the variable of integration $\frac{t^{1/\alpha}v}{|z-x|}$ to w in each term. This gives for all $k \in \mathbb{N}$,

$$(4.16) \quad D_z^a \tilde{p}(t, x, z) = \frac{C_{|a|}}{t^{\frac{|a|+d}{\alpha}}} \left\{ \sum_{m=0}^k \frac{(-1)^m}{(2m + \mathbb{I}_{|a| \text{ odd}})!} e_m^{|a|} \left(\frac{|z-x|}{t^{1/\alpha}} \right)^{2m + \mathbb{I}_{|a| \text{ odd}}} + R_{k+1}^{|a|} \right\}, \quad C_{|a|} = \frac{(-1)^{(|a| + \mathbb{I}_{|a| \text{ odd}})/2}}{(2\pi)^d},$$

where $\forall m \in \llbracket 1, k \rrbracket$,

$$e_m^{|a|} = \int_0^\infty dw \int_{-1}^1 d\tau \int_{[0, \pi]^{d-3} \times [0, 2\pi]} d\phi \exp \left\{ -w^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda(y, ds) \right\} \\ w^{|a|+2m+d-\mathbb{I}_{|a| \text{ even}}} \times \tau^{a_1+2m+\mathbb{I}_{|a| \text{ odd}}} (1-\tau^2)^{\frac{|a|-a_1+d-3}{2}} h(\phi, a), \\ |R_{k+1}^{|a|}| \leq \frac{|e_{k+1}^{|a|}|}{(2(k+1) + \mathbb{I}_{|a| \text{ odd}})!} \left(\frac{|z-x|}{t^{1/\alpha}} \right)^{2(k+1) + \mathbb{I}_{|a| \text{ odd}}}.$$

To simplify the notations we omit the dependence of the coefficients of our expansions on the direction $\zeta = z - x$. From **(A-1)**, **(A-2)** and (2.5) one then derives the following bound:

$$|e_m^{|a|}| \leq \frac{A_{d-2}}{\alpha C_1} \frac{\Gamma \left(\frac{|a| + 2m + d + \mathbb{I}_{|a| \text{ odd}}}{\alpha} \right)}{\alpha} \\ B \left(m + \frac{a_1 + 1 + \mathbb{I}_{|a| \text{ odd}}}{2}, \frac{|a| - a_1 + d - 1}{2} \right).$$

Here A_{d-2} denotes the area of the unit sphere S^{d-2} and B is the β -function. Note that the modulus of each term in the expansion (4.16) serves as estimate of the remainder in a finite Taylor expansion. From (4.16) we have

$$D_z^a \tilde{p}(t, x, z) = \frac{C_{|a|}}{t^{\frac{|a|+d}{\alpha}}} \left(e_0^{|a|} \left(\frac{|z-x|}{t^{1/\alpha}} \right)^{\mathbb{I}_{|a| \text{ odd}}} + R_1^{|a|} \right).$$

By Proposition 3.1 from [Kol00] for some \tilde{C} depending on \bar{C} , $\tilde{C}^{-1}t^{-d/\alpha} \leq \tilde{p}(t, x, z) \leq \tilde{C}t^{-d/\alpha}$. Hence,

$$(4.17) \quad |D_z^a \tilde{p}(t, x, z)| \leq \frac{C}{t^{\frac{|a|}{\alpha}}} \tilde{p}(t, x, z) \leq \frac{C\bar{C}^{|a|}}{|z-x|^{|a|}} \tilde{p}(t, x, z).$$

To estimate $D_z^a \tilde{p}(t, x, z)$ for $|z-x|/t^{1/\alpha} \geq (\bar{C})^{-1}$ we proceed as in Proposition 2.3 of [Kol00]. This gives the following representation $D_z^a \tilde{p}(t, x, z) =$

$[D_z^a \tilde{p}_\alpha(t, x, z)]_1 + [D_z^a \tilde{p}(t, x, z)]_2$ with

$$(4.18) \quad [D_z^a \tilde{p}(t, x, z)]_j = \frac{1}{(2\pi)^d} \int_0^\infty d\rho \rho^{|a|+d-1} \int_{-1}^1 d\tau \Psi(\rho|z-x|, \tau, |a|) \times \\ f_j(\tau) \int_{[0, \pi]^{d-3} \times [0, 2\pi]} d\phi \exp\{-t\rho^\alpha g_\lambda(\tau, \phi, y)\} h(\phi, a), \quad j = 1, 2.$$

