Reusable containers within reverse logistic context

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Abstract
Due to economic and environmental constraints, many organizations have started to ship their products in reusable containers such as plastic pallets, boxes and crates. Minimizing the total flow cost arising from reusable containers is a major problem for these organizations. In this paper we will focus on the modeling of this problem as a network flow, and the proposal of an appropriate resolution method. This resolution method will allow us to better understand the system behavior and can be an important tool for studying related problems, such as dimensioning and purchasing policy.

Keywords: Reusable transport packages, network flow, optimization, container management, supply chain.

1. Introduction

Since the mid-1980s, reusable containers have been adopted by many companies. In 1987, for example, the Bergen Brunswig Drug Company in California purchased 120,000 returnable plastic containers to replace one-use corrugated cartons. The company ships from its 37 distribution centers to its 10,000 pharmacies, located in 40 states (D.Saphire, 1994). Many other manufacturers in electronic goods and the automobile industry have also switched to reusable containers. This adoption is essentially due to potential economic benefits. It enables companies to save money by decreasing packaging material requirements: generally the cost of reusable containers is amortized over their lifecycle and the more the container is used the more the cost per trip is decreased. It reduces product damage due to shipping and handling because reusable containers are generally sturdier than one-way container and they are designed to withstand multiple uses thus providing better product protection. Other savings are possible. Warehouse utilization can be improved by reducing storage space requirements since reusable containers can be stacked higher than one-way containers. Improvements in worker safety can reduce costs since their ergonomic design reduces injuries from box cutters, staples, debris and stray packaging(D.Saphire, 1994). Furthermore, reusable containers are better for the environment since they reduce packaging waste. This can be a tool to meet the waste reduction requirements of government regulations, which are especially strict in EU countries (M.Kärkkäinen et al., 2004). However, reusing containers requires a more complex supply chain. The purchasing costs of reusable containers are significantly higher than those of one-way containers. A reusable container, such as a plastic box, can cost 10 times more than a one-way container, such as a corrugated box (D.Saphire, 1994). Therefore, for reusable containers to be of benefit, efficient container-management is a top priority.

Despite the importance of specific container management, there are few academic studies on the subject. Those that do exist can be classified under 4 headings:

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- *Organisational*: these studies deal with the different issues related to the implementation of reusable container systems, like collaboration and cooperation between supply chain members. They, generally, stress the role of information systems(Chan et al., 2005; Chan, 2007; Eelco de Jong et al., 2004; M.Kärkkäinen et al., 2004) .
- *Environmental*: These studies present the environmental advantages of reusable containers compared to one-way containers(Gassel and F.J.M., 1998; González-Torre et al., 2004; S. Paul Singh et al., 2006b).
- *Economic*: in these studies, the economic benefits of reusable containers are shown compared to one-way containers. Often, these studies present the environmental advantages linked to economic benefits(Lerpong Jarupan et al., 2003; Martin, 1996; Orbis, 2004; Stopwaste and RPA, 2008).
- *Operational*: these studies propose mathematical tools that can help companies optimize the different aspects of their reusable container systems. We can find studies that propose models which optimize the total cost generated by the use of reusable containers (transport storage, maintenance…). Others deal with the problem of container depot localisation(ED Castillo and Cochran, 1996; Erera Alan L et al., 2005a; Erera Alan L et al., 2005b; I. A. Karimi et al., 2005; Leo Kroon and Gaby Vrijens, 1995).

In this paper we will deal with the operational side of the problem. We will consider a system that contains three kinds of site: customer sites, supplier sites and depots:

A *customer site* is a facility that receives the loaded containers from a supplier site. After the containers have been emptied the customer can return them to depots or supplier sites for storage and reuse.

A *supplier site* is a facility that receives empty containers from depots or customer sites in order to load and send them to customers.

A *depot* is a place in which we store empty containers after they have been used by customers. Depots can supply suppliers with empty containers. We suppose that storage is also possible at customer and supplier sites.

Our purpose in this work is to propose a model that optimizes the total cost resulting from transportation between sites and storage and to propose an original resolution method for some particular cases. This method is based on simple recurring formulas that can be easily solved for the different decisions variables. This can help us to better understand the system behaviour and, in turn, can help us deal with other related problems, such as system dimensioning and purchasing policies for reusable containers.

The remainder of this paper is organized as follows. In section 2, we will present a literature review of the container management problem from the operational side. In this review we will concentrate on the work presented by Kroon and Vrijens (1995). In this work the authors present a step-by-step analysis of the operating policies of container management systems. This work will be a reference for our paper. In section 3, a description of our problem will be presented in detail and an optimization model, in the form of a network flow problem, will be developed. This model is a basic formulation of the case with several customer sites, several supplier sites and several depots under some simplifying assumptions. In section 4, we will simplify the model and treat the case of one customer site, one supplier site and one depot. In this section an original resolution method will be developed. In section 5, a result analysis and interpretation will be presented. In the final section, we will outline our future intentions for extension and enhancement of the presented model.

2. Literature Review

Most of the existing works have focused on sea container management. This is due to their large cost. One of the problems which has been studied is container repositioning. Karimi *et* *al.* (2005) presented a new linear programming methodology based on a continuous-time approach which they called the event-based “pull” approach. In this continuous-time representation, the principle is to fix all possible event times *a priori*, taking into account some simplifications and assumptions. Initially, a chronologically ordered superlist of possible instances is generated identifying which container movements (events) may occur and the types of movements that may occur at each such instance. This superlist of times and events is then used to develop a linear programming (LP) formulation whose solution will define the events that minimize the total logistic (flow) cost. Erera *et al.* (2005) proposed two different formulations, one deterministic(Erera Alan L et al., 2005a), one stochastic(Erera Alan L et al., 2005b), for the same problem. In the deterministic formulation, they integrated container booking and routing decisions with repositioning decisions in the same model. They believe that a container operator who uses this formulation via a global system may indeed be able to reduce costs and improve equipment utilization as compared to the standard approaches which ignore container routings. In the Stochastic formulation they presented a stochastic model in which uncertain parameters are assumed to fall within an interval around a nominal value and they established some conditions for robust flow under certain definitions.

Similar work has been developed in the soft drinks field for the returnable bottle. Del Castillo and Cochran (1996) modelled the reusable bottle production and distribution activities of a Soft Drink Company. They optimized the problem in terms of increasing the number of empty bottles available in order to make the comany more competitive. This is an alternative to the total cost optimization when cost calculation is not possible.

It is important to note that the majority of texts deal with the problem in specific cases (sea containers or returnable bottles). The only text that we found that considers the problem in the general case is the paper of Kroon and Vrijens (1995). In this paper, the authors classified returnable container systems into three groups depending on their operating policies: switch pool systems, systems with return logistics, and systems without return logistic.

In a *switch pool system*, participants have their own allotments of containers. Thus the cleaning, maintenance and storage are the responsibility of each pool-participant. Pool-participants may be the senders and recipients, or the senders, carriers, and recipients of the goods. If only the sender and the recipients are allotted containers, a transfer will take place when the goods are delivered to the recipient. In this case, the role of the carrier is to transport the loaded container to the recipients and to return the empty containers to the sender. Alternatively, the carrier is also allotted containers and replaces the loaded containers with empty containers when it picks up a load from a sender or recipient.

In a *system with return logistics*, the containers are owned by a central agency. It is responsible for returning the containers after they are unloaded. There are two variants of this system.

- *Transfer system*: the core of this system is that the sender always uses the same containers. He is responsible for tracking, tracing, maintaining, cleaning and storing them.
- *Depot system:* Under the depot system, the empty containers that are not in use are stored at container depots, which can supply the senders with empty containers on demand.

In the last system, *the system without return logistics*, the containers are owned by a central agency, and the sender rents them for fixed periods. The containers return to the agency after use. The sender is responsible for return logistics, cleaning, maintenance, storage and control. In the same paper, Kroon and Vrijens (1995) proposed a mathematical model for the design of a return logistics system for returnable containers with a depot system variant. The main purpose of the model is to determine the suitable number of containers, the appropriate number of container depots and their locations in order to minimize the total cost.

The model considered in our paper assumes the operator is using a system with return logistics with the depot system variant. We suppose that the central agency can be both the customer (recipient), suppliers (sender) or third parties.

3. Problem description

We assume that the transportation network consists of depots, customer sites and supplier sites (see figure 1).

For a period *t*, in order to satisfy customer demand, suppliers must have a sufficient number of empty containers at their disposal. These empty containers can be obtained from different customer sites, different depots or their own stocks. Once the containers are loaded they will be sent to their destinations (customers) during the same period. At customer sites the loaded containers will be emptied and can be returned to either depots or supplier sites as the best place for storage and reuse in subsequent periods. The different movements involved in this system generate large costs. Our purpose here is to optimize these costs and to propose the best plan for transportation and storage in each period.

We will model the problem as a network flow (see figure 2) but first we will make some assumptions for simplification purposes and present the different decision variables.

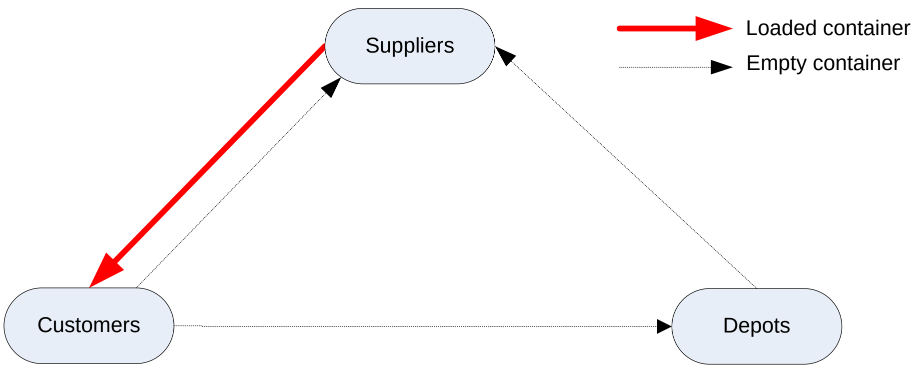

Figure 1 : Flow of container

Assumptions

- *The system is deterministic. The different costs for storing and moving the empty and loaded containers between sites and depots are known* a priori. *Similarly, the demands for each customer site are also known* a priori.
- *There is no loss in the system*
- *Container transition time between different sites and depots are negligible. Container loading and unloading times are also neglected.*
- *There is one type of container. All containers are homogeneous*
- *Two movements of empty containers can be done between each pair of sites/depots or customers sites/ suppliers sites during the same period. The first takes an empty container to be loaded and sent to customers. The second takes place when the containers are emptied and it attempts to return empty containers to suitable locations for the next usage.*
- *There are two types of inventory in sites and depots. The principal inventory, which is the inventory before the first movement of empty containers for loading, and the temporary inventory, which is the inventory after the movement of empty containers for loading. Only the cost of the principal inventory is considered in our model.*
- *The storage capacities of sites and depots are unlimited*
- *All demands are nonzero and are different*

Decisions variables

Let D be the number of depots ($d=1, \ldots, D$), I be the number of customers sites ($i=1,\ldots, I$), F be the number of Suppliers sites ($f=1,\ldots, F$) and T be the number of periods in the planning horizon ($t=1,\ldots, T$).
We define the different decisions variables as follows:

- x_{fi}^{t} : Number of loaded containers transported from suppliers f to a customer i at period t
- x_{i}^{t} : Number of containers needed in customer site i at period t
- y_{if}^{t} : Number of empty containers transported from customer i to supplier f at period t to be loaded and sent to customers.
- $y1_{if}^{t}$: Number of empty containers transported from customer i to supplier f at period t to be returned to the suitable location.
- z_{df}^{t} : Number of empty containers transported from depot d to suppliers f at period t to be loaded and sent to customers.
- $z1_{df}^{t}$: Number of empty containers transported from depot d to suppliers f at period t to be returned to the suitable location.
- u_{id}^{t} : Number of empty containers transported from customer i to depot d at period t to be loaded and sent to customers.
- $u1_{id}^{t}$: Number of empty containers transported from customer i to depot d at period t for them to be returned to the suitable location.
- a_{i}^{t} : the principal inventory of containers in customer site i at period t.
- $a1_{i}^{t}$: the temporary inventory of containers in customer site i at period t.
- d_{d}^{t} : the principal inventory of containers in depot d at period t.
- $d1_{d}^{t}$: the temporary inventory of containers in depot d at period t.
- b_{f}^{t} : the principal inventory of containers in supplier site f at period t.
- $b1_{f}^{t}$: the temporary inventory of containers in supplier site f at period t.

