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► **To cite this version:**

Youcef Askoura. About extension of upper semicontinuous multi-valued maps and applications. 2006.
hal-00331532

HAL Id: hal-00331532

<https://hal.science/hal-00331532>

Preprint submitted on 17 Oct 2008

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About extension of upper semicontinuous multi-valued maps and applications

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17th October 2008

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Abstract

We formulate a multi-valued version of the Tietze-Urysohn extension theorem. Precisely, we prove that any upper semicontinuous multi-valued map with nonempty closed convex values defined on a closed subset (resp. closed perfectly normal subset) of a completely normal (resp. of a normal) space X into the unit interval $[0, 1]$ can be extended to the whole space X . The extension is upper semicontinuous with nonempty closed convex values. We apply this result for the extension of real semicontinuous functions, the characterization of completely normal spaces, the existence of Gale-Mas-Colell and Shafer-Sonnenschein type fixed point theorems and the existence of equilibrium for qualitative games.

Mathematics Subject Classification: 54C60, 54C20

Keywords: Extension of multi-valued maps, Tietze-Urysohn extension theorem, Fixed points, Maximal elements, Qualitative equilibrium

1 Introduction

The aim of this paper is to extend upper semicontinuous (*usc* in short) multi-valued maps over spaces as large as possible. In particular, we want to avoid metrizability in definition domains. We are interested by extending *usc* multi-valued maps defined on a closed subset of a given topological space to another. Similar results are already obtained by Cellina [6], Brodskii [4, 5], Tan and Wu [21] and Ma [15], using the metrizability of the domain. A more general result, in this direction, is that of Borges [2], which established extensions for *usc* maps defined on closed subsets of stratifiable spaces to any topological space. Cite also results of Drozdovskii and Filippov [8] and Shishkov [20] which extended *usc* maps defined on closed subsets of paracompact completely normal spaces to completely metrizable ones. Another type of extensions, which is not concerned here, is to extend maps defined on dense subsets [11, 14].

In the sequel, when speaking about extension of maps, we signify extensions of the same type (single valued if the map is single valued and multi-valued if the map is multi-valued) and preserving the given continuity concept (continuity if the original map is single valued and continuous, upper semicontinuity if the original map is multi-valued and upper semicontinuous). If we try to compare the extension of usc multi-valued maps with continuous single-valued maps, two things appear. First, for usc maps, the extension need only be usc, then we are tempted to say that the first extension is easier. But, within the definition domains, for a map, to be continuous and single-valued is very constraining comparatively with the fact to be usc and multi-valued, then provides us additional properties. So the two problems are, a priori, quite different, and without evident comparison between them. The results of this paper (and some of the cited ones) prove, in fact, that the extension of multi-valued maps is more constraining. We consider an usc multi-valued map $T : A \subset E \rightarrow I$, where A is a closed subset of a topological space E and I the unit real interval. We obtain an extension of T when E is completely normal or E is normal and A is perfectly normal. As it is known, this type of results provides a characterization of completely normal spaces. An extension result (for multi-valued maps) in an infinite uncountable product of spaces gives directly some existence results of maximal elements and fixed points ([10], [19] and [12]) needed in game theory. The problem is to avoid in the proof, properties which are not satisfied in an uncountable products of usual (or a simply important class of) spaces (like metrizable). Unfortunately, the complete normality is not, in general, a property of an infinite uncountable product of spaces. So, the results characterizing the complete normality by extensions of usc multi-valued maps, suggest us to search maximal elements (resp. equilibrium for qualitative games) for an uncountable usc multi-valued maps (resp. with an uncountable set of players and usc preference correspondences) with additional requirements. Such applications are given in the second part of this paper. Our result can be seen as a multi-valued version of the Tietze-Urysohn extension theorem. As it is known, the original proof of this fundamental result uses the uniform convergence of a sequence of functions. However, some researchers ([1, 16, 17, 18, 22]) asked for a possibility to prove this result without the use of uniform convergence. The proof of our extension theorem is inspired particularly by the paper of Ossa [17], and proves that his technics are successful for usc multi-valued maps. It is direct and elementary. In this paper, *usc* means upper semicontinuous, *lsc* means lower semicontinuous, $co(A)$ means the convex hull of A . If Y is a topological space and X a subset of Y , then $int_Y(X)$ refer to the interior of X in Y and \overline{X} is the adherence (or the closure) of X in Y . For a multi-valued map $T : E_1 \rightarrow E_2$, we denote $Dom(T) = \{x \in E_1, T(x) \neq \emptyset\}$. In the whole of this document, the subsets are endowed with the induced topology.

