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# Uniform Growth of Groups Acting on Cartan-Hadamard Spaces

G. Besson, G. Courtois et S. Gallot

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## Preliminary version

### 1 Introduction

In this paper we investigate the growth of finitely generated groups. Given a group  $\Gamma$  generated by a finite set  $S$ , the word length  $l_S(\gamma)$  of an element  $\gamma \in \Gamma$  is the smallest integer  $m$  such that there exist elements  $\sigma_1, \dots, \sigma_m$  in  $S \cup S^{-1}$  with  $\gamma = \sigma_1 \dots \sigma_m$ . The entropy of  $\Gamma$  with respect to the generating set  $S$  is defined by

$$\text{Ent}_S(\Gamma) = \lim_{m \rightarrow \infty} \frac{1}{m} (\log |\{\gamma \in \Gamma / l_S(\gamma) \leq m\}|). \quad (1)$$

If  $\text{Ent}_S(\Gamma) > 0$  for some generating set  $S$ , it is true for all (finite) generating set and the group is said to have exponential growth. We now define the entropy of  $\Gamma$

$$\text{Ent } \Gamma = \inf_S \{\text{Ent}_S(\Gamma) / S \text{ finite generating set of } \Gamma\}. \quad (2)$$

We say that  $\Gamma$  has uniform exponential growth if  $\text{Ent } \Gamma > 0$ . In [11], remarque 5.12, M. Gromov raised the question whether exponential growth always implies uniform exponential growth. The answer is negative, indeed, in [14] J.S. Wilson gave examples of finitely generated groups of exponential growth and non uniform exponential growth. Nevertheless, exponential growth implies uniform exponential growth for hyperbolic groups [12], geometrically finite groups of isometries of Hadamard manifolds with pinched negative curvature [1], solvable groups [13] and linear groups [10], [4], [3]. For further references see the exposition paper [7].

We suppose that  $(X, g)$  is a  $n$ -dimensional Cartan Hadamard manifold of pinched sectional curvature  $-a^2 \leq K \leq -1$ . Our main result is the

**Theorem 1.1** *There exists a positive constant  $C(n, a)$  such that for any finitely generated discrete group  $\Gamma$  of isometries of  $(X, g)$ , then either  $\Gamma$  is virtually nilpotent or  $\text{Ent}(\Gamma) \geq C(n, a)$ .*

**Remark.** The difficulty is here to show that one can choose the constant  $C(n, a)$  not depending on the group  $\Gamma$ . In the linear setting, E. Breuillard obtained the same kind of uniformity proving the existence of a positive constant  $C(n)$  such that for any finitely generated group  $\Gamma$  of  $GL(n, K)$ ,  $K$  any field, then either  $\Gamma$  is virtually solvable or  $\text{Ent}(\Gamma) \geq C(n)$ .

The classical technique is to prove that "not too far" from any finite generating system one can exhibit a free group (in two generators). In this paper we do prove this in one of the cases under consideration, using the famous ping-pong lemma, however in the second case we use a different approach using natural Lipschitz maps from the Cayley graph into  $X$ . This is the new idea which is described in the following.

In a private communication M. Kapovich mentioned to us a different proof in the case when  $\Gamma$  acts without any elliptic element. One important issue in our proof is that we do not have this restriction, elliptic elements are permitted.

In the forthcoming paper [2] we shall use this result to prove a Margulis lemma without curvature; indeed, we shall replace the curvature assumptions by hypothesis on the growth of the fundamental group.

## 2 Preliminaries

Let  $(X, g)$  be a  $n$ -dimensional Cartan-Hadamard manifold with sectional curvature  $-a^2 \leq K_g \leq -1$ . Let us recall a few well-known facts about isometries. If  $\gamma$  is an isometry of  $(X, g)$ , the displacement of  $\gamma$  is defined by  $l(\gamma) = \inf_{x \in X} \rho(x, \gamma x)$ , where  $\rho$  is the distance associated to the metric  $g$  on  $X$ . We then have (see [9] p. 31):

1. The isometry  $\gamma$  is called hyperbolic (or axial) if  $l(\gamma) > 0$ , in which case there exists a geodesic  $a_\gamma$ , called the axis of  $\gamma$ , such that, for any  $x \in a_\gamma$ ,  $\rho(x, \gamma x) = l(\gamma)$ .
2. The isometry  $\gamma$  is called parabolic if  $l(\gamma) = 0$  and  $l(\gamma)$  is not achieved on  $X$ , in which case there exists a unique point  $\theta$  on the geometric boundary  $\partial X$  of  $X$  such that  $\gamma\theta = \theta$ .
3. The isometry  $\gamma$  is called elliptic if  $l(\gamma) = 0$  and  $l(\gamma)$  is achieved on  $X$ , in which case there exists a non empty convex subset  $F_\gamma$  of  $X$  such that, for any  $x \in F_\gamma$ ,  $\gamma x = x$ .

The following result, due to G. Margulis, describes the structure of discrete subgroups of isometries generated by elements with small displacement.

**Theorem 2.1 (G. Margulis, [5])** *There exists a constant  $\mu(n, a) > 0$  such that if  $\Gamma$  is a discrete subgroup of the isometry group of  $(X, g)$ , the subgroup  $\Gamma_\mu$  of  $\Gamma$  generated by,*

$$S_\mu = \{\gamma \in \Gamma / \rho(x, \gamma x) \leq \mu(n, a)\},$$

*is virtually nilpotent.*

Given a set of isometries  $S = \{\sigma_1, \dots, \sigma_p\}$  of  $(X, g)$ , we define the “minimal displacement” of  $S$  by

$$\mathbf{Definition\ 2.1} \quad L(S) = \inf_{x \in X} \max_{i=1, \dots, p} \rho(x, \sigma_i x)$$

When  $\Gamma$  is a finitely generated discrete subgroup of the isometry group of  $(X, g)$ , the above theorem 2.1 has the following

**Corollary 2.1** *There exists a constant  $\mu(n, a) > 0$  such that if  $\Gamma$  is a finitely generated not virtually nilpotent discrete subgroup of isometry of  $(X, g)$  and  $S = \{\sigma_1, \dots, \sigma_p\}$  a finite generating set of  $\Gamma$ , then*

$$L(S) \geq \mu(n, a).$$

In the following lemma we describe the structure of virtually nilpotent discrete subgroups of isometries of  $(X, g)$ . Here by discrete we mean that the orbits are discrete sets in  $(X, g)$ .

**Lemma 2.1** *Let  $G$  be a discrete virtually nilpotent group of isometries of  $(X, g)$ .*

- a) *If  $G$  contains an hyperbolic element  $\gamma$ , then  $G$  preserves the axis of  $\gamma$ .*
- b) *If  $G$  contains a parabolic element  $\gamma$  with fixed point  $\theta \in \partial X$ , then  $G$  fixes the point  $\theta$ .*
- c) *If all elements of  $G$  are elliptic, then  $G$  is finite.*

*Proof.* a) Let  $\gamma \in G$  be an hyperbolic element and  $\theta, \zeta \in \partial X$ , the end points of the axis  $a_\gamma$  of  $\gamma$ . We claim that for any  $\gamma' \in G$ , then  $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$  or  $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \emptyset$ . Indeed assume that  $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \{\theta\}$ . The isometry  $\gamma'\gamma\gamma'^{-1}$  is hyperbolic with axis  $a_{\gamma'\gamma\gamma'^{-1}} = \gamma'a_\gamma$  equal to the geodesic joining  $\theta$  and  $\zeta' \neq \zeta$ , where  $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta'\}$ . We may assume that  $\theta$  is the attractive fixed point of  $\gamma$  and  $\gamma'\gamma\gamma'^{-1}$  (replacing them by their inverse if necessary). Let  $x \in a_\gamma$ , then  $(\gamma'\gamma\gamma'^{-1})^{-N}\gamma^N x$  is a sequence of pairwise distinct points which converges to a point on the axis  $a_{\gamma'\gamma\gamma'^{-1}}$  of  $\gamma'\gamma\gamma'^{-1}$ . This contradicts the discreteness of  $G$ , proving thus the claim.

Now, if there exist  $\gamma' \in G$  such that  $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \emptyset$ , the two hyperbolic isometries  $\gamma$  and  $\gamma'\gamma\gamma'^{-1}$  would then have disjoint axis and therefore  $G$  would contain a free subgroup by a classical ping-pong argument. This would contradict the fact that  $G$  is virtually nilpotent. Consequently, for any  $\gamma' \in G$ ,  $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$ , which shows that  $G$  preserves the geodesic joining  $\theta$  and  $\zeta$ .

b) Let  $\gamma \in G$  be a parabolic element, and  $\theta \in \partial X$  its fixed point. If there exist  $\gamma' \in G$  such that  $\gamma'\theta \neq \theta$ , then  $\gamma$  and  $\gamma'\gamma\gamma'^{-1}$  would be to parabolic elements in  $G$  with distinct fixed point  $\theta$  and  $\gamma'\theta$  respectively. By a ping-pong argument,  $G$  would then contain a free subgroup, which contradicts the fact that  $G$  is virtually nilpotent. Thus  $G$  fixes  $\theta \in \partial X$ .

c) Let us now assume that all elements in  $G$  are elliptic. Let  $N \subset G$  be a nilpotent subgroup of  $G$  with finite index. If  $N = \{e\}$ , then  $G$  is finite. We thus assume that  $N \neq \{e\}$ , the center  $Z(N)$  of  $N$  is then not trivial. For

$g_1 \in Z(N) \setminus \{e\}$  let us denote  $F_{g_1} \subset X$  the set of fixed points of  $g_1$ . Let  $x_1 \in F_{g_1}$ ; by commutation of  $g_1$  and  $\exp_{x_1}$ , we have  $F_{g_1} = \exp_{x_1}(E_1)$ , where  $E_1$  is the eigenspace of  $d_{x_1}g_1$  corresponding to the eigenvalue  $+1$ . This shows that  $F_{g_1}$  is a totally geodesic submanifold of  $X$  satisfying  $\dim(F_{g_1}) < \dim(X)$ , since  $g_1 \neq e$ . As every  $\gamma \in N$  commutes with  $g_1$ , it satisfies  $\gamma(F_{g_1}) = F_{g_1}$ .

Let  $N_1$  be the subgroup of  $\text{Isom}(F_{g_1})$  obtained by restriction to  $F_{g_1}$  of the elements of  $N$ ; it is clearly nilpotent as the image of a nilpotent group. For  $\gamma \in N$ , the projection on  $F_{g_1}$  of any fixed point of  $\gamma$  is again a fixed point of  $\gamma$ ; consequently, the elements of  $N_1$  are elliptic elements of  $\text{Isom}(F_{g_1})$ .

If  $N_1 = \{e\}$ , then  $F_{g_1}$  is pointwise fixed by  $N$ , therefore  $N$  is finite (the group is discrete and all elements have a common fixed point).

If  $N_1 \neq \{e\}$ , we may iterate the process. Indeed, let us suppose that we have constructed the totally geodesic submanifold  $F_{g_i}$ , we then construct  $N_i$  as the set of restrictions of elements of  $N$  to  $F_{g_i}$ , and, either  $N_i = \{e\}$  in which case  $N$  is finite, or  $N_i$  is not trivial and, choosing  $g_{i+1} \in Z(N_i) \setminus \{e\}$ , we construct the totally geodesic submanifold  $F_{g_{i+1}} \subset F_{g_i}$  such that  $\dim(F_{g_{i+1}}) < \dim(F_{g_i})$ . This process stops for some  $i_0 \leq n$  and then  $N_{i_0} = \{e\}$  and  $F_{g_{i_0}}$  is pointwise fixed by  $N$  and not empty.  $\square$

**Lemma 2.2** *Let  $\Gamma$  be a finitely generated discrete group of isometries of  $(X, g)$ .*

- (i) *If there exist a point  $\theta \in \partial X$  fixed by  $\Gamma$ , then  $\Gamma$  is virtually nilpotent.*
- (ii) *If  $\Gamma$  preserves a geodesic in  $X$ , then  $\Gamma$  is virtually cyclic.*

*Proof.* Proof of (i). There are three cases: 1) there is an hyperbolic element in  $\Gamma$ , 2) there is no hyperbolic element, but there is a parabolic element in  $\Gamma$  and 3) all elements in  $\Gamma$  are elliptic.

1) Let  $\gamma$  be a hyperbolic element in  $\Gamma$ , and  $a_\gamma$  its axis. One of the endpoints of  $a_\gamma$  is  $\theta$ . As  $\Gamma$  is discrete, it follows from the argument in the proof of lemma 2.1 a) that for any  $\gamma' \in \Gamma$ ,  $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$  or  $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \emptyset$ , where  $\zeta$  is the other endpoint of  $a_\gamma$ . Therefore,  $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$  and  $\gamma'(\theta) = \theta$  and  $\gamma'(\zeta) = \zeta$ . The group  $\Gamma$  preserves  $a_\gamma$ . Let us note that  $\Gamma$  does not contain any parabolic element, since such an element would fix  $\theta$  and therefore also  $\zeta$  which is impossible. The elements in  $\Gamma$  are thus either hyperbolic or elliptic.

Now, the projection on  $a_\gamma$  being distance decreasing, any element  $\gamma' \in \Gamma$  achieves its displacement  $l(\gamma')$  on the axis  $a_\gamma$ , and  $\gamma'$  is elliptic (resp. hyperbolic) iff  $l(\gamma') = 0$  (resp.  $l(\gamma') \neq 0$ ). Moreover, since  $\gamma'(\theta) = \theta$ , any elliptic element fixes pointwise the axis  $a_\gamma$ . The restriction to the axis  $a_\gamma$  is thus a morphism from  $\Gamma$  into the group of translations of the axis, whose kernel is the set of elliptic elements, which fix all points of  $a_\gamma$  and hence is finite. The group  $\Gamma$  is then virtually abelian.

