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Uniform Growth of Groups Acting on Cartan-Hadamard Spaces

G. Besson, G. Courtois et S. Gallot

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Preliminary version

1 Introduction

In this paper we investigate the growth of finitely generated groups. Given a group Γ generated by a finite set S , the word length $l_S(\gamma)$ of an element $\gamma \in \Gamma$ is the smallest integer m such that there exist elements $\sigma_1, \dots, \sigma_m$ in $S \cup S^{-1}$ with $\gamma = \sigma_1 \dots \sigma_m$. The entropy of Γ with respect to the generating set S is defined by

$$\text{Ent}_S(\Gamma) = \lim_{m \rightarrow \infty} \frac{1}{m} (\log |\{\gamma \in \Gamma / l_S(\gamma) \leq m\}|). \quad (1)$$

If $\text{Ent}_S(\Gamma) > 0$ for some generating set S , it is true for all (finite) generating set and the group is said to have exponential growth. We now define the entropy of Γ

$$\text{Ent } \Gamma = \inf_S \{\text{Ent}_S(\Gamma) / S \text{ finite generating set of } \Gamma\}. \quad (2)$$

We say that Γ has uniform exponential growth if $\text{Ent } \Gamma > 0$. In [11], remarque 5.12, M. Gromov raised the question whether exponential growth always implies uniform exponential growth. The answer is negative, indeed, in [14] J.S. Wilson gave examples of finitely generated groups of exponential growth and non uniform exponential growth. Nevertheless, exponential growth implies uniform exponential growth for hyperbolic groups [12], geometrically finite groups of isometries of Hadamard manifolds with pinched negative curvature [1], solvable groups [13] and linear groups [10], [4], [3]. For further references see the exposition paper [7].

We suppose that (X, g) is a n -dimensional Cartan Hadamard manifold of pinched sectional curvature $-a^2 \leq K \leq -1$. Our main result is the

Theorem 1.1 *There exists a positive constant $C(n, a)$ such that for any finitely generated discrete group Γ of isometries of (X, g) , then either Γ is virtually nilpotent or $\text{Ent}(\Gamma) \geq C(n, a)$.*

Remark. The difficulty is here to show that one can choose the constant $C(n, a)$ not depending on the group Γ . In the linear setting, E. Breuillard obtained the same kind of uniformity proving the existence of a positive constant $C(n)$ such that for any finitely generated group Γ of $GL(n, K)$, K any field, then either Γ is virtually solvable or $\text{Ent}(\Gamma) \geq C(n)$.

The classical technique is to prove that "not too far" from any finite generating system one can exhibit a free group (in two generators). In this paper we do prove this in one of the cases under consideration, using the famous ping-pong lemma, however in the second case we use a different approach using natural Lipschitz maps from the Cayley graph into X . This is the new idea which is described in the following.

In a private communication M. Kapovich mentioned to us a different proof in the case when Γ acts without any elliptic element. One important issue in our proof is that we do not have this restriction, elliptic elements are permitted.

In the forthcoming paper [2] we shall use this result to prove a Margulis lemma without curvature; indeed, we shall replace the curvature assumptions by hypothesis on the growth of the fundamental group.

2 Preliminaries

Let (X, g) be a n -dimensional Cartan-Hadamard manifold with sectional curvature $-a^2 \leq K_g \leq -1$. Let us recall a few well-known facts about isometries. If γ is an isometry of (X, g) , the displacement of γ is defined by $l(\gamma) = \inf_{x \in X} \rho(x, \gamma x)$, where ρ is the distance associated to the metric g on X . We then have (see [9] p. 31):

1. The isometry γ is called hyperbolic (or axial) if $l(\gamma) > 0$, in which case there exists a geodesic a_γ , called the axis of γ , such that, for any $x \in a_\gamma$, $\rho(x, \gamma x) = l(\gamma)$.
2. The isometry γ is called parabolic if $l(\gamma) = 0$ and $l(\gamma)$ is not achieved on X , in which case there exists a unique point θ on the geometric boundary ∂X of X such that $\gamma\theta = \theta$.
3. The isometry γ is called elliptic if $l(\gamma) = 0$ and $l(\gamma)$ is achieved on X , in which case there exists a non empty convex subset F_γ of X such that, for any $x \in F_\gamma$, $\gamma x = x$.

The following result, due to G. Margulis, describes the structure of discrete subgroups of isometries generated by elements with small displacement.

Theorem 2.1 (G. Margulis, [5]) *There exists a constant $\mu(n, a) > 0$ such that if Γ is a discrete subgroup of the isometry group of (X, g) , the subgroup Γ_μ of Γ generated by,*

$$S_\mu = \{\gamma \in \Gamma / \rho(x, \gamma x) \leq \mu(n, a)\},$$

is virtually nilpotent.

Given a set of isometries $S = \{\sigma_1, \dots, \sigma_p\}$ of (X, g) , we define the “minimal displacement” of S by

$$\mathbf{Definition\ 2.1} \quad L(S) = \inf_{x \in X} \max_{i=1, \dots, p} \rho(x, \sigma_i x)$$

When Γ is a finitely generated discrete subgroup of the isometry group of (X, g) , the above theorem 2.1 has the following

Corollary 2.1 *There exists a constant $\mu(n, a) > 0$ such that if Γ is a finitely generated not virtually nilpotent discrete subgroup of isometry of (X, g) and $S = \{\sigma_1, \dots, \sigma_p\}$ a finite generating set of Γ , then*

$$L(S) \geq \mu(n, a).$$

In the following lemma we describe the structure of virtually nilpotent discrete subgroups of isometries of (X, g) . Here by discrete we mean that the orbits are discrete sets in (X, g) .

Lemma 2.1 *Let G be a discrete virtually nilpotent group of isometries of (X, g) .*

- a) *If G contains an hyperbolic element γ , then G preserves the axis of γ .*
- b) *If G contains a parabolic element γ with fixed point $\theta \in \partial X$, then G fixes the point θ .*
- c) *If all elements of G are elliptic, then G is finite.*

Proof. a) Let $\gamma \in G$ be an hyperbolic element and $\theta, \zeta \in \partial X$, the end points of the axis a_γ of γ . We claim that for any $\gamma' \in G$, then $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$ or $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \emptyset$. Indeed assume that $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \{\theta\}$. The isometry $\gamma'\gamma\gamma'^{-1}$ is hyperbolic with axis $a_{\gamma'\gamma\gamma'^{-1}} = \gamma'a_\gamma$ equal to the geodesic joining θ and $\zeta' \neq \zeta$, where $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta'\}$. We may assume that θ is the attractive fixed point of γ and $\gamma'\gamma\gamma'^{-1}$ (replacing them by their inverse if necessary). Let $x \in a_\gamma$, then $(\gamma'\gamma\gamma'^{-1})^{-N}\gamma^N x$ is a sequence of pairwise distinct points which converges to a point on the axis $a_{\gamma'\gamma\gamma'^{-1}}$ of $\gamma'\gamma\gamma'^{-1}$. This contradicts the discreteness of G , proving thus the claim.

Now, if there exist $\gamma' \in G$ such that $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \emptyset$, the two hyperbolic isometries γ and $\gamma'\gamma\gamma'^{-1}$ would then have disjoint axis and therefore G would contain a free subgroup by a classical ping-pong argument. This would contradict the fact that G is virtually nilpotent. Consequently, for any $\gamma' \in G$, $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$, which shows that G preserves the geodesic joining θ and ζ .

b) Let $\gamma \in G$ be a parabolic element, and $\theta \in \partial X$ its fixed point. If there exist $\gamma' \in G$ such that $\gamma'\theta \neq \theta$, then γ and $\gamma'\gamma\gamma'^{-1}$ would be to parabolic elements in G with distinct fixed point θ and $\gamma'\theta$ respectively. By a ping-pong argument, G would then contain a free subgroup, which contradicts the fact that G is virtually nilpotent. Thus G fixes $\theta \in \partial X$.

c) Let us now assume that all elements in G are elliptic. Let $N \subset G$ be a nilpotent subgroup of G with finite index. If $N = \{e\}$, then G is finite. We thus assume that $N \neq \{e\}$, the center $Z(N)$ of N is then not trivial. For

$g_1 \in Z(N) \setminus \{e\}$ let us denote $F_{g_1} \subset X$ the set of fixed points of g_1 . Let $x_1 \in F_{g_1}$; by commutation of g_1 and \exp_{x_1} , we have $F_{g_1} = \exp_{x_1}(E_1)$, where E_1 is the eigenspace of $d_{x_1}g_1$ corresponding to the eigenvalue $+1$. This shows that F_{g_1} is a totally geodesic submanifold of X satisfying $\dim(F_{g_1}) < \dim(X)$, since $g_1 \neq e$. As every $\gamma \in N$ commutes with g_1 , it satisfies $\gamma(F_{g_1}) = F_{g_1}$.

Let N_1 be the subgroup of $\text{Isom}(F_{g_1})$ obtained by restriction to F_{g_1} of the elements of N ; it is clearly nilpotent as the image of a nilpotent group. For $\gamma \in N$, the projection on F_{g_1} of any fixed point of γ is again a fixed point of γ ; consequently, the elements of N_1 are elliptic elements of $\text{Isom}(F_{g_1})$.

If $N_1 = \{e\}$, then F_{g_1} is pointwise fixed by N , therefore N is finite (the group is discrete and all elements have a common fixed point).

If $N_1 \neq \{e\}$, we may iterate the process. Indeed, let us suppose that we have constructed the totally geodesic submanifold F_{g_i} , we then construct N_i as the set of restrictions of elements of N to F_{g_i} , and, either $N_i = \{e\}$ in which case N is finite, or N_i is not trivial and, choosing $g_{i+1} \in Z(N_i) \setminus \{e\}$, we construct the totally geodesic submanifold $F_{g_{i+1}} \subset F_{g_i}$ such that $\dim(F_{g_{i+1}}) < \dim(F_{g_i})$. This process stops for some $i_0 \leq n$ and then $N_{i_0} = \{e\}$ and $F_{g_{i_0}}$ is pointwise fixed by N and not empty. \square

Lemma 2.2 *Let Γ be a finitely generated discrete group of isometries of (X, g) .*

- (i) *If there exist a point $\theta \in \partial X$ fixed by Γ , then Γ is virtually nilpotent.*
- (ii) *If Γ preserves a geodesic in X , then Γ is virtually cyclic.*

Proof. Proof of (i). There are three cases: 1) there is an hyperbolic element in Γ , 2) there is no hyperbolic element, but there is a parabolic element in Γ and 3) all elements in Γ are elliptic.

1) Let γ be a hyperbolic element in Γ , and a_γ its axis. One of the endpoints of a_γ is θ . As Γ is discrete, it follows from the argument in the proof of lemma 2.1 a) that for any $\gamma' \in \Gamma$, $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$ or $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \emptyset$, where ζ is the other endpoint of a_γ . Therefore, $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$ and $\gamma'(\theta) = \theta$ and $\gamma'(\zeta) = \zeta$. The group Γ preserves a_γ . Let us note that Γ does not contain any parabolic element, since such an element would fix θ and therefore also ζ which is impossible. The elements in Γ are thus either hyperbolic or elliptic.

Now, the projection on a_γ being distance decreasing, any element $\gamma' \in \Gamma$ achieves its displacement $l(\gamma')$ on the axis a_γ , and γ' is elliptic (resp. hyperbolic) iff $l(\gamma') = 0$ (resp. $l(\gamma') \neq 0$). Moreover, since $\gamma'(\theta) = \theta$, any elliptic element fixes pointwise the axis a_γ . The restriction to the axis a_γ is thus a morphism from Γ into the group of translations of the axis, whose kernel is the set of elliptic elements, which fix all points of a_γ and hence is finite. The group Γ is then virtually abelian.

2) In this case the elements of Γ are either elliptic or parabolic with fixed point θ . In particular, every element of Γ preserves each horospheres centred at θ . Indeed, this is clear for parabolic elements. Any elliptic element γ' fixes some point $x \in X$, and hence the whole geodesic c joining x to θ ; let H be any horosphere centred at θ and y be its intersection with c , then γ' maps H onto the horosphere centred at $\gamma'(\theta) = \theta$ containing $\gamma'(y) = y$. This shows that $\gamma'(H) = H$.