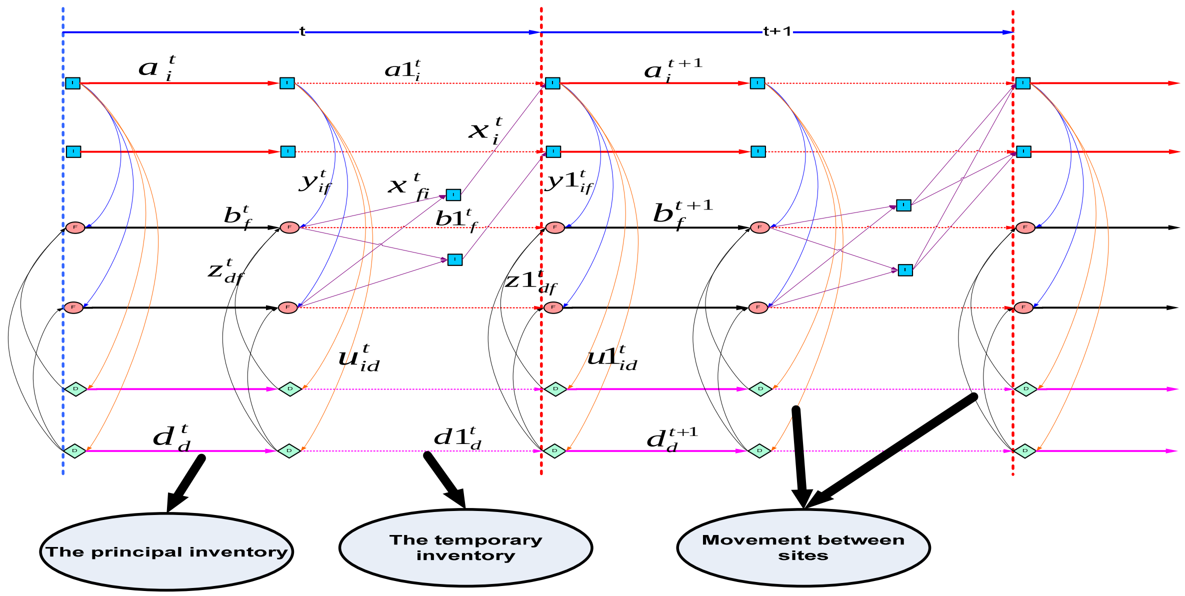

Figure 2 : Network flow

Constraints

The constraints of the system are essentially the flow conservation constraints. They can be classified into four categories:

- the first represents the conservation flow before sending the loaded containers to different sites and depots

$$a_i^t = \sum_f y_{if}^t + \sum_d u_{id}^t + a1_i^t \qquad \forall\, t \in [0,T], \forall\, i \in [0,I] \tag{1}$$

$$b_f^t = \sum_i x_{fi}^t + b1_f^t - \sum_i y_{if}^t - \sum_d z_{df}^t \qquad \forall\, t \in [0,T], \forall\, f \in [0,F] \tag{2}$$

$$d_d^t = \sum_f z_{df}^t + d1_d^t - \sum_i u_{id}^t \qquad \forall\, t \in [0,T], \forall\, d \in [0,D] \tag{3}$$

For example, constraint (1) represents the conservation flow at customer site *i* at period *t*; the principal inventory a_i^t is equal to the sum of departures from site *i* to depots $\sum_d u_{id}^t$ and supplier sites $\sum_f y_{if}^t$ plus the temporary inventory $a1_i^t$.

- The second category of constraints represents the conservation flow after the emptying of loaded containers and their return to different sites.

$$a_i^{t+1} = x_i^t + a1_i^t - \sum_f y1_{if}^t - \sum_d u1_{id}^t \qquad \forall\, t \in [0,T], \forall\, i \in [0,I] \tag{4}$$

$$b_f^{t+1} = \sum_d z1_{df}^{t-1} + \sum_i y1_{if}^{t-1} + b1_f^t \qquad \forall\, t \in [0,T], \forall\, f \in [0,F] \tag{5}$$

$$d_d^{t+1} = \sum_i u1_{id}^t + d1_d^t - \sum_f z1_{df}^t \qquad \forall\, t \in [0,T], \forall\, d \in [0,D] \tag{6}$$

- The third category represents the sum of containers arriving at a customer site *i* at period *t.*

$$\sum_{f,i} x_{fi}^t = x_i^t \qquad \forall\, t \in [0,T], \forall\, i \in [0,I] \tag{7}$$

- The last category of constraint is the lower band constraint, that is the sum of arriving containers must be greater than or equal to the demand at site *i*.

$$\delta_i^t \le x_i^t \qquad \forall\, t \in [0,T], \forall\, i \in [0,I] \tag{8}$$

We suppose that there is no capacity constraint on storage and transportation.
The objective function can be expressed as:

$$C = \sum_{t,f,i} C_{xfi}^{Tr} x_{fi}^t + \sum_{t,f,i} C_{yif}^{Tr} (y_{if}^t + y1_{if}^t) + \sum_{t,f,d} C_{zdf}^{Tr} (z_{df}^t + z1_{df}^t) + \sum_{t,i,d} C_{uid}^{Tr} (u_{id}^t + u1_{id}^t)$$
$$+ \sum_{t,i} C_{ai}^{st} a_i^t + \sum_{t,d} C_{dd}^{st} d_d^t + \sum_{t,f} C_{bf}^{st} b_f^t$$

where C_{xfi}^{Tr}, C_{yif}^{Tr}, C_{zdf}^{Tr} and C_{uid}^{Tr} are the transportation costs between sites and depots and C_{ai}^{st}, C_{dd}^{st} and C_{bf}^{st} are the storage cost in each site and depot.

In the following sections, we will simplify the model; we will deal with the case with only one customer site, one supplier site and one depot. We will show that it is possible to determine all the decision variables with a simple recurring formula.

4. Studied case

In this case we add the following assumptions to the system (see figure 3):

- *We have one customer site, one supplier site and one depot*
- *We assume that the depot has the lowest cost, followed by the customer site and the supplier site being the most expensive. The configuration of cost is as follow* $C_d^{st} < C_a^{st} < C_b^{st}$
- *the initial stock of empty containers at the supplier site is equal to zero*

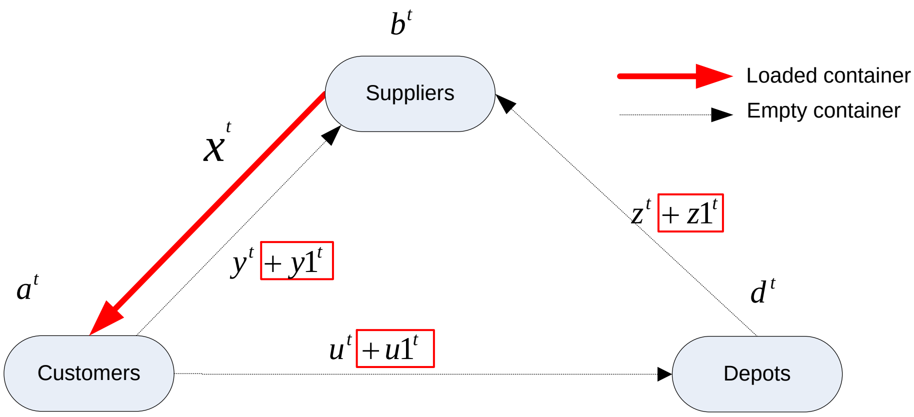

Figure 3 : Decision variables

The different constraint can be simplified as follows

$$a^t = y^t + u^t + a1^t$$

$$b^t = x^t + b1^t - y^t - z^t$$

$$d^t = z^t - d1^t - u^t$$

$$a^{t+1} = x^t + a1^t - y1^t - u1^t$$

$$b^{t+1} = z1^t + y1^t + b1^t$$

$$d^{t+1} = u1^t + d1^t - z1^t$$

$$x^t \geq \delta^t$$

The objective function is given by

$$C = \sum_t C_x^{tr} x_{fi}^t + \sum_t C_y^{tr}(y^t + y1^t) + \sum_t C_z^{tr}(z^t + z1^t) + \sum_t C_u^{tr}(u^t + u1^t) + \sum_t C_a^{st} a^t + \sum_t C_d^{st} d^t + \sum_t C_b^{st} b^t$$

Now we model the problem graphically as a network *R=(X, U, a, b, c)* for use in the different proofs.

X: Set of nodes that represents the three sites at each period *t*

U: Set of arcs joining nodes. These arcs can be storage arcs or transportations arcs. Each arc represent a decision variable and takes the same notation but in the capital form. For example, the arc which correspond to y^t take the notation of Y^t.

a (u) : represent the cost of an arc $u \in U$. For example, $a(Y^t) = C_y^{tr}$

b(u) : represent the lower band of arc $u \in U$. All arcs have a lower band equal to zero except X^t arcs which have δ^t as lower bands.

$$\begin{cases} b(u)=0 & \forall\ u \in U/(X^t) \\ b(u)=\delta^t & \forall\ u=(X^t) \end{cases} \quad t \in [0,T]$$

c(u) : represents the capacity of an arc. We suppose that it is infinite on all arcs $\forall\ u \in U\ c(u)=\infty$

The different pairs of variables and arcs are shown with the corresponding arrow on *figure 4* below.

In the remainder of this work, the majority of the properties will be demonstrated using the minimum cost network flow theorem. This theorem states that a necessary and sufficient condition for a feasible solution *f* to be optimal on a network *R=(X, U, a, b, c)* is that for each cycle $\Gamma=\Gamma^+ \cup \Gamma^-$ such that

$\gamma = Min\{ Min_{u\in\Gamma^+} [c(u)-f(u)]; Min_{u\in\Gamma^-} [f(u)-b(u)] \} > 0$

it is necessary that $\sum_{u\in\Gamma^+} a(u) - \sum_{u\in\Gamma^-} a(u) \geq 0$

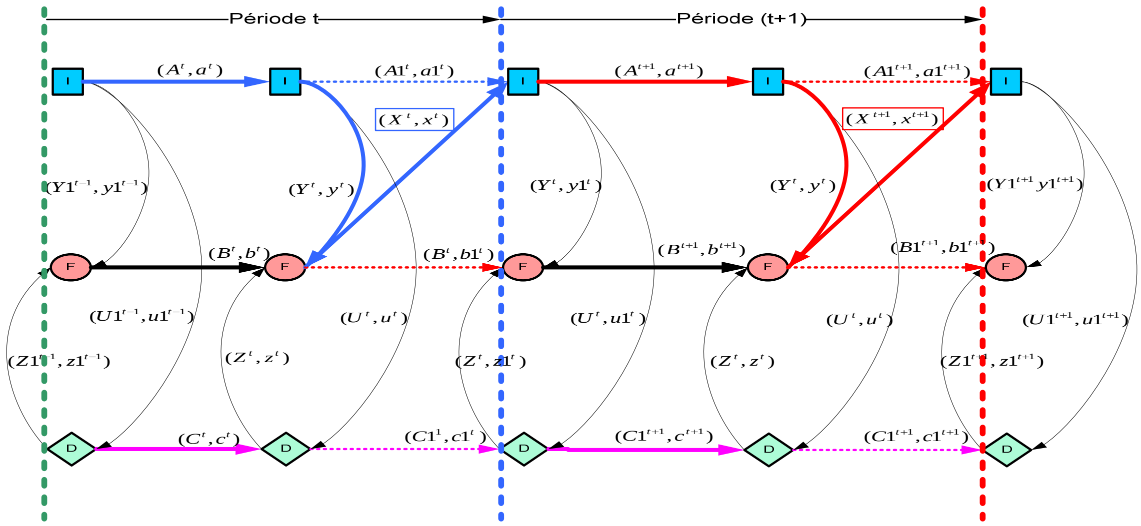

Figure 4 : Network flow illustration

Before resolving this problem we will define some new integer parameters, which will help us later when demonstrating properties. The first parameter is the principal threshold, N_d^a. This parameter will help us to decide whether or not to send surplus empty containers from the customer to the depot for storage and reuse in the following N_d^a periods. The second parameter is the secondary threshold, M_d^a. This parameter will show whether sending the container from the customer to the depot for storage without reuse is better then keeping it at the customer site in the following M_d^a periods.