2) In this case the elements of  $\Gamma$  are either elliptic or parabolic with fixed point  $\theta$ . In particular, every element of  $\Gamma$  preserves each horospheres centred at  $\theta$ . Indeed, this is clear for parabolic elements. Any elliptic element  $\gamma'$  fixes some point  $x \in X$ , and hence the whole geodesic  $c$  joining  $x$  to  $\theta$ ; let  $H$  be any horosphere centred at  $\theta$  and  $y$  be its intersection with  $c$ , then  $\gamma'$  maps  $H$  onto the horosphere centred at  $\gamma'(\theta) = \theta$  containing  $\gamma'(y) = y$ . This shows that  $\gamma'(H) = H$ .

Let  $S = \{\sigma_1, \dots, \sigma_p\}$  be a generating set of  $\Gamma$ , by the above discussion,  $\inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = 0$ . In fact, for any geodesic  $c$  such that  $c(+\infty) = \theta$ , let  $H_t$  be the horosphere centred at  $\theta$  and containing  $c(t)$ . The orthogonal projection from  $H_t$  to  $H_{t+t'}$  contracts distances, we then get that  $\rho(c(t), \gamma'(c(t)))$  decreases to zero when  $t$  goes to infinity, for any  $\gamma' \in \Gamma$ . The group  $\Gamma$  is then virtually nilpotent by theorem 2.1.

3) If  $\Gamma$  only contains elliptic elements, then for any finite generating set  $S = \{\sigma_1, \dots, \sigma_p\}$ ,  $\inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = 0$ , because each  $\sigma_i$  preserves each horosphere centred at  $\theta$ , by the above argument. The group  $\Gamma$  is again virtually nilpotent.

Proof of (ii). A subgroup of index two of  $\Gamma$  fixes each endpoint of the globally preserved geodesic. Then we conclude as in the case 1 of (i).  $\square$

For any two isometries  $\gamma, \gamma'$  acting on  $(X, g)$  we define,

$$L(\gamma, \gamma') = \inf_{x \in X} \max\{\rho(x, \gamma x), \rho(x, \gamma' x)\}.$$

We now prove the,

**Proposition 2.1** *Let  $\Gamma$  be a finitely generated discrete subgroup of  $\text{Isom}(X, g)$ , where  $(X, g)$  is a Cartan-Hadamard manifold of sectional curvature  $-a^2 \leq K_g \leq -1$ . Let  $S = \{\sigma_1, \dots, \sigma_p\}$  be a finite generating set of  $\Gamma$ . If  $\Gamma$  is not virtually nilpotent, we have*

- i) *either there exist  $\sigma_i, \sigma_j \in S$  such that the subgroup  $\langle \sigma_i, \sigma_j \rangle$  generated by these two elements is not virtually nilpotent and  $L(\sigma_i, \sigma_j) \geq \mu(n, a)$ ,*
- ii) *or all  $\sigma_i$  in  $S$  are elliptic and for all  $\sigma_i \neq \sigma_j \in S$ , either  $\langle \sigma_i, \sigma_j \rangle$  fixes some point in  $X$  and is finite, or it fixes a point  $\theta \in \partial X$ ,*
- iii) *or there exist  $\sigma_i, \sigma_j, \sigma_k \in S$  such that  $L(\sigma_i \sigma_j, \sigma_k) \geq \mu(n, a)$  and the group  $\langle \sigma_i \sigma_j, \sigma_k \rangle$  is not virtually nilpotent.*

*Proof.* There are again three cases: a) there is a hyperbolic element in  $S$ , say  $\sigma_1$ ; b) there is no hyperbolic element and there is a parabolic element in  $S$ , say  $\sigma_1$ ; c) all  $\sigma_i$ 's in  $S$  are elliptic.

a) Let us assume that  $\sigma_1$  is hyperbolic. Let us consider all pairs  $(\sigma_1, \sigma_i)$  with  $i = 2, \dots, p$ , and let us assume that  $L(\sigma_1, \sigma_i) < \mu(n, a)$  for  $i = 2, \dots, p$ . The groups  $\langle \sigma_1, \sigma_i \rangle$  are then virtually nilpotent. By lemma 2.1 a), every  $\sigma_i$  preserves the axis  $a_{\sigma_1}$  of  $\sigma_1$ , hence  $\Gamma$  preserves  $a_{\sigma_1}$  and is virtually nilpotent contradicting the assumption. Then there exist  $\sigma_i \in S$  such that  $L(\sigma_1, \sigma_i) \geq \mu(n, a)$  and  $\langle \sigma_1, \sigma_i \rangle$  is non virtually nilpotent.

b) Assume that  $\sigma_1$  is parabolic with fixed point  $\theta \in \partial X$ . Let us consider all pairs  $(\sigma_1, \sigma_i)$ ,  $i = 2, \dots, p$ , and assume that  $\langle \sigma_1, \sigma_i \rangle$  is virtually nilpotent (or that  $L(\sigma_1, \sigma_i) < \mu(n, a)$ , for all  $i = 2, \dots, p$ ). By lemma 2.1 b),  $\sigma_i$  fixes the point  $\theta \in \partial X$ , therefore  $\Gamma$  fixes  $\theta$  and is virtually nilpotent, by lemma 2.2, a contradiction. Consequently, if  $\sigma_1$  is parabolic, there exist  $\sigma_i \neq \sigma_1$  such that  $L(\sigma_1, \sigma_i) \geq \mu(n, a)$ .

c) Let us assume that all  $\sigma_i$ 's are elliptic, for  $i = 2, \dots, p$ , and that for all pairs  $(\sigma_i, \sigma_j)$  the groups  $\langle \sigma_i, \sigma_j \rangle$  are virtually nilpotent (or that  $L(\sigma_i, \sigma_j) < \mu(n, a)$ ). Let us denote  $G = \langle \sigma_i, \sigma_j \rangle$ . There are again three cases: 1) there is a hyperbolic element in  $G$ , 2) there is no hyperbolic element and there is a parabolic element in  $G$ , 3) all elements in  $G$  are elliptic.

In the case 1), let  $\gamma$  be a hyperbolic element in  $G$  with axis  $a_\gamma$ . By lemma 2.1 a),  $G$  preserves  $a_\gamma$ . Since  $\sigma_i, \sigma_j$  are elliptic, they fix points  $x_i$  and  $x_j$  (respectively) on  $a_\gamma$  (recall that the displacement of  $\sigma_i$  and  $\sigma_j$  are achieved on  $a_\gamma$  by the distance decreasing property of the projection onto  $a_\gamma$ ). If  $x_i = x_j$ , the  $G$  fixes  $x_i$  and it is thus finite. Let us now suppose that  $\sigma_i$  and  $\sigma_j$  do not fix the same point on  $a_\gamma$ , that is  $x_i \neq x_j$  and none of the restriction  $\tilde{\sigma}_i$  and  $\tilde{\sigma}_j$  of  $\sigma_i$  and  $\sigma_j$  to  $a_\gamma$  is the identity. In that case,  $\tilde{\sigma}_i$  and  $\tilde{\sigma}_j$  are both symmetries around  $x_i$  and  $x_j$ , and then  $\sigma_i\sigma_j$  is a hyperbolic element with axis  $a_\gamma$ . Let us then consider  $\langle \sigma_i\sigma_j, \sigma_l \rangle$  for  $l = 1, \dots, p$ . Assume that for all  $l = 1, \dots, p$ ,  $L(\sigma_i\sigma_j, \sigma_l) < \mu(n, a)$ , the groups  $\langle \sigma_i\sigma_j, \sigma_l \rangle$  are then virtually nilpotent, and by lemma 2.1 a), all  $\sigma_l$ 's preserve  $a_\gamma$  and hence  $\Gamma$  preserves  $a_\gamma$  and is thus virtually nilpotent which is a contradiction. Therefore, there exist  $\sigma_k \in S$  such that  $L(\sigma_i\sigma_j, \sigma_k) \geq \mu(n, a)$  and that  $\langle \sigma_i\sigma_j, \sigma_k \rangle$  is not virtually nilpotent.

In the case 2), let  $\gamma \in G$  be a parabolic element with fixed point  $\theta \in \partial X$ . By lemma 2.1 b),  $G$  fixes  $\theta$ .

In the case 3), all elements in  $G$  are elliptic and by lemma 2.1 c),  $G$  is finite. This ends the proof of the proposition.  $\square$

### 3 Algebraic length and $\eta$ -straight isometries

Let  $\Gamma$  be a finitely generated discrete group of isometries of  $(X, g)$  and  $S = \{\sigma_1, \dots, \sigma_p\}$  be a finite generating set of  $\Gamma$ .

Let us denote  $l_S$  and  $d_S$  the length and distance on the Cayley graph associated to  $S$ . Let  $x_0$  be a point in  $X$  and define  $L = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0)$ .

For any  $\gamma \in \Gamma$  it follows from the triangle inequality that

$$\rho(x_0, \gamma x_0) \leq l_S(\gamma)L. \quad (3)$$

Let  $\eta$  be a positive number such that  $0 < \eta < L$ .

**Definition 3.1** *An isometry  $\gamma$  of  $\Gamma$  is said to be  $(L, \eta)$ -straight if  $\rho(x_0, \gamma x_0) \geq (L - \eta)l_S(\gamma)$ .*

**Remark.** Notice that the above definition depends on the choice of  $x_0$  and of a generating set  $S$ .

When  $\Gamma$  is a finitely generated discrete group, for any finite generating set  $S = \{\sigma_1, \dots, \sigma_p\}$  we define,

$$L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x).$$

When  $\Gamma$  is not virtually nilpotent, by theorem 2.1, for any finite generating set  $S$ ,  $L = L(S) \geq \mu(n, a) > 0$ , where  $\mu(n, a)$  is the Margulis constant. We then have,

**Lemma 3.1** *Let  $\Gamma$  be a finitely generated non virtually nilpotent discrete group of isometries of  $(X, g)$ . For any finite generating set  $S = \{\sigma_1, \dots, \sigma_p\}$  of  $\Gamma$ , there exist  $x_0 \in X$  such that,*

$$L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0).$$

*Proof.* Let us assume that the infimum in the definition of  $L(S)$  is not achieved in  $X$ , then there exist a sequence of points  $x_k \in X$ , which satisfies  $\max_{i \in \{1, \dots, p\}} \rho(x_k, \sigma_i x_k) \rightarrow L(S)$  when  $k \rightarrow \infty$ , and  $x_k$  converges to a point, say  $\theta$ , in  $\partial X$ . For  $k$  large enough and  $i \in \{1, \dots, p\}$ , we then have  $\rho(x_k, \sigma_i x_k) \leq L+1$  and hence  $\sigma_i \theta = \theta$  for all  $i$ . This shows that  $\Gamma$  fixes  $\theta$  and is thus virtually nilpotent by lemma 2.2, which contradicts the hypothesis.  $\square$

In the sequel of this section, we shall show that if  $G$  is a finitely generated discrete group of isometries of  $(X, g)$ , for any finite generating set  $S = \{\sigma_1, \dots, \sigma_p\}$  of  $G$  such that each  $\sigma_i$  has a displacement  $l(\sigma_i)$  small compared to  $L(S)$ , then there exist many non- $(L(S), \eta)$ -straight elements in  $G$  for a constant  $\eta$  to be defined.

We need the following geometric lemmas.

**Lemma 3.2** *Let  $(x_1, x_2, x_3)$  be a geodesic triangle in  $(X, g)$ , where  $(X, g)$  is a Cartan-Hadamard manifold with  $K_g \leq -1$ . Let  $x'_2$  be the point in the segment  $[x_1, x_3]$  dividing it in two segments of length proportional to  $L_1 := \rho(x_1, x_2)$  and  $L_2 := \rho(x_2, x_3)$ . We have,*

$$\rho(x'_2, x_2) \leq \text{Argcosh} \left[ \exp \left( \alpha (\rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_1, x_3)) \right) \right],$$

where  $\alpha = \frac{\max(L_1, L_2)}{L_1 + L_2}$ .