Let $S = \{\sigma_1, \dots, \sigma_p\}$ be a generating set of Γ , by the above discussion, $\inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = 0$. In fact, for any geodesic c such that $c(+\infty) = \theta$, let H_t be the horosphere centred at θ and containing $c(t)$. The orthogonal projection from H_t to $H_{t+t'}$ contracts distances, we then get that $\rho(c(t), \gamma'(c(t)))$ decreases to zero when t goes to infinity, for any $\gamma' \in \Gamma$. The group Γ is then virtually nilpotent by theorem 2.1.

3) If Γ only contains elliptic elements, then for any finite generating set $S = \{\sigma_1, \dots, \sigma_p\}$, $\inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = 0$, because each σ_i preserves each horosphere centred at θ , by the above argument. The group Γ is again virtually nilpotent.

Proof of (ii). A subgroup of index two of Γ fixes each endpoint of the globally preserved geodesic. Then we conclude as in the case 1 of (i). \square

For any two isometries γ, γ' acting on (X, g) we define,

$$L(\gamma, \gamma') = \inf_{x \in X} \max\{\rho(x, \gamma x), \rho(x, \gamma' x)\}.$$

We now prove the,

Proposition 2.1 *Let Γ be a finitely generated discrete subgroup of $\text{Isom}(X, g)$, where (X, g) is a Cartan-Hadamard manifold of sectional curvature $-a^2 \leq K_g \leq -1$. Let $S = \{\sigma_1, \dots, \sigma_p\}$ be a finite generating set of Γ . If Γ is not virtually nilpotent, we have*

- i) *either there exist $\sigma_i, \sigma_j \in S$ such that the subgroup $\langle \sigma_i, \sigma_j \rangle$ generated by these two elements is not virtually nilpotent and $L(\sigma_i, \sigma_j) \geq \mu(n, a)$,*
- ii) *or all σ_i in S are elliptic and for all $\sigma_i \neq \sigma_j \in S$, either $\langle \sigma_i, \sigma_j \rangle$ fixes some point in X and is finite, or it fixes a point $\theta \in \partial X$,*
- iii) *or there exist $\sigma_i, \sigma_j, \sigma_k \in S$ such that $L(\sigma_i \sigma_j, \sigma_k) \geq \mu(n, a)$ and the group $\langle \sigma_i \sigma_j, \sigma_k \rangle$ is not virtually nilpotent.*

Proof. There are again three cases: a) there is a hyperbolic element in S , say σ_1 ; b) there is no hyperbolic element and there is a parabolic element in S , say σ_1 ; c) all σ_i 's in S are elliptic.

a) Let us assume that σ_1 is hyperbolic. Let us consider all pairs (σ_1, σ_i) with $i = 2, \dots, p$, and let us assume that $L(\sigma_1, \sigma_i) < \mu(n, a)$ for $i = 2, \dots, p$. The groups $\langle \sigma_1, \sigma_i \rangle$ are then virtually nilpotent. By lemma 2.1 a), every σ_i preserves the axis a_{σ_1} of σ_1 , hence Γ preserves a_{σ_1} and is virtually nilpotent contradicting the assumption. Then there exist $\sigma_i \in S$ such that $L(\sigma_1, \sigma_i) \geq \mu(n, a)$ and $\langle \sigma_1, \sigma_i \rangle$ is non virtually nilpotent.

b) Assume that σ_1 is parabolic with fixed point $\theta \in \partial X$. Let us consider all pairs (σ_1, σ_i) , $i = 2, \dots, p$, and assume that $\langle \sigma_1, \sigma_i \rangle$ is virtually nilpotent (or that $L(\sigma_1, \sigma_i) < \mu(n, a)$, for all $i = 2, \dots, p$). By lemma 2.1 b), σ_i fixes the point $\theta \in \partial X$, therefore Γ fixes θ and is virtually nilpotent, by lemma 2.2, a contradiction. Consequently, if σ_1 is parabolic, there exist $\sigma_i \neq \sigma_1$ such that $L(\sigma_1, \sigma_i) \geq \mu(n, a)$.

c) Let us assume that all σ_i 's are elliptic, for $i = 2, \dots, p$, and that for all pairs (σ_i, σ_j) the groups $\langle \sigma_i, \sigma_j \rangle$ are virtually nilpotent (or that $L(\sigma_i, \sigma_j) < \mu(n, a)$). Let us denote $G = \langle \sigma_i, \sigma_j \rangle$. There are again three cases: 1) there is a hyperbolic element in G , 2) there is no hyperbolic element and there is a parabolic element in G , 3) all elements in G are elliptic.

In the case 1), let γ be a hyperbolic element in G with axis a_γ . By lemma 2.1 a), G preserves a_γ . Since σ_i, σ_j are elliptic, they fix points x_i and x_j (respectively) on a_γ (recall that the displacement of σ_i and σ_j are achieved on a_γ by the distance decreasing property of the projection onto a_γ). If $x_i = x_j$, the G fixes x_i and it is thus finite. Let us now suppose that σ_i and σ_j do not fix the same point on a_γ , that is $x_i \neq x_j$ and none of the restriction $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$ of σ_i and σ_j to a_γ is the identity. In that case, $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$ are both symmetries around x_i and x_j , and then $\sigma_i \sigma_j$ is a hyperbolic element with axis a_γ . Let us then consider $\langle \sigma_i \sigma_j, \sigma_l \rangle$ for $l = 1, \dots, p$. Assume that for all $l = 1, \dots, p$, $L(\sigma_i \sigma_j, \sigma_l) < \mu(n, a)$, the groups $\langle \sigma_i \sigma_j, \sigma_l \rangle$ are then virtually nilpotent, and by lemma 2.1 a), all σ_l 's preserve a_γ and hence Γ preserves a_γ and is thus virtually nilpotent which is a contradiction. Therefore, there exist $\sigma_k \in S$ such that $L(\sigma_i \sigma_j, \sigma_k) \geq \mu(n, a)$ and that $\langle \sigma_i \sigma_j, \sigma_k \rangle$ is not virtually nilpotent.

In the case 2), let $\gamma \in G$ be a parabolic element with fixed point $\theta \in \partial X$. By lemma 2.1 b), G fixes θ .

In the case 3), all elements in G are elliptic and by lemma 2.1 c), G is finite. This ends the proof of the proposition. \square

3 Algebraic length and η -straight isometries

Let Γ be a finitely generated discrete group of isometries of (X, g) and $S = \{\sigma_1, \dots, \sigma_p\}$ be a finite generating set of Γ .

Let us denote l_S and d_S the length and distance on the Cayley graph associated to S . Let x_0 be a point in X and define $L = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0)$.

For any $\gamma \in \Gamma$ it follows from the triangle inequality that

$$\rho(x_0, \gamma x_0) \leq l_S(\gamma)L. \quad (3)$$

Let η be a positive number such that $0 < \eta < L$.

Definition 3.1 *An isometry γ of Γ is said to be (L, η) -straight if $\rho(x_0, \gamma x_0) \geq (L - \eta)l_S(\gamma)$.*

Remark. Notice that the above definition depends on the choice of x_0 and of a generating set S .

When Γ is a finitely generated discrete group, for any finite generating set $S = \{\sigma_1, \dots, \sigma_p\}$ we define,

$$L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x).$$

When Γ is not virtually nilpotent, by theorem 2.1, for any finite generating set S , $L = L(S) \geq \mu(n, a) > 0$, where $\mu(n, a)$ is the Margulis constant. We then have,

Lemma 3.1 *Let Γ be a finitely generated non virtually nilpotent discrete group of isometries of (X, g) . For any finite generating set $S = \{\sigma_1, \dots, \sigma_p\}$ of Γ , there exist $x_0 \in X$ such that,*

$$L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0).$$

Proof. Let us assume that the infimum in the definition of $L(S)$ is not achieved in X , then there exist a sequence of points $x_k \in X$, which satisfies $\max_{i \in \{1, \dots, p\}} \rho(x_k, \sigma_i x_k) \rightarrow L(S)$ when $k \rightarrow \infty$, and x_k converges to a point, say θ , in ∂X . For k large enough and $i \in \{1, \dots, p\}$, we then have $\rho(x_k, \sigma_i x_k) \leq L+1$ and hence $\sigma_i \theta = \theta$ for all i . This shows that Γ fixes θ and is thus virtually nilpotent by lemma 2.2, which contradicts the hypothesis. \square

In the sequel of this section, we shall show that if G is a finitely generated discrete group of isometries of (X, g) , for any finite generating set $S = \{\sigma_1, \dots, \sigma_p\}$ of G such that each σ_i has a displacement $l(\sigma_i)$ small compared to $L(S)$, then there exist many non- $(L(S), \eta)$ -straight elements in G for a constant η to be defined.

We need the following geometric lemmas.

Lemma 3.2 *Let (x_1, x_2, x_3) be a geodesic triangle in (X, g) , where (X, g) is a Cartan-Hadamard manifold with $K_g \leq -1$. Let x'_2 be the point in the segment $[x_1, x_3]$ dividing it in two segments of length proportional to $L_1 := \rho(x_1, x_2)$ and $L_2 := \rho(x_2, x_3)$. We have,*

$$\rho(x'_2, x_2) \leq \text{Argcosh} \left[\exp \left(\alpha (\rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_1, x_3)) \right) \right],$$

where $\alpha = \frac{\max(L_1, L_2)}{L_1 + L_2}$.

Proof. We consider a comparison geodesic triangle (y_1, y_2, y_3) in the Poincaré disk (\mathbb{H}^2, d) of constant curvature -1 such that $d(y_i, y_j) = \rho(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. Let y'_2 be the point of the segment $[y_1, y_3]$ dividing it in two segments of length proportional to L_1 and L_2 . Since (X, g) is a $CAT(-1)$ space we have

$$\rho(x_2, x'_2) \leq d(y_2, y'_2). \quad (4)$$

One of the two triangles (y_1, y'_2, y_2) , (y_3, y'_2, y_2) has angle at y'_2 greater than or equal to $\pi/2$, therefore from hyperbolic trigonometry formulae we get the existence of $i \in \{1, 2\}$ such that

$$\cosh L_i \geq \cosh [d(y_2, y'_2)] \cosh \left[\frac{L_i}{(L_1 + L_2)} d(y_1, y_3) \right] \quad (5)$$

Let us denote $\Delta = \rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_1, x_3)$. We have

$$\frac{L_i}{L_1 + L_2} d(y_1, y_3) \geq L_i - \alpha \Delta, \quad (6)$$

where $\alpha = \frac{\max(L_1, L_2)}{L_1 + L_2}$. Therefore from (4) and (5) we get

$$\cosh [\rho(x'_2, x_2)] \leq \frac{\cosh L_i}{\cosh(L_i - \alpha\Delta)}, \quad (7)$$

hence

$$\cosh [\rho(x'_2, x_2)] \leq e^{\alpha\Delta}. \quad (8)$$

□

Lemma 3.3 *Let (X, g) be a Cartan-Hadamard manifold with sectional curvature $K_g \leq -1$. Let δ, L be any positive numbers such that $L > \text{Argcosh}(e^\delta)$. Then, for any isometry γ of (X, g) such that its displacement $l(\gamma)$ satisfies $l(\gamma) \leq \delta$, and for any point $x_0 \in X$ such that $\rho(x_0, \gamma x_0) \geq L$, we have*

$$\rho(x_0, \gamma^2 x_0) \leq 2\rho(x_0, \gamma x_0) - \left(1 - \frac{e^\delta}{\cosh L}\right)^2,$$