Definition 1. *in the case when we have* $C_y^{Tr} < C_u^{Tr} + C_z^{Tr}$, *we define the principal-storing-threshold of the customer to the depot, the integer* N_d^a, *such that:*

$$\begin{cases} N_d^a \times C_a^{st} + C_y^{Tr} < C_u^{Tr} + N_d^a \times C_d^{st} + C_z^{Tr} \\ (N_d^a+1)\times C_a^{st} + C_y^{Tr} > C_u^{Tr} + (N_d^a+1)\times C_d^{st} + C_z^{Tr} \end{cases}$$

Remark 1. *Figure 5* represent an illustration of *definition 1*. The first term of the first inequality $N_d^a \times C_a^{st} + C_y^{Tr}$ represents the cost of path $\Gamma^+_{t,N_d^a} = (A^t, A1^t, ..., A^{t+N_d^a-1}, Y^{t+N_d^a-1})$, in other words this cost is the cost of storing one unit of container during N_d^a periods and transporting it from the customer to the supplier. The second term, $C_u^{Tr} + N_d^a \times C_c^{st} + C_z^{Tr}$, corresponds to the cost of path $\Gamma^-_{t,N_d^a} = (U1^{t-1}, D^t, D1^t, ..., D^{t+N_d^a-1}, Z^{t+N_d^a-1})$ which represents the total cost of sending one container from the customer to the depot, storing it during N_d^a in the depot and sending it on to the supplier.

The intention here is to note that storing containers more than N_d^a at the customer site and transporting them subsequently to suppliers can be more expensive than transporting them to the depot, storing during more than N_d^a, and sending them on. It is easy to remark that $\Gamma^-_{t,N_d^a} \cup \Gamma^+_{t,N_d^a}$ constitutes a cycle

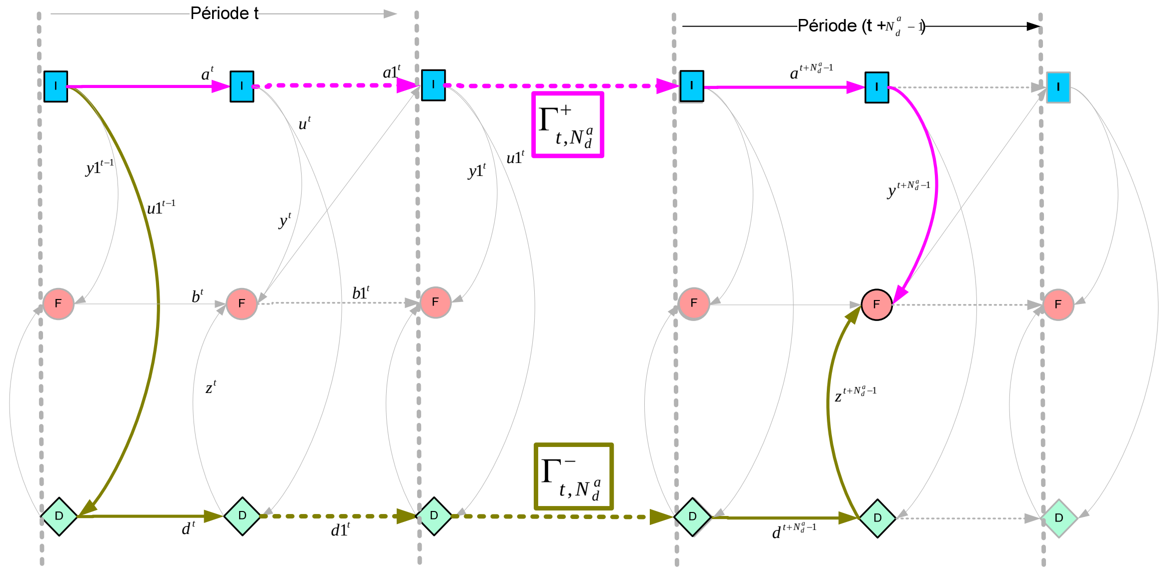

Figure 5 : Cycle $\Gamma^-_{t,N_d^a} \cup \Gamma^+_{t,N_d^a}$

Remark 2. It is easy to verify that for all integers *i*, such that $i \le N_d^a$, $i \times C_a^{st} + C_y^{Tr} < C_u^{Tr} + i \times C_d^{st} + C_z^{Tr}$ and that for all $i > N_d^a$, $i \times C_a^{st} + C_y^{Tr} > C_u^{Tr} + i \times C_d^{st} + C_z^{Tr}$.

This can be proved as follows:

Let *i* be an integer such that $i \le N_d^a$. *i* can be written in the form $i = N_d^a - j$ where *j* is an integer. Since $j \times C_a^{st} > j \times C_d^{st} \quad \Rightarrow \quad -j \times C_a^{st} < -j \times C_d^{st}$ summing this with $N_d^a \times C_a^{st} + C_y^{Tr} < C_u^{Tr} + N_d^a \times C_d^{st} + C_z^{Tr}$ gives $(N_d^a - j) \times C_a^{st} + C_y^{Tr} < C_u^{Tr} + (N_d^a - j) \times C_d^{st} + C_z^{Tr}$ which is exactly $i \times C_a^{st} + C_y^{Tr} < C_u^{Tr} + i \times C_d^{st} + C_z^{Tr}$.

Now suppose that $i > N_d^a + 1$. i can be written in the form $i = N_d^a + 1 + j$ where $j > 0$. Since $j \times C_a^{st} > j \times C_d^{st}$ summing this with $(N_d^a + 1) \times C_a^{st} + C_y^{Tr} > C_u^{Tr} + (N_d^a + 1) \times C_c^{st} + C_z^{Tr}$ gives $(N_d^a + 1 + j) \times C_a^{st} + C_y^{Tr} > C_u^{Tr} + (N_d^a + j + 1) \times C_d^{st} + C_z^{Tr}$ which is exactly $i \times C_a^{st} + C_y^{Tr} > C_u^{Tr} + i \times C_d^{st} + C_z^{Tr}$.

Remark 3. The first inequality ($N_d^a \times C_a^{st} + C_y^{Tr} < C_u^{Tr} + N_d^a \times C_c^{st} + C_z^{Tr}$) is valid only if $C_y^{Tr} < C_u^{Tr} + C_z^{Tr}$. In fact, the first inequality can be written as:

$N_d^a \times (C_a^{st} - C_d^{st}) < C_u^{Tr} + C_z^{Tr} - C_y^{Tr}$. Given that $N_d^a \times (C_a^{st} - C_d^{st}) > 0$ consequently $C_y^{Tr} < C_u^{Tr} + C_z^{Tr}$. Otherwise, $C_y^{Tr} > C_u^{Tr} + C_z^{Tr}$ would give $C_y^{Tr} + C_a^{st} > C_u^{Tr} + C_d^{st} + C_z^{Tr}$. In this case it is simple to see that if we have stock at the customer site at the end of period *t* it will be better to send all of it to the depot for storage and reuse.

Definition 2. *We define the secondary-storing-threshold of the customer to the depot, the integer* M_d^a*, such that:*

$$\begin{cases} M_d^a \times C_a^{st} < C_u^{Tr} + M_d^a \times C_d^{st} \\ (M_d^a + 1) \times C_a^{st} > C_u^{Tr} + (M_d^a + 1) \times C_d^{st} \end{cases}$$

Remark 4. This definition (see figure 6) shows that for a number of periods, i.e. greater than M_d^a, storing stock at the customer site will be more expensive than sending it to the depot for storage without reuse. The two parameters M_d^a and N_d^a will help us to establish the different properties.

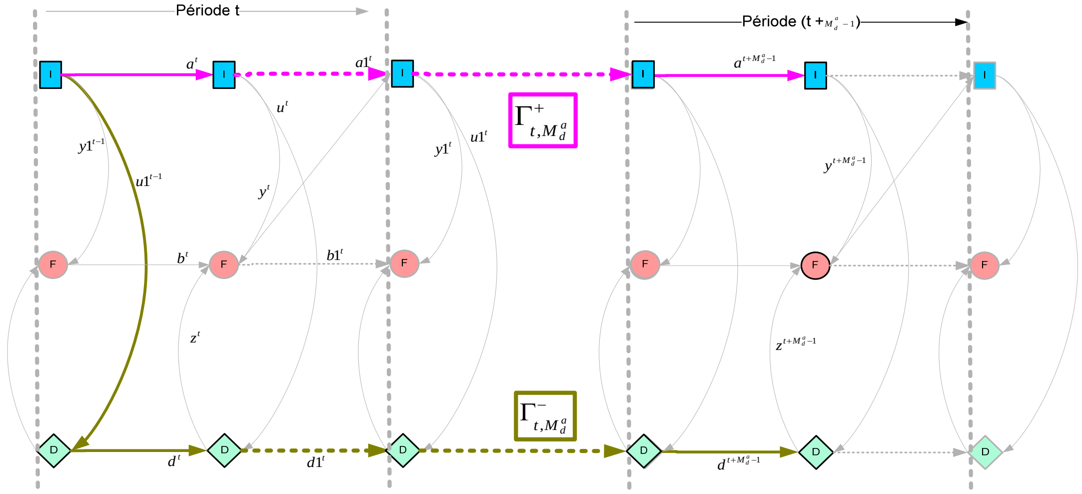

Figure 6 : Γ^+_{t,M_d^a} **and** Γ^-_{t,M_d^a}

Property 1. *For all* $t \in [0,T]$, $y1^t = z1^t = 0$

Proof. Suppose there exists a $t \in [0,T]$ such that $z1^t \neq 0$, and hence $b^{t+1} \neq 0$. Let $\Gamma_t = \Gamma_t^+ \cup \Gamma_t^-$ be a cycle composed of 4 arcs (see Figure 7) such that:

$\Gamma_t^- = (Z1^t, B^{t+1})$

$\Gamma_t^+ = (D^{t+1}, Z^{t+1})$

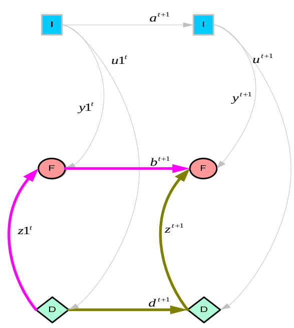

Figure 7 : Cycle $\Gamma_t^+ \cup \Gamma_t^-$

We have $Min_{u \in \Gamma_t^-} [f(u) - b(u)]$ because $z1^t > 0$ and $b^{t+1} > 0$. In addition, $Min_{u \in \Gamma_t^+} [c(u) - f(u)] > 0$ because the capacities of arcs are unlimited. Therefore, $\gamma = Min\{ Min_{u \in \Gamma_t^-} [f(u) - b(u)]; Min_{u \in \Gamma_t^+} [c(u) - f(u)] \} > 0$ and according to *the minimum cost network flow theorem* we must have that $\sum_{u \in \Gamma_t^+} a(u) - \sum_{u \in \Gamma_t^-} a(u) \geq 0$. However, in this case $\sum_{u \in \Gamma_t^+} a(u) = C_d^{st} + C_z^{tr} < \sum_{u \in \Gamma_t^-} a(u) = C_z^{tr} + C_b^{st}$ which is absurd, therefore $z1^t$ must equal zero, i.e. for all $t \in [0,T] \Rightarrow z1^t = 0$.