*Proof.* We consider a comparison geodesic triangle  $(y_1, y_2, y_3)$  in the Poincaré disk  $(\mathbb{H}^2, d)$  of constant curvature  $-1$  such that  $d(y_i, y_j) = \rho(x_i, x_j)$  for all  $i, j \in \{1, 2, 3\}$ . Let  $y'_2$  be the point of the segment  $[y_1, y_3]$  dividing it in two segments of length proportional to  $L_1$  and  $L_2$ . Since  $(X, g)$  is a  $CAT(-1)$  space we have

$$\rho(x_2, x'_2) \leq d(y_2, y'_2). \quad (4)$$

One of the two triangles  $(y_1, y'_2, y_2)$ ,  $(y_3, y'_2, y_2)$  has angle at  $y'_2$  greater than or equal to  $\pi/2$ , therefore from hyperbolic trigonometry formulae we get the existence of  $i \in \{1, 2\}$  such that

$$\cosh L_i \geq \cosh [d(y_2, y'_2)] \cosh \left[ \frac{L_i}{(L_1 + L_2)} d(y_1, y_3) \right] \quad (5)$$

Let us denote  $\Delta = \rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_1, x_3)$ . We have

$$\frac{L_i}{L_1 + L_2} d(y_1, y_3) \geq L_i - \alpha \Delta, \quad (6)$$



where  $\alpha = \frac{\max(L_1, L_2)}{L_1 + L_2}$ . Therefore from (4) and (5) we get

$$\cosh [\rho(x'_2, x_2)] \leq \frac{\cosh L_i}{\cosh(L_i - \alpha\Delta)}, \quad (7)$$

hence

$$\cosh [\rho(x'_2, x_2)] \leq e^{\alpha\Delta}. \quad (8)$$

□

**Lemma 3.3** *Let  $(X, g)$  be a Cartan-Hadamard manifold with sectional curvature  $K_g \leq -1$ . Let  $\delta, L$  be any positive numbers such that  $L > \text{Argcosh}(e^\delta)$ . Then, for any isometry  $\gamma$  of  $(X, g)$  such that its displacement  $l(\gamma)$  satisfies  $l(\gamma) \leq \delta$ , and for any point  $x_0 \in X$  such that  $\rho(x_0, \gamma x_0) \geq L$ , we have*

$$\rho(x_0, \gamma^2 x_0) \leq 2\rho(x_0, \gamma x_0) - \left(1 - \frac{e^\delta}{\cosh L}\right)^2,$$

*Proof.* Let us consider  $\Delta = 2\rho(x_0, \gamma x_0) - \rho(x_0, \gamma^2 x_0)$ . We want to prove that  $\Delta \geq \left(1 - \frac{e^\delta}{\cosh L}\right)^2$ . By assumption there is a point  $y \in X$  such that  $\rho(y, \gamma y) \leq \delta$ . Let us write  $L_1 =: \rho(x_0, \gamma y)$ ,  $L_2 =: \rho(\gamma^2 x_0, \gamma y)$  and  $L' =: \rho(x_0, y)$ . By the triangle inequality we have for  $i = 1, 2$

$$L' - \delta \leq L_i \leq L' + \delta. \quad (9)$$

Let us associate to the triangle  $(x_0, \gamma y, \gamma^2 x_0)$  the comparison triangle  $(z_1, z_2, z_3)$  in the hyperbolic plane  $(\mathbb{H}^2, d)$  such that  $d(z_1, z_2) = L_1$ ,  $d(z_2, z_3) = L_2$  and  $d(z_1, z_3) = \rho(x_0, \gamma^2 x_0)$ . Let  $x$  [resp.  $z$ ] be the middle point of the segment  $(x_0, \gamma^2 x_0)$  [resp.  $(z_1, z_3)$ ]. One of the two triangles  $(z_2, z, z_1)$  or  $(z_2, z, z_3)$  has angle at  $z$  greater than or equal to  $\pi/2$ . Let us assume without restriction that this triangle is  $(z_2, z, z_1)$ , then the hyperbolic trigonometric formulas give

$$\cosh L_1 \geq \cosh [d(z_2, z)] \cosh \left[\frac{1}{2}d(z_1, z_3)\right]$$

therefore from (9) we get

$$\cosh(L' + \delta) \geq \cosh [d(z_2, z)] \cosh \left[\frac{1}{2}d(z_1, z_3)\right]$$

and since  $(X, g)$  is a  $CAT(-1)$  space we have  $\rho(x, \gamma y) \leq d(z_2, z)$ , thus we obtain

$$\cosh(L' + \delta) \geq \cosh [\rho(x, \gamma y)] \cosh \left[\frac{1}{2}\rho(x_0, \gamma^2 x_0)\right]. \quad (10)$$

Let us write  $L_0 = \rho(x_0, \gamma x_0)$ . By the triangle inequality we have

$$\rho(x, \gamma y) \geq |\rho(\gamma y, \gamma x_0) - \rho(\gamma x_0, x)|,$$

therefore, since  $\rho(\gamma y, \gamma x_0) = \rho(y, x_0) = L'$  and  $\frac{1}{2}\rho(\gamma^2 x_0, x_0) = L_0 - \frac{\Delta}{2}$ , we get from 10

$$\cosh(L' + \delta) \geq \cosh\left(L' - \rho(\gamma x_0, x)\right) \cosh\left(L_0 - \frac{\Delta}{2}\right). \quad (11)$$

We get from (11),

$$(\cosh \delta + \sinh \delta) \cosh L' \geq \left(\cosh [\rho(\gamma x_0, x)] - \sinh [\rho(\gamma x_0, x)]\right) (\cosh L') \cosh\left(L_0 - \frac{\Delta}{2}\right)$$

hence

$$e^\delta \geq \left(\cosh [\rho(\gamma x_0, x)] - \sinh [\rho(\gamma x_0, x)]\right) \cosh\left(L_0 - \frac{\Delta}{2}\right). \quad (12)$$

Now applying the inequality 7 in the proof of lemma 3.2 we have

$$\cosh [\rho(\gamma x_0, x)] \leq \frac{\cosh L_0}{\cosh\left(L_0 - \frac{\Delta}{2}\right)},$$

and since  $\cosh r - \sinh r = e^{-r}$  is a decreasing function of  $r$  we get from (12)

$$e^\delta \geq \cosh L_0 - \left(\cosh^2 L_0 - \cosh^2\left(L_0 - \frac{\Delta}{2}\right)\right)^{\frac{1}{2}}. \quad (13)$$

But we can check that  $\cosh^2 L_0 - \cosh^2\left(L_0 - \frac{\Delta}{2}\right) \leq \Delta \cosh^2 L_0$  so we get from (13)

$$e^\delta \geq \cosh(L_0)(1 - \Delta^{\frac{1}{2}})$$

and therefore

$$\Delta \geq \left(1 - \frac{e^\delta}{\cosh L_0}\right)^2,$$

when  $e^\delta < \cosh L$ . The lemma now follows whenever  $L_0 \geq L$ . □

**Lemma 3.4** *Let  $(X, g)$  be a Cartan-Hadamard manifold with sectional curvature  $K_g \leq -1$ . Let us consider four points  $y_0, y_1, y_2, y_3$  such that*

$$\rho(y_0, y_1) + \rho(y_1, y_2) - \rho(y_0, y_2) \leq \eta_1$$

and

$$\rho(y_1, y_2) + \rho(y_2, y_3) - \rho(y_1, y_3) \leq \eta_2$$

then

$$\rho(y_0, y_1) + \rho(y_1, y_2) + \rho(y_2, y_3) - \rho(y_0, y_3) \leq \left(1 + \frac{\rho(y_2, y_3)}{\rho(y_1, y_2)}\right) \left(\eta_1 + \text{Argcosh } e^{\eta_2}\right).$$

*Proof.* For  $i = 1, 2, 3$  let us write  $L_i = \rho(y_{i-1}, y_i)$ . Let  $y'_2$  be the point on the segment  $(y_1, y_3)$  dividing it in two segments of length proportional to  $L_2$  and  $L_3$ . By lemma 3.2 we have

$$\rho(y_2, y'_2) \leq \text{Argcosh}(e^{\eta_2}). \quad (14)$$

Since  $\rho(y_0, y_1) + \rho(y_1, y_2) - \rho(y_0, y_2) \leq \eta_1$  by assumption we get from (14) and the triangle inequality

$$\rho(y_0, y'_2) \geq \rho(y_0, y_2) - \rho(y_2, y'_2) \geq \rho(y_0, y_1) + \rho(y_1, y_2) - [\eta_1 + \text{Argcosh}(e^{\eta_2})] \quad (15)$$

On the other hand by convexity of the distance function on  $(X, g)$  we get

$$\rho(y_0, y'_2) \leq \frac{L_3}{L_2 + L_3} \rho(y_0, y_1) + \frac{L_2}{L_2 + L_3} \rho(y_0, y_3) \quad (16)$$

The inequalities (15) and (16) give

$$\rho(y_0, y_3) \geq \rho(y_0, y_1) + L_2 + L_3 - \left(\frac{L_2 + L_3}{L_2}\right) (\eta_1 + \text{Argcosh}(e^{\eta_2}))$$

and the lemma follows.  $\square$

**Lemma 3.5** *Let  $L$  and  $\eta$  be two positive numbers such that,*

$$\eta < \min\left(\frac{L}{4}, \frac{1}{2} \log\left[\frac{1}{2}\left(\cosh\left(\frac{L}{2}\right) + \frac{1}{\cosh\left(\frac{L}{2}\right)}\right)\right]\right).$$

*Let  $(X, g)$  be a Cartan-Hadamard manifold with sectional curvature  $K_g \leq -1$ . We consider two elliptic isometries  $\gamma_1, \gamma_2$  of  $(X, g)$  with a common fixed point  $y \in X \cup \partial X$ . If we assume that  $L - \eta \leq \rho(x_0, \gamma_1 x_0) \leq L$  and that  $L - \eta \leq \rho(x_0, \gamma_2 x_0) \leq L$ , then*

$$\rho(x_0, \gamma_1 \gamma_2 x_0) < 2(L - \eta).$$

*Proof.* We first claim that in both cases,  $y \in X$  and  $y \in \partial X$ , there exist some sequence  $(u_k)_{k \in \mathbf{N}}$  of points in  $X$  converging to  $y$  such that  $\rho(u_k, \gamma_1 \gamma_2 x_0) = \rho(u_k, x_0) = l_k$  and that the quantity  $\epsilon_k = |\rho(u_k, \gamma_1 x_0) - l_k|$  goes to zero when  $k$  goes to  $+\infty$ ; in fact, when  $\gamma_1$  and  $\gamma_2$  fix some point  $y \in X$  we may choose  $u_k = y$  for every  $k$ . If  $\gamma_1$  and  $\gamma_2$  fix  $y \in \partial X$ , they also preserve each horosphere centred at  $y$  (see the proof of lemma 2.2 2)), and thus  $x_0, \gamma_1 x_0$  and  $\gamma_1 \gamma_2 x_0$  lie on the same horosphere centred at  $y$ . Approximating this horosphere by a sequence  $(S_k)_{k \in \mathbf{N}}$  of spheres passing through  $x_0$  and  $\gamma_1 \gamma_2 x_0$  and denoting  $u_k$  the centre of  $S_k$ , we get that  $\rho(u_k, \gamma_1 x_0) - \rho(O, u_k)$  and  $\rho(u_k, x_0) - \rho(O, u_k)$  simultaneously go to  $B(\gamma_1 x_0, y) = B(x_0, y)$  (where  $O$  is some fixed origin in  $X$  and  $B$  the Busemann function normalised at  $O$ ). This proves the claim.

Consider the triangle  $(u_k, v, w) = (u_k, x_0, \gamma_1 \gamma_2 x_0)$  and  $z$  the point of the geodesic segment  $[v, w]$  which divides it in two segments of length proportional to  $L_1 =: \rho(v, \gamma_1 x_0)$  and  $L_2 =: \rho(w, \gamma_1 x_0)$ . Let us recall that by assumption we have  $L - \eta \leq L_i \leq L$ .

We consider the comparison triangle  $(\bar{u}_k, \bar{v}, \bar{w})$  on the two-dimensional hyperbolic space  $\mathbf{H}^2$  such that  $d(\bar{u}_k, \bar{v}) = \rho(u_k, v) = l_k = \rho(u_k, w) = d(\bar{u}_k, \bar{w})$  and  $d(\bar{v}, \bar{w}) = \rho(v, w)$ , where  $d$  is the hyperbolic distance on  $\mathbf{H}^2$ . Let  $\bar{z}$  be the point of the segment  $[\bar{u}, \bar{v}]$  dividing it in two segments of length proportional to  $L_1$  and  $L_2$ . Let us write  $L'_1 = \rho(v, z)$  and  $L'_2 = \rho(w, z)$ . We now consider the triangle  $(\bar{u}_k, \bar{v}, \bar{z})$  or  $(\bar{u}_k, \bar{w}, \bar{z})$ , namely the one which has angle at  $\bar{z}$  larger than or equal to  $\pi/2$ . We can assume without restriction that this triangle is  $(\bar{u}_k, \bar{v}, \bar{z})$ . The hyperbolic trigonometry formulas then show that the point  $\bar{z}$  satisfies,

$$\cosh(l_k) \geq \cosh(L'_1) \cosh(d(\bar{u}_k, \bar{z})).$$

Since  $(X, g)$  is a CAT(-1)-space, we get that,

$$\rho(u_k, z) \leq d(\bar{u}_k, \bar{z}),$$

and thus that,

$$\cosh(\rho(u_k, z)) \leq \frac{\cosh(l_k)}{\cosh(L'_1)}. \quad (17)$$

On the other hand, the triangle inequality implies that  $\rho(u_k, z) \geq l_k - \epsilon_k - \rho(\gamma_1 x_0, z)$  and thus that

$$\cosh(\rho(u_k, z)) \geq e^{-(\rho(\gamma_1 x_0, z) + \epsilon_k)} \cosh(l_k).$$

Plugging this in formula 17 and letting  $\epsilon_k \rightarrow 0$ , we get:

$$e^{\rho(\gamma_1 x_0, z)} \geq \cosh(L'_1). \quad (18)$$

On the other hand, by lemma 3.2, we have

$$\cosh(\rho(\gamma_1 x_0, z)) \leq \exp \left( \max \left\{ \rho(v, \gamma_1 x_0), \rho(w, \gamma_1 x_0) \right\} \left( 1 - \frac{\rho(v, w)}{\rho(v, \gamma_1 x_0) + \rho(w, \gamma_1 x_0)} \right) \right).$$

and hence

$$\cosh(\rho(\gamma_1 x_0, z)) \leq e^{(L - \frac{\rho(v, w)}{2})}. \quad (19)$$

Let us now assume, by contradiction, that

$$\rho(v, w) = \rho(x_0, \gamma_1 \gamma_2 x_0) > 2(L - \eta).$$

Plugging this in the inequalities (18) and (19) we obtain, using the fact that  $x \rightarrow x + 1/x$  is an increasing function for  $x > 1$ :

$$\cosh(L'_1) + \frac{1}{\cosh(L'_1)} \leq 2 \cosh(\rho(\gamma_1 x_0, z)) \leq 2e^\eta. \quad (20)$$