Proof. Let us consider $\Delta = 2\rho(x_0, \gamma x_0) - \rho(x_0, \gamma^2 x_0)$. We want to prove that $\Delta \geq \left(1 - \frac{e^\delta}{\cosh L}\right)^2$. By assumption there is a point $y \in X$ such that $\rho(y, \gamma y) \leq \delta$. Let us write $L_1 =: \rho(x_0, \gamma y)$, $L_2 =: \rho(\gamma^2 x_0, \gamma y)$ and $L' =: \rho(x_0, y)$. By the triangle inequality we have for $i = 1, 2$

$$L' - \delta \leq L_i \leq L' + \delta. \quad (9)$$

Let us associate to the triangle $(x_0, \gamma y, \gamma^2 x_0)$ the comparison triangle (z_1, z_2, z_3) in the hyperbolic plane (\mathbb{H}^2, d) such that $d(z_1, z_2) = L_1$, $d(z_2, z_3) = L_2$ and $d(z_1, z_3) = \rho(x_0, \gamma^2 x_0)$. Let x [resp. z] be the middle point of the segment $(x_0, \gamma^2 x_0)$ [resp. (z_1, z_3)]. One of the two triangles (z_2, z, z_1) or (z_2, z, z_3) has angle at z greater than or equal to $\pi/2$. Let us assume without restriction that this triangle is (z_2, z, z_1) , then the hyperbolic trigonometric formulas give

$$\cosh L_1 \geq \cosh [d(z_2, z)] \cosh \left[\frac{1}{2}d(z_1, z_3)\right]$$

therefore from (9) we get

$$\cosh(L' + \delta) \geq \cosh [d(z_2, z)] \cosh \left[\frac{1}{2}d(z_1, z_3)\right]$$

and since (X, g) is a $CAT(-1)$ space we have $\rho(x, \gamma y) \leq d(z_2, z)$, thus we obtain

$$\cosh(L' + \delta) \geq \cosh [\rho(x, \gamma y)] \cosh \left[\frac{1}{2}\rho(x_0, \gamma^2 x_0)\right]. \quad (10)$$

Let us write $L_0 = \rho(x_0, \gamma x_0)$. By the triangle inequality we have

$$\rho(x, \gamma y) \geq |\rho(\gamma y, \gamma x_0) - \rho(\gamma x_0, x)|,$$

therefore, since $\rho(\gamma y, \gamma x_0) = \rho(y, x_0) = L'$ and $\frac{1}{2}\rho(\gamma^2 x_0, x_0) = L_0 - \frac{\Delta}{2}$, we get from 10

$$\cosh(L' + \delta) \geq \cosh\left(L' - \rho(\gamma x_0, x)\right) \cosh\left(L_0 - \frac{\Delta}{2}\right). \quad (11)$$

We get from (11),

$$(\cosh \delta + \sinh \delta) \cosh L' \geq \left(\cosh [\rho(\gamma x_0, x)] - \sinh [\rho(\gamma x_0, x)]\right) (\cosh L') \cosh\left(L_0 - \frac{\Delta}{2}\right)$$

hence

$$e^\delta \geq \left(\cosh [\rho(\gamma x_0, x)] - \sinh [\rho(\gamma x_0, x)]\right) \cosh\left(L_0 - \frac{\Delta}{2}\right). \quad (12)$$

Now applying the inequality 7 in the proof of lemma 3.2 we have

$$\cosh [\rho(\gamma x_0, x)] \leq \frac{\cosh L_0}{\cosh\left(L_0 - \frac{\Delta}{2}\right)},$$

and since $\cosh r - \sinh r = e^{-r}$ is a decreasing function of r we get from (12)

$$e^\delta \geq \cosh L_0 - \left(\cosh^2 L_0 - \cosh^2\left(L_0 - \frac{\Delta}{2}\right)\right)^{\frac{1}{2}}. \quad (13)$$

But we can check that $\cosh^2 L_0 - \cosh^2\left(L_0 - \frac{\Delta}{2}\right) \leq \Delta \cosh^2 L_0$ so we get from (13)

$$e^\delta \geq \cosh(L_0)(1 - \Delta^{\frac{1}{2}})$$

and therefore

$$\Delta \geq \left(1 - \frac{e^\delta}{\cosh L_0}\right)^2,$$

when $e^\delta < \cosh L$. The lemma now follows whenever $L_0 \geq L$. □

Lemma 3.4 *Let (X, g) be a Cartan-Hadamard manifold with sectional curvature $K_g \leq -1$. Let us consider four points y_0, y_1, y_2, y_3 such that*

$$\rho(y_0, y_1) + \rho(y_1, y_2) - \rho(y_0, y_2) \leq \eta_1$$

and

$$\rho(y_1, y_2) + \rho(y_2, y_3) - \rho(y_1, y_3) \leq \eta_2$$

then

$$\rho(y_0, y_1) + \rho(y_1, y_2) + \rho(y_2, y_3) - \rho(y_0, y_3) \leq \left(1 + \frac{\rho(y_2, y_3)}{\rho(y_1, y_2)}\right) \left(\eta_1 + \text{Argcosh } e^{\eta_2}\right).$$

Proof. For $i = 1, 2, 3$ let us write $L_i = \rho(y_{i-1}, y_i)$. Let y'_2 be the point on the segment (y_1, y_3) dividing it in two segments of length proportional to L_2 and L_3 . By lemma 3.2 we have

$$\rho(y_2, y'_2) \leq \text{Argcosh}(e^{\eta_2}). \quad (14)$$

Since $\rho(y_0, y_1) + \rho(y_1, y_2) - \rho(y_0, y_2) \leq \eta_1$ by assumption we get from (14) and the triangle inequality

$$\rho(y_0, y'_2) \geq \rho(y_0, y_2) - \rho(y_2, y'_2) \geq \rho(y_0, y_1) + \rho(y_1, y_2) - [\eta_1 + \text{Argcosh}(e^{\eta_2})] \quad (15)$$

On the other hand by convexity of the distance function on (X, g) we get

$$\rho(y_0, y'_2) \leq \frac{L_3}{L_2 + L_3} \rho(y_0, y_1) + \frac{L_2}{L_2 + L_3} \rho(y_0, y_3) \quad (16)$$

The inequalities (15) and (16) give

$$\rho(y_0, y_3) \geq \rho(y_0, y_1) + L_2 + L_3 - \left(\frac{L_2 + L_3}{L_2}\right) (\eta_1 + \text{Argcosh}(e^{\eta_2}))$$

and the lemma follows. \square

Lemma 3.5 *Let L and η be two positive numbers such that,*

$$\eta < \min\left(\frac{L}{4}, \frac{1}{2} \log\left[\frac{1}{2}\left(\cosh\left(\frac{L}{2}\right) + \frac{1}{\cosh\left(\frac{L}{2}\right)}\right)\right]\right).$$

Let (X, g) be a Cartan-Hadamard manifold with sectional curvature $K_g \leq -1$. We consider two elliptic isometries γ_1, γ_2 of (X, g) with a common fixed point $y \in X \cup \partial X$. If we assume that $L - \eta \leq \rho(x_0, \gamma_1 x_0) \leq L$ and that $L - \eta \leq \rho(x_0, \gamma_2 x_0) \leq L$, then

$$\rho(x_0, \gamma_1 \gamma_2 x_0) < 2(L - \eta).$$

Proof. We first claim that in both cases, $y \in X$ and $y \in \partial X$, there exist some sequence $(u_k)_{k \in \mathbf{N}}$ of points in X converging to y such that $\rho(u_k, \gamma_1 \gamma_2 x_0) = \rho(u_k, x_0) = l_k$ and that the quantity $\epsilon_k = |\rho(u_k, \gamma_1 x_0) - l_k|$ goes to zero when k goes to $+\infty$; in fact, when γ_1 and γ_2 fix some point $y \in X$ we may choose $u_k = y$ for every k . If γ_1 and γ_2 fix $y \in \partial X$, they also preserve each horosphere centred at y (see the proof of lemma 2.2 2)), and thus $x_0, \gamma_1 x_0$ and $\gamma_1 \gamma_2 x_0$ lie on the same horosphere centred at y . Approximating this horosphere by a sequence $(S_k)_{k \in \mathbf{N}}$ of spheres passing through x_0 and $\gamma_1 \gamma_2 x_0$ and denoting u_k the centre of S_k , we get that $\rho(u_k, \gamma_1 x_0) - \rho(O, u_k)$ and $\rho(u_k, x_0) - \rho(O, u_k)$ simultaneously go to $B(\gamma_1 x_0, y) = B(x_0, y)$ (where O is some fixed origin in X and B the Busemann function normalised at O). This proves the claim.

Consider the triangle $(u_k, v, w) = (u_k, x_0, \gamma_1 \gamma_2 x_0)$ and z the point of the geodesic segment $[v, w]$ which divides it in two segments of length proportional to $L_1 =: \rho(v, \gamma_1 x_0)$ and $L_2 =: \rho(w, \gamma_1 x_0)$. Let us recall that by assumption we have $L - \eta \leq L_i \leq L$.

We consider the comparison triangle $(\bar{u}_k, \bar{v}, \bar{w})$ on the two-dimensional hyperbolic space \mathbf{H}^2 such that $d(\bar{u}_k, \bar{v}) = \rho(u_k, v) = l_k = \rho(u_k, w) = d(\bar{u}_k, \bar{w})$ and $d(\bar{v}, \bar{w}) = \rho(v, w)$, where d is the hyperbolic distance on \mathbf{H}^2 . Let \bar{z} be the point of the segment $[\bar{u}, \bar{v}]$ dividing it in two segments of length proportional to L_1 and L_2 . Let us write $L'_1 = \rho(v, z)$ and $L'_2 = \rho(w, z)$. We now consider the triangle $(\bar{u}_k, \bar{v}, \bar{z})$ or $(\bar{u}_k, \bar{w}, \bar{z})$, namely the one which has angle at \bar{z} larger than or equal to $\pi/2$. We can assume without restriction that this triangle is $(\bar{u}_k, \bar{v}, \bar{z})$. The hyperbolic trigonometry formulas then show that the point \bar{z} satisfies,

$$\cosh(l_k) \geq \cosh(L'_1) \cosh(d(\bar{u}_k, \bar{z})).$$

Since (X, g) is a CAT(-1)-space, we get that,

$$\rho(u_k, z) \leq d(\bar{u}_k, \bar{z}),$$

and thus that,

$$\cosh(\rho(u_k, z)) \leq \frac{\cosh(l_k)}{\cosh(L'_1)}. \quad (17)$$

On the other hand, the triangle inequality implies that $\rho(u_k, z) \geq l_k - \epsilon_k - \rho(\gamma_1 x_0, z)$ and thus that

$$\cosh(\rho(u_k, z)) \geq e^{-(\rho(\gamma_1 x_0, z) + \epsilon_k)} \cosh(l_k).$$

Plugging this in formula 17 and letting $\epsilon_k \rightarrow 0$, we get:

$$e^{\rho(\gamma_1 x_0, z)} \geq \cosh(L'_1). \quad (18)$$

On the other hand, by lemma 3.2, we have

$$\cosh(\rho(\gamma_1 x_0, z)) \leq \exp \left(\max \left\{ \rho(v, \gamma_1 x_0), \rho(w, \gamma_1 x_0) \right\} \left(1 - \frac{\rho(v, w)}{\rho(v, \gamma_1 x_0) + \rho(w, \gamma_1 x_0)} \right) \right).$$

and hence

$$\cosh(\rho(\gamma_1 x_0, z)) \leq e^{(L - \frac{\rho(v, w)}{2})}. \quad (19)$$

Let us now assume, by contradiction, that

$$\rho(v, w) = \rho(x_0, \gamma_1 \gamma_2 x_0) > 2(L - \eta).$$

Plugging this in the inequalities (18) and (19) we obtain, using the fact that $x \rightarrow x + 1/x$ is an increasing function for $x > 1$:

$$\cosh(L'_1) + \frac{1}{\cosh(L'_1)} \leq 2 \cosh(\rho(\gamma_1 x_0, z)) \leq 2e^\eta. \quad (20)$$

Now since $\frac{L'_1}{L'_2} = \frac{L_1}{L_2}$, we also obtain

$$L'_1 = (L'_1 + L'_2) \left(\frac{L_1}{L_1 + L_2} \right) \geq \frac{2(L - \eta)(L - \eta)}{2L} \geq L - 2\eta$$

which gives by inequalities (20)

$$\cosh(L - 2\eta) + \frac{1}{\cosh(L - 2\eta)} \leq 2 \cosh(\rho(\gamma_1 x_0, z)) \leq 2e^\eta. \quad (21)$$

we then get a contradiction when

$$\eta < \min \left(\frac{L}{4}, \frac{1}{2} \log \left[\frac{1}{2} \left(\cosh\left(\frac{L}{2}\right) + \frac{1}{\cosh\left(\frac{L}{2}\right)} \right) \right] \right).$$

□

Let Γ be a finitely generated discrete group of isometries of (X, g) and $S = \{\sigma_1, \dots, \sigma_p\}$ be a finite generating set. Let us assume that Γ is not virtually nilpotent and recall that $L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x)$. By lemma 3.1 we have $L(S) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0)$, for some point $x_0 \in X$, and by corollary 2.1, $L(S) \geq \mu(n, a) > 0$. Let us recall that for $0 \leq \eta \leq L$, an element $\gamma \in \Gamma$ is said to be (L, η) -straight if

$$\rho(x_0, \gamma x_0) > (L - \eta)l_S(\gamma).$$

In the following two propositions we give conditions under which there are many non (L, η) -straight elements in Γ .