The same reasoning can be applied to $y1^t$.

Suppose there exists a $t \in [0,T]$ such that $y1^t \neq 0$, and hence $b^{t+1} \neq 0$. Let $\Gamma_t = \Gamma_t^+ \bigcup \Gamma_t^-$ be a cycle composed of 4 arcs (see Figure 8) such that:

$\Gamma_t^- = (Y1^t, B^{t+1})$

$\Gamma_t^+ = (A^{t+1}, Y^{t+1})$

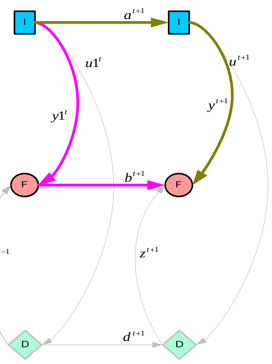

Figure 8 : Cycle $\Gamma_t^+ \bigcup \Gamma_t^-$

We have $Min_{u \in \Gamma_t^-} [f(u) - b(u)]$ because $y1^t > 0$ et $b^{t+1} > 0$. In addition, $Min_{u \in \Gamma_t^+} [c(u) - f(u)] > 0$ because the capacities of arcs are unlimited. Therefore, $\gamma = Min\{ Min_{u \in \Gamma_t^-} [f(u) - b(u)]; Min_{u \in \Gamma_t^+} [c(u) - f(u)] \} > 0$ and according to *the minimum cost network flow theorem* we must have that $\sum_{u \in \Gamma_t^+} a(u) - \sum_{u \in \Gamma_t^-} a(u) \geq 0$. However, in this case $\sum_{u \in \Gamma_t^+} a(u) = C_a^{st} + C_y^{tr} < \sum_{u \in \Gamma_t^-} a(u) = C_y^{tr} + C_b^{st}$ which is absurd, therefore $y1^t = 0$ for all $t \in [0,T]$.

Remark 5. *Property 1* implies that the stock at the supplier site will remain equal to zero if we begin with a stock equal to zero at *t=0* .

Property 2. *For* $t \in [0,T]$*,if* $u1^t \neq 0$ *then* $\forall\ k \in [1, N_d^a]$, $z^{t+k} = 0$

Proof. Suppose there exists a $t \in [0,T]$ such that $u1^t \neq 0$.

Let $\Gamma_{t,k}$ be a cycle with $t \in [0,T]$ and $k \in [1, N_d^a]$ (see figure 9). Suppose that $\Gamma_{t,k} = \Gamma_{t,k}^+ \cup \Gamma_{t,k}^-$, where $\Gamma_{t,k}^+ = (A^{t+1}, ..., A^{t+k}, Y^{t+k})$, is the path composed of the storage arcs between *t+1* and *t+k* and the transportation arc between the customer and the supplier. $\Gamma_{t,k}^- = (U1^t, D^{t+1}, .., D^{t+k}, Z^{t+k})$ is the path composed of the transportation arc from the customer to the depot at period *t*, the storage arcs at depot between *t+1* and *t+k* and the transportation arc from the depot to the supplier at period *t+k* .

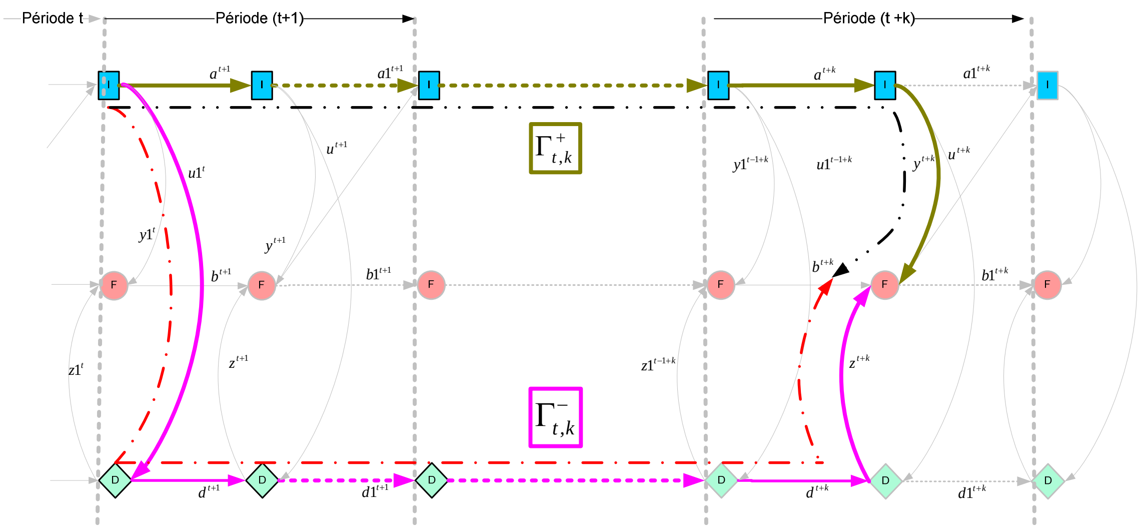

Figure 9 : Cycle $\Gamma^{+}_{t,k} \cup \Gamma^{-}_{t,k}$

Suppose there exists a $z^{t+k} \neq 0$ which is the first non-zero occurrence, that is if $k > 1$ then for all $i \in [1, k-1]$ we have $z^{t+i} = 0$. For, the case when *k=1*, z^{t+1} is the first and only non-zero occurrence. Therefore, all d^{t+i} and $d1^{t+j}$ will be non-zero for $i \in [1, k]$ for all *k* and $j \in [1, k-1]$ for *k>1* because there are no units leaving the depot since, according to *property 1*, $z1^{t} = 0$ and $y1^{t} = 0$ for all *t.*

Consequently, all arcs of path $\Gamma^{-}_{t,k} = (U1^{t}, D^{t+1}, ..., D^{t+k}, Z^{t+k})$ will have non-zero values, and thus $Min_{u\in\Gamma^{-}_{t,k}} [f(u) - b(u)] > 0$ and since $Min_{u\in\Gamma^{+}_{t,k}} [c(u) - f(u)] > 0$ it follows that:

$\gamma = Min\{ Min_{u\in\Gamma^{-}_{t,k}} [f(u) - b(u)]; Min_{u\in\Gamma^{+}_{t,k}} [c(u) - f(u)] \} > 0$. We Know that that

$\sum_{u\in\Gamma^{+}_{t,k}} a(u) = k \times C^{st}_{a} + C^{Tr}_{y}$ and $\sum_{u\in\Gamma^{-}_{t,k}} a(u) = C^{Tr}_{u} + k \times C^{st}_{d} + C^{Tr}_{z}$ this implies that

$\sum_{u\in\Gamma^{+}_{t,k}} a(u) - \sum_{u\in\Gamma^{-}_{t,k}} a(u) < 0$ which is absurd because according to *definition 1* for all $k \leq N^{a}_{d}$, $k \times C^{st}_{a} + C^{Tr}_{y} < C^{Tr}_{u} + k \times C^{st}_{d} + C^{Tr}_{z}$. Therefore, $z^{t+k} \neq 0$ for all $k \in [1, N^{a}_{d}]$.

Property 3. *If the initial stock at the supplier site is equal to zero, then for all* $t \in [0, T]$ $b^{t} = b1^{t} = 0$

Proof. We have that $b^{0} = 0$ because we began with zero stock at the supplier site and that $y1^{t} = z1^{t} = 0 \ \ \forall\, t$, therefore, there is no entry at the supplier site at *t=0.* Assume that there exists a $t \in [1, T]$ such that $b^{t} \neq 0$ which is the first non-zero occurrence, i.e. that for all $i \in [0, t-1]$ $b^{i} = 0$. According to *property 1*, $y1^{t-1} = z1^{t-1} = 0$ and therefore, $b1^{t-1} \neq 0$. Given that $\delta^{t-1} \neq 0$ and $b^{t-1} = 0$, it follows that $(y^{t-1}, z^{t-1}) \neq (0,0)$

<u>1st case</u> : *if* $y^{t-1} \neq 0$

Let $\Gamma_{y,t} = \Gamma^{+}_{y,t} \cup \Gamma^{-}_{y,t}$ (see figure 10) be a cycle such that:

$\Gamma^{-}_{y,t} = (Y^{t-1}, B1^{t-1}, B^{t})$

$\Gamma^{+}_{y,t} = (A1^{t-1}, A^{t}, Y^{t})$

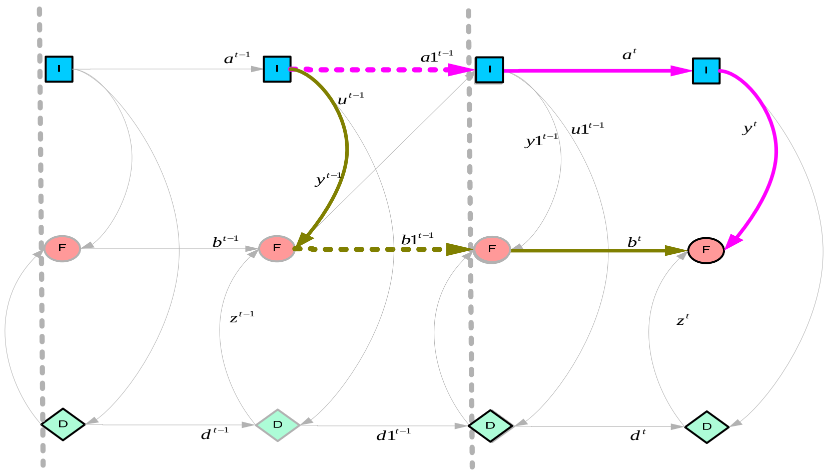

Figure 10 : Cycle $\Gamma^{+}_{y,t} \cup \Gamma^{-}_{y,t}$

The path $\Gamma^{-}_{y,t} = (Y^{t-1}, B1^{t-1}, B^{t})$ does not contain an arc of zero value, given that $y^{t-1} \neq 0$ and $b1^{t-1} \neq 0$. Consequently, $Min_{\Gamma^{-}_{y,t}}[f(u)-b(u)] > 0$ and given that $Min_{\Gamma^{+}_{y,t}}[c(u)-f(u)] > 0$ it follows that
$\gamma = Min\{\ Min_{u\in\Gamma^{-}_{y,t}}[f(u)-b(u)]\,;\, Min_{u\in\Gamma^{+}_{y,t}}[c(u)-f(u)]\ \} > 0$.
Since $\sum_{u\in\Gamma^{+}_{y,t}} a(u) = C^{st}_{a} + C^{tr}_{y} < \sum_{u\in\Gamma^{-}_{y,t}} a(u) = C^{tr}_{y} + C^{st}_{b}$ (given that $C^{st}_{b} > C^{st}_{a}$) $\Rightarrow$
$\sum_{u\in\Gamma^{+}_{y,t}} a(u) - \sum_{u\in\Gamma^{-}_{y,t}} a(u) < 0$ this is absurd, therefore b^{t} can never non-zero. Consequently, given that $y1^{t-1} = z1^{t-1} = 0$ it also follows that $b1^{t-1} = 0$.