Now since  $\frac{L'_1}{L'_2} = \frac{L_1}{L_2}$ , we also obtain

$$L'_1 = (L'_1 + L'_2) \left( \frac{L_1}{L_1 + L_2} \right) \geq \frac{2(L - \eta)(L - \eta)}{2L} \geq L - 2\eta$$

which gives by inequalities (20)

$$\cosh(L - 2\eta) + \frac{1}{\cosh(L - 2\eta)} \leq 2 \cosh(\rho(\gamma_1 x_0, z)) \leq 2e^\eta. \quad (21)$$

we then get a contradiction when

$$\eta < \min \left( \frac{L}{4}, \frac{1}{2} \log \left[ \frac{1}{2} \left( \cosh\left(\frac{L}{2}\right) + \frac{1}{\cosh\left(\frac{L}{2}\right)} \right) \right] \right).$$

□

Let  $\Gamma$  be a finitely generated discrete group of isometries of  $(X, g)$  and  $S = \{\sigma_1, \dots, \sigma_p\}$  be a finite generating set. Let us assume that  $\Gamma$  is not virtually nilpotent and recall that  $L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x)$ . By lemma 3.1 we have  $L(S) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0)$ , for some point  $x_0 \in X$ , and by corollary 2.1,  $L(S) \geq \mu(n, a) > 0$ . Let us recall that for  $0 \leq \eta \leq L$ , an element  $\gamma \in \Gamma$  is said to be  $(L, \eta)$ -straight if

$$\rho(x_0, \gamma x_0) > (L - \eta)l_S(\gamma).$$

In the following two propositions we give conditions under which there are many non  $(L, \eta)$ -straight elements in  $\Gamma$ .

**Proposition 3.1** *Let  $(X, g)$  be a Cartan Hadamard manifold whose sectional curvature satisfies  $-a^2 \leq K \leq -1$  and  $\Gamma$  a discrete non virtually nilpotent group of isometries of  $(X, g)$  generated by  $S = \{\sigma_1, \dots, \sigma_p\}$ . Let us assume that all  $\sigma_i$ 's are elliptic and that for all  $\sigma_i \neq \sigma_j \in S$ , the group  $\langle \sigma_i, \sigma_j \rangle$  fixes a point  $y \in X$  or  $\theta \in \partial X$ . Let  $\eta$  be a positive number such that*

$$\eta < \min \left( \frac{L}{4}, \frac{1}{2} \left( 1 - \frac{1}{\cosh(L)} \right)^2, \frac{1}{2} \log \left[ \frac{1}{2} \left( \cosh \frac{L}{2} + \frac{1}{\cosh \frac{L}{2}} \right) \right] \right),$$

where  $L = L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0)$ , then any  $\gamma \in \Gamma$  with  $l_S(\gamma) = 2$ , i.e  $\gamma = \sigma_i^2$  or  $\gamma = \sigma_i \sigma_j$ , is not  $(L, \eta/2)$ -straight, that is,  $\rho(x_0, \gamma x_0) \leq 2(L - \eta/2)$ .

*Proof.* Consider the case where  $\gamma = \sigma_i^2$ . If  $\sigma_i$  is not  $(L, \eta)$ -straight, we have, by the triangle inequality,  $\rho(x_0, \sigma_i^2 x_0) \leq 2(L - \eta)$ . If  $\sigma_i$  is  $(L, \eta)$ -straight, we have by lemma 3.3,

$$\rho(x_0, \sigma_i^2 x_0) \leq 2L - \left( 1 - \frac{1}{\cosh L} \right)^2 \leq 2(L - \eta).$$

Let us now consider the case where  $\gamma = \sigma_i \sigma_j$ , for  $i \neq j$ . If  $\sigma_i$  or  $\sigma_j$  is not  $(L, \eta)$ -straight, we have, by the triangle inequality,

$$\rho(x_0, \sigma_i \sigma_j x_0) \leq \rho(x_0, \sigma_i x_0) + \rho(x_0, \sigma_j x_0) \leq L + (L - \eta),$$

therefore,  $\rho(x_0, \sigma_i \sigma_j x_0) \leq 2(L - \eta/2)$ .

If  $\sigma_i$  and  $\sigma_j$  are  $(L, \eta)$ -straight, lemma 3.5 implies that  $\rho(x_0, \sigma_i \sigma_j x_0) \leq 2(L - \eta)$ .  $\square$

In the next proposition we will assume that all elements  $\gamma \in \Gamma$  whose algebraic length is less than or equal to 4 have a displacement smaller than  $\delta$  where

$$\delta = \log \left[ \cosh\left(\frac{L}{4}\right) \right], \quad (22)$$

and we set

$$\eta = 10^{-3} \left( 1 - \frac{\cosh\left(\frac{L}{4}\right)}{\cosh\left(\frac{L}{2}\right)} \right)^4. \quad (23)$$

We will find in that case many non  $(L, \eta)$ -straight elements.

**Proposition 3.2** *Let  $(X, g)$  be a Cartan-Hadamard manifold whose sectional curvature satisfies  $a^2 \leq K_g \leq -1$  and  $G$  a discrete non virtually nilpotent group of isometries of  $(X, g)$  generated by a set of two isometries  $\Sigma = \{\sigma_1, \sigma_2\}$ . Let  $L = \inf_{x \in X} \max\{\rho(x, \sigma_1 x), \rho(x, \sigma_2 x)\}$ , and  $\Sigma = \{\sigma_1, \sigma_2\}$ . Let  $\eta$  and  $\delta$  be the numbers defined in (23) and (22). We assume that  $l(\gamma') < \delta$  for all  $\gamma' \in G$  such that  $l_\Sigma(\gamma') < 4$ . Then all elements  $\gamma \in G$  such that  $l_\Sigma(\gamma) = 6$  are not  $(L, \eta)$ -straight.*

We will need the following lemmas.

**Lemma 3.6** *Let  $\gamma = a\gamma'b \in G$  be such that  $l_\Sigma(\gamma) = l_\Sigma(a) + l_\Sigma(\gamma') + l_\Sigma(b)$ . If  $\gamma$  is  $(L, \eta)$ -straight, then  $\gamma'$  is  $(L, C\eta)$ -straight where  $C = \frac{l_\Sigma(\gamma)}{l_\Sigma(\gamma')}$ .*

*Proof.* Let us note that by definition of  $L = L(\Sigma)$ , we have for any  $\gamma \in G$

$$\rho(x_0, \gamma x_0) \leq L \cdot l_\Sigma(\gamma).$$

By triangle inequality we have

$$\rho(x_0, \gamma x_0) \leq \rho(x_0, ax_0) + \rho(x_0, \gamma' x_0) + \rho(x_0, bx_0),$$

hence by assumption on  $\gamma$  we get

$$(L - \eta) l_\Sigma(a\gamma'b) \leq L (l_\Sigma(a) + l_\Sigma(b)) + \rho(x_0, \gamma' x_0)$$

and therefore,

$$\rho(x_0, \gamma' x_0) \geq L l_\Sigma(\gamma') - \eta l_\Sigma(\gamma) \geq (L - C\eta) l_\Sigma(\gamma').$$

$\square$

**Lemma 3.7** *Let  $\alpha, \beta$  be two elements of  $G$  distinct of the neutral element and such that  $l_\Sigma(\alpha) \leq 2$ ,  $l_\Sigma(\beta) \leq 2$ . Under the assumptions of the proposition 3.2, if  $\gamma$  is  $(L, \eta)$ -straight with  $l_\Sigma(\gamma) = 6$ , then any reduced word representing  $\gamma$  does not contain (i)  $\alpha^2$  or (ii)  $\alpha\beta\alpha$ .*

Assuming the lemma 3.7, the proof of the proposition 3.2 can be finished as follows:

*Proof.* Let  $\gamma \in G$  of length  $l_\Sigma(\gamma) = 6$ . Let us write  $\gamma$  as a reduced word in the generators of  $\Sigma$ ,  $\gamma = \sigma_{i_1}^{p_1} \dots \sigma_{i_k}^{p_k}$ , where  $\sigma_{i_j} = \sigma_1$  or  $\sigma_{i_j} = \sigma_2$ ,  $p_j \in \mathbb{Z}^*$ ,  $i_j \neq i_{j+1}$  and  $i_j = i_{j+2}$ . Arguing by contradiction we assume that  $\gamma$  is  $\eta$ -straight. Then, by lemma 3.7 (i), all  $p_j$  are equal to  $+1$  or  $-1$  and in particular we have  $k = 6$ . Therefore  $\gamma = \sigma_{i_1}^{p_1} \cdot \sigma_{i_2}^{p_2} \cdot \sigma_{i_3}^{p_3} \cdot \sigma_{i_4}^{p_4} \cdot \sigma_{i_5}^{p_5} \cdot \sigma_{i_6}^{p_6}$ . By lemma 3.7 (ii) we also have  $p_{j+2} \neq p_j$  hence  $p_{j+2} = -p_j$  so  $\gamma = \sigma_{i_1}^{p_1} \cdot \sigma_{i_2}^{p_2} \cdot \sigma_{i_1}^{-p_1} \cdot \sigma_{i_2}^{-p_2} \cdot \sigma_{i_1}^{p_1} \cdot \sigma_{i_2}^{p_2}$ , which is impossible by lemma 3.7 (ii) with  $\alpha = \sigma_{i_1}^{p_1} \cdot \sigma_{i_2}^{p_2}$  and  $\beta = \sigma_{i_1}^{-p_1} \cdot \sigma_{i_2}^{-p_2}$ . This finishes the proof of proposition 3.2.  $\square$

Let us now prove the lemma 3.7:

*Proof.* We first claim that if  $L$ ,  $\eta$  and  $\delta$  are chosen as in the proposition 3.2 then we have

$$\eta \leq \frac{L}{4000} \quad (24)$$

and

$$12\eta + \text{Argcosh}(e^{12\eta}) \leq \frac{1}{4} \left( 1 - \frac{e^\delta}{\cosh(L/2)} \right). \quad (25)$$

Proof of the claim. By definition of  $\eta$ , (cf. (23)), we have

$$1000\eta = \left( 1 - \frac{\cosh(L/4)}{\cosh(L/2)} \right)^4$$

therefore

$$1000\eta < \frac{\cosh(L/2) - \cosh(L/4)}{\cosh(L/2)}$$

and

$$1000\eta < \frac{\sinh(L/2) \cdot L/4}{\cosh(L/2)} < \frac{L}{4},$$

which proves the first inequality of the claim. On the other hand, let  $x \in ]0, 1[$ , then  $e^x \leq 1+2x \leq \cosh(2\sqrt{x})$ . Choosing  $x = 12\eta$  we obtain, using the inequality  $\eta < \frac{1}{1000}$ , that

$$12\eta + \text{Argcosh}(e^{12\eta}) \leq 12\eta + 2\sqrt{12\eta} < \frac{1}{4}\sqrt{1000\eta}$$

therefore we get

$$12\eta + \text{Argcosh}(e^{12\eta}) \leq \frac{1}{4} \left( 1 - \frac{\cosh(L/4)}{\cosh(L/2)} \right)^2 \leq \frac{1}{4} \left( 1 - \frac{e^\delta}{\cosh(L/2)} \right),$$

which ends the proof of the claim.

*Proof of lemma 3.7 (i).* Let us assume that  $\gamma = a\alpha^2b$  is  $(L, \eta)$ -straight, and  $l_\Sigma(\gamma) = 6$ . Then by lemma 3.6  $\alpha$  and  $\alpha^2$  are  $(L, 3\eta)$ -straight. We then get with (24)

$$\rho(x_0, \alpha x_0) > (L - 3\eta)l_\Sigma(\alpha) > \frac{L}{2}.$$

On the other hand since  $l(\alpha) \leq \delta$  and  $\text{Argcosh}[e^\delta] = \frac{L}{4} < \frac{L}{2}$ , we can apply lemma 3.3 to  $\alpha$  replacing  $L$  by  $L/2$  and get

$$\rho(x_0, \alpha^2 x_0) < 2\rho(x_0, \alpha x_0) - \left(1 - \frac{e^\delta}{\cosh(L/2)}\right)^2.$$

We then get by the choice of  $\eta$ , cf. (23),

$$\rho(x_0, \alpha^2 x_0) < (L - 3\eta) l_\Sigma(\alpha^2)$$

which contradicts the fact that  $\alpha^2$  is  $(L, 3\eta)$ -straight and concludes the proof of lemma 3.7 (i).