2 Multi-valued version of the Tietze-Urysohn extension theorem

Recall two definitions :

A separated topological space X is said to be completely normal if it is hereditarily normal, that is : every subspace of X is normal. This definition is equivalent to the following : X is completely normal if and only if all subsets A and B of X satisfying $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ can be separated by open sets, *i.e.* there exists two open subsets of X , U and V , $A \subset U$, $B \subset V$ such that $U \cap V = \emptyset$.

A separated topological space X is said to be perfectly normal if each closed subset of X is a G_δ -set, *i.e.* intersection of a countable open sets. Or equivalently, X is perfectly normal if each open subset of X is an F_σ -set, *i.e.* a countable union of closed sets.

The perfect normality imply complete normality and the converse is false. See [9] for more details about these notions.

The following theorem is the main result in this work.

Theorem 1. *Let X be a separated topological space, A a closed subset of X and $T : A \rightarrow [0, 1]$, a usc multi-valued map with closed convex values. Suppose that one of the two conditions holds :*

C1) X is completely normal,

C2) X is normal and A is perfectly normal,

*Then, there exists a usc extension of T with closed convex values defined on X into $[0, 1]$. *i.e.* $\exists \tilde{T} : X \rightarrow [0, 1]$ usc with closed convex values such that $\tilde{T} \upharpoonright_A \equiv T$.*

Proof. Let $B_0 = \{0, 1\}, \dots, B_n = \{i/2^n, i \in \{0, \dots, 2^n\}\}$. Define the set $B = \cup_{n \in \mathbb{N}} B_n$ the set of all dyadic numbers of $[0, 1]$. We have, $B_{n+1} = B_n \cup \{(r_i + r_{i+1})/2, i \in \{0, \dots, 2^n - 1\}\}, r_i = i/2^n \in B_n$. It is well known that B is dense in $[0, 1]$.

Let for every $r \in B$, $A_r = T^{-1}([0, r]) = \{x \in A, \exists y \in T(x), y \leq r\}$. Since T is usc on A , for all $r \in [0, 1]$, A_r is closed.

After this, we construct closed subsets $X_r, r \in B$, of X satisfying the following three conditions :

- 1) $X_r \cap A = A_r$,
- 2) $int_A(A_r) \subset int_X(X_r)$,
- 3) $X_r \subset X_s$, if $s, r \in B$ and $r < s$.

Put $X_1 = X$. In the following step, the condition C1) or C2) of the theorem is needed. We illustrate the use of each of them.

Begin by the condition C1) : the space X is completely normal. We have, $\overline{int_A(A_0)} \cap \mathfrak{C}_A A_0 = int_A(A_0) \cap \overline{\mathfrak{C}_A A_0} = \emptyset$. Then, we can separate $int_A(A_0)$ and $\mathfrak{C}_A A_0$ by open sets (in X) O_0 and O_c respectively. This gives $\overline{O_0} \cap A \subset \mathfrak{C}_A(O_c \cap A) \subset A_0$. We conclude the two relations $O_0 \cap A = int_A(A_0)$ and $\overline{O_0} \cap A \subset A_0$, put $X_0 = A_0 \cup \overline{O_0}$. Then, the sets $X_r, r \in B_0$, satisfying 1) – 3) are defined.