2^nd^ case: if $z^{t-1} \neq 0$:

Let $\Gamma_{z,t} = \Gamma^{+}_{z,t} \cup \Gamma^{-}_{z,t}$ (see figure 11) such that:

$\Gamma^{-}_{z,t} = (Z^{t-1}, B1^{t-1}, B^{t})$

$\Gamma^{+}_{z,t} = (D1^{t-1}, D^{t}, Z^{t})$

Given that $z^{t-1} \neq 0$, $b1^{t-1} \neq 0$ and $b^{t} \neq 0$, it follows that $Min_{\Gamma^{-}_{z,t}}[f(u)-b(u)] > 0$ on $\Gamma^{-}_{z,t} = (Z^{t-1}, B1^{t-1}, B^{t})$. In addition, since $Min_{\Gamma^{+}_{z,t}}[c(u)-f(u)] > 0$ it follows that $\gamma = Min\{\ Min_{u\in\Gamma^{-}_{z,t}}[f(u)-b(u)]\,;\, Min_{u\in\Gamma^{+}_{z,t}}[c(u)-f(u)]\ \} > 0$. In this case
$\sum_{u\in\Gamma^{+}_{z,t}} a(u) = C^{st}_{d} + C^{tr}_{z} < \sum_{u\in\Gamma^{-}_{y,t}} a(u) = C^{tr}_{z} + C^{st}_{b}$ (given that $C^{st}_{b} > C^{st}_{d}$) $\Rightarrow$
$\sum_{u\in\Gamma^{+}_{z,t}} a(u) - \sum_{u\in\Gamma^{-}_{z,t}} a(u) < 0$ which is also absurd, therefore b^{t} can never be null. Therefore, given that $y1^{t-1} = z1^{t-1} = 0$ it follows that $b1^{t-1} = b^{t} = 0$ ($b^{t} = z1^{t-1} + y1^{t-1} + b1^{t-1}$).

Finally, for all $t \in [0, T]$, $b^{t} = b1^{t} = 0$

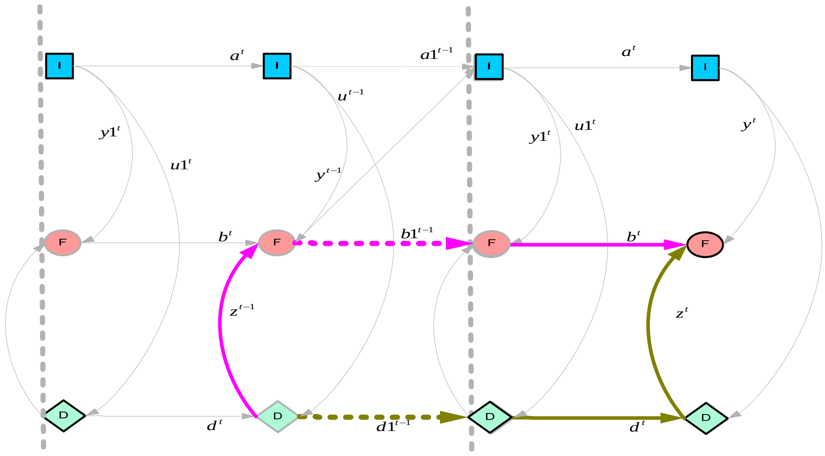

Figure 11 : Cycle $\Gamma^+_{z,t} \cup \Gamma^-_{z,t}$

Property 4. *For all* $t \in [0,T]$, $u^t = 0$

Proof. Assume there exists a $t \in [0,T]$ such that $u^t \neq 0$. Therefore, $a^t \neq 0$. Let $\Gamma_{u,t} = \Gamma^+_{u,t} \cup \Gamma^-_{u,t}$ be a cycle (see figure 12) such that:

$\Gamma^-_{u,t} = (A^t, U^t)$

$\Gamma^+_{u,t} = (U1^{t-1}, D^t)$

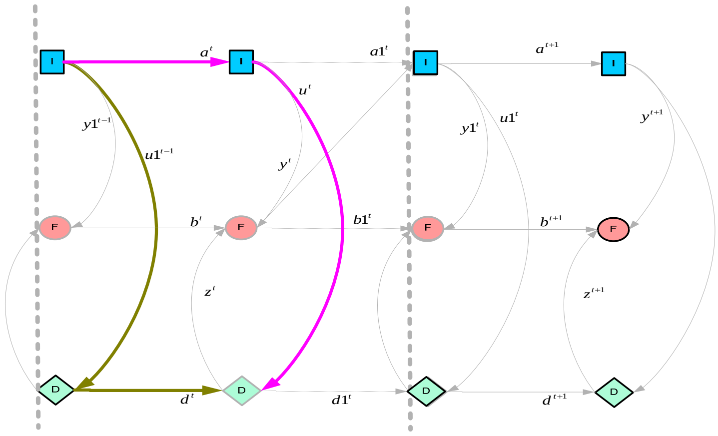

Figure 12 : Cycle $\Gamma^+_{u,t} \cup \Gamma^-_{u,t}$

Since $u^t \neq 0$, it also follows that $a^t \neq 0$, and as a consequence that $Min_{\Gamma^-_{u,t}}[f(u) - b(u)] > 0$.

In addition, since $Min_{\Gamma^+_{u,t}}[c(u) - f(u)] > 0$ it follows that

$\gamma = Min\{\ Min_{u \in \Gamma^-_{u,t}}[f(u) - b(u)]\,;\ Min_{u \in \Gamma^+_{u,t}}[c(u) - f(u)]\ \} > 0$.

However, since

$\sum_{u \in \Gamma^+_{u,t}} a(u) = C^{tr}_u + C^{st}_d < \sum_{u \in \Gamma^-_{u,t}} a(u) = C^{tr}_u + C^{tr}_a \Rightarrow \sum_{u \in \Gamma^+_{u,t}} a(u) - \sum_{u \in \Gamma^-_{u,t}} a(u) < 0$ this is in contradiction with the *minimum cost network flow theorem*, and therefore for all $t \in [0,T]$ $u^t = 0$.

Property 5. *For all* $t \in [0,T]$, $x^t = \delta^t$

Proof. Assume there exists a $t \in [0,T]$ such that $x^t > \delta^t$. According to *property 5* , for all $t \in [0,T]$, $b^t = b1^t = 0$. Given that $\delta^t \neq 0$ (see assumptions above) it follows that $(y^t, z^t) \neq (0,0)$

1^st^ case: if $y^t \neq 0$

Let $\Gamma_{y,t} = \Gamma^+_{y,t} \cup \Gamma^-_{y,t}$ (see figure 13) be a cycle such that:

$\Gamma^-_{y,t} = (Y^t, X^t)$

$\Gamma^+_{y,t} = (A1^t)$

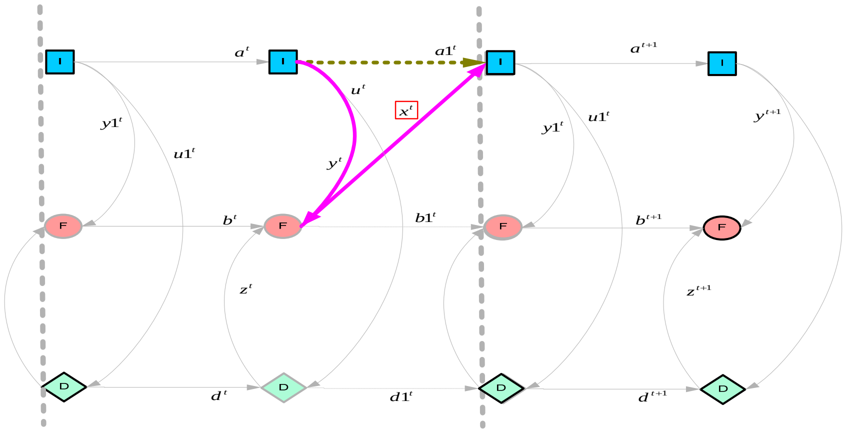

Figure 13 : Cycle $\Gamma^+_{y,t} \cup \Gamma^-_{y,t}$

Given that $x^t > \delta^t$ it follows that $Min_{\Gamma^-_{y,t}}[f(u) - b(u)] > 0$ and since $Min_{\Gamma^+_{y,t}}[c(u) - f(u)] > 0$ it also follows that
$\gamma = Min\{ \; Min_{u \in \Gamma^-_{y,t}}[f(u) - b(u)]; Min_{u \in \Gamma^+_{y,t}}[c(u) - f(u)] \; \} > 0$
According to the *minimum cost network flow theorem* we must have that $\sum_{u \in \Gamma^+_{y,t}} a(u) - \sum_{u \in \Gamma^-_{y,t}} a(u) \geq 0$ but given that $\sum_{u \in \Gamma^+_{y,t}} a(u) = 0 < \sum_{u \in \Gamma^-_{y,i}} a(u) = C^{tr}_y + C^{tr}_x$ (the cost of arc $A1^t$ is null) there is a contradiction. Therefore, in this case $x^t = \delta^t$.

2^nd^ case: if $z^t \neq 0$ and $y^t = 0$ (if $y^t \neq 0$ we return to *case 1*)

Let y^{t+i}, $i \in [1, T-t]$, or $u1^{t+j}$, $j \in [0, T-t]$, and be the first non-zero variable. For example, if y^{t+i} is the first non-zero variable then all y^{t+m} and $u1^{t+n}$ (with $m \in [1, i-1]$ and $n \in [0, i-1]$) are zero.
1^st^sub-case: if y^{t+i} the first non-zero variable:

Let $\Gamma_{z,t} = \Gamma^+_{z,t} \cup \Gamma^-_{z,t}$ (see figure 14) such that:

$\Gamma^-_{z,t} = (Z^t, X^t, A^{t+1}, \ldots, A^{t+i}, Y^{t+i})$

$\Gamma^+_{z,t} = (D1^t, \ldots, D^{t+i}, Z^{t+i})$

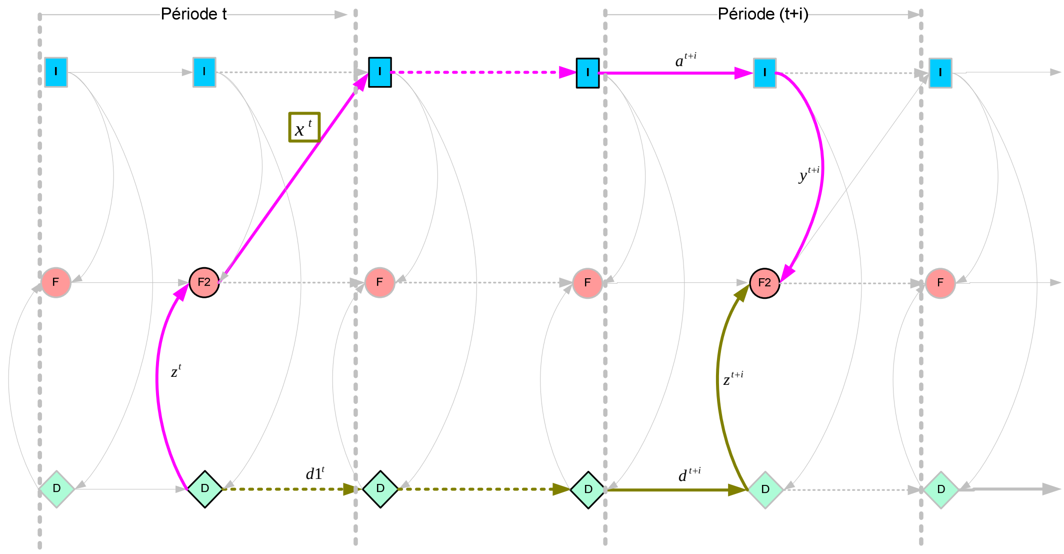

Figure 14 : Cycle $\Gamma^+_{z,t} \cup \Gamma^-_{z,t}$

Given that y^{t+i} is the first non-zero variable, it follows that between *t+1* and *t+i* no containers depart from the customer site, i.e. all arcs of $\Gamma^-_{z,t}$ are nonzero. In addition, $x^t > \delta^t$ which simply means that $Min_{\Gamma^-_{z,t}}[f(u) - b(u)] > 0$. Therefore,

$\gamma = Min\{\ Min_{u\in\Gamma^-_{z,t}}[f(u) - b(u)];\ Min_{u\in\Gamma^+_{z,t}}[c(u) - f(u)]\ \} > 0$ and given that

$\sum_{u\in\Gamma^+_{z,t}} a(u) = i \times C^{st}_d + C^{tr}_z < \sum_{u\in\Gamma^-_{z,i}} a(u) = C^{tr}_z + C^{tr}_x + i \times C^{st}_a + C^{tr}_y$ (Because $C^{st}_d < C^{st}_a$)

this is in contradiction with *minimum cost network flow theorem*. Therefore $x^t > \delta^t$ is impossible $\Rightarrow$ $x^t = \delta^t$ in this case.