*Proof of lemma 3.7 (ii).* Let us assume that  $\gamma = \alpha\beta\alpha b$  is  $(L, \eta)$ -straight, and  $l_\Sigma(\gamma) = 6$ . The Lemma 3.6 says that  $\alpha\beta\alpha$  is  $(L, 2\eta)$ -straight and that  $\alpha\beta$  is  $(L, C'\eta)$ -straight where  $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$ . Since  $\alpha\beta\alpha$  is  $(L, 2\eta)$ -straight, we have by triangle inequality

$$(L - 2\eta)l_\Sigma(\alpha\beta\alpha) \leq 2\rho(x_0, \alpha x_0) + L l_\Sigma(\beta)$$

and therefore

$$2\rho(x_0, \alpha x_0) \geq (L - 2\eta)l_\Sigma(\alpha\beta\alpha) - L l_\Sigma(\beta) = L l_\Sigma(\alpha^2) - 2\eta l_\Sigma(\alpha\beta\alpha)$$

hence we obtain

$$\rho(x_0, \alpha x_0) \geq L l_\Sigma(\alpha) - \eta l_\Sigma(\alpha\beta\alpha),$$

and since  $l_\Sigma(\alpha) \leq 2$  and  $l_\Sigma(\beta) \leq 2$ , we deduce that

$$\rho(x_0, \alpha x_0) \geq (L - 4\eta) l_\Sigma(\alpha),$$

that is  $\alpha$  is  $(L, 4\eta)$ -straight. We set  $x_1 = \alpha\beta x_0$ ,  $x_2 = \alpha\beta\alpha x_0$  and  $x_3 = (\alpha\beta)^2 x_0 = \alpha\beta\alpha\beta x_0$ . We get, since  $\alpha\beta\alpha$  is  $(L, 2\eta)$ -straight,

$$\begin{aligned} \rho(x_0, x_1) + \rho(x_1, x_2) - \rho(x_0, x_2) &= \rho(x_0, \alpha\beta x_0) + \rho(x_0, \alpha x_0) - \rho(x_0, \alpha\beta\alpha x_0) \\ &\leq L[l_\Sigma(\alpha\beta) + l_\Sigma(\alpha)] - (L - 2\eta)l_\Sigma(\alpha\beta\alpha) \\ &\leq 12\eta. \end{aligned}$$

In the same way, since  $\alpha\beta$  is  $(L, C'\eta)$ -straight with  $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$ , we have

$$\begin{aligned} \rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_1, x_3) &= \rho(x_0, \alpha x_0) + \rho(x_0, \beta x_0) - \rho(x_0, \alpha\beta x_0) \\ &\leq L[l_\Sigma(\alpha) + l_\Sigma(\beta)] - (L - C'\eta)l_\Sigma(\alpha\beta) \\ &\leq 2\eta l_\Sigma(\alpha\beta\alpha) \\ &\leq 12\eta. \end{aligned}$$

We can therefore apply the lemma 3.4 and get

$$\begin{aligned} 2\rho(x_0, \alpha\beta x_0) - \rho(x_0, (\alpha\beta)^2 x_0) &\leq \rho(x_0, \alpha\beta x_0) + \rho(x_0, \alpha x_0) + \rho(x_0, \beta x_0) - \rho(x_0, (\alpha\beta)^2 x_0) \\ &= \rho(x_0, x_1) + \rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_0, x_3) \\ &\leq \left(1 + \frac{\rho(x_0, \beta x_0)}{\rho(x_0, \alpha x_0)}\right) (12\eta + \text{Argcosh}[e^{12\eta}]) \\ &\leq \left(1 + \frac{L l_\Sigma(\beta)}{(L - 4\eta)l_\Sigma(\alpha)}\right) \cdot \frac{1}{4} \left(1 - \frac{e^\delta}{\cosh(L/2)}\right)^2, \end{aligned}$$



the last inequality coming from (25) and the fact that  $\alpha$  is  $(L, 4\eta)$ -straight. From (24) we therefore get

$$\rho(x_0, (\alpha\beta)^2 x_0) \geq 2\rho(x_0, \alpha\beta x_0) - \left(1 - \frac{e^\delta}{\cosh(L/2)}\right)^2. \quad (26)$$

On the other hand we have seen that  $\alpha\beta$  is  $(L, C'\eta)$ -straight with  $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$ , so that

$$\rho(x_0, \alpha\beta x_0) \geq (L - C'\eta) l_\Sigma(\alpha\beta) \geq 2L - 2\eta l_\Sigma(\alpha\beta\alpha),$$

and since  $l_\Sigma(\alpha\beta\alpha) \leq 6$  the above inequality gives with (24)

$$\rho(x_0, \alpha\beta x_0) \geq L. \quad (27)$$

By assumption, since  $l_\Sigma(\alpha\beta) \leq 4$ , the displacement of  $\alpha\beta$  satisfies  $l(\alpha\beta) \leq \delta$ , and with (27) we can apply the lemma 3.3 to get

$$\rho(x_0, (\alpha\beta)^2 x_0) \leq 2\rho(x_0, \alpha\beta x_0) - \left(1 - \frac{e^\delta}{\cosh(L/2)}\right)^2$$

which contradicts (26). This concludes the proof of the lemma 3.7 and the proposition 3.2.  $\square$

## 4 Mapping the Cayley graph of $G$ into $X$ .

Let  $G$  be a finitely generated discrete group of isometries of  $(X, g)$  a Cartan Hadamard manifold of sectional curvature  $-a^2 \leq K \leq -1$ . We consider  $S$  a finite generating set of  $G$  and the Cayley graph  $\mathcal{G}_S$  of  $G$  associated to  $S$ . We define a distance  $d_S$  on  $\mathcal{G}_S$  in the following way: each edge is isometric to the segment  $[0, 1] \subset \mathbb{R}$  and the distance  $d_S(\gamma, \gamma')$  between two vertices  $\gamma, \gamma'$  of  $\mathcal{G}_S$  is the word distance  $d_S(\gamma, \gamma') = l_S(\gamma^{-1}\gamma')$ . The group  $G$  acts by isometries on  $(\mathcal{G}_S, d_S)$  and on  $(X, g)$ . The goal of this section is to construct for each number  $c$  large enough an equivariant map  $f_c : \mathcal{G}_S \rightarrow X$  such that  $f_c$  is Lipschitzian of Lipschitz constant  $c$ .

### 4.1 Poincaré series, measures and convexity.

We first consider the Poincaré series,

$$P_c(s, x, y) = \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \cosh[\rho(x, \gamma y)] \quad (28)$$

where  $c \in \mathbb{R}_+$ ,  $s \in \mathcal{G}_S$  and  $x, y \in X$ .

**Lemma 4.1** *For all  $s \in \mathcal{G}_S$ ,  $x, y, x_0, y_0 \in X$ ,  $c \in \mathbb{R}$  and  $\gamma_0 \in G$  we have*

- (i)  $P_c(\gamma_0 s, \gamma_0 x, y) = P_c(s, x, y)$
- (ii)  $P_c(s, x, y) \leq P_c(s, x_0, y_0) e^{\rho(x_0, x) + \rho(y_0, y)}$ .

*In particular the convergence of the series is independant of the choice of the points  $x, y \in X$ .*

*Proof.* The equivariance property of the Poincaré series is straightforward. On the other hand by triangle inequality we have

$$\begin{aligned} P_c(s, x, y) &= \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \cosh [\rho(x, \gamma y)] \\ &\leq \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \cosh [\rho(x_0, \gamma y_0) + \rho(x_0, x) + \rho(y_0, y)] \end{aligned}$$

hence we get

$$P_c(s, x, y) \leq P_c(s, x_0, y_0) \cdot e^{\rho(x_0, x) + \rho(y_0, y)}.$$

□

The critical exponent of this series is defined as

$$c_0 =: \inf \{ c > 0 \mid P_c(s, x, y) < \infty \}.$$

Let  $x_0$  be the point of  $X$  such that  $L(S) = \max_i \{ \rho(x_0, \sigma_i x_0) \}$ . By the triangle inequality we have for all  $\gamma \in \Gamma$ ,  $\rho(x_0, \gamma x_0) \leq L(S) l_S(\gamma)$ , therefore

$$P_c(e, x_0, x_0) \leq \sum_{\gamma \in \Gamma} e^{(c-L(S)) l_S(\gamma)}.$$

On the other hand, by definition of  $\text{Ent}_S \Gamma$ , we have  $\sum_{\gamma \in \Gamma} e^{-t l_S(\gamma)} < \infty$  for all  $t > \text{Ent}_S \Gamma$ , hence we have proved that

$$c_0 \leq \text{Ent}_S \Gamma + L(S). \quad (29)$$

**We now consider until the end of this section a  $c \in \mathbb{R}_+$  such that  $P_c(s, x, y) < \infty$ .**

Let us choose a probability measure  $\mu$  with smooth density and compact support on  $X$ . For each  $s \in \mathcal{G}_S$  let us define the measure on  $X$

$$\mu_s^c = \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \gamma_* \mu \quad (30)$$

and the function  $\mathcal{B}^c : G \times X \rightarrow \mathbb{R}$ ,

$$\mathcal{B}^c(s, x) = \int_X \cosh [\rho(x, z)] d\mu_s^c(z). \quad (31)$$

In the following lemmas 4.2, 4.3 and corollary 4.1 we show that  $x \rightarrow \mathcal{B}^c(s, x)$  is a strictly convex  $C^2$  function such that

$$\lim_{x \rightarrow \infty} \mathcal{B}^c(s, x) = +\infty.$$

**Lemma 4.2** *Let  $c$  be such that  $P_c(s, x, y) < \infty$ . For all  $s \in \mathcal{G}_S$  and  $x \in X$ , we have  $\mathcal{B}^c(s, x) < \infty$ . Moreover, the function  $x \rightarrow \mathcal{B}^c(s, x)$  is strictly convex and  $\lim_{x \rightarrow \infty} \mathcal{B}^c(s, x) = +\infty$ .*

*Proof.* By definition of  $\mu_s^c$ ,

$$\mathcal{B}^c(s, x) = \int_X \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \cosh [\rho(x, \gamma z)] d\mu(z) = \int_X P_c(s, x, z) d\mu(z),$$

so we get  $\mathcal{B}^c(s, x) < \infty$  by lemma 4.1 (ii) since the support of  $\mu$  is compact. For any geodesic  $c(t)$  and  $z$  in  $X$ ,  $t \rightarrow d(c(t), z)$  is a convex function since  $(X, g)$  has negative sectional curvature, therefore  $t \rightarrow \cosh [\rho(c(t), z)]$  is strictly convex and so is  $x \rightarrow \mathcal{B}^c(s, x) = \int_X \cosh [\rho(x, z)] d\mu_s^c(z)$ . On the other hand we have

$$\mathcal{B}^c(s, x) = \int_X \cosh [\rho(x, z)] d\mu_s^c(z) \geq \frac{1}{2} e^{\rho(x, x_0)} \int_X e^{-\rho(x_0, z)} d\mu_s^c(z),$$

so  $\mathcal{B}^c(s, x) \rightarrow +\infty$  whenever  $x$  tends to infinity in  $X$ .  $\square$

In the above lemma we proved that  $x \rightarrow \mathcal{B}^c(s, x)$  is a convex function which tends to  $+\infty$  when  $x$  tend to infinity. We shall now prove that  $x \rightarrow \mathcal{B}^c(s, x)$  is a strictly convex  $C^2$  function. We will also give estimates of the second derivative of  $x \rightarrow \mathcal{B}^c(s, x)$ .

**Lemma 4.3** *Let  $c$  be such that  $P_c(s, x, y) < \infty$ . The function  $x \rightarrow \mathcal{B}^c(s, x)$  is  $C^2$  and for any  $s \in \mathcal{G}_S$ ,  $x \in X$  and any tangent vectors  $v, w \in T_x X$  we have*

$$d\mathcal{B}^c(s, x)(v) = \int_X d\rho(x, z)(v) \sinh [\rho(x, z)] d\mu_s^c(z)$$

and

$$Dd\mathcal{B}^c(s, x)(v, w) = \int_X \left( \sinh [\rho(x, z)] Dd\rho(x, z)(v, w) + \cosh [\rho(x, z)] d\rho(x, z) \otimes d\rho(x, z)(v, w) \right) d\mu_s^c(z)$$

*Proof.* Let  $v \in T_x X$  be a unit tangent vector at a point  $x \in X$ . For each point  $z \neq x$  in  $X$ , we have

$$d(\cosh [\rho(x, z)])(v) = d\rho(x, z)(v) \sinh [\rho(x, z)],$$

hence we get

$$|d(\cosh [\rho(x, z)])(v)| = |d\rho(x, z)(v) \sinh [\rho(x, z)]| \leq \cosh [\rho(x, z)], \quad (32)$$

therefore,  $\cosh [\rho(x, z)] \leq 2 \cosh [\rho(x_1, z)]$  for  $x$  in a sufficiently small neighbourhood of an arbitrary point  $x_1$ . Since  $z \rightarrow 2 \cosh [\rho(x_1, z)]$  is  $\mu_s^c$ -integrable, we can differentiate  $x \rightarrow \mathcal{B}^c(s, x)$  applying Lebesgue derivation theorem. Let us now compute the second derivative. Let  $v, w \in T_x X$  be unit tangent vectors at  $x \in X$ . Let  $\alpha(t)$  the geodesic such that  $\alpha(0) = x$  and  $\alpha'(0) = v$ . We denote  $W(t)$  the parallel vector field along  $\alpha$  such that  $W(0) = w$  and write  $\rho_{(z, \alpha(t))}$  instead of  $\rho(z, \alpha(t))$ . Let us denote

$$h(t, z) = \frac{1}{t} \left( d\rho_{(z, \alpha(t))}(W(t)) \sinh [\rho_{(z, \alpha(t))}] - d\rho_{(z, \alpha(0))}(W(0)) \sinh [\rho_{(z, \alpha(0))}] \right).$$

When  $z \neq x$  we have

$$\lim_{t \rightarrow 0} h(t, z) = \sinh [\rho_{(x, z)}] Dd\rho_{(x, z)}(v, w) + \cosh [\rho_{(x, z)}] d\rho_{(x, z)} \otimes d\rho_{(x, z)}(v, w). \quad (33)$$