Here

$$f_1(\tau) = \tau^{a_1} \left(1 - \tau^2\right)^{\frac{|a|-a_1+d-3}{2}} \chi(\tau), \quad f_2(\tau) = \tau^{a_1} \left(1 - \tau^2\right)^{\frac{|a|-a_1+d-3}{2}} (1 - \chi(\tau))$$

where $\chi(\tau)$ is a C^∞ even truncation function $\mathbb{R} \rightarrow [0, 1]$ that equals 1 for $|\tau| \leq 1 - 2\varepsilon$, and 0 for $|\tau| \geq 1 - \varepsilon$ for some $\varepsilon \in (0, \frac{1}{2})$. Because of the symmetry in τ , it is easy to see that the integral in (4.18) is non-zero only if a_1 and $|a|$ are both even or odd. Expanding the exponential at order 2 in (4.18) and making the change of variables $\rho|z-x| = v$ we get

$$(4.19) \quad [D_z^a \tilde{u}_\alpha(t, x, z)]_1 = \frac{C_{|a|}}{|z-x|^{|a|+d}} \sum_{m=0}^2 \frac{1}{m!} b_m^{|a|} \left(\frac{t}{|z-x|^\alpha}\right)^m,$$

where for $m \in \llbracket 0, 1 \rrbracket$,

$$b_m^{|a|} = (-1)^m \int_0^\infty F_m^{|a|}(v) v^{|a|+m\alpha+d-1} dv, \\ F_m^{|a|}(v) = [\mathbb{I}_{|a| \text{ even Re}} - \mathbb{I}_{|a| \text{ odd Im}}] \left[\int_{-\infty}^\infty \exp(-iv\tau) \varphi_m(\tau) d\tau \right], \\ \varphi_m(\tau) = f_1(\tau) \int_{[0, \pi]^{d-3} \times [0, 2\pi]} g_\lambda^m(\tau, \phi, y) h(\phi, a) d\phi.$$

and

$$b_2^{|a|} = 2 \int_0^1 (1 - \delta) \int_0^\infty F_{2, \delta}^{|a|}(v) v^{|a|+2\alpha+d-1} dv d\delta, \\ F_{2, \delta}^{|a|}(v) = [\mathbb{I}_{|a| \text{ even Re}} - \mathbb{I}_{|a| \text{ odd Im}}] \left[\int_{-\infty}^\infty \exp(-iv\tau) \varphi_{2, \delta}(\tau) d\tau \right], \\ \varphi_{2, \delta}(\tau) = f_1(\tau) \int_{[0, \pi]^{d-3} \times [0, 2\pi]} g_\lambda^2(\tau, \phi, y) \exp\left\{-\delta t \left(\frac{v}{|z-x|}\right)^\alpha g_\lambda(\tau, \phi, y)\right\} \\ \times h(\phi, a) d\phi.$$

Now Assumption **(A-1)** implies that $\varphi_m(\tau)$, $m = 0, 1$, and $\varphi_{2, \delta}(\tau)$ are C^q functions of τ with compact support. Indeed, for the unit vectors $\vec{p}(\tau + \Delta\tau, \phi)$

and $\bar{p}(\tau, \phi)$, from elementary algebra there exists an orthogonal matrix $A := A(\Delta\tau)$ s.t. $\bar{p}(\tau + \Delta\tau, \phi) = A\bar{p}(\tau, \phi)$. Hence, if $\lambda(x, ds) = \Theta(x, s)ds$ where Θ has the previous smoothness one can show

$$\begin{aligned} & \lim_{\Delta\tau \rightarrow 0} \frac{g_\lambda(\tau + \Delta\tau, \phi, x) - g_\lambda(\tau, \phi, x)}{\Delta\tau} \\ &= \lim_{\Delta\tau \rightarrow 0} \frac{\int_{S^{d-1}} |\langle \bar{p}(\tau, \phi), A^*s \rangle|^\alpha - |\langle \bar{p}(\tau, \phi), s \rangle|^\alpha \lambda(x, ds)}{\Delta\tau} \\ &= \int_{S^{d-1}} |\langle \bar{p}(\tau, \phi), s \rangle|^\alpha \lim_{\Delta\tau \rightarrow 0} \frac{[\Theta(x, As) - \Theta(x, s)]}{\Delta\tau} ds \\ &= \int_{S^{d-1}} |\langle \bar{p}(\tau, \phi), s \rangle|^\alpha \Theta'_s(x, s) \beta(\tau, \phi, s) ds, \end{aligned}$$

where $\beta(\tau, \phi, s)$ is C^∞ function in τ uniformly bounded in (τ, ϕ, s) in our region. The process can then be iterated other $q - 1$ times.