<u>*2nd sub-case*:</u> if $u1^{t+j}$ is the first non-zero variable with $j > 0$. In this case all y^{t+m} and $u1^{t+n}$, for $m \in [1, j]$ et $n \in [0, j-1]$, are zero.

Let $\Gamma_{z,t} = \Gamma^+_{z,t} \cup \Gamma^-_{z,t}$ (See figure 15) such that :

$\Gamma^-_{z,t} = (Z^t, X^t, A^{t+1}, \ldots, A1^{t+j}, U1^{t+j})$

$\Gamma^+_{z,t} = (D1^t, \ldots, D^{t+j}, D1^{t+j})$

In the case where *j=0* the tow paths are reduced to

$\Gamma^-_{z,t} = (Z^t, X^t, U1^t)$

$\Gamma^+_{z,t} = (D1^t)$

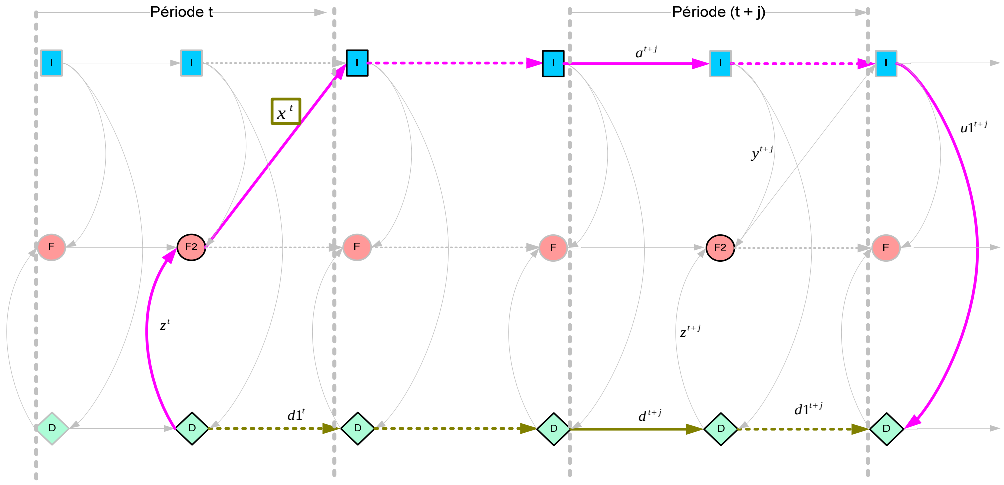

Figure 15 : Cycle $\Gamma^+_{z,t} \bigcup \Gamma^-_{z,t}$

Because it is supposed that $x^t > \delta^t$ then, given that no containers depart from the customer until *t+j*, there are no zero arcs on the path $\Gamma^-_{z,t}$ and the X^t arcs have a value greater than the lower band ($x^t > \delta^t$). Therefore, $Min_{\Gamma^-_{z,t}}[f(u)-b(u)]>0$ and since $Min_{\Gamma^+_{z,t}}[c(u)-f(u)]>0$ it follows that

$\gamma = Min\{\ Min_{u\in\Gamma^-_{z,t}}[f(u)-b(u)]\,;Min_{u\in\Gamma^+_{z,t}}[c(u)-f(u)]\ \}>0$

Since, according to *minimum cost network flow theorem*, $\sum_{u\in\Gamma^+_{z,t}} a(u) - \sum_{u\in\Gamma^-_{z,t}} a(u) \geq 0$ is a condition for optimality, and given that

$\sum_{u\in\Gamma^+_{z,t}} a(u) = j\times C_d < \sum_{u\in\Gamma^-_{z,i}} a(u) = C^{tr}_z + C^{tr}_x + j\times C_a + C_u$, this is absurd. Therefore, in this case $x^t = \delta^t$.

3[rd] *sub-case*: if all y^{t+i} and $u1^{t+j}$, $i\in[1,T-t]$ and $j\in[0,T-t]$, are equal to zero, then after *t* all transportation variables will be equal to zero except z^k *and* x^k , $k\in[t,T]$. Now, suppose that the feasible flow *f* found in this case is optimal. Let *f'* be another feasible flow obtained by transforming *f* by subtracting $\lambda = x^t - \delta^t$ from $\Gamma = (Z^t, X^t, A^t, A1^t, ..., A^T, A1^T)$ and adding it to $\Gamma' = (D^t, D1^t, ..., D^T, D1^T)$. This gives a gain of $\lambda\times((T-t)\times C_a + C_x + C_z - (T-t)\times C_d)$ which implies that *f* is not optimal. Therefore, $x^t > \delta^t$ is impossible $\Rightarrow$ $x^t = \delta^t$.

Property 6. *If for* $t\in[0,T]$, $a1^t + \delta^t \geq \max_{k\in[1,N^a_d]}(\delta^{t+k})$ *then*

$u1^t \leq a1^t + \delta^t - \max_{k\in[1,N^a_d]}(\delta^{t+k})$

Proof. Assume there exists a $t\in[0,T]$ such that $u1^t > a1^t + \delta^t - \max_{k\in[1,N^a_d]}(\delta^{t+k})$ and let $i\in[1,N^a_d]$ be the index corresponding to the position of maximum demand, in other words $\max_{k\in[1,N^a_d]}(\delta_{t+k}) = \delta_{t+i}$. So, according to *property 2*, if $u1^t \neq 0$ then $\forall\ k\in[1,N^a_d]$ z^{t+k} =0. In other words, in the interval $[t+1, t+N^a_d]$ the system uses only the stock available at the customer site. However, this stock is strictly lower than δ_{t+i} because $a^{t+1} = a1^t + \delta^t - u1^t < \max_{k\in[1,N^a_d]}(\delta^{t+k})$ and consequently the demand at *t+i* cannot be satisfied, which is absurd. Therefore, $u1^t \leq a1^t + \delta^t - \max_{k\in[1,N^a_d]}(\delta^{t+k})$.

Property 7. *Let* $t \in [0,T]$ *and* $i \in [1, N_b^a]$ *represent the position index of maximal demand: this means that* $\max_{k\in[1,N_d^a]}(\delta^{t+k}) = \delta^{t+i}$ *with i>1. If* $a1^t + \delta^t > \max_{k\in[1,N_d^a]}(\delta^{t+k})$ *then we have:* $u1^{t+j} = 0$ *for all* $j \in [1, i-1]$, $a^{t+k} \neq 0$ *for all* $k \in [1,i]$ *and* $a1^{t+l} \neq 0$ *for all* $l \in [1, i-1]$

Remark 6. *If* i=1 *(i is at the position of maximum demand), then* $a^{t+1} \neq 0$

Proof. Let $t \in [0,T]$ and $i \in [1, N_b^a]$ such that $\max_{k\in[1,N_d^a]}(\delta^{t+k}) = \delta^{t+i}$ and assume that $a1^t + \delta^t > \max_{k\in[1,N_d^a]}(\delta^{t+k})$. $u1^t \leq a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$ which means that the customer stock at period $a^{t+1} = a1^t + \delta^t - u1^t$ is greater than or equal to $\max_{k\in[1,N_d^a]}(\delta^{t+k})$, i.e. $a^{t+1} = a1^t + \delta^t - u1^t \geq \max_{k\in[1,N_d^a]}(\delta^{t+k})$. Assume that there is a non-zero departure $u1^{t+j}$ for $j \in [1, i-1]$ which is the first non-zero occurrence, i.e. all departures $u1^{t+m}$ for $m \in [1, j-1]$ are null. Let $\Gamma = \Gamma^- \cup \Gamma^+$ be a cycle (see figure 16) such that

$\Gamma^- = (A^{t+1}, A1^{t+1}, ..., U1^{t+j})$

$\Gamma^+ = (U1^t, D^{t+1}, ..., D1^{t+j})$

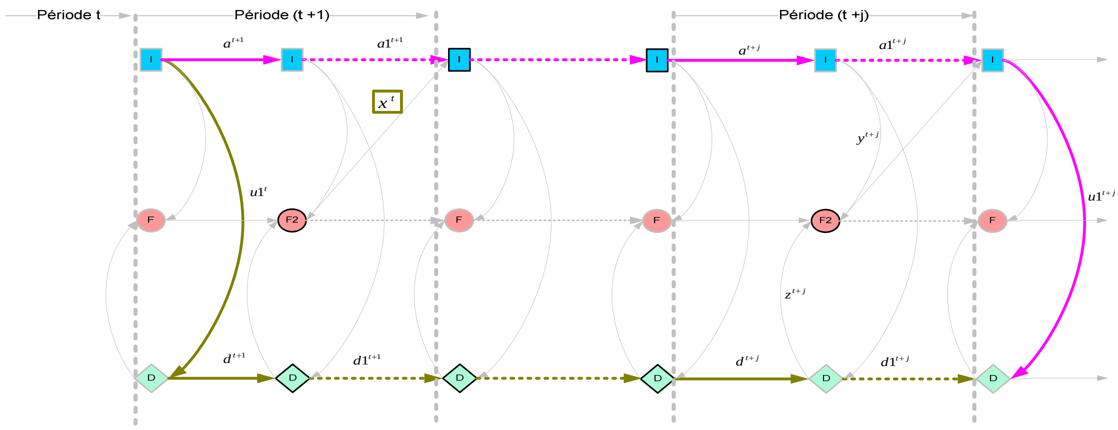

Figure 16 : Cycle $\Gamma^+ \cup \Gamma^-$

Since all $u1^{t+m}$ are null, there are no departures from the customer site, this means that stock a^{t+l} for $l \in [1, j]$ can never decrease and will remain greater than $\max_{k\in[1,N_d^a]}(\delta^{t+k})$. In addition, for all $k \in [1, j]$ y^{t+k} can never exceed $\max_{k\in[1,N_d^a]}(\delta^{t+k}) \Rightarrow a^{t+k} - y^{t+k} = a1^{t+k} > 0$.

Therefore, the path $\Gamma^- = (A^{t+1}, A1^{t+1}, ..., U1^{t+j})$ cannot contain a null arc, i.e. $Min_{\Gamma^-}[f(u) - b(u)] > 0$, and because $Min_{\Gamma^+}[c(u) - f(u)] > 0$, it follows that:

$\gamma = Min\{ Min_{u\in\Gamma^-}[f(u) - b(u)]; Min_{u\in\Gamma^+}[c(u) - f(u)] \} > 0$

Since, $\sum_{u\in\Gamma^+} a(u) = C_u^{tr} + j \times C_d^{st} < \sum_{u\in\Gamma^-} a(u) = j \times C_a^{st} + C_u^{tr}$ (because $C_d^{st} < C_a^{st}$) this is absurd and therefore there is no departure $u1^{t+j}$ for $j \in [1, i-1]$, and the stock a^{t+k} for $k \in [1, i]$ can never decrease, i.e. $a^{t+k} \geq \max_{n\in[1,N_d^a]}(\delta^{t+n})$. In addition, given that $\max_{n\in[1,N_d^a]}(\delta^{t+n}) > \delta^{t+l} \geq y^{t+l}$ for $l \in [1, i-1]$, it follows that $a1^{t+l} = a^{t+l} - y^{t+l} > 0$.