We will write

$$h_0(z) =: \lim_{t \rightarrow 0} h(t, z). \quad (34)$$

The formula which gives  $Dd\mathcal{B}^c(s, x)(v, w)$  in lemma 4.3 is equivalent to

$$Dd\mathcal{B}^c(s, x)(v, w) = \int_X h_0(z) d\mu_s^c(z) \quad (35)$$

and will be a consequence of the dominated convergence theorem of Lebesgue with the existence of a  $\mu_s^c(z)$ -integrable function  $H(z)$  such that for any  $z \notin \alpha([0, t])$  then  $h(t, z) \leq H(z)$ . Let us now prove the existence of such a function  $H$ . For each  $z \notin \alpha([0, t])$  we have

$$\begin{aligned} |h(t, z)| \leq & \sup_{t' \in [0, t]} \left| \sinh[\rho_{(z, \alpha(t'))}] Dd\rho_{(z, \alpha(t'))}(\dot{\alpha}(t'), W(t')) + \dots \right. \\ & \left. \dots + \cosh[\rho_{(z, \alpha(t'))}] d\rho_{(z, \alpha(t'))} \otimes d\rho_{(z, \alpha(t'))}(\dot{\alpha}(t'), W(t')) \right|. \end{aligned}$$

Since the curvature of  $(X, g)$  satisfies  $-a^2 \leq K \leq -1$ , the Rauch comparison theorem shows that for each  $x, y \in X$

$$Dd\rho_{(x, y)} \leq a \frac{\cosh[a\rho_{(x, y)}]}{\sinh[a\rho_{(x, y)}]} \left( g - d\rho_{(x, y)} \otimes d\rho_{(x, y)} \right),$$

hence we get from the previous inequality

$$\begin{aligned} |h(t, z)| \leq & \left[ a \sinh[\rho_{(z, \alpha(t'))}] \frac{\cosh[a\rho_{(z, \alpha(t'))}]}{\sinh[a\rho_{(z, \alpha(t'))}]} \left( g - d\rho_{(z, \alpha(t'))} \otimes d\rho_{(z, \alpha(t'))} \right) + \dots \right. \\ & \left. \dots + \cosh[\rho_{(z, \alpha(t'))}] d\rho_{(z, \alpha(t'))} \otimes d\rho_{(z, \alpha(t'))} \right] (\dot{\alpha}(t'), W(t')). \end{aligned}$$

But since  $a \geq 1$  the concavity of the function  $\tanh$  on  $\mathbb{R}_+$  gives

$$\frac{a}{\tanh a\rho} \geq \frac{1}{\tanh \rho}$$

therefore we get

$$|h(t, z)| \leq a \sinh[\rho_{(z, \alpha(t'))}] \frac{\cosh[a\rho_{(z, \alpha(t'))}]}{\sinh[a\rho_{(z, \alpha(t'))}]} \quad (36)$$

Since  $\sinh \rho \leq \frac{1}{a} \sinh a\rho$  by convexity of  $\sinh$ , we then get that  $|h(t, z)| \leq H(z)$  from 36 for all  $|t| \leq \frac{1}{a}$  and all  $z \notin \alpha([0, t])$  where

$$H(z) = \begin{cases} a \frac{\cosh 1}{\sinh 1} \sinh[\rho_{(z, \alpha(0))} + 1], & \rho_{(z, \alpha(0))} \geq \frac{2}{a} \\ \cosh[a\rho_{(z, \alpha(0))} + 1], & \rho_{(z, \alpha(0))} < \frac{2}{a} \end{cases}$$

This concludes the proof of lemma 4.3.  $\square$

The above lemma 4.3 has the following

**Corollary 4.1** *Under the assumptions of lemma 4.3 we have*

$$Dd\mathcal{B}^c \geq \mathcal{B}^c .g,$$

*in particular,  $\mathcal{B}^c$  is strictly convex.*

*Proof.* Since the sectional curvature of  $(X, g)$  satisfies  $K \leq -1$  Rauch's theorem shows that

$$Dd\rho \geq \frac{1}{\tanh \rho} (g - d\rho \otimes d\rho).$$

From this inequality and lemma 4.3 we therefore get, for all  $x \in X$  and any unit tangent vector  $v \in T_x X$ ,

$$Dd\mathcal{B}^c(v, v) \geq \left( \int_X \cosh[\rho_{(z,x)}] d\mu_s^c(z) \right) g(v, v) = \mathcal{B}^c(x) .g(v, v).$$

□

## 4.2 Construction of Lipschitzian maps $f_c : \mathcal{G}_S \rightarrow X$ .

So far we have shown that for any  $s \in \mathcal{G}_S$  the function  $x \rightarrow \mathcal{B}^c(s, x)$  is strictly convex and tends to  $+\infty$  when  $x$  tend to infinity. We then can define the map  $f_c : \mathcal{G}_S \rightarrow X$  as follows. For  $s \in \mathcal{G}_S$  we define  $f_c(s)$  as the unique point  $x \in X$  which achieves the strict minimum of the function  $x \rightarrow \mathcal{B}^c(s, x)$ . The end of this section is devoted to proving the following

**Proposition 4.1** *Let  $c$  be such that  $P_c(s, x, y) < \infty$ . Let  $f_c : (\mathcal{G}_S, d_S) \rightarrow (X, g)$  which associates to  $s \in \mathcal{G}_S$  the unique point  $x \in X$  which achieves the unique minimum of the function  $x \rightarrow \mathcal{B}^c(s, x)$ . Then,  $f_c$  is Lipschitzian of Lipschitz constant equal to  $c$ .*

The proof of the proposition 4.1 relies on the following technical lemmas.

**Lemma 4.4** *Let  $c$  be such that  $P_c(s, x, y) < \infty$ . For all  $x \in X$  and all tangent vector  $v \in T_x X$  the function  $\alpha : s \rightarrow d\mathcal{B}^c(s, x)(v)$  is differentiable at each point  $s \in \mathcal{G}_S$  distinct from a vertex or a middle point of an edge. Moreover, for such an  $s$  we have*

$$\alpha'(s) = -c \int_X d\rho_{(x,z)}(v) \sinh[\rho(x, z)] \sum_{\gamma \in G} \frac{d}{ds} (d_S(s, \gamma)) e^{-cd_S(s, \gamma)} d(\gamma_* \mu)(z)$$

*Proof.* Let us denote by  $[g, g']$  the edge containing  $s$  and parametrize it by  $t \in [0, 1]$ . We first observe that for all  $\gamma \in G$  then

$$d_S(s, \gamma) = \min [d_S(g, \gamma) + t, d_S(g', \gamma) + 1 - t],$$

therefore  $s \rightarrow d_S(s, \gamma)$  is differentiable at each  $s \in ]g, g'[$  distinct of the middle point of  $]g, g'[$ . On the other hand we have by lemma 4.3

$$d\mathcal{B}^c(s, x)(v) = \int_X d\rho_{(x,z)}(v) \sinh[\rho(x, z)] d\mu_s^c(z),$$

so that we can write

$$\frac{1}{t}(\alpha(s+t) - \alpha(s)) = \sum_{\gamma \in G} \int_X d\rho_{(x,\gamma z)}(v) \sinh[\rho(x,\gamma z)] \cdot \frac{1}{t} [e^{-cd_S(s+t,\gamma)} - e^{-cd_S(s,\gamma)}] d\mu(z) \quad ,$$

where we have identified the point  $s$  in the edge  $[g, g']$  with its parameter. Let us observe that for  $|t|$  small enough,

$$\left| \frac{1}{t} [e^{-cd_S(s+t,\gamma)} - e^{-cd_S(s,\gamma)}] \right| \leq 2c e^{-cd_S(s,\gamma)} \quad ,$$

and that

$$2c \sum_{\gamma \in G} \int_X |d\rho_{(x,\gamma z)}(v)| \sinh[\rho(x,\gamma z)] \cdot e^{-cd_S(s,\gamma)} d\mu(z) < \infty \quad ,$$

hence if  $s \in \mathcal{G}_S$  is distinct from a vertex or a middle point of an edge we get

$$\lim_{t \rightarrow 0} \frac{1}{t}(\alpha(s+t) - \alpha(s)) = -c \int_X d\rho_{(x,z)}(v) \sinh[\rho(x,z)] \sum_{\gamma \in G} \frac{d}{ds}(d_S(s,\gamma)) e^{-cd_S(s,\gamma)} d(\gamma_*\mu)(z)$$

by Lebesgue's theorem. □

**Lemma 4.5** *Let  $c$  be such that  $P_c(s, x, y) < \infty$ . Let  $s_0 \in \mathcal{G}_S$  be a point distinct from a vertex or a middle point of an edge, and  $u$  a unit vector tangent at  $s_0$  to the edge containing  $s_0$ . Then, we have  $\|df_c(u)\| \leq c$ .*

*Proof.* Let us fix a smooth moving frame  $\{E_1, \dots, E_n\}$  of  $TX$  and define the function  $\Phi : X \times \mathcal{G}_S \rightarrow \mathbb{R}^n$  by

$$\Phi(x, s) = (d\mathcal{B}^c(s, x)(E_1), \dots, d\mathcal{B}^c(s, x)(E_n)) \quad .$$

By definition, the point  $f_c(s)$  is characterized by the implicit equation

$$\Phi(f_c(s), s) = 0 \quad ,$$

or equivalently,

$$d\mathcal{B}^c(s, f_c(s)) = 0 \quad .$$

For all  $x \in X$  and  $s \in \mathcal{G}_S$  in a neighbourhood of  $s_0$  the function  $\Phi$  is differentiable by lemma 4.3 and 4.4. Moreover since  $x = f_c(s)$  is a critical point of the function  $x \rightarrow \mathcal{B}^c(s, x)$ , we have, for  $j = 1, \dots, n$ ,

$$\frac{\partial \Phi}{\partial x}(f_c(s), s)(E_j) = (Dd\mathcal{B}^c(s, f_c(s))(E_j, E_1), \dots, Dd\mathcal{B}^c(s, f_c(s))(E_j, E_n)) \quad ,$$

thus  $\frac{\partial \Phi}{\partial x}(f_c(s), s)$  is invertible by corollary 4.1. By the implicit function theorem, the function  $f_c$  is then differentiable at  $s$  in a neighbourhood of  $s_0$  and we have,

if  $u$  is a unit vector tangent at  $s_0$  to the edge containing  $s_0$  and  $v$  a tangent vector in  $T_{f_c(s)}X$ ,

$$Dd\mathcal{B}^c(s_0, f_c(s_0))(df_c(u), v) = -\frac{d}{ds}\Big|_{s=s_0} d\mathcal{B}^c(s, f_c(s_0))(v). \quad (37)$$

From corollary 4.1 and lemma 4.4 we obtain, setting  $v = \frac{df_c(u)}{\|df_c(u)\|}$

$$g(df_c(u), v) \mathcal{B}^c(s_0, f_c(s_0)) \leq c \int_X |d\rho_{(f_c(s_0), z)}(v)| \sinh[\rho(f_c(s_0), z)] \Sigma_{\gamma \in G} \Big| \frac{d}{ds}\Big|_{s=s_0} (d_S(s, \gamma)) \Big| e^{-cd_S(s_0, \gamma)} d(\gamma_*\mu)(z)$$

therefore

$$|g(df_c(u), v) \mathcal{B}^c(s_0, f_c(s_0))| \leq c \int_X \sinh[\rho(f_c(s_0), z)] d\mu_{s_0}^c(z). \quad (38)$$

hence

$$\|df_c(u)\| \leq c \frac{\int_X \sinh[\rho(f_c(s_0), z)] d\mu_{s_0}^c(z)}{\int_X \cosh[\rho(f_c(s_0), z)] d\mu_{s_0}^c(z)} \leq c,$$

which completes the proof of lemma 4.5.  $\square$

The proposition 4.1 follows then from the

**Corollary 4.2** *Let  $c$  be such that  $P_c(s, x, y) < \infty$ . The map  $f_c$  is Lipschitzian of Lipschitz constant equal to  $c$ .*

*Proof.* Let us consider a segment  $[s_1, s_2] \subset \mathcal{G}_S$  which contains no edges nor middle points. It directly follows from lemma 4.5 that

$$\rho(f_c(s_1), f_c(s_2)) \leq c d_S(s_1, s_2). \quad (39)$$

We now want to extend the inequality (39) for all points  $s_1, s_2 \in \mathcal{G}_S$ . For that purpose we first consider a segment  $[s_1, s_2] \subset \mathcal{G}_S$  where  $s_1$  is a midpoint of an edge  $e$  and  $s_2$  a vertex of the same edge  $e$  and the inequality (39) for these points  $s_1, s_2$  derives from the continuity of  $f_c$  at  $x_1$  and  $x_2$ . The corollary 4.2 will then follow from the fact that any segment  $[s_1, s_2] \subset \mathcal{G}_S$  can be decomposed in a finite sequence of adjacent intervals  $[y_1^k, y_2^k]$  where  $y_1^k$  is a midpoint and  $y_2^k$  a vertex of one same edge or the other way around. Let us prove the continuity of  $f_c$  at a vertex or a midpoint  $s$  of an edge. Given such a point  $s$ , let  $\{s_k\}_{k \in \mathbb{N}}$  be a sequence converging to  $s$  and staying in a single mid-edge containing  $s$ . The sequence  $x_k =: f_c(s_k)$  is a Cauchy sequence in  $X$  by (39) whose limit is a point  $x = \lim_k x_k$ . We want to prove that  $f_c(s) = x$ . For all  $z \in X$  and  $k \in \mathbb{N}$  we have

$$\mathcal{B}^c(s_k, z) \geq \mathcal{B}^c(s_k, x_k) \quad (40)$$

by definition of  $x_k = f_c(s_k)$ . We claim that  $\lim_k \mathcal{B}^c(s_k, x_k) = \mathcal{B}^c(s, x)$  and that  $\lim_k \mathcal{B}^c(s_k, z) = \mathcal{B}^c(s, z)$ . Assuming the claim and passing to the limit in (40) when  $k$  tends to infinity gives for all  $z \in X$ ,

$$\mathcal{B}^c(s, z) \geq \mathcal{B}^c(s, x), \quad (41)$$

therefore  $x = f_c(s)$ . Let us prove the claim. By definition (31) and (30), we have

$$\begin{aligned}\mathcal{B}^c(s_k, x_k) &= \int_X \cosh[\rho(x_k, z)] d\mu_{s_k}^c(z) \\ &= \int_X \sum_{\gamma \in G} e^{-cd_S(s_k, \gamma)} \cosh[\rho(x_k, \gamma z)] d\mu(z),\end{aligned}$$

Since  $e^{-cd_S(s_k, \gamma)} \cosh[\rho(x_k, \gamma z)] \leq e^c e^{-cd_S(s, \gamma)} \cosh[\rho(x, \gamma z) + 1]$  for  $k$  large enough, we get  $\lim_k \mathcal{B}^c(s_k, x_k) = \mathcal{B}^c(s, x)$  by the Lebesgue's theorem. By the way we get  $\lim_k \mathcal{B}^c(s_k, z) = \mathcal{B}^c(s, z)$  which concludes the proof of the claim, the corollary 4.2 and the proposition 4.1.  $\square$

## 5 Algebraic Entropy and $\eta$ -straight isometries.

Let  $G$  be a finitely generated discrete group of isometries of  $(X, g)$  whose sectional curvature satisfies  $-a^2 \leq K_g \leq -1$ , and  $S = \{\sigma_1, \dots, \sigma_p\}$  be a finite generating set.