Thus all coefficients b_m are well defined.

Next, analogously to Proposition 2.3 in [Kol00] (where the case $|a| = 0$ was considered) and with the same rotations of the integration contours for $\alpha \in (0, 1]$, $\alpha \in (1, 2)$, we obtain for all $k \in \mathbb{N}$

$$(4.20) \quad [D_z^a \tilde{p}(t, x, z)]_2 = \frac{C_{|a|}}{|z - x|^{|a|+d}} \left\{ \sum_{m=0}^k \frac{1}{m!} c_m^{|a|} \left(\frac{t}{|z - x|^\alpha} \right)^m + R_{2,k+1}^{|a|} \right\},$$

$$c_m^{|a|} = 2[\mathbb{I}_{|a| \text{ even}} \text{Re} - \mathbb{I}_{|a| \text{ odd}} \text{Im}] \left[\int_{1-2\varepsilon}^1 d\tau \int_{[0, \pi]^{d-3} \times [0, 2\pi]} d\phi h(\phi, a) (-g_\lambda(\tau, \phi))^m \right. \\ \left. \times \exp\left(-\frac{i\pi\alpha m}{2}\right) (-i)^{|a|+d} \tau^{-(\alpha m + d + |a|)} \Gamma(\alpha m + d + |a|) f_2(\tau) \right],$$

and $|R_{2,k+1}^{|a|}| \leq \frac{|c_{k+1}^{|a|}|}{(k+1)!} \left(\frac{t}{|x-z|^\alpha} \right)^{k+1}$. Note that the coefficients $c_m^{|a|}$ are also well defined because τ does not approach zero. Precisely $|c_m^{|a|}| \leq 2A_{d-2} C_2^m (1 - 2\varepsilon)^{-\alpha m + d + |a|} \Gamma(\alpha m + d + |a|)$.

Now the sum of expansions (4.19) and (4.20) gives the expansion for $D_z^a \tilde{p}(t, x, z)$. Note that by construction, the first coefficient $b_0^{|a|} + c_0^{|a|}$ does not depend on the spectral measure $\lambda(y, \cdot)$ and it vanishes when the spectral measure is uniform (that is $C_1 = C_2 = 1$ in (2.5)). This can be shown by means of representations involving Bessel and Whittaker functions and the same rotations of the integration contours as in Proposition 2.2 of [Kol00], see Appendix B for details. Thus, for all $k \in \mathbb{N}^*$, we get a representation

$$(4.21) \quad D_z^a \tilde{p}(t, x, z) = \frac{C_{|a|}}{|z - x|^{|a|+d}} \left\{ \sum_{m=1}^k \frac{1}{m!} d_m^{|a|} \left(\frac{t}{|z - x|^\alpha} \right)^m + R_{k+1}^{|a|} \right\},$$

where $d_m^{|\alpha|} = b_m^{|\alpha|} + c_m^{|\alpha|}$ with $b_m = 0$ for $m \geq 3$ and $|R_{k+1}^{|\alpha|}| \leq \frac{|d_{k+1}^{|\alpha|}|}{(k+1)!} \left(\frac{t}{|x-z|^\alpha} \right)^{k+1}$. Now, from Proposition 3.1 in [Kol00] $d_1^0 > 0$. Equation (4.21) yields

$$D_z^a \tilde{p}(t, x, z) = \frac{C_{|a|} d_1^0 t}{|z-x|^{|\alpha|+d+\alpha}} \left(\frac{d_1^{|\alpha|}}{d_1^0} + \tilde{R}_2^{|\alpha|} \right), \quad |\tilde{R}_2^{|\alpha|}| \leq \frac{|d_2^{|\alpha|}|}{2d_1^0} \frac{t}{|x-z|^\alpha},$$

$$\tilde{p}(t, x, z) = \frac{C_{|a|}}{|z-x|^d} \left(\frac{d_1^0 t}{|z-x|^\alpha} + R_2^0 \right) \geq \frac{C_{|a|} d_1^0 t}{2|z-x|^{d+\alpha}}$$

for sufficiently small \bar{C} . Hence, we have

$$(4.22) \quad \begin{aligned} |D_z^a \tilde{p}(t, x, z)| &\leq \frac{CC_{|a|}}{|z-x|^{|\alpha|}} \frac{d_1^0 t}{|z-x|^{d+\alpha}} \leq \frac{C}{|z-x|^{|\alpha|}} \tilde{p}(t, x, z) \\ &\leq \frac{C}{\bar{C}^{|\alpha|} t^{|\alpha|/\alpha}} \tilde{p}(t, x, z). \end{aligned}$$