Proposition 3.1 *Let (X, g) be a Cartan Hadamard manifold whose sectional curvature satisfies $-a^2 \leq K \leq -1$ and Γ a discrete non virtually nilpotent group of isometries of (X, g) generated by $S = \{\sigma_1, \dots, \sigma_p\}$. Let us assume that all σ_i 's are elliptic and that for all $\sigma_i \neq \sigma_j \in S$, the group $\langle \sigma_i, \sigma_j \rangle$ fixes a point $y \in X$ or $\theta \in \partial X$. Let η be a positive number such that*

$$\eta < \min \left(\frac{L}{4}, \frac{1}{2} \left(1 - \frac{1}{\cosh(L)} \right)^2, \frac{1}{2} \log \left[\frac{1}{2} \left(\cosh \frac{L}{2} + \frac{1}{\cosh \frac{L}{2}} \right) \right] \right),$$

where $L = L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0)$, then any $\gamma \in \Gamma$ with $l_S(\gamma) = 2$, i.e $\gamma = \sigma_i^2$ or $\gamma = \sigma_i \sigma_j$, is not $(L, \eta/2)$ -straight, that is, $\rho(x_0, \gamma x_0) \leq 2(L - \eta/2)$.

Proof. Consider the case where $\gamma = \sigma_i^2$. If σ_i is not (L, η) -straight, we have, by the triangle inequality, $\rho(x_0, \sigma_i^2 x_0) \leq 2(L - \eta)$. If σ_i is (L, η) -straight, we have by lemma 3.3,

$$\rho(x_0, \sigma_i^2 x_0) \leq 2L - \left(1 - \frac{1}{\cosh L} \right)^2 \leq 2(L - \eta).$$

Let us now consider the case where $\gamma = \sigma_i \sigma_j$, for $i \neq j$. If σ_i or σ_j is not (L, η) -straight, we have, by the triangle inequality,

$$\rho(x_0, \sigma_i \sigma_j x_0) \leq \rho(x_0, \sigma_i x_0) + \rho(x_0, \sigma_j x_0) \leq L + (L - \eta),$$

therefore, $\rho(x_0, \sigma_i \sigma_j x_0) \leq 2(L - \eta/2)$.

If σ_i and σ_j are (L, η) -straight, lemma 3.5 implies that $\rho(x_0, \sigma_i \sigma_j x_0) \leq 2(L - \eta)$. \square

In the next proposition we will assume that all elements $\gamma \in \Gamma$ whose algebraic length is less than or equal to 4 have a displacement smaller than δ where

$$\delta = \log \left[\cosh\left(\frac{L}{4}\right) \right], \quad (22)$$

and we set

$$\eta = 10^{-3} \left(1 - \frac{\cosh\left(\frac{L}{4}\right)}{\cosh\left(\frac{L}{2}\right)} \right)^4. \quad (23)$$

We will find in that case many non (L, η) -straight elements.

Proposition 3.2 *Let (X, g) be a Cartan-Hadamard manifold whose sectional curvature satisfies $a^2 \leq K_g \leq -1$ and G a discrete non virtually nilpotent group of isometries of (X, g) generated by a set of two isometries $\Sigma = \{\sigma_1, \sigma_2\}$. Let $L = \inf_{x \in X} \max\{\rho(x, \sigma_1 x), \rho(x, \sigma_2 x)\}$, and $\Sigma = \{\sigma_1, \sigma_2\}$. Let η and δ be the numbers defined in (23) and (22). We assume that $l(\gamma') < \delta$ for all $\gamma' \in G$ such that $l_\Sigma(\gamma') < 4$. Then all elements $\gamma \in G$ such that $l_\Sigma(\gamma) = 6$ are not (L, η) -straight.*

We will need the following lemmas.

Lemma 3.6 *Let $\gamma = a\gamma'b \in G$ be such that $l_\Sigma(\gamma) = l_\Sigma(a) + l_\Sigma(\gamma') + l_\Sigma(b)$. If γ is (L, η) -straight, then γ' is $(L, C\eta)$ -straight where $C = \frac{l_\Sigma(\gamma)}{l_\Sigma(\gamma')}$.*

Proof. Let us note that by definition of $L = L(\Sigma)$, we have for any $\gamma \in G$

$$\rho(x_0, \gamma x_0) \leq L \cdot l_\Sigma(\gamma).$$

By triangle inequality we have

$$\rho(x_0, \gamma x_0) \leq \rho(x_0, ax_0) + \rho(x_0, \gamma' x_0) + \rho(x_0, bx_0),$$

hence by assumption on γ we get

$$(L - \eta) l_\Sigma(a\gamma'b) \leq L (l_\Sigma(a) + l_\Sigma(b)) + \rho(x_0, \gamma' x_0)$$

and therefore,

$$\rho(x_0, \gamma' x_0) \geq L l_\Sigma(\gamma') - \eta l_\Sigma(\gamma) \geq (L - C\eta) l_\Sigma(\gamma').$$

\square

Lemma 3.7 *Let α, β be two elements of G distinct of the neutral element and such that $l_\Sigma(\alpha) \leq 2$, $l_\Sigma(\beta) \leq 2$. Under the assumptions of the proposition 3.2, if γ is (L, η) -straight with $l_\Sigma(\gamma) = 6$, then any reduced word representing γ does not contain (i) α^2 or (ii) $\alpha\beta\alpha$.*

Assuming the lemma 3.7, the proof of the proposition 3.2 can be finished as follows:

Proof. Let $\gamma \in G$ of length $l_\Sigma(\gamma) = 6$. Let us write γ as a reduced word in the generators of Σ , $\gamma = \sigma_{i_1}^{p_1} \dots \sigma_{i_k}^{p_k}$, where $\sigma_{i_j} = \sigma_1$ or $\sigma_{i_j} = \sigma_2$, $p_j \in \mathbb{Z}^*$, $i_j \neq i_{j+1}$ and $i_j = i_{j+2}$. Arguing by contradiction we assume that γ is η -straight. Then, by lemma 3.7 (i), all p_j are equal to $+1$ or -1 and in particular we have $k = 6$. Therefore $\gamma = \sigma_{i_1}^{p_1} \cdot \sigma_{i_2}^{p_2} \cdot \sigma_{i_3}^{p_3} \cdot \sigma_{i_4}^{p_4} \cdot \sigma_{i_5}^{p_5} \cdot \sigma_{i_6}^{p_6}$. By lemma 3.7 (ii) we also have $p_{j+2} \neq p_j$ hence $p_{j+2} = -p_j$ so $\gamma = \sigma_{i_1}^{p_1} \cdot \sigma_{i_2}^{p_2} \cdot \sigma_{i_1}^{-p_1} \cdot \sigma_{i_2}^{-p_2} \cdot \sigma_{i_1}^{p_1} \cdot \sigma_{i_2}^{p_2}$, which is impossible by lemma 3.7 (ii) with $\alpha = \sigma_{i_1}^{p_1} \cdot \sigma_{i_2}^{p_2}$ and $\beta = \sigma_{i_1}^{-p_1} \cdot \sigma_{i_2}^{-p_2}$. This finishes the proof of proposition 3.2. \square

Let us now prove the lemma 3.7:

Proof. We first claim that if L , η and δ are chosen as in the proposition 3.2 then we have

$$\eta \leq \frac{L}{4000} \quad (24)$$

and

$$12\eta + \text{Argcosh}(e^{12\eta}) \leq \frac{1}{4} \left(1 - \frac{e^\delta}{\cosh(L/2)} \right). \quad (25)$$

Proof of the claim. By definition of η , (cf. (23)), we have

$$1000\eta = \left(1 - \frac{\cosh(L/4)}{\cosh(L/2)} \right)^4$$

therefore

$$1000\eta < \frac{\cosh(L/2) - \cosh(L/4)}{\cosh(L/2)}$$

and

$$1000\eta < \frac{\sinh(L/2) \cdot L/4}{\cosh(L/2)} < \frac{L}{4},$$

which proves the first inequality of the claim. On the other hand, let $x \in]0, 1[$, then $e^x \leq 1+2x \leq \cosh(2\sqrt{x})$. Choosing $x = 12\eta$ we obtain, using the inequality $\eta < \frac{1}{1000}$, that

$$12\eta + \text{Argcosh}(e^{12\eta}) \leq 12\eta + 2\sqrt{12\eta} < \frac{1}{4}\sqrt{1000\eta}$$

therefore we get

$$12\eta + \text{Argcosh}(e^{12\eta}) \leq \frac{1}{4} \left(1 - \frac{\cosh(L/4)}{\cosh(L/2)} \right)^2 \leq \frac{1}{4} \left(1 - \frac{e^\delta}{\cosh(L/2)} \right),$$

which ends the proof of the claim.

Proof of lemma 3.7 (i). Let us assume that $\gamma = a\alpha^2b$ is (L, η) -straight, and $l_\Sigma(\gamma) = 6$. Then by lemma 3.6 α and α^2 are $(L, 3\eta)$ -straight. We then get with (24)

$$\rho(x_0, \alpha x_0) > (L - 3\eta)l_\Sigma(\alpha) > \frac{L}{2}.$$

On the other hand since $l(\alpha) \leq \delta$ and $\text{Argcosh}[e^\delta] = \frac{L}{4} < \frac{L}{2}$, we can apply lemma 3.3 to α replacing L by $L/2$ and get

$$\rho(x_0, \alpha^2 x_0) < 2\rho(x_0, \alpha x_0) - \left(1 - \frac{e^\delta}{\cosh(L/2)}\right)^2.$$

We then get by the choice of η , cf. (23),

$$\rho(x_0, \alpha^2 x_0) < (L - 3\eta) l_\Sigma(\alpha^2)$$

which contradicts the fact that α^2 is $(L, 3\eta)$ -straight and concludes the proof of lemma 3.7 (i).

Proof of lemma 3.7 (ii). Let us assume that $\gamma = \alpha\beta\alpha b$ is (L, η) -straight, and $l_\Sigma(\gamma) = 6$. The Lemma 3.6 says that $\alpha\beta\alpha$ is $(L, 2\eta)$ -straight and that $\alpha\beta$ is $(L, C'\eta)$ -straight where $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$. Since $\alpha\beta\alpha$ is $(L, 2\eta)$ -straight, we have by triangle inequality

$$(L - 2\eta)l_\Sigma(\alpha\beta\alpha) \leq 2\rho(x_0, \alpha x_0) + L l_\Sigma(\beta)$$

and therefore

$$2\rho(x_0, \alpha x_0) \geq (L - 2\eta)l_\Sigma(\alpha\beta\alpha) - L l_\Sigma(\beta) = L l_\Sigma(\alpha^2) - 2\eta l_\Sigma(\alpha\beta\alpha)$$

hence we obtain

$$\rho(x_0, \alpha x_0) \geq L l_\Sigma(\alpha) - \eta l_\Sigma(\alpha\beta\alpha),$$

and since $l_\Sigma(\alpha) \leq 2$ and $l_\Sigma(\beta) \leq 2$, we deduce that

$$\rho(x_0, \alpha x_0) \geq (L - 4\eta) l_\Sigma(\alpha),$$

that is α is $(L, 4\eta)$ -straight. We set $x_1 = \alpha\beta x_0$, $x_2 = \alpha\beta\alpha x_0$ and $x_3 = (\alpha\beta)^2 x_0 = \alpha\beta\alpha\beta x_0$. We get, since $\alpha\beta\alpha$ is $(L, 2\eta)$ -straight,

$$\begin{aligned} \rho(x_0, x_1) + \rho(x_1, x_2) - \rho(x_0, x_2) &= \rho(x_0, \alpha\beta x_0) + \rho(x_0, \alpha x_0) - \rho(x_0, \alpha\beta\alpha x_0) \\ &\leq L[l_\Sigma(\alpha\beta) + l_\Sigma(\alpha)] - (L - 2\eta)l_\Sigma(\alpha\beta\alpha) \\ &\leq 12\eta. \end{aligned}$$