We assume that the minimal displacement  $L(S) = \inf_{x \in X} \max_{i=1, \dots, p} \rho(x, \sigma_i x)$  of  $S$  (cf. definition 2.1) satisfies  $L(S) > 0$ . By lemma 3.1 there exist a point  $x_0 \in X$  such that

$$L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0).$$

The goal of this section is to prove that if all elements of  $G$  are “almost non  $\eta$ -straight” for some  $\eta$  such that  $L(S) > \eta > 0$ , then the entropy of  $G$  with respect to  $S$  is bounded below by  $\eta$ . By “almost non  $\eta$ -straight” elements we mean isometries  $\gamma$  such that  $\rho(x_0, \gamma x_0) \leq (L(S) - \eta)l_S(\gamma) + D$ , for some positive number  $D$ .

**Theorem 5.1** *Let  $G$  be a finitely generated discrete group of isometries of  $(X, g)$  whose sectional curvature satisfies  $-a^2 \leq K_g \leq -1$ , and  $S = \{\sigma_1, \dots, \sigma_p\}$  be a finite generating set of  $G$  with  $L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0) > 0$ . Let us assume that there exist  $D \geq 0$  and  $\eta$ ,  $0 < \eta < L(S)$ , such that for all  $\gamma \in G$ ,*

$$\rho(x_0, \gamma x_0) \leq (L(S) - \eta)l_S(\gamma) + D, \quad (42)$$

then  $\text{Ent}_S(G) \geq \eta$ .

*Proof.* The proof relies on the construction made in section 3 of an equivariant Lipschitzian map of Lipschitz constant  $c > \text{Ent}_S(G) + L(S) - \eta$ .

Let us prove that under the assumption (42) then for any  $c > \text{Ent}_S(G) + L(S) - \eta$  we have  $P_c(s, x, y) < \infty$ . By triangle inequality we have

$$e^{-cd_S(s, \gamma)} \leq e^{cd_S(s, e)} e^{-cd_S(\gamma, e)},$$



and for any  $x_0 \in X$

$$\cosh [\rho(x, \gamma y)] \leq e^{\rho(x, \gamma y)} \leq e^{\rho(x, x_0) + \rho(x_0, y) + \rho(x_0, \gamma x_0)}.$$

Therefore for  $x_0$ ,  $D$  and  $\eta$  chosen such that (42) holds, we get

$$P_c(s, x, y) \leq e^{D+c d_S(e, s) + \rho(x, x_0) + \rho(x_0, y) \sum_{\gamma \in G} e^{[L(S) - \eta - c] d_S(e, \gamma)}},$$

and so  $P_c(s, x, y) < \infty$  for each  $c > \text{Ent}_S(G) + L(S) - \eta$ .

Hence by section 3, proposition 4.1 there exists an equivariant Lipschitzian map  $f_c : (\mathcal{G}_S, d_S) \rightarrow (X, g)$  of Lipschitz constant  $c$  for any  $c > \text{Ent}_S(G) + L(S) - \eta$ . We consider the point  $x = f_c(e)$ , where  $e$  is the neutral element of  $G$ . By definition of  $L(S)$ , there is a  $\sigma_i \in S$  such that  $\rho(x, \sigma_i x) \geq L(S)$ . Therefore, by equivariance,

$$\rho(f_c(e), \sigma_i(f_c(e))) = \rho(f_c(e), f_c(\sigma_i(e))) \geq L(S).$$

On the other hand, since  $f_c$  is  $c$ -Lipschitzian we have

$$\rho(f_c(e), f_c(\sigma_i(e))) \leq c d_S(e, \sigma_i(e)) = c.$$

The two above inequalities give

$$c \geq L(S)$$

and since  $c$  is any number such that  $c > \text{Ent}_S(G) + L(S) - \eta$ , we get  $\text{Ent}_S(G) \geq \eta$ .

□

## 6 Proof of the main theorem.

In this section we shall first prove that the entropy of a group with respect to a set of two generators with displacement  $L > 0$  is bounded below. Then we shall prove the main theorem.

**Proposition 6.1** *Let  $(X, g)$  be a Cartan-Hadamard manifold whose sectional curvature satisfies  $a^2 \leq K_g \leq -1$  and  $G$  a discrete group of isometries of  $(X, g)$  generated by two isometries  $\{\sigma_1, \sigma_2\}$ . Let us assume*

$$L = \inf_{x \in X} \max\{\rho(x, \sigma_1 x), \rho(x, \sigma_2 x)\} > 0.$$

*Then the entropy of  $G$  relatively to the set of generators  $\Sigma = \{\sigma_1, \sigma_2\}$  satisfies*

$$\text{ent}_\Sigma G \geq \min \left[ \frac{\log(\cosh(\frac{L}{4}))}{5 + \log(\cosh(\frac{L}{4}))} \frac{\log 2}{6}, \frac{1}{1000} \left( 1 - \frac{\cosh(\frac{L}{4})}{\cosh(\frac{L}{2})} \right)^4 \right].$$

*Proof.* Let  $\delta = \log \cosh(\frac{L}{4})$ . The proof divides into two cases. In the first case we can find two elements in  $G$  of bounded length  $l_\Sigma$  which are hyperbolic with distinct axes and displacement larger than  $\delta$ . In that case, a ping-pong argument

shows that the semigroup generated by these two elements (or their inverse) is free with corresponding entropy bounded below by a constant depending on  $\delta$ . In the second case, when we cannot find such a free semigroup, then we can show that all elements of  $G$  are almost non  $\eta$ -straight for some  $\eta = \eta(\delta, L)$  and we will conclude by theorem 5.1.

**Case 1.** There exists an element  $\gamma \in G$  of algebraic length  $l_\Sigma(\gamma) \leq 4$  whose displacement  $l(\gamma)$  in  $X$  is bounded below  $l(\gamma) > \delta$ .

**Case 2.** The displacement of all elements  $\gamma \in G$  of algebraic length  $l_\Sigma(\gamma) \leq 4$  satisfies  $l(\gamma) \leq \delta$ .

In the case 1, let us consider an element  $\gamma \in G$  of algebraic length  $l_\Sigma(\gamma) \leq 4$  and whose displacement  $l(\gamma)$  in  $X$  satisfies  $l(\gamma) > \delta$ . We note that  $\gamma$  is then an hyperbolic isometry of  $X$ . Since  $G$  is not virtually nilpotent one of the generators  $\sigma_1$  or  $\sigma_2$ , say  $\sigma_1$  does not preserve the axis of  $\gamma$ . Indeed if both  $\sigma_1$  and  $\sigma_2$  were preserving the axis of  $\gamma$ , then  $G$  would preserve the axis of  $\gamma$  and hence would be virtually abelian by lemma 2.2 (ii), contradiction. Then, if  $(\theta, \eta)$  are the endpoints of the axis of  $\gamma$ ,  $\sigma_1(\{\theta, \eta\}) \cap \{\theta, \eta\} = \emptyset$  by the proof of lemma 2.1, a). We can then apply the effective ping-pong lemma proved in the appendix to the two hyperbolic elements  $\gamma$  and  $\sigma_1\gamma\sigma_1^{-1}$  which have disjoint fixed-point-sets. This shows that the algebraic entropy of the subgroup generated by  $\gamma$  and  $\sigma_1\gamma\sigma_1^{-1}$  is bounded below by  $\frac{\delta}{5+\delta} \log 2$ . We then deduce that,

$$\text{Ent}_\Sigma(\Gamma) \geq \frac{\delta}{5+\delta} \frac{\log 2}{6}.$$

In the case 2, proposition 3.2 tells that all elements  $\gamma \in G$  of length  $l_\Sigma(\gamma) = 6$  are not  $(L, \eta)$ -straight where  $\eta$  is given by (23),  $\eta = 10^{-3} \left(1 - \frac{\cosh(\frac{L}{2})}{\cosh(\frac{L}{2})}\right)^4$ . Then every element  $g \in \Gamma$  of algebraic length 6 satisfies,

$$\rho(x_0, gx_0) \leq (L - \eta)l_\Sigma(g).$$

Hence, one obtain that every element  $\gamma \in \Gamma$ , satisfies,

$$\rho(x_0, \gamma x_0) \leq (L - \eta)(l_\Sigma(\gamma) - 5) + 5L.$$

Therefore we get from theorem 5.1 that  $\text{Ent}_\Sigma G \geq \eta = 10^{-3} \left(1 - \frac{\cosh(\frac{L}{2})}{\cosh(\frac{L}{2})}\right)^4$ .  $\square$

We may now prove the main theorem which we recall below,

**Theorem 6.1 (Main theorem)** *Let  $(X, g)$  be a Cartan-Hadamard manifold whose sectional curvature satisfies  $-a^2 \leq K_g \leq -1$ . Let  $\Gamma$  be a discrete and finitely generated subgroup of the isometry group of  $(X, g)$ , then either  $\Gamma$  is virtually nilpotent or its algebraic entropy is bounded below by an explicit constant  $C(n, a)$ .*

**Remark.** The constant is

$$C(n, a) = \min \left[ \frac{\log(\cosh(\frac{\mu(n, a)}{4}))}{5 + \log(\cosh(\frac{\mu(n, a)}{4}))} \frac{\log 2}{12}, \frac{1}{2000} \left(1 - \frac{\cosh(\frac{\mu(n, a)}{4})}{\cosh(\frac{\mu(n, a)}{2})}\right)^4, \frac{\mu(n, a)}{4}, \right]$$

$$\frac{1}{4} \left(1 - \frac{1}{\cosh \mu(n, a)}\right)^2, \frac{1}{2} \log \left( \frac{1}{2} \left( \cosh \frac{\mu(n, a)}{2} + \frac{1}{\cosh \frac{\mu(n, a)}{2}} \right) \right) \Bigg].$$

*Proof.* If  $S = \{\sigma_1, \dots, \sigma_p\}$  is a finite generating set of  $\Gamma$ , proposition 2.1 allows to reduce to the following three cases,

i) there exist  $\sigma_i, \sigma_j \in S$  such that  $L(\langle \sigma_i, \sigma_j \rangle) \geq \mu(n, a)$  and such that the subgroup  $\langle \sigma_i, \sigma_j \rangle$  is not virtually nilpotent.

ii) There exist  $\sigma_i, \sigma_j, \sigma_k \in S$  such that  $L(\langle \sigma_i \sigma_j, \sigma_k \rangle) \geq \mu(n, a)$  and such that  $\langle \sigma_i \sigma_j, \sigma_k \rangle$  is not virtually nilpotent.

iii) All  $\sigma_i$ 's are elliptic and, for all  $i \neq j$ , the subgroup  $\langle \sigma_i, \sigma_j \rangle$  fixes a point  $y \in X$  or a point  $\theta \in \partial X$ .

In the first case (resp. the second case) proposition 6.1 gives a lower bound of the algebraic entropy of  $\langle \sigma_i, \sigma_j \rangle$  (resp.  $\langle \sigma_i \sigma_j, \sigma_k \rangle$ ) with respect to the generating set  $\{\sigma_i, \sigma_j\}$  (resp.  $\{\sigma_i \sigma_j, \sigma_k\}$ ), by the number,

$$\min \left[ \frac{\log(\cosh(\frac{\mu(n, a)}{4}))}{5 + \log(\cosh(\frac{\mu(n, a)}{4}))} \frac{\log 2}{6}, \frac{1}{1000} \left(1 - \frac{\cosh(\frac{\mu(n, a)}{4})}{\cosh(\frac{\mu(n, a)}{2})}\right)^4 \right],$$

using the fact that  $L(\sigma_i, \sigma_j) \geq \mu(n, a)$ , (resp.  $L(\sigma_i \sigma_j, \sigma_k) \geq \mu(n, a)$ ). We conclude in the two first cases (i) and (ii) by noticing that the entropy of  $\Gamma$  with respect to  $S$  is bounded below by  $\text{Ent}_{\{\sigma_i, \sigma_j\}}(\langle \sigma_i, \sigma_j \rangle)$  (resp. by  $\frac{1}{2} \text{Ent}_{\{\sigma_i \sigma_j, \sigma_k\}}(\langle \sigma_i \sigma_j, \sigma_k \rangle)$ ), since  $d_{\{\sigma_i, \sigma_j\}} \geq d_S$  (resp.  $d_{\{\sigma_i \sigma_j, \sigma_k\}} \geq \frac{1}{2} d_S$ ).