For the same result, let us use the condition C2) : X is normal and A is perfectly normal. In this case, $int_A(A_0)$ and $\mathfrak{C}_A A_0$ are F_σ -sets, then, it can be written us a countable unions of closed sets, let $int_A(A_0) = \cup_{i \in \mathbb{N}} F_i$ and $\mathfrak{C}_A A_0 = \cup_{i \in \mathbb{N}} G_i$. Furthermore, we have $\overline{int_A(A_0)} \cap \mathfrak{C}_A A_0 = int_A(A_0) \cap \overline{\mathfrak{C}_A A_0} = \emptyset$. We apply a Bonan's lemma [1] to separate $int_A(A_0)$

and $\mathbb{C}_A A_0$ by open sets, and we define X_0 by the same way.

Let the sets $X_r, r \in \cup_{k \leq n} B_k$ satisfying 1) – 3) be given. Then we obtain the sets $X_r, r \in B_{n+1}$ like this : Let $r \in B_{n+1} \setminus B_n$ and $i \in \{0, \dots, 2^n - 1\}$ such that $r = (r_i + r_{i+1})/2$, with $r_i = i/2^n, r_{i+1} = (i + 1)/2^n$ are elements of B_n .

We proceed as previously by the use of condition C1) or C2), when constructing X_0 , with the set A_r in the place of A_0 . We obtain an open set O'_r of X such that $\overline{O'_r} \cap A \subset A_r$ and $O'_r \cap A = \text{int}_A(A_r)$. Since $\text{int}_A(A_r) \subset \text{int}_A(A_{r_{i+1}}) \subset \text{int}_X(X_{r_{i+1}})$, the set $O_r = O'_r \cap \text{int}_X(X_{r_{i+1}})$ is open in X and the two relations $\overline{O_r} \cap A \subset A_r$ and $O_r \cap A = \text{int}_A(A_r)$ are obtained. Put $X_r = A_r \cup \overline{O_r} \cup \overline{X_{r_i}}$. We easily verify that conditions 1) – 3) are satisfied (the verification is down for the sets X_{r_i}, X_r and $X_{r_{i+1}}$). At this moment, the recursive process for the construction of all the sets $X_r, r \in B$, satisfying conditions 1)-3) is given.

Thereafter, we define the map $F : X \rightarrow [0, 1]$, as follows :

$$F(x) = \begin{cases} T(x) & \text{if } x \in A, \\ \inf\{r, x \in X_r\} & \text{otherwise.} \end{cases}$$

Verify that F , as a map defined on X is usc at each point of A (*i.e.* with respect to the topology of X). Let $x_0 \in A$ and $a, b \in [0, 1]$ such that $T(x_0) = [t_1, t_2] \subset]a, b[$. The other cases are analogous and explained next. Note also that the convexity of the values of T is considered here. Let $r_1, r_2 \in B$ such that $t_2 < r_1 < b$ and $a < r_2 < t_1$.

In one hand, $x_0 \in \text{int}_A(A_{r_1})$ (because T is usc on A and $T(x_0) \subset [0, r_1[$) and since $\text{int}_A(A_{r_1}) \subset \text{int}_X(X_{r_1})$, there exists an open neighborhood O_1 of x_0 (in X) such that $\forall x \in O_1 \setminus A, F(x) \leq r_1$. In other hand, $x_0 \notin A_{r_2}$. Since x_0 is an element of $A, x_0 \notin X_{r_2}$. Let O_2 be an open neighborhood of x_0 in X such that $O \cap X_{r_2} = \emptyset$. We have, $\forall x \in O_2 \setminus A, F(x) \geq r_2$. We take in the last time an open set O_3 (in X) such that $\forall x \in O_3 \cap A, T(x) \subset]a, b[$ and define $O = O_1 \cap O_2 \cap O_3$. We obtain $\forall x \in O, F(x) \subset]a, b[$, which gives the fact that F is usc on A . The case of $t_2 = b = 1$ and $T(x_0) \subset]a, b[$ (resp. $t_1 = a = 0$ and $T(x_0) \subset [a, b[$) is a simple particular case where it suffices to consider only O_2 (resp. O_1). In the other case, we put $O = X$.

Denote by H the graph of F in $X \times [0, 1]$. The desired map is \tilde{T} given as follows : $\tilde{T}(x) = \text{co}\{y, (x, y) \in \overline{H}\}$. This map is usc, because its graph is closed and the image space $([0, 1])$ is compact. We end by applying the upper semicontinuity of F on A to prove that its values are not affected on A when passing through the closure of its graph.

