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SEMIDEFINITE PROGRAMMING FOR MIN-MAX PROBLEMS AND GAMES

R. LARAKI AND J.B. LASSERRE

Abstract. We consider two min-max problems: (1) minimizing the supremum of finitely many rational functions over a compact basic semi-algebraic set and (2) solving a 2-player zero-sum polynomial game in randomized strategies with compact basic semi-algebraic sets of pure strategies. In both problems the optimal value can be approximated by solving a hierarchy of semidefinite relaxations, in the spirit of the moment approach developed in Lasserre [24, 26]. This provides a unified approach and a class of algorithms to compute Nash equilibria and min-max strategies of several static and dynamic games. Each semidefinite relaxation can be solved in time which is polynomial in its input size and practice on a sample of experiments reveals that few relaxations are needed for a good approximation (and sometimes even for finite convergence), a behavior similar to what was observed in polynomial optimization.

1. Introduction

Initially, this paper was motivated by developing a unified methodology for solving several types of (neither necessarily finite nor zero-sum) N-player games. But it is also of self-interest in optimization for minimizing the maximum of finitely many rational functions (whence min-max) on a compact basic semi-algebraic set. Briefly, the moment-s.o.s. approach developed in [24, 26] is extended to two large classes of min-max problems. This allows to obtain a new numerical scheme based on semidefinite programming to compute approximately and sometimes exactly (1) Nash equilibria and the min-max of any finite game[1] and (2) the value and the optimal strategies of any polynomial two-player zero-sum game[2]. In particular, the approach can be applied to the so-called Loomis games defined in [32] and to some dynamic games described in Kolhberg [21].

Background. Nash equilibrium [33] is a central concept in non-cooperative game theory. It is a profile of mixed strategies (a strategy for each player) such that each player is best-responding to the strategies of the opponents. An important problem is to compute numerically a Nash equilibrium, an approximate Nash equilibrium for a given precision or all Nash equilibria in mixed strategies of a finite game.

Key words and phrases. N-player games; Nash equlibria; min-max optimization problems; semidefinite programming.

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[1]Games with finitely many players where the set of pure actions of each player is finite.

[2]Games with two players where the set of pure actions of each player is a compact basic semi-algebraic-set and the payoff function is polynomial.

It is well known that any two-player *zero-sum finite* game (in mixed strategies) is reducible to a linear program (Dantzig [5]) and hence equilibria could be computed in polynomial time (Khachiyan [20]).

Lemke and Howson [29] provided a famous algorithm that computes a Nash-equilibrium (in mixed strategies) of any 2-player *non-zero-sum* finite game. The algorithm has been extended to N-player finite games by Rosenmüller [39], Wilson [50] and Govindan and Wilson [12].

An alternative to the Lemke-Howson algorithm for 2-player games is provided in van den Elzen and Talman [9] and has been extended to n-player games by Herings and van den Elzen [16]. As shown in the recent survey of Herings and Peeters [17], all these algorithms (including the Lemke-Howson) are homotopy-based and converge (only) when the game is non-degenerate.

Recently, Savani and von Stengel [41] proved that the Lemke-Howson algorithm for 2-player games may be exponential. One may expect that this result extends to all known homotopy methods. Daskalakis, Goldberg and Papadimitriou [4] proved that solving numerically 3-player finite games is hard[3]. The result has been extended to 2-player finite games by Chen and Deng [6]. Hence, computing a Nash equilibrium is as hard as finding a Brouwer-fixed point. For a recent and deep survey on the complexity of Nash equilibria see Papadimitriou [35]. For the complexity of computing equilibria on game theory in general, see [34] and [40].

A different approach to solve the problem is to view the set of Nash equilibria as the set of real nonnegative solutions to a system of polynomial equations. Methods of computational algebra (e.g. using Gröbner bases) can be applied as suggested and studied in e.g. Datta [8], Lipton [30] and Sturmfels [47]. However, in this approach, one first computes *all* complex solutions to sort out all real nonnegative solutions afterwards. Interestingly, polynomial equations can also be solved via homotopy-based methods (see e.g. Verschelde [48]).

Another important concept in game theory is the min-max payoff of some player i, $\underline{v}_i$. This is the level at which the team of players (other than i) can punish player i. The concept plays an important role in repeated games and the famous folk theorem of Aumann and Shapley [2]. To our knowledge, no algorithm in the literature deals with this problem. However, it has been proved recently that computing the min-max for 3 or more player games is NP-hard [3].

The algorithms described above concern finite games. In the class of polynomial games introduced by Dresher, Karlin and Shapley (1950), the set of pure strategies S^i of player i is a product of real intervals and the payoff function of each player is polynomial. When the game is zero-sum and $S^i = [0,1]$ for each player $i = 1,2$, Parrilo [37] showed that finding an optimal solution is equivalent to solving a single semidefinite program. In the same framework but with several players, Stein, Parrilo and Ozdaglar [46] propose several algorithms to compute correlated equilibria, one among them using SDP relaxations. Shah and Parrilo [44] extended the methodology in [37] to discounted zero-sum stochastic games in which the transition is controlled by one of the two players. Finally, it is worth noticing recent algorithms designed to solve some specific classes of infinite games (not necessarily polynomial). For instance, Gürkan and Pang [13].