W.l.o.g. we can assume $\bar{C} < 1$. It remains to consider the case $|x-z|/t^{1/\alpha} \in]\bar{C}, \bar{C}^{-1}[:= I(\bar{C})$. It follows from (4.15) that $|z-x|^d \tilde{p}(t, x, z)$ and $|z-x|^{d+|\alpha|} D_z^a \tilde{p}(t, x, z)$ are continuous functions of $|x-z|/t^{1/\alpha}$. Since the stable density is also strictly positive, we deduce that there exists \tilde{C} s.t. on $I(\bar{C})$, $|D_z^a \tilde{p}(t, x, z)| \leq \frac{\tilde{C}}{|z-x|^{|\alpha|}} \leq \frac{C}{|z-x|^{|\alpha|}} \tilde{p}(t, x, z) \leq \frac{C\bar{C}^{-a}}{t^{|\alpha|/\alpha}} \tilde{p}(t, x, z)$ which concludes the proof. \square

Lemma 4.5 *For a constant $C > 1$ and for $|a| + |b| < (q-d-4)$:*

$$(4.23) \quad \left| D_y^a D_x^b H(t, x, y) \right| \leq \frac{C}{t^{\frac{|a|+|b|}{\alpha}}} \tilde{p}(t, x, y) \left(1 + \frac{\min(1, |y-x|)}{t} \right),$$

$$(4.24) \quad \left| D_x^b H(t, x, x+v) \right| \leq C \tilde{p}(t, x, x+v) \left(1 + \frac{\min(1, |v|)}{t} \right),$$

$$(4.25) \quad \left| D_y^a D_x^b \tilde{p}(t, x, y) \right| \leq \frac{C}{|y-\gamma(y)t-x|^{|\alpha|+|b|}} \tilde{p}(t, x, y).$$

Proof. Inequalities (4.23) and (4.24) follow from the representation

$$(4.26) \quad \begin{aligned} H(t, x, y) &= \langle \gamma(x) - \gamma(y), \nabla_x \tilde{p}(t, x, y) \rangle + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |p|^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \times \\ &\quad (\lambda(y, ds) - \lambda(x, ds)) \exp \left\{ -t |p|^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda(y, ds) \right\} \times \\ &\quad \exp \{ -i \langle p, y - \gamma(y)t - x \rangle \} dp \end{aligned}$$

analogously to the proof of Proposition 2.3 in [Kol00]. Inequality (4.24) contains in (3.23') p.748 of that reference. Inequality (4.25) can be derived following the proof of Lemma 4.4. \square

The proof of Lemma 4.3 can then be achieved from Lemmas 4.4 and 4.5 adapting the arguments in Appendix A concerning the control in terms of the frozen density for the "formal" series appearing in (3.5). See also the proof of Theorem 2.3 in [KM02] or Theorem 3.1 in [Kol00].

5. Extensions and conclusion. A careful examination of the previous proofs shows that the absolute continuity of λ w.r.t. to the Lebesgue measure of S^{d-1} can be removed in **(A-1)** provided the function

$$\zeta(t, x, y) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\int_{S^{d-1}} |\langle p, s \rangle|^\alpha \mu(x, ds) \right) \exp \left(-t \int_{S^{d-1}} |\langle p, s \rangle|^\alpha \lambda(y, ds) \right) \times \exp(-i\langle p, x \rangle) dp$$