Consequently, for $u1^{t+j}$, $j \in [1, i-1]$, $a^{t+k} \neq 0$ for all $k \in [1, i]$ and $a1^{t+k} \neq 0$ for all $l \in [1, i-1]$.

Property 8. *For a* $t \in [0, T - \max(N_d^a, M_d^a)]$, *if* $a1^t + \delta^t > \max_{k\in[1,N_d^a]}(\delta^{t+k})$, *then*

$u1^t = a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$

Proof. Let $t \in [0, T - \max(N_d^a, M_d^a)]$ such that $a1^t + \delta^t > \max_{k\in[1,N_d^a]}(\delta^{t+k})$ and let $i \in [1, N_b^a]$ such that the maximum demand position be defined as $\max_{k\in[1,N_d^a]}(\delta^{t+k}) = \delta^{t+i}$. Assume that $u1^t < a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$. In other words, for the customer stock at period *t+1*, $a^{t+1} > \max_{k\in[1,N_d^a]}(\delta^{t+k})$ and because there is no departure (according to *property 8*, $u1^{t+j} = 0$ *for all* $j \in [1, i-1]$), the stock will stay the same or increase at the customer site in $[t+1, t+i]$. Therefore, we will find that at period $t+i$, $a^{t+i} > \max_{k\in[1,N_d^a]}(\delta^{t+k})$ and given that $y^{t+i} \le \delta^{t+i}$, it follows that $a1^{t+i} = a^{t+i} - y^{t+i} > 0$. Therefore, for all $k \in [1, i]$, a^{t+k} and $a1^{t+k}$ will be nonzero.

1^st^ case: assume there exists a $j \in [i, N_d^a]$ such that $u1^{t+j} \ne 0$ which is the first non-zero occurrence of $u1^{t+j}$. In other words, for all $m \in [i, j-1]$, $u1^{t+m}$ are null. Let $\Gamma = \Gamma^- \cup \Gamma^+$ be a cycle (see figure 17) such that:

$\Gamma^- = (A^{t+1}, A1^{t+1}, ..., U1^{t+j})$

$\Gamma^+ = (U1^t, D^{t+1}, ..., D1^{t+j})$

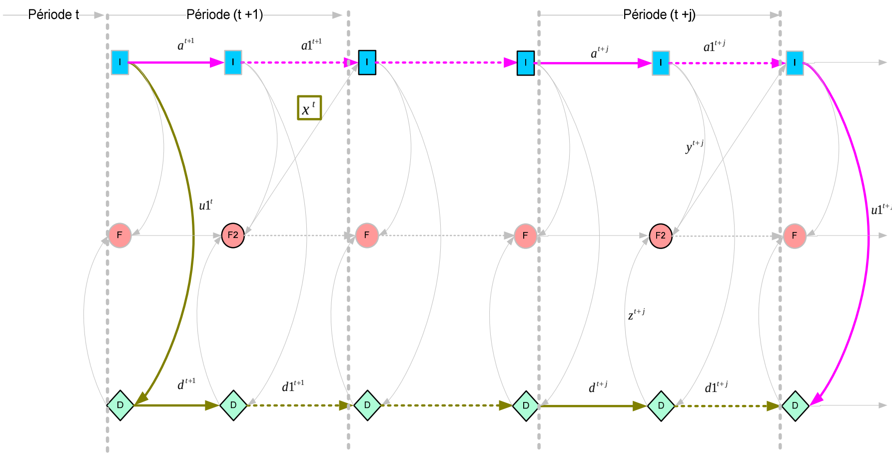

Figure 17 : Cycle $\Gamma^+ \cup \Gamma^-$

Given that the stock at the customer site will stay the same until the period *t+j* (the first departure is $u1^{t+j}$, $j \in [i, N_d^a]$), it follows that all a^{t+m} and $a1^{t+m}$ for $m \in [i, j]$ are nonzero. Thus, $Min_{\Gamma^-}[f(u) - b(u)] > 0$ and since $Min_{\Gamma^+}[c(u) - f(u)] > 0$ this implies that

$\gamma = Min\{ Min_{u\in\Gamma^-}[f(u) - b(u)]; Min_{u\in\Gamma^+}[c(u) - f(u)] \} > 0$. Given that:

$\sum_{u\in\Gamma^+} a(u) = C_u^{tr} + j \times C_d^{st} < \sum_{u\in\Gamma^-} a(u) = j \times C_a^{st} + C_u^{tr}$ (because $C_d^{st} < C_a^{st}$) this is absurd.

Therefore, in this case $u1^t < a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$ is impossible.

2^nd^ Case: suppose that there is no period $j \in [i, N_d^a]$ such that $u1^{t+j} \ne 0$.

1^st^ sub-case: assume there exists a $k > N_d^a$ such that $u1^{t+k} \ne 0$ or $y^{t+k} \ne 0$ which is the first non-zero occurrence.

1) If $u1^{t+k} \neq 0$ is the first non-zero occurrence, then all $u1^{t+l}$ for $l \in [N_d^a + 1, k-1]$ and all y^{t+m} for $m \in [N_d^a + 1, k]$ are equal to zero. In this case, let $\Gamma = \Gamma^- \cup \Gamma^+$ be a cycle such that:

$\Gamma^- = (A^{t+1}, A1^{t+1}, ..., U1^{t+k})$

$\Gamma^+ = (U1^t, D^{t+1}, ..., D1^{t+k})$

(the same form of cycle as figure 16)

Given that for all $n \in [1, N_d^a]$, $a1^{t+n} \neq 0$ and that there are no departures between periods $t+ N_d^a$ and $t+k$, it follows that, for all $m \in [N_d^a + 1, k]$, a^{t+m} and $a1^{t+m}$ are non-zero and the path $\Gamma^- = (A^{t+1}, A1^{t+1}, ..., U1^{t+k})$ does not contain a null arc, i.e. $Min_{\Gamma^-}[f(u) - b(u)] > 0$. Since, $Min_{\Gamma^+}[c(u) - f(u)] > 0$, it follows that

$\gamma = Min\{\ Min_{u\in\Gamma^-}[f(u) - b(u)]; Min_{u\in\Gamma^+}[c(u) - f(u)]\ \} > 0$

Given that $\sum_{u\in\Gamma^+} a(u) = C_u^{tr} + k \times C_d^{st} < \sum_{u\in\Gamma^-} a(u) = k \times C_a^{st} + C_u^{tr}$ this is absurd. Therefore, the inequality $u1^t < a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$ is impossible $\Rightarrow u1^t \geq a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$.

2) If y^{t+k} is the first non-zero occurrence, then all the other $u1^{t+m}$ and y^{t+m} values, for $m \in [N_d^a + 1, k-1]$, are equal to zero. Let $\Gamma = \Gamma^- \cup \Gamma^+$ be a cycle such that (see figure 18):

$\Gamma^- = (A^{t+1}, A1^{t+1}, ..., Y^{t+k})$

$\Gamma^+ = (U1^t, D^{t+1}, ..., D^{t+k}, Z^{t+k})$

Given that $a1^{t+n} \neq 0$ for $n \in [1, N_d^a]$ and that there are no departures between periods $t+ N_d^a$ and $t+k$, it follows that stocks a^{t+m}, for $m \in [N_d^a + 1, k]$, and $a1^{t+n}$, for all $n \in [N_d^a + 1, k-1]$, will be non-zero and consequently $Min_{\Gamma^-}[f(u) - b(u)] > 0$. Since, $Min_{\Gamma^+}[c(u) - f(u)] > 0$ it follows that $\gamma = Min\{\ Min_{u\in\Gamma^-}[f(u) - b(u)]; Min_{u\in\Gamma^+}[c(u) - f(u)]\ \} > 0$. Since, $\sum_{u\in\Gamma^+} a(u) = C_u^{tr} + k \times C_d^{st} + C_z^{tr} < \sum_{u\in\Gamma^-} a(u) = k \times C_a^{st} + C_y^{tr}$ for $k > N_d^a$ this is absurd and in contradiction with *minimum cost network flow theorem*.

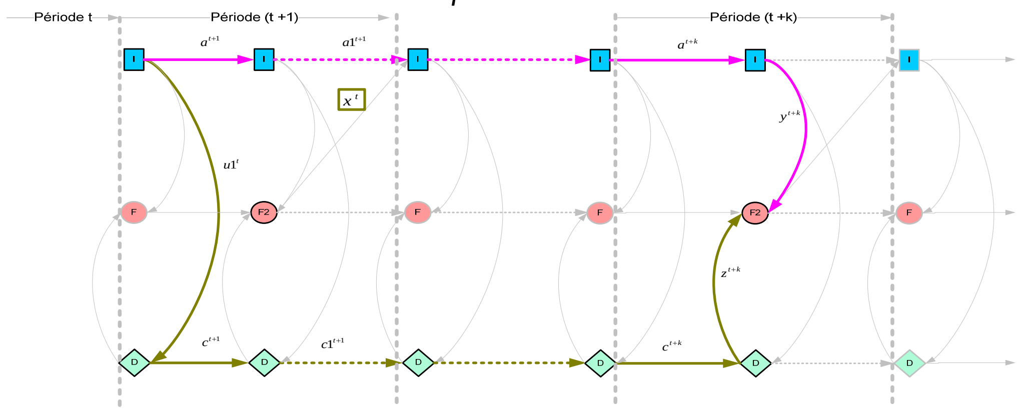

Figure 18 : Cycle $\Gamma^+ \cup \Gamma^-$

<u>*2nd sub-case*</u>: Assume that all $u1^{t+k}$ and y^{t+k} are equal to zero for $k > N_d^a$. Let *f* be the flow found in this case and suppose that it is optimal with the assumption that $u1^t < a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$.

Let $\alpha = a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k}) - u1^t > 0$ and let *f'* be a new flow obtained from *f* by subtracting α from the path $\Gamma = (A^{t+1}, A1^{t+1},..., A^T)$ and adding it to the path $\Gamma' = (U1^t, D^{t+1}, D1^{t+1},..., D^T)$. The total cost resulting from *f'* has a variation of $\alpha \times (C_u^{st} + (T-t)C_d^{st} - (T-t)C_a^{st})$ which is negative given that $T - t > \max(N_d^a, M_d^a) \Rightarrow T - t > M_d^a$. According to definition 2, for all $j > M_d^a$, $j \times C_a^{st} > C_u^{Tr} + j \times C_d^{st}$ and consequently:
$(T-t) \times C_a^{st} > C_u^{Tr} + (T-t) \times C_d^{st} \Rightarrow \alpha \times ((T-t)C_a^{st} - C_u^{st} - (T-t)C_d^{st}) < 0$. Therefore, the cost of *f'* is less than *f* and consequently if $u1^t < a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$ the flow cannot be optimal and, therefore, the inequality $u1^t < a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$ is impossible $\Rightarrow u1^t \geq a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$.
According to *property 7* $u1^t \leq a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$, and, therefore, $u1^t = a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k})$.

Property 9. *If,* $a1^t + \delta^t \leq \max_{k\in[1,N_d^a]}(\delta^{t+k})$ *for a* $t \in [0, T - \max(N_d^a, M_d^a)]$, *then* $u1^t = 0$

Proof. Let $t \in [0, T - \max(N_d^a, M_d^a)]$ such that $a1^t + \delta^t \leq \max_{k\in[1,N_d^a]}(\delta^{t+k})$ and assume that $u1^t \neq 0$. According, to *property 2* if $u1^t \neq 0$ then for all $k \in [1, N_d^a]$, $z^{t+k} = 0$. Given that $a1^t + \delta^t \leq \max_{k\in[1,N_d^a]}(\delta^{t+k})$, the customer stock cannot satisfy the maximum demand $\max_{k\in[1,N_d^a]}(\delta^{t+k})$. Consequently, there will exist a $k \in [1, N_d^a]$ such that $z^{t+k} \neq 0$, which is absurd according to *property 2.* Therefore, for all $t \in [0, T - \max(N_d^a, M_d^a)]$ if $a1^t + \delta^t \leq \max_{k\in[1,N_d^a]}(\delta^{t+k})$ then $u1^t = 0$.