Let $x_0 \in A$. We can separate $F(x_0) = T(x_0)$ and any point $y \notin T(x_0)$ of $[0, 1]$ by open sets V_0, V_y respectively. Then, there exists an open neighborhood O_{x_0} of x_0 such that $F(O_{x_0}) \subset V_0$. We have obtained, $(x_0, y) \in O_{x_0} \times V_y, F(O_{x_0}) \subset V_0$ and $V_y \cap V_0 = \emptyset$, which means that $O_{x_0} \times V_y \cap H = \emptyset$. That is (x_0, y) is not a point of the adherence of H . We can finally affirm that : $\tilde{T} \upharpoonright_A \equiv F \upharpoonright_A \equiv T$ □

Note that the previous theorem, stated only with condition C1), is proved differently by Shishkov [20].

3 Applications

Now we give applications of Theorem 1 in different domains. In the first time, analogously to the characterization by Tietze-Urysohn extension theorem of normal spaces, we give a characterization of completely normal spaces (after Gutev [13] and Shishkov [20], using Theorem 1). This proves that the last theorem, stated only with condition C1), can not be improved by the relaxation of the complete normality imposed to X .

Corollary 1. *Let X be a separated topological space. Then, X is completely normal if and only if every usc multi-valued map T defined on a closed subset of X into $[0, 1]$ with nonempty closed convex values has a multi-valued usc extension defined on the whole of X into $[0, 1]$ with nonempty closed convex values.*

Proof. The necessity is a straightforward consequence of Theorem 1. The sufficiency is proved easily as follows : Let A and B be subsets of X such that $A \cap \overline{B} = B \cap \overline{A} = \emptyset$. Define

$$\text{the multi-valued map } T : \overline{A \cup B} \rightarrow [0, 1], \text{ by : } T(x) = \begin{cases} [0, 1] & \text{if } x \in \overline{A \cap B}, \\ 0 & \text{if } x \in \overline{A} \setminus \overline{A \cap B}, \\ 1 & \text{if } x \in \overline{B} \setminus \overline{A \cap B}. \end{cases}$$

Let us verify that T is usc on $\overline{A \cup B}$. Let $x_0 \in \overline{A \cup B}$ and V an open subset of $[0, 1]$ containing $T(x_0)$. If $x_0 \in \overline{A \cap B}$, then $V = [0, 1]$, we can choose, in this case, $O = \overline{A \cup B}$ as a neighborhood of x_0 such that $T(O) \subset V$. If $x_0 \in \overline{A} \setminus \overline{A \cap B} = \mathfrak{C}_{\overline{A \cup B}} \overline{B}$, choose $O = \mathfrak{C}_{\overline{A \cup B}} \overline{B}$ as a neighborhood of x_0 such that $T(O) \subset V$. The other case is similar. Then, T is usc on $\overline{A \cup B}$.

From the hypothesis, there exists a usc extension \tilde{T} of T defined on the whole of X , with closed convex values. The sets $\tilde{T}^{-1}(]1/2, 1]) = \{x \in X, \tilde{T}(x) \subset]1/2, 1]\}$ and $\tilde{T}^{-1}([0, 1/2[)$ are open and separate A and B . Then X is completely normal \square

Theorem 1 can be applied to extend real semicontinuous functions. In what follows, we use the same notations *usc* and *lsc* for corresponding semicontinuity concepts for single-valued real functions.

Corollary 2. *Let X be a separated topological space and A a closed subset of X . Suppose that one of the conditions C1) or C2) is satisfied. Given two functions $f, g : A \rightarrow [0, 1]$, such that f is lsc, g is usc and $f(x) \leq g(x)$, for every $x \in A$.*

Then, there exists extensions \tilde{f} and \tilde{g} of f and g respectively, such that \tilde{f} is lsc, \tilde{g} is usc and $\tilde{f}(x) \leq \tilde{g}(x)$, for every $x \in X$.