In the third case, proposition 3.1, implies that,

$$\rho(x_0, \gamma x_0) \leq (L(S) - \eta/2)(l_S(\gamma) - 1) + L(S),$$

where  $\eta$  is given in proposition 3.1. We conclude by applying theorem 5.1, which gives  $\text{Ent}_S(\Gamma) \geq \eta$ , and then bounding below  $\eta$  using  $L(S) \geq \mu(n, a)$ .  $\square$

## 7 Appendix

In this section  $(X, g)$  is a Cartan-Hadamard manifold of sectional curvature  $K \leq -1$ . It is well known that if  $\alpha, \beta$  are two hyperbolic isometries of  $(X, g)$  with disjoint axes, then a sufficiently large power  $\alpha^N$  and  $\beta^N$  of  $\alpha$  and  $\beta$  generates a non abelian free group of  $\text{Isom } X$ . In [6], [8], it was shown that if  $\Gamma$  is an hyperbolic group then  $N$  can be chosen independantly of  $\alpha$  and  $\beta$  in  $\Gamma$  and under the same assumptions the  $N$  was shown to depend only on the number of generators and the constant of hyperbolicity of  $\Gamma$  [6]. In what follows we show that  $N = N(\delta)$  can be chosen independantly of  $\alpha$  and  $\beta$  two hyperbolic isometries of  $(X, g)$  with disjoint set of fixed points and displacement greater than or equal to a positive number  $\delta$ .

**Proposition 7.1** *Let  $(X, g)$  be a Cartan Hadamard manifold of sectional curvature  $K \leq -1$  and  $\Gamma$  a discrete group of isometries in  $\text{Isom}(X, g)$ . We assume that  $\alpha$  and  $\beta$  have disjoint set of fixed point and their displacement satisfy  $l(\alpha) \geq \delta$  and  $l(\beta) \geq \delta$ , where  $\delta$  is a positive number. Then,  $(\alpha^N, \beta^N)$  or  $(\alpha^N, \beta^{-N})$  generates a non abelian free semi-group, where  $N = E(\frac{5}{\delta}) + 1$ .*

Before going to the proof of the proposition 7.1 let us set some notations. Let us write  $x = x(t)$  and  $y = y(t)$ ,  $t \in \mathbb{R}$ , the axes of  $\alpha$  and  $\beta$ . The points  $\theta^\pm = \lim_{t \rightarrow \pm\infty} x(t)$  and  $\zeta^\pm = \lim_{t \rightarrow \pm\infty} y(t)$  are the fixed points of  $\alpha$  and  $\beta$  on the ideal boundary  $\partial X$  of  $X$ . Let us denote  $x^+$  and  $x^-$  the projections of  $\zeta^+$  and  $\zeta^-$  on the axis of  $\alpha$ . We can assume that  $x^+$  is closer to  $\theta^+$  than  $x^-$ , (if it is not the case, we replace  $\beta$  by  $\beta^{-1}$ ). Let us also denote  $y_0$  the projection of  $x^+$  on the axis of  $\beta$ . We now parametrize  $x$  and  $y$  in such a way that  $x(0) = x^+$  and  $y(0) = y_0$ . We set  $t_1 = Nl(\alpha) = l(\alpha^N)$  and  $t_2 = Nl(\beta) = l(\beta^N)$ , where  $N = E(\frac{5}{\delta}) + 1$  is chosen as in the proposition. We define  $U^\pm$  as the set of points  $p \in X$  such that  $\rho(p, x(\pm t_1)) \leq \rho(p, x(o))$ . In the same way we define  $V^\pm$  as the set of points  $p \in X$  such that  $\rho(p, y(\pm t_2)) \leq \rho(p, y(o))$ . For a unit tangent vector  $u \in T_x X$  at a point  $x \in X$  and  $\alpha \in [0, \pi[$  we denote  $\mathcal{C}(u, \alpha) = \{exp_x v : v \in T_x X, \angle(u, v) \in [0, \alpha[ \}$  the cone of angle  $\alpha$  at  $x$ , where  $exp_x$  is the exponential map at  $x$ .

We further need the following geometric lemmas. For a triangle  $ABC$  in  $(X, g)$ , we will write  $\hat{A}$  the angle at  $A$ , and  $a, b, c$  the length of the sides opposite to  $A, B$  and  $C$ .

**Lemma 7.1** *Let  $ABC$  be a triangle in  $(X, g)$  such that  $\frac{\pi}{6} \leq \hat{A} \leq \pi$ , then  $\rho(B, C) > \rho(A, B) + \rho(A, C) - 4$ . Moreover, if  $\hat{A} \geq \frac{\pi}{2}$ , then  $\rho(B, C) > \rho(A, B) + \rho(A, C) - 1$ .*

*Proof.* Since the curvature  $K \leq -1$ , we have

$$\cosh a \geq \cosh b \cosh c - \cos \hat{A} \sinh b \sinh c. \quad (43)$$

The first inequality of lemma 7.1 will therefore be a consequence of the fact that if  $b + c > 4$  then

$$\cosh(b + c - 4) - \cosh b \cosh c + \cos \hat{A} \sinh b \sinh c < 0. \quad (44)$$

Setting  $X = e^{-(b+c)}$  we have

$$\begin{aligned} & \cosh(b + c - 4) - \cosh b \cosh c + \cos \hat{A} \sinh b \sinh c = \\ & e^{(b+c)} \left[ (2e^4 - 1 + \cos \hat{A})X^2 - (e^{-2b} + e^{-2c})(1 + \cos \hat{A}) - (1 - \cos \hat{A} - 2e^{-4}) \right]. \end{aligned}$$

Since  $e^{-2b} + e^{-2c} \geq 2e^{-(b+c)}$ , we then get

$$\cosh(b + c - 4) - \cosh b \cosh c + \cos \hat{A} \sinh b \sinh c \leq e^{(b+c)} P(X)$$

where

$$P(X) = (2e^4 - 1 + \cos \hat{A})X^2 - (1 + \cos \hat{A})X - (1 - \cos \hat{A} - 2e^{-4})$$

and  $P(X)$  is negative when  $P(0) < 0$  and  $P(e^{-4}) < 0$  which is the case if  $\cos \hat{A} < 1 - 2e^{-4}$  and so when  $\hat{A} \geq \pi/6$ . This proves the first inequality of the lemma. The second inequality is proved similarly when  $\cos \hat{A} < 1 - 2e^{-1}$ .  $\square$

**Lemma 7.2** *The sets  $U^+$  and  $U^-$  are contained in  $\mathcal{C}(\dot{x}(0), \pi/6)$  and  $\mathcal{C}(-\dot{x}(0), \pi/6)$  respectively.*

*Proof.* We recall that  $x(0) = x^+$ . Let  $c(t)$  be a geodesic ray starting at  $x^+$  such that  $\angle(\dot{x}(0), \dot{c}(0)) \geq \pi/6$ . Since  $t_1 \geq 5$ , the lemma 7.1 implies

$$\rho(c(t), x(t_1)) > \rho(x^+, c(t)) + \rho(x^+, x(t_1)) - 4 \geq \rho(c(t), x^+)$$

therefore  $c(t) \notin U^+$ . The same argument holds for  $U^-$ .  $\square$

Let us denote by  $z_t$  the geodesic joining  $x^+$  and  $y(t)$  and  $z_{\pm\infty}$  the geodesic joining  $x^+$  and  $y(\pm\infty) = \zeta^{\pm\infty}$ .

**Lemma 7.3** *The set  $V^\pm$  is contained in  $\mathcal{C}(\dot{z}_{\pm\infty}(0), \pi/3)$ .*

*Proof.* Let us recall that the angle at  $y(0) = y_0$  between  $z_0$  and  $y$  is equal to  $\pi/2$ , so that the lemma 7.1 says that

$$\text{length}(z_t) > \text{length}(z_0) + t - 1, \quad (45)$$

and in particular,

$$\text{length}(z_{t_2}) > \text{length}(z_0) + t_2 - 1. \quad (46)$$

Let us now show that  $\angle(\dot{z}_{t_2}(0), \dot{z}_{+\infty}(0)) \leq \pi/6$ . Assume by contradiction that  $\angle(\dot{z}_{t_2}(0), \dot{z}_{+\infty}(0)) > \pi/6$ , then by lemma 7.1 we have, when  $t$  tends to  $+\infty$ ,

$$t - t_2 > \text{length}(z_{t_2}) + \text{length}(z_t) - 4, \quad (47)$$

but summing up (45) and (46), in (47) leads to a contradiction since  $t_2 \geq 5$ . Therefore we have  $\angle(\dot{z}_{t_2}(0), \dot{z}_{+\infty}(0)) \leq \pi/6$ . Let now consider a geodesic ray  $c$  starting at  $x^+$  such that  $\angle(\dot{c}(0), \dot{z}_{+\infty}(0)) \geq \pi/3$ . Thus,  $\angle(\dot{c}(0), \dot{z}_{t_2}(0)) \geq \pi/6$  and by lemma 7.1 we get

$$\rho(c(t), y(t_2)) > \rho(c(t), x^+) + \text{length}(z_{t_2}) - 4,$$

and applying again the lemma 7.1,

$$\rho(c(t), y(t_2)) > \rho(c(t), x^+) + \text{length}(z_0) + t_2 - 5.$$

The last inequality becomes by triangle inequality,

$$\rho(c(t), y(t_2)) > \rho(c(t), y_0) + t_2 - 5,$$

therefore  $\rho(c(t), y(t_2)) > \rho(c(t), y_0)$  since  $t_2 \geq 5$ .

We have proved that a geodesic ray  $c$  starting at  $x^+$  such that  $\angle(\dot{c}(0), \dot{z}_{+\infty}(0)) \geq \pi/3$  does not intersect  $V^+$ . This proves that  $V^+ \subset \mathcal{C}(\dot{z}_{+\infty}(0), \pi/3)$ . By the same argument we also have  $V^- \subset \mathcal{C}(\dot{z}_{-\infty}(0), \pi/3)$ , which ends the proof of the lemma.  $\square$

**Lemma 7.4** *We have  $U^\pm \cap V^\pm = \emptyset$ ,  $U^+ \cap U^- = \emptyset$ ,  $V^+ \cap V^- = \emptyset$ .*

*Proof.* From the angle relations,  $\angle(\dot{x}(0), \dot{z}_{+\infty}(0)) = \pi/2$ ,  $\angle(\dot{x}(0), \dot{z}_{-\infty}(0)) \geq \pi/2$ ,  $\angle(\dot{x}(0), -\dot{x}(0)) = \pi$ , and the relative position of  $x^+$ ,  $x^-$  and  $\theta^+$ , it follows that  $\mathcal{C}(\dot{x}(0), \pi/6)$  does not intersect  $\mathcal{C}(\dot{z}_{+\infty}(0), \pi/3)$ ,  $\mathcal{C}(-\dot{x}(0), \pi/6)$  and  $\mathcal{C}(\dot{z}_{-\infty}(0), \pi/3)$ . Therefore by the lemmas 7.2, 7.3 we conclude that  $U^+$  does not intersect  $U^-$ ,  $V^+$  and  $V^-$ . Now since  $\angle(-\dot{x}(0), \dot{z}_{+\infty}(0)) = \pi/2$ , we have  $\mathcal{C}(\dot{z}_{+\infty}(0), \pi/3) \cap \mathcal{C}(-\dot{x}(0), \pi/6) = \emptyset$ , hence  $V^+ \cap U^- = \emptyset$ . If  $p \in V^+ \cap V^-$ , we have  $\rho(p, y(0)) \geq \rho(p, y(-t_2))$  and  $\rho(p, y(0)) \geq \rho(p, y(t_2))$  which contradicts the convexity of the function  $t \rightarrow \rho(x(t), p)$ . Therefore  $V^+ \cap V^- = \emptyset$ .  $\square$

**Lemma 7.5** *We have  $\alpha^N(V^+) \subset U^+$  and  $\beta^N(U^+) \subset V^+$ .*

*Proof.* Since  $x$  and  $y$  are the axes of  $\alpha^N$  and  $\beta^N$  respectively we have  $\alpha^N(x(-t_1)) = x(0)$ ,  $\beta^N(y(-t_1)) = y(0)$ ,  $\alpha^N(x(0)) = x(t_1)$  and  $\beta^N(y(0)) = y(t_2)$ . Therefore for any  $p \in X - U^-$ , we have  $\alpha^N(p) \in U^+$  and similarly for any  $p \in X - V^-$ , we have  $\beta^N(p) \in V^+$  by definition of  $N$ . On the other hand, by the lemma 7.4, we have  $V^+ \subset X - U^-$  and  $U^+ \subset X - V^-$ , which concludes.  $\square$

The proof of the proposition 7.1 is a direct application of the lemma 7.5 by a standard ping-pong argument.

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