Proprerty 10. *For a* $t \in [0, T - \max(N_d^a, M_d^a)]$, *if* $a1^t + \delta^t \leq \max_{k\in[1,N_d^a]}(\delta^{t+k})$ *then*
$z^{t+1} = \max(0, \delta^{t+1} - (a1^t + \delta^t))$
$y^{t+1} = \min(a1^t + \delta^t, \delta^{t+1})$

Proof. Let $t \in [0, T - \max(N_d^a, M_d^a)]$ such that $a1^t + \delta^t \leq \max_{k\in[1,N_d^a]}(\delta^{t+k})$, which, according to *property 10,* implies that $u1^t = 0$. Suppose that $z^{t+1} > \max\{0, \delta^{t+1} - (a1^t + \delta^t)\}$. Since $\delta^{t+1} = y^{t+1} + z^{t+1}$, it follows that $a1^{t+1} = (a1^t + \delta^t) - y^{t+1} = (a1^t + \delta^t) - \delta^{t+1} + z^{t+1}$ and, given that $z^{t+1} > \max\{0, \delta^{t+1} - (a1^t + \delta^t)\}$, it follows that
$a1^{t+1} = (a1^t + \delta^t) - \delta^{t+1} + z^{t+1} > (a1^t + \delta^t) - \delta^{t+1} + \delta^{t+1} - (a1^t + \delta^t) = 0 \Rightarrow a1^{t+1} > 0$

1^st^ case : Assume there exists $u1^{t+k} \neq 0$, for $k > 0$ and $t + k \leq T$, or $y^{t+l} \neq 0$, for $l > 0$ and $t + l \leq T$ and that one of the two variables are non-zero occurrences.

1st sub-case: if $u1^{t+k} \neq 0$ and $t + k \in [t+1, T]$ is the first to be non-zero, which implies that there are no departures between period *t+1* and *t+k* (unless y^{t+1}), then the stock at customer site will be non-zero between these two periods. Let $\Gamma = \Gamma^- \cup \Gamma^+$ be a cycle such that:
$\Gamma^- = (A^{t+1}, A1^{t+1},..., A1^{t+k}, U1^{t+k})$
$\Gamma^+ = (U1^t, D^{t+1},..., D1^{t+k})$

Given that there are no departures between period *t+1* and *t+k* (unless y^{t+1}), and that $u1^{t+k} \neq 0$, it follows that all arcs of path $\Gamma^- = (A^{t+1}, A1^{t+1}, ..., A^{t+k}, U1^{t+k})$ are non-zero$\Rightarrow Min_{\Gamma^-}[f(u) - b(u)] > 0$ and

thus $\gamma = Min\{ Min_{u\in\Gamma^-}[f(u) - b(u)]; Min_{u\in\Gamma^+}[c(u) - f(u)] \} > 0$.

Given that $\sum_{u\in\Gamma^+} a(u) = C_u^{tr} + k \times C_d^{st} < \sum_{u\in\Gamma^-} a(u) = k \times C_a^{st} + C_u^{tr}$, this is absurd. Therefore, in this case the inequality $z^{t+1} > \max(0, \delta^{t+1} - (a1^t + \delta^t))$ is impossible.

<u>*2^{nd} sub-case :*</u> if $y^{t+l} \neq 0$ for $t + l \in [t+2, T]$ is the first to be non-zero, then there are no departures from the customer site between periods *t+2* and *t+l*. Let $\Gamma = \Gamma^- \cup \Gamma^+$ be a cycle such that

$\Gamma^- = (Z^{t+1}, A1^{t+1}, ..., A^{t+l}, Y^{t+l})$

$\Gamma^+ = (Y^{t+1}, D1^{t+1}, ..., D^{t+l}, Z^{t+l})$

On the path $\Gamma^- = (Z^{t+1}, A1^{t+1}, ..., A^{t+l}, Y^{t+l})$ there are no zero arcs, i.e. $Min_{\Gamma^-}[f(u) - b(u)] > 0$ and since $Min_{\Gamma^+}[c(u) - f(u)] > 0$, it follows that

$\gamma = Min\{ Min_{u\in\Gamma^-}[f(u) - b(u)]; Min_{u\in\Gamma^+}[c(u) - f(u)] \} > 0$.

Given that $\sum_{u\in\Gamma^+} a(u) = C_y^{tr} + (l-1) \times C_d^{st} + C_z^{tr} < \sum_{u\in\Gamma^-} a(u) = C_z^{tr} + (l-1) \times C_a^{st} + C_y^{tr}$, this is absurd and consequently $z^{t+1} > \max(0, \delta^{t+1} - (a1^t + \delta^t))$ is also impossible in this case.

<u>***2^{nd} case:***</u> if all $u1^{t+k}$ and y^{t+k} are zero for $t + k \leq T$. Let *f* be the optimal flow found in this case and let $\beta = a1^{t+1} = a1^t + \delta^t - y^{t+1} - u1^t > 0$ be an integer that is subtracted from the path $\Gamma = (A^{t+1}, ..., A^T)$ and added to $\Gamma' = (U1^t, D^t, ..., D^T)$. Calling the obtained flow from this transformation *f'*, the cost variation from this transformation is $\beta \times [C_u^{tr} + (T-t) \times C_d^{st} - (T-t) \times C_a^{st}]$. Given that $T - t > \max(N_d^a, M_d^a) \Rightarrow T - t > M_d^a$ and that, according to definition 2, $(T-t) \times C_a^{st} > C_u^{Tr} + (T-t) \times C_d^{st} \Rightarrow \beta \times (C_u^{tr} + (T-t) \times C_d^{st} - (T-t) \times C_a^{st}) < 0$ this implies that *f'* is better than *f*. Therefore, taking $z^{t+1} > \max(0, \delta^{t+1} - (a1^t + \delta^t))$ cannot optimize the flow.

We have seen that in all cases the inequality $z^{t+1} > \max(0, \delta^{t+1} - (a1^t + \delta^t)) \Rightarrow z^{t+1} \leq \max(0, \delta^{t+1} - (a1^t + \delta^t))$. In addition, $z^{t+1} < \max(0, \delta^{t+1} - (a1^t + \delta^t))$ is also impossible. We have also seen that when $\max(0, \delta^{t+1} - (a1^t + \delta^t)) = 0$, z^{t+1} cannot be negative and that when $\max(0, \delta^{t+1} - (a1^t + \delta^t)) = \delta^{t+1} - (a1^t + \delta^t)$ taking z^{t+1} less than $\delta^{t+1} - (a1^t + \delta^t)$ cannot satisfy the demand at period *t+1*. Therefore:

$$\begin{cases} z^{t+1} \geq \max(0, \delta^{t+1} - (a1^t + \delta^t)) \\ z^{t+1} \leq \max(0, \delta^{t+1} - (a1^t + \delta^t)) \end{cases} \Leftrightarrow z^{t+1} = \max(0, \delta^{t+1} - (a1^t + \delta^t))$$

And because $z^{t+1} + y^{t+1} = \delta^{t+1} \Rightarrow y^{t+1} = \min(a1^t + \delta^t, \delta^{t+1})$

Finally we can summarise this work as follow:

For all $t \in [0, T - \max(N_d^a, M_d^a)]$

$z^t = \max(0, \delta^t - (a1^{t-1} + \delta^{t-1}))$

$y^t = \min(a1^{t-1} + \delta^{t-1}, \delta^t)$

$u1^t = \max(0, a1^t + \delta^t - \max_{k\in[1,N_d^a]}(\delta^{t+k}))$

and for all t

$x^t = \delta^t$

$y1^t = z1^t = 0$

$u^t = 0$

5. Interpretation and Discussion

In our case, the supplier site has the largest storage cost. We have seen that storage at the supplier site should be avoided. This is obvious; in fact there is no interest in sending empty containers for storage at the supplier site, neither from the customer site nor the depot site. Keeping empty containers and sending them as needed is better then sending them before they are needed and suffering the additional costs involved. Concerning movement between the customer and the depot; at the end of each period there are always deliveries of loaded containers from supplier site δ^t. After they have been emptied the customer is able to return them to the best place for storage. Although the depot has the lowest storage cost, we have seen that the quantity sent from the customer, to the depot in each period is different to the total quantity in customer possession ($a1^t + \delta^t$). In fact, the decision of how many containers to send from the customer site to the depot must not only take into account the different storage costs but also the different transportation costs. The transportation cost from the customer to the supplier C_y^{Tr} the transportation cost from the customer to the depot and the transportation cost from the depot to the supplier $(C_u^{Tr} + C_z^{Tr})$ must all be taken into consideration. At the customer site, the system holds a quantity exactly equal to the maximum demand on the horizon of the next N_d^a periods for direct use (in the case when the entries ($a1^t + \delta^t$) are greater than maximum demand). This is because if we send a quantity to the depot such that the remaining stock becomes $a^t < \max_{k\in[1,N_d^a]}(\delta^{t+k})$ we will be obliged to reuse it in the interval $[t, t+N_d^a]$ and since $i \times C_a^{st} + C_y^{Tr} < C_u^{Tr} + i \times C_d^{st} + C_z^{Tr}$ for all $i \le N_d^a$, this transportation will cost more than storage. Similarly, if we keep a quantity more than $\max_{k\in[1,N_d^a]}(\delta^{t+k})$, it will not be used in the interval $[t, t+N_d^a]$ and storage costs may outweigh the transportation costs of sending it to the depot; $j \times C_a^{st} + C_y^{Tr} > C_u^{Tr} + j \times C_d^{st} + C_z^{Tr}$ for all $j > N_d^a$. Thus, the system sends only the surplus of containers compared to $\max_{k\in[1,N_d^a]}(\delta^{t+k})$ to the depot.

In order to satisfy the demand for empty containers at the supplier site, the system always uses the customer inventory first and uses depot inventory to make up any shortfall. This is because if we use the depot inventory such that $z^t > \max\{0, \delta^{t+1} - (a1^t + \delta^t)\}$ this entails that $y^t < \min\{a1^{t-1} + \delta^{t-1}, \delta^t\}$ (given that $y^t + z^t = \delta^t$); in other words, we will obtain a surplus at the customer site which will create additional costs.

6. Conclusion

Throughout this paper we have demonstrated that in this cost configuration ($C_d^{st} < C_a^{st} < C_b^{st}$) a simple recurring formula solves the model. This result can be extended to other configurations costs such as:

- the supplier has the lowest cost followed by the depot ($C_d^{st} < C_b^{st} < C_a^{st}$).
- the customer site has the lowest cost independently from whether the depot or the supplier has the second lowest cost.
- the supplier site has the lowest cost, independently from whether the depot or the customer has the second lowest cost.

The same reasoning can be applied to each of these situations. This shows that the problem can be divided into four categories and easily understood.
This result is important for two reasons. Firstly, it simplifies the resolution of the container management problem. Secondly, it permits greater understanding of system behaviour and facilitates the interpretation of all movements. This can help us understand other situations, such as when there are several customer sites, depots and supplier sites. In addition, this new resolution method opens new horizons for the treatment of other problems related to flow optimization, such as system container dimensioning and container purchasing policies. The last two problems have been examined for certain particular cases. The cases have only looked at one site systems and have neglected the different movements in the transportation network flow (D.J. Buchanan and Abad, 1998). It is our intention in future studies to investigate this problem from the global aspect and demonstrate possible simplifications.

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