has bounded derivatives w.r.t. x up to order q (see Appendix A and the statement of Theorem 3.1 in [Kol00]). Also up to a standard perturbative argument, similar controls on the density can be obtained when we consider (1.1) driven by $(Z_s + P_s)_{s \geq 0}$ where $(P_s)_{s \geq 0}$ is a compound Poisson process with Lévy measure $\nu_P(dz) = f(z)dz$ and $|f(z)| \leq \frac{C}{1+|z|^{d+\beta}}$, $\beta > 0$, see Theorem 4.1 in [Kol00]. In that case our main results remain valid up to a modification of the remainder. Indeed, it is the smaller exponent that leads the asymptotic behavior of $p(t, x, y)$ when $|x - y|$ is large. Thus $\rho_\alpha(y - x)$ has to be replaced by $\rho_{\min(\alpha, \beta)}$ in Theorem 2.1. Eventually, good controls have been obtained on p for stable-like processes, i.e. when the stability index in the generator $A\psi(x)$ in (2.4) can depend on the spatial position x , i.e. α turns to $\alpha(x) \in [\underline{\alpha}, \bar{\alpha}]$ strictly included in $(0, 2]$ (see Section 5 in [Kol00]). Anyhow the processes associated to those generators cannot be approximated by a usual Euler scheme and the previous analysis breaks down. The approximation of such processes will concern further research.

APPENDIX A: CONTROL OF THE PARAMETRIX SERIES OF THE DENSITY

For the sake of completeness we provide in this section a complete proof of the control for the r.h.s of (3.5) under our standing Assumptions **(A-1)**-**(A-3)**.

We first sum up the various ingredients needed for the proof coming from Section 3 in Kolokoltsov and that can also be directly obtained following the computations of Appendix B.

Proposition A.1 *For all K sufficiently large, there exists $C > 0$ s.t. the following estimates hold uniformly for α in any compact subset of $(0, 2)$.*

$$\begin{aligned} C^{-1}t^{-d/\alpha} &\leq \tilde{p}(t, x, y) \leq Ct^{-d/\alpha}, \quad |x - y| \leq Kt^{1/\alpha}, \\ \frac{C^{-1}t}{|x - y|^{d+\alpha}} &\leq \tilde{p}(t, x, y) \leq \frac{Ct}{|x - y|^{d+\alpha}}, \quad |x - y| \geq Kt^{1/\alpha}. \end{aligned}$$

Also, there exists $C > 0$ s.t. $\forall (t, x, y) \in [0, T] \times (\mathbb{R}^d)^2$,

$$\int dz \min(1, |z|) \tilde{p}(t, z; \lambda(y), \gamma(y)) \leq Ct^\omega, \quad \omega := \min(1, 1/\alpha).$$

For all $s \in (0, t)$

$$\begin{aligned} &\int dz \tilde{p}(t - s, z - x; \lambda(z), \gamma(z)) \min(1, |y - z|) s^{-1} \tilde{p}(s, y - z; \lambda(y), \gamma(y)) \\ &\leq C(t^{-1} \min(1, |y - x|) + s^{\omega-1}) \tilde{p}(t, y - x; \lambda(y), \gamma(y)), \\ &\quad \int dz \min(1, |z - x|) \tilde{p}(t - s, z - x; \lambda(z), \gamma(z)) \\ &\times \min(1, |y - z|) s^{-1} \tilde{p}(s, y - z; \lambda(y), \gamma(y)) \leq Cs^{\omega-1} \tilde{p}(t, y - x; \lambda(y), \gamma(y)), \\ &\quad \int dz \tilde{p}(t - s, z - x; \lambda(z), \gamma(z)) \tilde{p}(s, y - z; \lambda(y), \gamma(y)) \\ &\leq C \tilde{p}(t, y - x; \lambda(y), \gamma(y)). \end{aligned}$$

Introduce now for a given bounded measure η on S^{d-1} the function

$$\begin{aligned} \varphi_\eta(t, z, \lambda(y)) &:= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dp |p|^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \eta(ds) \\ &\times \exp\left(-t|p|^\alpha \int_{S^{d-1}} |\langle \bar{p}, s \rangle|^\alpha \lambda(y, ds)\right) \exp(-i\langle p, z \rangle). \end{aligned}$$

With this notation and (4.26) we get

$$H(t, x, y) = \langle \gamma(x) - \gamma(y), \tilde{p}(t, x, y) \rangle + (\varphi_{\lambda(y)} - \varphi_{\lambda(x)})(t, y - x - \gamma(y)t, \lambda(y)).$$

Now Proposition 2.5 in [Kol00] gives that for a bounded measure η ,

$$\varphi_\eta(t, z, \lambda(y)) \leq Ct^{-1} \tilde{p}(t, z; \lambda(y)).$$

From Lemmas 4.4, 4.5 and the above control one deduces $|H(t, x, y)| \leq C \tilde{p}(t, y - x; \lambda(y)) (1 + t^{-1} \min(1, |x - y|)) := Cv(t, x, y)$ (which actually gives (4.23) for $a = b = 0$).