In the same way, since $\alpha\beta$ is $(L, C'\eta)$ -straight with $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$, we have

$$\begin{aligned} \rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_1, x_3) &= \rho(x_0, \alpha x_0) + \rho(x_0, \beta x_0) - \rho(x_0, \alpha\beta x_0) \\ &\leq L[l_\Sigma(\alpha) + l_\Sigma(\beta)] - (L - C'\eta)l_\Sigma(\alpha\beta) \\ &\leq 2\eta l_\Sigma(\alpha\beta\alpha) \\ &\leq 12\eta. \end{aligned}$$

We can therefore apply the lemma 3.4 and get

$$\begin{aligned} 2\rho(x_0, \alpha\beta x_0) - \rho(x_0, (\alpha\beta)^2 x_0) &\leq \rho(x_0, \alpha\beta x_0) + \rho(x_0, \alpha x_0) + \rho(x_0, \beta x_0) - \rho(x_0, (\alpha\beta)^2 x_0) \\ &= \rho(x_0, x_1) + \rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_0, x_3) \\ &\leq \left(1 + \frac{\rho(x_0, \beta x_0)}{\rho(x_0, \alpha x_0)}\right)(12\eta + \text{Argcosh}[e^{12\eta}]) \\ &\leq \left(1 + \frac{L l_\Sigma(\beta)}{(L - 4\eta)l_\Sigma(\alpha)}\right) \cdot \frac{1}{4} \left(1 - \frac{e^\delta}{\cosh(L/2)}\right)^2, \end{aligned}$$

the last inequality coming from (25) and the fact that α is $(L, 4\eta)$ -straight. From (24) we therefore get

$$\rho(x_0, (\alpha\beta)^2 x_0) \geq 2\rho(x_0, \alpha\beta x_0) - \left(1 - \frac{e^\delta}{\cosh(L/2)}\right)^2. \quad (26)$$

On the other hand we have seen that $\alpha\beta$ is $(L, C'\eta)$ -straight with $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$, so that

$$\rho(x_0, \alpha\beta x_0) \geq (L - C'\eta) l_\Sigma(\alpha\beta) \geq 2L - 2\eta l_\Sigma(\alpha\beta\alpha),$$

and since $l_\Sigma(\alpha\beta\alpha) \leq 6$ the above inequality gives with (24)

$$\rho(x_0, \alpha\beta x_0) \geq L. \quad (27)$$

By assumption, since $l_\Sigma(\alpha\beta) \leq 4$, the displacement of $\alpha\beta$ satisfies $l(\alpha\beta) \leq \delta$, and with (27) we can apply the lemma 3.3 to get

$$\rho(x_0, (\alpha\beta)^2 x_0) \leq 2\rho(x_0, \alpha\beta x_0) - \left(1 - \frac{e^\delta}{\cosh(L/2)}\right)^2$$

which contradicts (26). This concludes the proof of the lemma 3.7 and the proposition 3.2. \square

4 Mapping the Cayley graph of G into X .

Let G be a finitely generated discrete group of isometries of (X, g) a Cartan Hadamard manifold of sectional curvature $-a^2 \leq K \leq -1$. We consider S a finite generating set of G and the Cayley graph \mathcal{G}_S of G associated to S . We define a distance d_S on \mathcal{G}_S in the following way: each edge is isometric to the segment $[0, 1] \subset \mathbb{R}$ and the distance $d_S(\gamma, \gamma')$ between two vertices γ, γ' of \mathcal{G}_S is the word distance $d_S(\gamma, \gamma') = l_S(\gamma^{-1}\gamma')$. The group G acts by isometries on (\mathcal{G}_S, d_S) and on (X, g) . The goal of this section is to construct for each number c large enough an equivariant map $f_c : \mathcal{G}_S \rightarrow X$ such that f_c is Lipschitzian of Lipschitz constant c .

4.1 Poincaré series, measures and convexity.

We first consider the Poincaré series,

$$P_c(s, x, y) = \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \cosh[\rho(x, \gamma y)] \quad (28)$$

where $c \in \mathbb{R}_+$, $s \in \mathcal{G}_S$ and $x, y \in X$.

Lemma 4.1 *For all $s \in \mathcal{G}_S$, $x, y, x_0, y_0 \in X$, $c \in \mathbb{R}$ and $\gamma_0 \in G$ we have*

- (i) $P_c(\gamma_0 s, \gamma_0 x, y) = P_c(s, x, y)$
- (ii) $P_c(s, x, y) \leq P_c(s, x_0, y_0) e^{\rho(x_0, x) + \rho(y_0, y)}$.

In particular the convergence of the series is independant of the choice of the points $x, y \in X$.

Proof. The equivariance property of the Poincaré series is straightforward. On the other hand by triangle inequality we have

$$\begin{aligned} P_c(s, x, y) &= \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \cosh [\rho(x, \gamma y)] \\ &\leq \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \cosh [\rho(x_0, \gamma y_0) + \rho(x_0, x) + \rho(y_0, y)] \end{aligned}$$

hence we get

$$P_c(s, x, y) \leq P_c(s, x_0, y_0) \cdot e^{\rho(x_0, x) + \rho(y_0, y)}.$$

□

The critical exponent of this series is defined as

$$c_0 =: \inf \{ c > 0 \mid P_c(s, x, y) < \infty \}.$$

Let x_0 be the point of X such that $L(S) = \max_i \{ \rho(x_0, \sigma_i x_0) \}$. By the triangle inequality we have for all $\gamma \in \Gamma$, $\rho(x_0, \gamma x_0) \leq L(S) l_S(\gamma)$, therefore

$$P_c(e, x_0, x_0) \leq \sum_{\gamma \in \Gamma} e^{(c-L(S)) l_S(\gamma)}.$$

On the other hand, by definition of $\text{Ent}_S \Gamma$, we have $\sum_{\gamma \in \Gamma} e^{-t l_S(\gamma)} < \infty$ for all $t > \text{Ent}_S \Gamma$, hence we have proved that

$$c_0 \leq \text{Ent}_S \Gamma + L(S). \quad (29)$$

We now consider until the end of this section a $c \in \mathbb{R}_+$ such that $P_c(s, x, y) < \infty$.

Let us choose a probability measure μ with smooth density and compact support on X . For each $s \in \mathcal{G}_S$ let us define the measure on X

$$\mu_s^c = \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \gamma_* \mu \quad (30)$$

and the function $\mathcal{B}^c : G \times X \rightarrow \mathbb{R}$,

$$\mathcal{B}^c(s, x) = \int_X \cosh [\rho(x, z)] d\mu_s^c(z). \quad (31)$$

In the following lemmas 4.2, 4.3 and corollary 4.1 we show that $x \rightarrow \mathcal{B}^c(s, x)$ is a strictly convex C^2 function such that

$$\lim_{x \rightarrow \infty} \mathcal{B}^c(s, x) = +\infty.$$

Lemma 4.2 *Let c be such that $P_c(s, x, y) < \infty$. For all $s \in \mathcal{G}_S$ and $x \in X$, we have $\mathcal{B}^c(s, x) < \infty$. Moreover, the function $x \rightarrow \mathcal{B}^c(s, x)$ is strictly convex and $\lim_{x \rightarrow \infty} \mathcal{B}^c(s, x) = +\infty$.*

Proof. By definition of μ_s^c ,

$$\mathcal{B}^c(s, x) = \int_X \sum_{\gamma \in G} e^{-cd_S(s, \gamma)} \cosh [\rho(x, \gamma z)] d\mu(z) = \int_X P_c(s, x, z) d\mu(z),$$

so we get $\mathcal{B}^c(s, x) < \infty$ by lemma 4.1 (ii) since the support of μ is compact. For any geodesic $c(t)$ and z in X , $t \rightarrow d(c(t), z)$ is a convex function since (X, g) has negative sectional curvature, therefore $t \rightarrow \cosh [\rho(c(t), z)]$ is strictly convex and so is $x \rightarrow \mathcal{B}^c(s, x) = \int_X \cosh [\rho(x, z)] d\mu_s^c(z)$. On the other hand we have

$$\mathcal{B}^c(s, x) = \int_X \cosh [\rho(x, z)] d\mu_s^c(z) \geq \frac{1}{2} e^{\rho(x, x_0)} \int_X e^{-\rho(x_0, z)} d\mu_s^c(z),$$

so $\mathcal{B}^c(s, x) \rightarrow +\infty$ whenever x tends to infinity in X . \square

In the above lemma we proved that $x \rightarrow \mathcal{B}^c(s, x)$ is a convex function which tends to $+\infty$ when x tend to infinity. We shall now prove that $x \rightarrow \mathcal{B}^c(s, x)$ is a strictly convex C^2 function. We will also give estimates of the second derivative of $x \rightarrow \mathcal{B}^c(s, x)$.

Lemma 4.3 *Let c be such that $P_c(s, x, y) < \infty$. The function $x \rightarrow \mathcal{B}^c(s, x)$ is C^2 and for any $s \in \mathcal{G}_S$, $x \in X$ and any tangent vectors $v, w \in T_x X$ we have*

$$d\mathcal{B}^c(s, x)(v) = \int_X d\rho(x, z)(v) \sinh [\rho(x, z)] d\mu_s^c(z)$$

and

$$Dd\mathcal{B}^c(s, x)(v, w) = \int_X \left(\sinh [\rho(x, z)] Dd\rho(x, z)(v, w) + \cosh [\rho(x, z)] d\rho(x, z) \otimes d\rho(x, z)(v, w) \right) d\mu_s^c(z)$$

Proof. Let $v \in T_x X$ be a unit tangent vector at a point $x \in X$. For each point $z \neq x$ in X , we have

$$d(\cosh [\rho(x, z)])(v) = d\rho(x, z)(v) \sinh [\rho(x, z)],$$

hence we get

$$|d(\cosh [\rho(x, z)])(v)| = |d\rho(x, z)(v) \sinh [\rho(x, z)]| \leq \cosh [\rho(x, z)], \quad (32)$$

therefore, $\cosh [\rho(x, z)] \leq 2 \cosh [\rho(x_1, z)]$ for x in a sufficiently small neighbourhood of an arbitrary point x_1 . Since $z \rightarrow 2 \cosh [\rho(x_1, z)]$ is μ_s^c -integrable, we can differentiate $x \rightarrow \mathcal{B}^c(s, x)$ applying Lebesgue derivation theorem. Let us now compute the second derivative. Let $v, w \in T_x X$ be unit tangent vectors at $x \in X$. Let $\alpha(t)$ the geodesic such that $\alpha(0) = x$ and $\alpha'(0) = v$. We denote $W(t)$ the parallel vector field along α such that $W(0) = w$ and write $\rho_{(z, \alpha(t))}$ instead of $\rho(z, \alpha(t))$. Let us denote

$$h(t, z) = \frac{1}{t} \left(d\rho_{(z, \alpha(t))}(W(t)) \sinh [\rho_{(z, \alpha(t))}] - d\rho_{(z, \alpha(0))}(W(0)) \sinh [\rho_{(z, \alpha(0))}] \right).$$

When $z \neq x$ we have

$$\lim_{t \rightarrow 0} h(t, z) = \sinh [\rho_{(x, z)}] Dd\rho_{(x, z)}(v, w) + \cosh [\rho_{(x, z)}] d\rho_{(x, z)} \otimes d\rho_{(x, z)}(v, w). \quad (33)$$

We will write

$$h_0(z) =: \lim_{t \rightarrow 0} h(t, z). \quad (34)$$

The formula which gives $Dd\mathcal{B}^c(s, x)(v, w)$ in lemma 4.3 is equivalent to

$$Dd\mathcal{B}^c(s, x)(v, w) = \int_X h_0(z) d\mu_s^c(z) \quad (35)$$

and will be a consequence of the dominated convergence theorem of Lebesgue with the existence of a $\mu_s^c(z)$ -integrable function $H(z)$ such that for any $z \notin \alpha([0, t])$ then $h(t, z) \leq H(z)$. Let us now prove the existence of such a function H . For each $z \notin \alpha([0, t])$ we have

$$\begin{aligned} |h(t, z)| \leq & \sup_{t' \in [0, t]} \left| \sinh[\rho_{(z, \alpha(t'))}] Dd\rho_{(z, \alpha(t'))}(\dot{\alpha}(t'), W(t')) + \dots \right. \\ & \left. \dots + \cosh[\rho_{(z, \alpha(t'))}] d\rho_{(z, \alpha(t'))} \otimes d\rho_{(z, \alpha(t'))}(\dot{\alpha}(t'), W(t')) \right|. \end{aligned}$$