Proof. Define the multi-valued map $T : A \rightarrow [0, 1]$ by $T(x) = [f(x), g(x)]$, for every $x \in A$. Let us verify that T is usc on A . Let $x_0 \in A$ and $a, b \in [0, 1]$ such that $T(x_0) \subset]a, b[$. The other possibilities are particular cases. We have $f(x_0) > a$ and $g(x_0) < b$. Then, there exists two neighborhoods V_1 and V_2 of x_0 (in A) such that $f(x) > a$, for every $x \in V_1$ and $g(x) < b$, for every $x \in V_2$. Put $V = V_1 \cap V_2$. This gives $T(V) \subset]a, b[$. If $a = 0, b \neq 1$ and $T(x_0) \subset [a, b[$ (resp. $b = 1, a \neq 0$ and $T(x_0) \subset]a, b]$), we do not need V_1 (resp. V_2). If $a = 0, b = 1$ and $T(x_0) \subset [a, b]$, we put $V = A$.

We infer, by Theorem 1, an usc extension \tilde{T} of T with nonempty convex compact values. Define \tilde{f} and \tilde{g} as follows : $\tilde{f}(x) = \min\{y, y \in T(x)\}$ and $\tilde{g}(x) = \max\{y, y \in T(x)\}$, for every $x \in X$.

It remains to verify the desired properties of \tilde{f} and \tilde{g} . It is clear that $\tilde{f}|_A = f$, $\tilde{g}|_A = g$ and $\tilde{f}(x) \leq \tilde{g}(x)$, for every $x \in X$. Let $\lambda \in [0, 1]$. Then, $\{x \in X, \tilde{f}(x) \leq \lambda\} = \tilde{T}^{-1}([0, \lambda])$ and $\{x \in X, \tilde{g}(x) \geq \lambda\} = \tilde{T}^{-1}([\lambda, 1])$. Since \tilde{T} is usc, the last level sets are closed. This proves that \tilde{f} is lsc on X and \tilde{g} is usc on X \square

We apply Theorem 1 to prove a version of the Gale-Mas-Colell's [10], Shafer-Sonnenschein's [19] and Gourdel's [12] fixed point theorems with an arbitrary number (possibly uncountable) of multi-valued maps.

Theorem 2. *Let $X_\alpha, \alpha \in I$, I is arbitrary and $X_\alpha = [0, 1]$. Let for every α in I , $T_\alpha : \prod_{\lambda \in I} X_\lambda \rightarrow X_\alpha$ an usc multi-valued map with empty or nonempty closed convex values such that $\text{Dom}(T_\alpha)$ is perfectly normal.*

Then, $\exists x \in \prod_{\lambda \in I} X_\lambda$ such that : $\forall \alpha \in I$,

$$\text{either } x_\alpha \in T_\alpha(x) \text{ or } T_\alpha(x) = \emptyset.$$

Proof. We apply Theorem 1 to find extensions \tilde{T}_α of T_α , $\alpha \in I$. Consider the map $F : \prod_{\lambda \in I} X_\lambda \rightarrow \prod_{\lambda \in I} X_\lambda$, defined by $F(x) = \prod_{\lambda \in I} \tilde{T}_\lambda(x)$. Any fixed point of F (apply any fixed point theorem for Kakutani maps in locally convex spaces, for example Glicksberg's fixed point theorem) ensure the result \square

Now, we give an application to qualitative games with an infinitely number of players. A qualitative game is a pair $(X_i, P_i)_{i \in I}$, I is the set of players, X_i is the set of strategies of the player $i \in I$ and P_i is his preference correspondence. For a literature in qualitative games, see [3, 21, 7].

Theorem 3. *Let $G = (X_i, P_i)_{i \in I}$ be a qualitative game. For every $i \in I$, $X_i = [0, 1]$, $P_i : X = \prod_{j \in I} X_j \rightarrow X_i$ is usc with empty or nonempty closed convex values such that $\text{Dom}(P_i)$ is perfectly normal and I is an arbitrary set of indices. If $\forall i \in I, \forall x \in X, x_i \notin P_i(x)$, then G has an equilibrium, that is : $\exists y \in X$, such that $\forall i \in I, P_i(y) = \emptyset$.*

Proof. Is a straightforward consequence of Theorem 2 \square

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