Introduce now $\beta \circ \psi(t, s, x, y) = \int_{\mathbb{R}^d} \beta(t-s, x, z) \psi(s, z, y) dz$, i.e. \circ is the spatial part of the convolution operator \otimes , and set $\tilde{v}(t, x, y) := tv(t, x, y)$. From Proposition A.1 one derives

$$\begin{aligned}\tilde{p} \circ v(t, s, x, y) &\leq C(v(t, x, y) + s^{\omega-1} \tilde{p}(t, x, y)) \\ \tilde{v} \circ v(t, s, x, y) &\leq C(\tilde{v}(t, x, y) + s^{\omega-1} \tilde{p}(t, x, y)).\end{aligned}$$

Recalling $|H(t, x, y)| \leq Cv(t, x, y)$, one gets:

$$|\tilde{p} \otimes H|(t, x, y) \leq C(\tilde{v}(t, x, y) + t^\omega \tilde{p}(t, x, y)), \quad |\tilde{p} \otimes H \otimes H|(t, x, y) \leq C^2 t^\omega (\tilde{p} + \tilde{v})(t, x, y).$$

An induction yields, for all $k \in \mathbb{N}^*$:

$$\begin{aligned}|\tilde{p} \otimes H^{(2k)}|(t, x, y) &\leq \frac{C^{2k} t^{k\omega}}{(k!)^2} (\tilde{p} + \tilde{v})(t, x, y), \\ |\tilde{p} \otimes H^{(2k+1)}|(t, x, y) &\leq \frac{C^{2k+1} t^{k\omega}}{k!(k + \mathbb{I}_{\alpha \in (0,1]})!} (tv(t, x, y) + t^\omega \tilde{p}(t, x, y) \mathbb{I}_{\alpha > 1}),\end{aligned}$$

and the the required control, i.e. $p(t, x, y) \leq C\tilde{p}(t, x, y)$. The controls on the derivatives can be proved in a similar way, up to suitable rearrangements of the variable of integration, see p.747 and 748 in [Kol00]. Also the whole proof can be carried out for p^d, p^N . \square

Remark A.1 *To conclude, note that by arguments similar to those used to prove Proposition A.1, one gets*

$$|H \otimes H(t, x, y)| \leq Ct^{\omega-1} \tilde{p}(t, x, y),$$

which turns to be a useful estimate to derive (4.8) following the above proof.

APPENDIX B: ADDITIONAL COMPUTATIONS CONCERNING THE DERIVATIVES OF THE DENSITY

In this section we prove that $b_0^{|a|} + c_0^{|a|} = 0$, justifying that the first index in (4.21) is one.

Odd dimensions d . From the definitions in the proof of Lemma 4.4 , it is enough to show

$$\begin{aligned}\left[\mathbb{I}_{|a| \text{ even}} \text{Re} - \mathbb{I}_{|a| \text{ odd}} \text{Im} \right] \left\{ \int_0^\infty \left[\int_{\mathbb{R}} \exp(-iv\tau) f_1(\tau) d\tau \right] v^{|a|+d-1} dv \right. \\ \left. + 2 \int_{1-2\varepsilon}^1 (i\tau)^{-(d+|a|)} \Gamma(d+|a|) f_2(\tau) d\tau \right\} = 0.\end{aligned}$$

Note that by a rotation of angle $-\frac{\pi}{2}$ of our contour integration and Cauchy's theorem

$$\begin{aligned} (i\tau)^{-(d+|a|)} \Gamma(d+|a|) &= i^{-(d+|a|)} \int_0^\infty \exp(-\tau x) x^{|a|+d-1} dx \\ &= \int_0^\infty \exp(-i\tau w) w^{|a|+d-1} dw. \end{aligned}$$

Hence, the previous condition writes

$$(B.1) \quad \left[\mathbb{I}_{|a| \text{ even Re}} - \mathbb{I}_{|a| \text{ odd Im}} \right] \left\{ \int_0^\infty \left[\int_{\mathbb{R}} \exp(-iv\tau) f_1(\tau) d\tau \right. \right. \\ \left. \left. + 2 \int_{1-2\varepsilon}^1 \exp(-i\tau v) f_2(\tau) d\tau \right] v^{|a|+d-1} dv \right\} = 0.$$

Denote $F_j(v) = \int_{\mathbb{R}} \exp(-iv\tau) f_j(\tau) d\tau$, $j \in \{1, 2\}$. Remind that $|a|$ and a_1 have the same parity, see proof of Lemma 4.4.