Since the curvature of (X, g) satisfies $-a^2 \leq K \leq -1$, the Rauch comparison theorem shows that for each $x, y \in X$

$$Dd\rho_{(x, y)} \leq a \frac{\cosh[a\rho_{(x, y)}]}{\sinh[a\rho_{(x, y)}]} \left(g - d\rho_{(x, y)} \otimes d\rho_{(x, y)} \right),$$

hence we get from the previous inequality

$$\begin{aligned} |h(t, z)| \leq & \left[a \sinh[\rho_{(z, \alpha(t'))}] \frac{\cosh[a\rho_{(z, \alpha(t'))}]}{\sinh[a\rho_{(z, \alpha(t'))}]} \left(g - d\rho_{(z, \alpha(t'))} \otimes d\rho_{(z, \alpha(t'))} \right) + \dots \right. \\ & \left. \dots + \cosh[\rho_{(z, \alpha(t'))}] d\rho_{(z, \alpha(t'))} \otimes d\rho_{(z, \alpha(t'))} \right] (\dot{\alpha}(t'), W(t')). \end{aligned}$$

But since $a \geq 1$ the concavity of the function \tanh on \mathbb{R}_+ gives

$$\frac{a}{\tanh a\rho} \geq \frac{1}{\tanh \rho}$$

therefore we get

$$|h(t, z)| \leq a \sinh[\rho_{(z, \alpha(t'))}] \frac{\cosh[a\rho_{(z, \alpha(t'))}]}{\sinh[a\rho_{(z, \alpha(t'))}]} \quad (36)$$

Since $\sinh \rho \leq \frac{1}{a} \sinh a\rho$ by convexity of \sinh , we then get that $|h(t, z)| \leq H(z)$ from 36 for all $|t| \leq \frac{1}{a}$ and all $z \notin \alpha([0, t])$ where

$$H(z) = \begin{cases} a \frac{\cosh 1}{\sinh 1} \sinh[\rho_{(z, \alpha(0))} + 1], & \rho_{(z, \alpha(0))} \geq \frac{2}{a} \\ \cosh[a\rho_{(z, \alpha(0))} + 1], & \rho_{(z, \alpha(0))} < \frac{2}{a} \end{cases}$$

This concludes the proof of lemma 4.3. \square

The above lemma 4.3 has the following

Corollary 4.1 *Under the assumptions of lemma 4.3 we have*

$$Dd\mathcal{B}^c \geq \mathcal{B}^c .g,$$

in particular, \mathcal{B}^c is strictly convex.

Proof. Since the sectional curvature of (X, g) satisfies $K \leq -1$ Rauch's theorem shows that

$$Dd\rho \geq \frac{1}{\tanh \rho} (g - d\rho \otimes d\rho).$$

From this inequality and lemma 4.3 we therefore get, for all $x \in X$ and any unit tangent vector $v \in T_x X$,

$$Dd\mathcal{B}^c(v, v) \geq \left(\int_X \cosh[\rho_{(z,x)}] d\mu_s^c(z) \right) g(v, v) = \mathcal{B}^c(x) .g(v, v).$$

□

4.2 Construction of Lipschitzian maps $f_c : \mathcal{G}_S \rightarrow X$.

So far we have shown that for any $s \in \mathcal{G}_S$ the function $x \rightarrow \mathcal{B}^c(s, x)$ is strictly convex and tends to $+\infty$ when x tend to infinity. We then can define the map $f_c : \mathcal{G}_S \rightarrow X$ as follows. For $s \in \mathcal{G}_S$ we define $f_c(s)$ as the unique point $x \in X$ which achieves the strict minimum of the function $x \rightarrow \mathcal{B}^c(s, x)$. The end of this section is devoted to proving the following

Proposition 4.1 *Let c be such that $P_c(s, x, y) < \infty$. Let $f_c : (\mathcal{G}_S, d_S) \rightarrow (X, g)$ which associates to $s \in \mathcal{G}_S$ the unique point $x \in X$ which achieves the unique minimum of the function $x \rightarrow \mathcal{B}^c(s, x)$. Then, f_c is Lipschitzian of Lipschitz constant equal to c .*

The proof of the proposition 4.1 relies on the following technical lemmas.

Lemma 4.4 *Let c be such that $P_c(s, x, y) < \infty$. For all $x \in X$ and all tangent vector $v \in T_x X$ the function $\alpha : s \rightarrow d\mathcal{B}^c(s, x)(v)$ is differentiable at each point $s \in \mathcal{G}_S$ distinct from a vertex or a middle point of an edge. Moreover, for such an s we have*

$$\alpha'(s) = -c \int_X d\rho_{(x,z)}(v) \sinh[\rho(x, z)] \sum_{\gamma \in G} \frac{d}{ds} (d_S(s, \gamma)) e^{-cd_S(s, \gamma)} d(\gamma_* \mu)(z)$$

Proof. Let us denote by $[g, g']$ the edge containing s and parametrize it by $t \in [0, 1]$. We first observe that for all $\gamma \in G$ then

$$d_S(s, \gamma) = \min [d_S(g, \gamma) + t, d_S(g', \gamma) + 1 - t],$$

therefore $s \rightarrow d_S(s, \gamma)$ is differentiable at each $s \in]g, g'[$ distinct of the middle point of $]g, g'[$. On the other hand we have by lemma 4.3

$$d\mathcal{B}^c(s, x)(v) = \int_X d\rho_{(x,z)}(v) \sinh[\rho(x, z)] d\mu_s^c(z),$$

so that we can write

$$\frac{1}{t}(\alpha(s+t) - \alpha(s)) = \Sigma_{\gamma \in G} \int_X d\rho_{(x,\gamma z)}(v) \sinh[\rho(x,\gamma z)] \cdot \frac{1}{t} [e^{-cd_S(s+t,\gamma)} - e^{-cd_S(s,\gamma)}] d\mu(z) \quad ,$$

where we have identified the point s in the edge $[g, g']$ with its parameter. Let us observe that for $|t|$ small enough,

$$\left| \frac{1}{t} [e^{-cd_S(s+t,\gamma)} - e^{-cd_S(s,\gamma)}] \right| \leq 2c e^{-cd_S(s,\gamma)} \quad ,$$

and that

$$2c \Sigma_{\gamma \in G} \int_X |d\rho_{(x,\gamma z)}(v)| \sinh[\rho(x,\gamma z)] \cdot e^{-cd_S(s,\gamma)} d\mu(z) < \infty \quad ,$$

hence if $s \in \mathcal{G}_S$ is distinct from a vertex or a middle point of an edge we get

$$\lim_{t \rightarrow 0} \frac{1}{t}(\alpha(s+t) - \alpha(s)) = -c \int_X d\rho_{(x,z)}(v) \sinh[\rho(x,z)] \Sigma_{\gamma \in G} \frac{d}{ds}(d_S(s,\gamma)) e^{-cd_S(s,\gamma)} d(\gamma_*\mu)(z)$$

by Lebesgue's theorem. □

Lemma 4.5 *Let c be such that $P_c(s, x, y) < \infty$. Let $s_0 \in \mathcal{G}_S$ be a point distinct from a vertex or a middle point of an edge, and u a unit vector tangent at s_0 to the edge containing s_0 . Then, we have $\|df_c(u)\| \leq c$.*

Proof. Let us fix a smooth moving frame $\{E_1, \dots, E_n\}$ of TX and define the function $\Phi : X \times \mathcal{G}_S \rightarrow \mathbb{R}^n$ by

$$\Phi(x, s) = (d\mathcal{B}^c(s, x)(E_1), \dots, d\mathcal{B}^c(s, x)(E_n)) \quad .$$

By definition, the point $f_c(s)$ is characterized by the implicit equation

$$\Phi(f_c(s), s) = 0 \quad ,$$

or equivalently,

$$d\mathcal{B}^c(s, f_c(s)) = 0 \quad .$$

For all $x \in X$ and $s \in \mathcal{G}_S$ in a neighbourhood of s_0 the function Φ is differentiable by lemma 4.3 and 4.4. Moreover since $x = f_c(s)$ is a critical point of the function $x \rightarrow \mathcal{B}^c(s, x)$, we have, for $j = 1, \dots, n$,

$$\frac{\partial \Phi}{\partial x}(f_c(s), s)(E_j) = (Dd\mathcal{B}^c(s, f_c(s))(E_j, E_1), \dots, Dd\mathcal{B}^c(s, f_c(s))(E_j, E_n)) \quad ,$$

thus $\frac{\partial \Phi}{\partial x}(f_c(s), s)$ is invertible by corollary 4.1. By the implicit function theorem, the function f_c is then differentiable at s in a neighbourhood of s_0 and we have,

if u is a unit vector tangent at s_0 to the edge containing s_0 and v a tangent vector in $T_{f_c(s)}X$,

$$Dd\mathcal{B}^c(s_0, f_c(s_0))(df_c(u), v) = -\frac{d}{ds}\Big|_{s=s_0} d\mathcal{B}^c(s, f_c(s_0))(v). \quad (37)$$

From corollary 4.1 and lemma 4.4 we obtain, setting $v = \frac{df_c(u)}{\|df_c(u)\|}$

$$g(df_c(u), v) \mathcal{B}^c(s_0, f_c(s_0)) \leq c \int_X |d\rho_{(f_c(s_0), z)}(v)| \sinh[\rho(f_c(s_0), z)] \Sigma_{\gamma \in G} \Big| \frac{d}{ds}\Big|_{s=s_0} (d_S(s, \gamma)) \Big| e^{-cd_S(s_0, \gamma)} d(\gamma_*\mu)(z)$$

therefore

$$|g(df_c(u), v) \mathcal{B}^c(s_0, f_c(s_0))| \leq c \int_X \sinh[\rho(f_c(s_0), z)] d\mu_{s_0}^c(z). \quad (38)$$

hence

$$\|df_c(u)\| \leq c \frac{\int_X \sinh[\rho(f_c(s_0), z)] d\mu_{s_0}^c(z)}{\int_X \cosh[\rho(f_c(s_0), z)] d\mu_{s_0}^c(z)} \leq c,$$

which completes the proof of lemma 4.5. \square

The proposition 4.1 follows then from the

Corollary 4.2 *Let c be such that $P_c(s, x, y) < \infty$. The map f_c is Lipschitzian of Lipschitz constant equal to c .*

Proof. Let us consider a segment $[s_1, s_2] \subset \mathcal{G}_S$ which contains no edges nor middle points. It directly follows from lemma 4.5 that

$$\rho(f_c(s_1), f_c(s_2)) \leq c d_S(s_1, s_2). \quad (39)$$

We now want to extend the inequality (39) for all points $s_1, s_2 \in \mathcal{G}_S$. For that purpose we first consider a segment $[s_1, s_2] \subset \mathcal{G}_S$ where s_1 is a midpoint of an edge e and s_2 a vertex of the same edge e and the inequality (39) for these points s_1, s_2 derives from the continuity of f_c at x_1 and x_2 . The corollary 4.2 will then follow from the fact that any segment $[s_1, s_2] \subset \mathcal{G}_S$ can be decomposed in a finite sequence of adjacent intervals $[y_1^k, y_2^k]$ where y_1^k is a midpoint and y_2^k a vertex of one same edge or the other way around. Let us prove the continuity of f_c at a vertex or a midpoint s of an edge. Given such a point s , let $\{s_k\}_{k \in \mathbb{N}}$ be a sequence converging to s and staying in a single mid-edge containing s . The sequence $x_k =: f_c(s_k)$ is a Cauchy sequence in X by (39) whose limit is a point $x = \lim_k x_k$. We want to prove that $f_c(s) = x$. For all $z \in X$ and $k \in \mathbb{N}$ we have

$$\mathcal{B}^c(s_k, z) \geq \mathcal{B}^c(s_k, x_k) \quad (40)$$

by definition of $x_k = f_c(s_k)$. We claim that $\lim_k \mathcal{B}^c(s_k, x_k) = \mathcal{B}^c(s, x)$ and that $\lim_k \mathcal{B}^c(s_k, z) = \mathcal{B}^c(s, z)$. Assuming the claim and passing to the limit in (40) when k tends to infinity gives for all $z \in X$,

$$\mathcal{B}^c(s, z) \geq \mathcal{B}^c(s, x), \quad (41)$$

therefore $x = f_c(s)$. Let us prove the claim. By definition (31) and (30), we have

$$\begin{aligned}\mathcal{B}^c(s_k, x_k) &= \int_X \cosh[\rho(x_k, z)] d\mu_{s_k}^c(z) \\ &= \int_X \sum_{\gamma \in G} e^{-cd_S(s_k, \gamma)} \cosh[\rho(x_k, \gamma z)] d\mu(z),\end{aligned}$$

Since $e^{-cd_S(s_k, \gamma)} \cosh[\rho(x_k, \gamma z)] \leq e^c e^{-cd_S(s, \gamma)} \cosh[\rho(x, \gamma z) + 1]$ for k large enough, we get $\lim_k \mathcal{B}^c(s_k, x_k) = \mathcal{B}^c(s, x)$ by the Lebesgue's theorem. By the way we get $\lim_k \mathcal{B}^c(s_k, z) = \mathcal{B}^c(s, z)$ which concludes the proof of the claim, the corollary 4.2 and the proposition 4.1. \square

5 Algebraic Entropy and η -straight isometries.

Let G be a finitely generated discrete group of isometries of (X, g) whose sectional curvature satisfies $-a^2 \leq K_g \leq -1$, and $S = \{\sigma_1, \dots, \sigma_p\}$ be a finite generating set.