a) For even $|a|, a_1$, $(F_j(v))_{j \in \{1, 2\}}$ are even and belong to a Schwartz space of functions. Since d is odd, by inverse Fourier transform, Equation (B.1) reduces to

$$\begin{aligned} \sum_{j \in \{1, 2\}} \int_0^\infty F_j(v) v^{|a|+d-1} dv &= \frac{1}{2} \sum_{j \in \{1, 2\}} \int_{\mathbb{R}} F_j(v) v^{|a|+d-1} dv \\ &= \frac{(-i)^{|a|+d-1} (2\pi)^d}{2} \sum_{j \in \{1, 2\}} f_j^{(|a|+d-1)}(0) = 0. \end{aligned}$$

Now $f_2^{(|a|+d-1)}(0) = 0$ by definition of f_2 and the equality $f_1^{(|a|+d-1)}(0) = 0$ follows from the Leibniz differentiation rule for the product $\tau^{a_1} \times (1 - \tau^2)^{\frac{|a|-a_1+d-3}{2}}$ and the identity $|a| + d - 1 = a_1 + (|a| - a_1 + d - 3) + 2$. Thus (B.1) holds in this case.

b) Analogously, for odd $|a|, a_1$, $-\text{Im}(F_j(v))_{j \in \{1, 2\}}$ are odd and belong to a Schwartz space of functions. The functions $(-\text{Im}F_j(v)v^{|a|+d-1})_{j \in \{1, 2\}}$ is even. Thus $\sum_{j \in \{1, 2\}} \int_0^\infty (-\text{Im}F_j(v))v^{|a|+d-1} dv = \frac{(-i)^{|a|+d}}{2} \sum_{j \in \{1, 2\}} f_j^{(|a|+d-1)}(0) = 0$ for the same previous reasons and equation (B.1) holds in this case as well.

Even dimensions d . We assume in this section that the spectral measure is uniform. For $|a|$ and $a_1 = 2m$ even, equation (4.15) can be rewritten as

$$(B.2) \quad D_z^a \tilde{p}(t, x, z) = \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \\ \times \int_0^\infty dv v^{|a|+d-1} \exp\left(-t \frac{v^\alpha}{|z-x|^\alpha}\right) \int_{-1}^1 (1-\tau^2)^{N_j-1/2} \cos(v\tau) d\tau$$

where $A_{d-2}^a = \int_{[0,\pi]^{d-3} \times [0,2\pi]} h(\phi, a) d\phi$ and $N_j = \frac{|a| - a_1 + d - 2 + 2j}{2}$, $j \in \llbracket 0, m \rrbracket$. Now recalling the definition of the Bessel function $J_n(z) := \frac{(z/2)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^1 (1-t^2)^{n-1/2} \cos(zt) dt$ which is well defined for $n > 1/2$ on $\mathbb{C} \setminus (-\infty, 0)$, we get

$$(B.3) \quad \begin{aligned} D_z^a \tilde{p}(t, x, z) &= \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j 2^{N_j} \Gamma(N_j + \frac{1}{2}) \sqrt{\pi} \\ &\quad \times \int_0^\infty v^{\frac{|a|+a_1+d-2j}{2}} \exp\left(-t \frac{v^\alpha}{|z-x|^\alpha}\right) J_{N_j}(v) dv \\ &= \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \Gamma(N_j + \frac{1}{2}) 2^{N_j+1/2} \end{aligned}$$

$$\times \operatorname{Re} \int_0^\infty \exp\left(-t \frac{v^\alpha}{|z-x|^\alpha}\right) \exp\left[\left(\frac{1}{2}N_j + \frac{1}{4}\right)\pi i\right] W_{0,N_j}(2iv) v^{N_j'} dv,$$

where $W_{0,n}(z) = \frac{\exp(-z/2)}{\Gamma(n+\frac{1}{2})} \int_0^\infty [t(1+t/z)]^{n-1/2} e^{-t} dt$, $n > 1/2$, $z \in \mathbb{C} \setminus (-\infty, 0)$, is the Whittaker function and for $z > 0$,

$$J_n(z) = 2\operatorname{Re} \left[\frac{1}{\sqrt{2\pi z}} \exp\left(\frac{1}{2}(n + \frac{1}{2})\pi i\right) W_{0,n}(-2iz) \right]$$