We assume that the minimal displacement $L(S) = \inf_{x \in X} \max_{i=1, \dots, p} \rho(x, \sigma_i x)$ of S (cf. definition 2.1) satisfies $L(S) > 0$. By lemma 3.1 there exist a point $x_0 \in X$ such that

$$L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0).$$

The goal of this section is to prove that if all elements of G are “almost non η -straight” for some η such that $L(S) > \eta > 0$, then the entropy of G with respect to S is bounded below by η . By “almost non η -straight” elements we mean isometries γ such that $\rho(x_0, \gamma x_0) \leq (L(S) - \eta)l_S(\gamma) + D$, for some positive number D .

Theorem 5.1 *Let G be a finitely generated discrete group of isometries of (X, g) whose sectional curvature satisfies $-a^2 \leq K_g \leq -1$, and $S = \{\sigma_1, \dots, \sigma_p\}$ be a finite generating set of G with $L(S) = \inf_{x \in X} \max_{i \in \{1, \dots, p\}} \rho(x, \sigma_i x) = \max_{i \in \{1, \dots, p\}} \rho(x_0, \sigma_i x_0) > 0$. Let us assume that there exist $D \geq 0$ and η , $0 < \eta < L(S)$, such that for all $\gamma \in G$,*

$$\rho(x_0, \gamma x_0) \leq (L(S) - \eta)l_S(\gamma) + D, \quad (42)$$

then $\text{Ent}_S(G) \geq \eta$.

Proof. The proof relies on the construction made in section 3 of an equivariant Lipschitzian map of Lipschitz constant $c > \text{Ent}_S(G) + L(S) - \eta$.

Let us prove that under the assumption (42) then for any $c > \text{Ent}_S(G) + L(S) - \eta$ we have $P_c(s, x, y) < \infty$. By triangle inequality we have

$$e^{-cd_S(s, \gamma)} \leq e^{cd_S(s, e)} e^{-cd_S(\gamma, e)},$$

and for any $x_0 \in X$

$$\cosh [\rho(x, \gamma y)] \leq e^{\rho(x, \gamma y)} \leq e^{\rho(x, x_0) + \rho(x_0, y) + \rho(x_0, \gamma x_0)}.$$

Therefore for x_0 , D and η chosen such that (42) holds, we get

$$P_c(s, x, y) \leq e^{D+c d_S(e, s) + \rho(x, x_0) + \rho(x_0, y) \sum_{\gamma \in G} e^{[L(S) - \eta - c] d_S(e, \gamma)}},$$

and so $P_c(s, x, y) < \infty$ for each $c > \text{Ent}_S(G) + L(S) - \eta$.

Hence by section 3, proposition 4.1 there exists an equivariant Lipschitzian map $f_c : (\mathcal{G}_S, d_S) \rightarrow (X, g)$ of Lipschitz constant c for any $c > \text{Ent}_S(G) + L(S) - \eta$. We consider the point $x = f_c(e)$, where e is the neutral element of G . By definition of $L(S)$, there is a $\sigma_i \in S$ such that $\rho(x, \sigma_i x) \geq L(S)$. Therefore, by equivariance,

$$\rho(f_c(e), \sigma_i(f_c(e))) = \rho(f_c(e), f_c(\sigma_i(e))) \geq L(S).$$

On the other hand, since f_c is c -Lipschitzian we have

$$\rho(f_c(e), f_c(\sigma_i(e))) \leq c d_S(e, \sigma_i(e)) = c.$$

The two above inequalities give

$$c \geq L(S)$$

and since c is any number such that $c > \text{Ent}_S(G) + L(S) - \eta$, we get $\text{Ent}_S(G) \geq \eta$. \square

6 Proof of the main theorem.

In this section we shall first prove that the entropy of a group with respect to a set of two generators with displacement $L > 0$ is bounded below. Then we shall prove the main theorem.

Proposition 6.1 *Let (X, g) be a Cartan-Hadamard manifold whose sectional curvature satisfies $a^2 \leq K_g \leq -1$ and G a discrete group of isometries of (X, g) generated by two isometries $\{\sigma_1, \sigma_2\}$. Let us assume*

$$L = \inf_{x \in X} \max\{\rho(x, \sigma_1 x), \rho(x, \sigma_2 x)\} > 0.$$

Then the entropy of G relatively to the set of generators $\Sigma = \{\sigma_1, \sigma_2\}$ satisfies

$$\text{ent}_\Sigma G \geq \min \left[\frac{\log(\cosh(\frac{L}{4}))}{5 + \log(\cosh(\frac{L}{4}))} \frac{\log 2}{6}, \frac{1}{1000} \left(1 - \frac{\cosh(\frac{L}{4})}{\cosh(\frac{L}{2})} \right)^4 \right].$$

Proof. Let $\delta = \log \cosh(\frac{L}{4})$. The proof divides into two cases. In the first case we can find two elements in G of bounded length l_Σ which are hyperbolic with distinct axes and displacement larger than δ . In that case, a ping-pong argument

shows that the semigroup generated by these two elements (or their inverse) is free with corresponding entropy bounded below by a constant depending on δ . In the second case, when we cannot find such a free semigroup, then we can show that all elements of G are almost non η -straight for some $\eta = \eta(\delta, L)$ and we will conclude by theorem 5.1.

Case 1. There exists an element $\gamma \in G$ of algebraic length $l_\Sigma(\gamma) \leq 4$ whose displacement $l(\gamma)$ in X is bounded below $l(\gamma) > \delta$.

Case 2. The displacement of all elements $\gamma \in G$ of algebraic length $l_\Sigma(\gamma) \leq 4$ satisfies $l(\gamma) \leq \delta$.

In the case 1, let us consider an element $\gamma \in G$ of algebraic length $l_\Sigma(\gamma) \leq 4$ and whose displacement $l(\gamma)$ in X satisfies $l(\gamma) > \delta$. We note that γ is then an hyperbolic isometry of X . Since G is not virtually nilpotent one of the generators σ_1 or σ_2 , say σ_1 does not preserve the axis of γ . Indeed if both σ_1 and σ_2 were preserving the axis of γ , then G would preserve the axis of γ and hence would be virtually abelian by lemma 2.2 (ii), contradiction. Then, if (θ, η) are the endpoints of the axis of γ , $\sigma_1(\{\theta, \eta\}) \cap \{\theta, \eta\} = \emptyset$ by the proof of lemma 2.1, a). We can then apply the effective ping-pong lemma proved in the appendix to the two hyperbolic elements γ and $\sigma_1\gamma\sigma_1^{-1}$ which have disjoint fixed-point-sets. This shows that the algebraic entropy of the subgroup generated by γ and $\sigma_1\gamma\sigma_1^{-1}$ is bounded below by $\frac{\delta}{5+\delta} \log 2$. We then deduce that,

$$\text{Ent}_\Sigma(\Gamma) \geq \frac{\delta}{5+\delta} \frac{\log 2}{6}.$$

In the case 2, proposition 3.2 tells that all elements $\gamma \in G$ of length $l_\Sigma(\gamma) = 6$ are not (L, η) -straight where η is given by (23), $\eta = 10^{-3} \left(1 - \frac{\cosh(\frac{L}{2})}{\cosh(\frac{L}{2})}\right)^4$. Then every element $g \in \Gamma$ of algebraic length 6 satisfies,

$$\rho(x_0, gx_0) \leq (L - \eta)l_\Sigma(g).$$

Hence, one obtain that every element $\gamma \in \Gamma$, satisfies,

$$\rho(x_0, \gamma x_0) \leq (L - \eta)(l_\Sigma(\gamma) - 5) + 5L.$$

Therefore we get from theorem 5.1 that $\text{Ent}_\Sigma G \geq \eta = 10^{-3} \left(1 - \frac{\cosh(\frac{L}{2})}{\cosh(\frac{L}{2})}\right)^4$. \square

We may now prove the main theorem which we recall below,

Theorem 6.1 (Main theorem) *Let (X, g) be a Cartan-Hadamard manifold whose sectional curvature satisfies $-a^2 \leq K_g \leq -1$. Let Γ be a discrete and finitely generated subgroup of the isometry group of (X, g) , then either Γ is virtually nilpotent or its algebraic entropy is bounded below by an explicit constant $C(n, a)$.*

Remark. The constant is

$$C(n, a) = \min \left[\frac{\log(\cosh(\frac{\mu(n, a)}{4}))}{5 + \log(\cosh(\frac{\mu(n, a)}{4}))} \frac{\log 2}{12}, \frac{1}{2000} \left(1 - \frac{\cosh(\frac{\mu(n, a)}{4})}{\cosh(\frac{\mu(n, a)}{2})}\right)^4, \frac{\mu(n, a)}{4}, \right]$$

$$\frac{1}{4} \left(1 - \frac{1}{\cosh \mu(n, a)}\right)^2, \frac{1}{2} \log \left(\frac{1}{2} \left(\cosh \frac{\mu(n, a)}{2} + \frac{1}{\cosh \frac{\mu(n, a)}{2}} \right) \right) \Bigg].$$

Proof. If $S = \{\sigma_1, \dots, \sigma_p\}$ is a finite generating set of Γ , proposition 2.1 allows to reduce to the following three cases,

i) there exist $\sigma_i, \sigma_j \in S$ such that $L(\langle \sigma_i, \sigma_j \rangle) \geq \mu(n, a)$ and such that the subgroup $\langle \sigma_i, \sigma_j \rangle$ is not virtually nilpotent.

ii) There exist $\sigma_i, \sigma_j, \sigma_k \in S$ such that $L(\langle \sigma_i \sigma_j, \sigma_k \rangle) \geq \mu(n, a)$ and such that $\langle \sigma_i \sigma_j, \sigma_k \rangle$ is not virtually nilpotent.

iii) All σ_i 's are elliptic and, for all $i \neq j$, the subgroup $\langle \sigma_i, \sigma_j \rangle$ fixes a point $y \in X$ or a point $\theta \in \partial X$.

In the first case (resp. the second case) proposition 6.1 gives a lower bound of the algebraic entropy of $\langle \sigma_i, \sigma_j \rangle$ (resp. $\langle \sigma_i \sigma_j, \sigma_k \rangle$) with respect to the generating set $\{\sigma_i, \sigma_j\}$ (resp. $\{\sigma_i \sigma_j, \sigma_k\}$), by the number,

$$\min \left[\frac{\log(\cosh(\frac{\mu(n, a)}{4}))}{5 + \log(\cosh(\frac{\mu(n, a)}{4}))} \frac{\log 2}{6}, \frac{1}{1000} \left(1 - \frac{\cosh(\frac{\mu(n, a)}{4})}{\cosh(\frac{\mu(n, a)}{2})}\right)^4 \right],$$

using the fact that $L(\sigma_i, \sigma_j) \geq \mu(n, a)$, (resp. $L(\sigma_i \sigma_j, \sigma_k) \geq \mu(n, a)$). We conclude in the two first cases (i) and (ii) by noticing that the entropy of Γ with respect to S is bounded below by $\text{Ent}_{\{\sigma_i, \sigma_j\}}(\langle \sigma_i, \sigma_j \rangle)$ (resp. by $\frac{1}{2} \text{Ent}_{\{\sigma_i \sigma_j, \sigma_k\}}(\langle \sigma_i \sigma_j, \sigma_k \rangle)$), since $d_{\{\sigma_i, \sigma_j\}} \geq d_S$ (resp. $d_{\{\sigma_i \sigma_j, \sigma_k\}} \geq \frac{1}{2} d_S$).

In the third case, proposition 3.1, implies that,

$$\rho(x_0, \gamma x_0) \leq (L(S) - \eta/2)(l_S(\gamma) - 1) + L(S),$$

where η is given in proposition 3.1. We conclude by applying theorem 5.1, which gives $\text{Ent}_S(\Gamma) \geq \eta$, and then bounding below η using $L(S) \geq \mu(n, a)$. \square

7 Appendix

In this section (X, g) is a Cartan-Hadamard manifold of sectional curvature $K \leq -1$. It is well known that if α, β are two hyperbolic isometries of (X, g) with disjoint axes, then a sufficiently large power α^N and β^N of α and β generates a non abelian free group of $\text{Isom } X$. In [6], [8], it was shown that if Γ is an hyperbolic group then N can be chosen independantly of α and β in Γ and under the same assumptions the N was shown to depend only on the number of generators and the constant of hyperbolicity of Γ [6]. In what follows we show that $N = N(\delta)$ can be chosen independantly of α and β two hyperbolic isometries of (X, g) with disjoint set of fixed points and displacement greater than or equal to a positive number δ .

Proposition 7.1 *Let (X, g) be a Cartan Hadamard manifold of sectional curvature $K \leq -1$ and Γ a discrete group of isometries in $\text{Isom}(X, g)$. We assume that α and β have disjoint set of fixed point and their displacement satisfy $l(\alpha) \geq \delta$ and $l(\beta) \geq \delta$, where δ is a positive number. Then, (α^N, β^N) or (α^N, β^{-N}) generates a non abelian free semi-group, where $N = E(\frac{5}{\delta}) + 1$.*

Before going to the proof of the proposition 7.1 let us set some notations. Let us write $x = x(t)$ and $y = y(t)$, $t \in \mathbb{R}$, the axes of α and β . The points $\theta^\pm = \lim_{t \rightarrow \pm\infty} x(t)$ and $\zeta^\pm = \lim_{t \rightarrow \pm\infty} y(t)$ are the fixed points of α and β on the ideal boundary ∂X of X . Let us denote x^+ and x^- the projections of ζ^+ and ζ^- on the axis of α . We can assume that x^+ is closer to θ^+ than x^- , (if it is not the case, we replace β by β^{-1}). Let us also denote y_0 the projection of x^+ on the axis of β . We now parametrize x and y in such a way that $x(0) = x^+$ and $y(0) = y_0$. We set $t_1 = Nl(\alpha) = l(\alpha^N)$ and $t_2 = Nl(\beta) = l(\beta^N)$, where $N = E(\frac{5}{\delta}) + 1$ is chosen as in the proposition. We define U^\pm as the set of points $p \in X$ such that $\rho(p, x(\pm t_1)) \leq \rho(p, x(o))$. In the same way we define V^\pm as the set of points $p \in X$ such that $\rho(p, y(\pm t_2)) \leq \rho(p, y(o))$. For a unit tangent vector $u \in T_x X$ at a point $x \in X$ and $\alpha \in [0, \pi[$ we denote $\mathcal{C}(u, \alpha) = \{exp_x v : v \in T_x X, \angle(u, v) \in [0, \alpha[\}$ the cone of angle α at x , where exp_x is the exponential map at x .

We further need the following geometric lemmas. For a triangle ABC in (X, g) , we will write \hat{A} the angle at A , and a, b, c the length of the sides opposite to A, B and C .

Lemma 7.1 *Let ABC be a triangle in (X, g) such that $\frac{\pi}{6} \leq \hat{A} \leq \pi$, then $\rho(B, C) > \rho(A, B) + \rho(A, C) - 4$. Moreover, if $\hat{A} \geq \frac{\pi}{2}$, then $\rho(B, C) > \rho(A, B) + \rho(A, C) - 1$.*

Proof. Since the curvature $K \leq -1$, we have

$$\cosh a \geq \cosh b \cosh c - \cos \hat{A} \sinh b \sinh c. \quad (43)$$

The first inequality of lemma 7.1 will therefore be a consequence of the fact that if $b + c > 4$ then

$$\cosh(b + c - 4) - \cosh b \cosh c + \cos \hat{A} \sinh b \sinh c < 0. \quad (44)$$

Setting $X = e^{-(b+c)}$ we have

$$\begin{aligned} & \cosh(b + c - 4) - \cosh b \cosh c + \cos \hat{A} \sinh b \sinh c = \\ & e^{(b+c)} \left[(2e^4 - 1 + \cos \hat{A})X^2 - (e^{-2b} + e^{-2c})(1 + \cos \hat{A}) - (1 - \cos \hat{A} - 2e^{-4}) \right]. \end{aligned}$$

Since $e^{-2b} + e^{-2c} \geq 2e^{-(b+c)}$, we then get

$$\cosh(b + c - 4) - \cosh b \cosh c + \cos \hat{A} \sinh b \sinh c \leq e^{(b+c)} P(X)$$

where

$$P(X) = (2e^4 - 1 + \cos \hat{A})X^2 - (1 + \cos \hat{A})X - (1 - \cos \hat{A} - 2e^{-4})$$

and $P(X)$ is negative when $P(0) < 0$ and $P(e^{-4}) < 0$ which is the case if $\cos \hat{A} < 1 - 2e^{-4}$ and so when $\hat{A} \geq \pi/6$. This proves the first inequality of the lemma. The second inequality is proved similarly when $\cos \hat{A} < 1 - 2e^{-1}$. \square

Lemma 7.2 *The sets U^+ and U^- are contained in $\mathcal{C}(\dot{x}(0), \pi/6)$ and $\mathcal{C}(-\dot{x}(0), \pi/6)$ respectively.*

Proof. We recall that $x(0) = x^+$. Let $c(t)$ be a geodesic ray starting at x^+ such that $\angle(\dot{x}(0), \dot{c}(0)) \geq \pi/6$. Since $t_1 \geq 5$, the lemma 7.1 implies

$$\rho(c(t), x(t_1)) > \rho(x^+, c(t)) + \rho(x^+, x(t_1)) - 4 \geq \rho(c(t), x^+)$$

therefore $c(t) \notin U^+$. The same argument holds for U^- . \square

Let us denote by z_t the geodesic joining x^+ and $y(t)$ and $z_{\pm\infty}$ the geodesic joining x^+ and $y(\pm\infty) = \zeta^{\pm\infty}$.

Lemma 7.3 *The set V^\pm is contained in $\mathcal{C}(\dot{z}_{\pm\infty}(0), \pi/3)$.*

Proof. Let us recall that the angle at $y(0) = y_0$ between z_0 and y is equal to $\pi/2$, so that the lemma 7.1 says that

$$\text{length}(z_t) > \text{length}(z_0) + t - 1, \quad (45)$$

and in particular,

$$\text{length}(z_{t_2}) > \text{length}(z_0) + t_2 - 1. \quad (46)$$

Let us now show that $\angle(\dot{z}_{t_2}(0), \dot{z}_{+\infty}(0)) \leq \pi/6$. Assume by contradiction that $\angle(\dot{z}_{t_2}(0), \dot{z}_{+\infty}(0)) > \pi/6$, then by lemma 7.1 we have, when t tends to $+\infty$,

$$t - t_2 > \text{length}(z_{t_2}) + \text{length}(z_t) - 4, \quad (47)$$

but summing up (45) and (46), in (47) leads to a contradiction since $t_2 \geq 5$. Therefore we have $\angle(\dot{z}_{t_2}(0), \dot{z}_{+\infty}(0)) \leq \pi/6$. Let now consider a geodesic ray c starting at x^+ such that $\angle(\dot{c}(0), \dot{z}_{+\infty}(0)) \geq \pi/3$. Thus, $\angle(\dot{c}(0), \dot{z}_{t_2}(0)) \geq \pi/6$ and by lemma 7.1 we get

$$\rho(c(t), y(t_2)) > \rho(c(t), x^+) + \text{length}(z_{t_2}) - 4,$$

and applying again the lemma 7.1,

$$\rho(c(t), y(t_2)) > \rho(c(t), x^+) + \text{length}(z_0) + t_2 - 5.$$

The last inequality becomes by triangle inequality,

$$\rho(c(t), y(t_2)) > \rho(c(t), y_0) + t_2 - 5,$$

therefore $\rho(c(t), y(t_2)) > \rho(c(t), y_0)$ since $t_2 \geq 5$.

We have proved that a geodesic ray c starting at x^+ such that $\angle(\dot{c}(0), \dot{z}_{+\infty}(0)) \geq \pi/3$ does not intersect V^+ . This proves that $V^+ \subset \mathcal{C}(\dot{z}_{+\infty}(0), \pi/3)$. By the same argument we also have $V^- \subset \mathcal{C}(\dot{z}_{-\infty}(0), \pi/3)$, which ends the proof of the lemma. \square

Lemma 7.4 *We have $U^\pm \cap V^\pm = \emptyset$, $U^+ \cap U^- = \emptyset$, $V^+ \cap V^- = \emptyset$.*

Proof. From the angle relations, $\angle(\dot{x}(0), \dot{z}_{+\infty}(0)) = \pi/2$, $\angle(\dot{x}(0), \dot{z}_{-\infty}(0)) \geq \pi/2$, $\angle(\dot{x}(0), -\dot{x}(0)) = \pi$, and the relative position of x^+ , x^- and θ^+ , it follows that $\mathcal{C}(\dot{x}(0), \pi/6)$ does not intersect $\mathcal{C}(\dot{z}_{+\infty}(0), \pi/3)$, $\mathcal{C}(-\dot{x}(0), \pi/6)$ and $\mathcal{C}(\dot{z}_{-\infty}(0), \pi/3)$. Therefore by the lemmas 7.2, 7.3 we conclude that U^+ does not intersect U^- , V^+ and V^- . Now since $\angle(-\dot{x}(0), \dot{z}_{+\infty}(0)) = \pi/2$, we have $\mathcal{C}(\dot{z}_{+\infty}(0), \pi/3) \cap \mathcal{C}(-\dot{x}(0), \pi/6) = \emptyset$, hence $V^+ \cap U^- = \emptyset$. If $p \in V^+ \cap V^-$, we have $\rho(p, y(0)) \geq \rho(p, y(-t_2))$ and $\rho(p, y(0)) \geq \rho(p, y(t_2))$ which contradicts the convexity of the function $t \rightarrow \rho(x(t), p)$. Therefore $V^+ \cap V^- = \emptyset$. \square

Lemma 7.5 *We have $\alpha^N(V^+) \subset U^+$ and $\beta^N(U^+) \subset V^+$.*

Proof. Since x and y are the axes of α^N and β^N respectively we have $\alpha^N(x(-t_1)) = x(0)$, $\beta^N(y(-t_1)) = y(0)$, $\alpha^N(x(0)) = x(t_1)$ and $\beta^N(y(0)) = y(t_2)$. Therefore for any $p \in X - U^-$, we have $\alpha^N(p) \in U^+$ and similarly for any $p \in X - V^-$, we have $\beta^N(p) \in V^+$ by definition of N . On the other hand, by the lemma 7.4, we have $V^+ \subset X - U^-$ and $U^+ \subset X - V^-$, which concludes. \square

The proof of the proposition 7.1 is a direct application of the lemma 7.5 by a standard ping-pong argument.

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