(relation (2.10) from [Kol00]) and $N_j' = \frac{|a|+a_1+d-2j-1}{2}$, $j \in \llbracket 0, m \rrbracket$. For $\alpha \in (0, 1]$, from Cauchy's theorem we can change the integration path in (B.3) to the negative imaginary half line. Setting then $v = -i\xi$ we obtain

$$\begin{aligned} D_z^a \tilde{p}(t, x, z) &= \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \Gamma(N_j + \frac{1}{2}) 2^{N_j+1/2} \\ &\quad \times (-1)^{j-m} \operatorname{Re} \left[-i \int_0^\infty \exp\left(-t \frac{\xi^\alpha}{|z-x|^\alpha} \exp\left(-\frac{i\pi\alpha}{2}\right)\right) W_{0,N_j}(2\xi) \xi^{N_j'} d\xi \right]. \end{aligned}$$

Recalling the definition of W_{0,N_j} , we conclude expanding the exponential in power series that the first term is 0.

For $\alpha \in (1, 2)$, using the same arguments we can rotate the initial contour through the angle $-\pi/(2\alpha)$. Setting then $\eta = e^{i\frac{\pi}{2\alpha}} v$ we get

$$\begin{aligned} D_z^a \tilde{p}(t, x, z) &= \frac{(-1)^{|a|/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \Gamma(N_j + \frac{1}{2}) 2^{N_j+1/2} \\ &\quad \times \operatorname{Re} \int_0^\infty \exp\left(it \frac{\eta^\alpha}{|z-x|^\alpha} + \left(\frac{1}{2}N_j + \frac{1}{4}\right)\pi i - \frac{i\pi}{2\alpha}(N_j' + 1)\right) \end{aligned}$$

$$\times W_{0,N_j}(2\eta \exp\left\{\frac{i\pi(\alpha-1)}{2\alpha}\right\})\eta^{N'_j}d\eta.$$

Taylor's formula for $\exp(it\frac{\eta^\alpha}{|z-x|^{|a|+d}})$ yields for the first term, $\forall j \in \llbracket 0, m \rrbracket$, $I_\alpha^{1,j} := \operatorname{Re} \left\{ \exp \left[\left(\frac{1}{2}N_j + \frac{1}{4} \right) \pi i - \frac{i\pi}{2\alpha}(N'_j + 1) \right] \int_0^\infty W_{0,N_j}(2\eta \exp\left\{\frac{i\pi(\alpha-1)}{2\alpha}\right\})\eta^{N'_j} d\eta \right\}$. At last, we rotate the contour through the angle $-\frac{\pi(\alpha-1)}{2\alpha}$. Setting $\xi = \eta \exp\left(\frac{i\pi(\alpha-1)}{2\alpha}\right)$ we obtain $I_\alpha^{1,j} = \operatorname{Re} \left\{ -i(-1)^{j-m} \int_0^\infty W_{0,N_j}(2\xi)\xi^{N'_j} d\xi \right\} = 0$.

For $|a|$ and $a_1 = 2m + 1$ odd,

$$\begin{aligned} D_z^a \tilde{p}(t, x, z) &= \frac{(-1)^{(|a|+1)/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \\ &\times \int_0^\infty dv v^{|a|+d-1} \exp\left(-t \frac{v^\alpha}{|z-x|^\alpha}\right) \int_{-1}^1 (1-\tau^2)^{N_j-1/2} \tau \sin(v\tau) d\tau \\ &= \frac{(-1)^{(|a|+1)/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m+1-j} C_m^j \int_0^\infty dv v^{|a|+d-2} \exp\left(-t \frac{v^\alpha}{|z-x|^\alpha}\right) \\ &\times \int_{-1}^1 (1-\tau^2)^{N_j-1/2} \tau d(\cos(v\tau)) \\ &= \frac{(-1)^{(|a|+1)/2} A_{d-2}^a}{(2\pi)^d |z-x|^{|a|+d}} \sum_{j=0}^m (-1)^{2m-j} C_m^j \int_0^\infty dv v^{|a|+d-2} \exp\left(-t \frac{v^\alpha}{|z-x|^\alpha}\right) \\ &\times \int_{-1}^1 \cos(v\tau) \times \left[(1-2N_j)\tau^2(1-\tau^2)^{N_j-3/2} + (1-\tau^2)^{N_j-1/2} \right] d\tau. \end{aligned}$$

The above integrals have the same form as in (B.1) and can be estimated similarly.

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