[3]More precisely, it is complete in the PPAD class of all search problems that are guaranteed to exist by means of a direct graph argument. This class was introduced by Papadimitriou in [36] and is between P and NP.

Contribution. In the first part we consider what we call the **MRF** problem which consists of minimizing the supremum of finitely many *rational functions* over a compact basic semi-algebraic set. In the spirit of the moment approach developed in Lasserre [24, 26] for polynomial optimization, we define a hierarchy of semidefinite relaxations (in short SDP relaxations). Each SDP relaxation is a semidefinite program which, up to arbitrary (but fixed) precision, can be solved in polynomial time and the monotone sequence of optimal values associated with the hierarchy converges to the optimal value of **MRF**. Sometimes the convergence is finite and a sufficient condition permits to detect whether a certain relaxation in the hierarchy is exact (i.e. provides the optimal value), and to extract optimal solutions. It is shown that computing the min-max or a Nash equilibrium in mixed strategies for static finite games or dynamic finite absorbing games reduces to solving an **MRF** problem. For zero-sum finite games in mixed strategies the hierarchy of SDP relaxations for the associated **MRF** reduces to the first one of the hierarchy, which in turn reduces to a linear program. This is in support of the claim that the above **MRF** formulation is a natural extension to the non linear case of the well-known LP-approach [5] as it reduces to the latter for finite zero-sum games. In addition, if the SDP solver uses primal-dual interior points methods and if the convergence is finite then the algorithm returns *all* Nash equilibria (if of course there are finitely many).

To compute all Nash equilibria, homotopy algorithms are developed in Kostreva and Kinard [22] and Herings and Peeters [18]. They apply numerical techniques to obtain all solutions of a system of polynomial equations.

In the second part, we consider general 2-player zero-sum polynomial games in mixed strategies (whose action sets are basic compact semi-algebraic sets of $\mathbb{R}^n$ and payoff functions are polynomials). We show that the common value of max-min and min-max problems can be approximated as closely as desired, again by solving a certain hierarchy of SDP relaxations. Moreover, if a certain rank condition is satisfied at an optimal solution of some relaxation of the hierarchy, then this relaxation is exact and one may extract optimal strategies. Interestingly and not surprisingly, as this hierarchy is derived from a min-max problem over sets of measures, it is a subtle combination of moment and sums of squares (s.o.s.) constraints whereas the hierarchy for polynomial optimization is based on either moments (primal formulation) or s.o.s. (dual formulation) but not both. This is a multivariate extension of Parrilo's [37] result for the univariate case where one needs to solve a single semidefinite program (as opposed to a hierarchy). The approach may be extended to dynamic absorbing games[4] with discounted rewards, and in the univariate case, one can construct a polynomial time algorithm that combines a binary search with a semidefinite program.

Hence the first main contribution is to formulate several game problems as a particular instance of the **MRF** problem while the second main contribution extends the moment-s.o.s. approach of [24, 26] in two directions. The first extension (the **MRF** problem) is essentially a non trivial adaptation of Lasserre's approach [24] to the problem of minimizing the sup of finitely many rational functions. Notice that the sup of finitely many rational functions is *not* a rational function. However one reduces the initial problem to that of minimizing a *single* rational function

[4]In dynamic absorbing games, transitions are controlled by *both* players, in contrast with Parrilo and Shah [44] where only one player controls the transitions.

(but now of $n+1$ variables) on an appropriate set of $\mathbb{R}^{n+1}$. As such, this can also be viewed as an extension of Jibetean and De Klerk's result [19] for minimizing a single rational function. The second extension generalizes (to the multivariate case) Parrilo's approach [37] for the univariate case, and provides a hierarchy of mixed moment-s.o.s. SDP relaxations. The proof of convergence is delicate as one has to consider simultaneously moment constraints as well as s.o.s.-representation of positive polynomials. (In particular, and in contrast to polynomial optimization, the converging sequence of optimal values associated with the hierarchy of SDP relaxations is *not* monotone anymore.)

To conclude, within the game theory community the rather negative computational complexity results ([3], [4], [6], [40], [41]) have reinforced the idea that solving a game is computationally hard. On a more positive tone our contribution provides a unified semidefinite programming approach to many game problems. It shows that optimal value and optimal strategies can be approximated as closely as desired (and sometimes obtained exactly) by solving a hierarchy of semidefinite relaxations, very much in the spirit of the moment approach described in [24] for solving polynomial minimization problems (a particular instance of the Generalized Problem of Moments [26]). Moreover, the methodology is consistent with previous results of [5] and [37] as it reduces to a linear program for finite zero-sum games and to a single semidefinite program for univariate polynomial zero-sum games.

Finally, even if practice from problems in polynomial optimization seems to reveal that this approach is efficient, of course the size of the semidefinite relaxations grows rapidly with the initial problem size. Therefore, in view of the present status of public SDP solvers available, its application is limited to small and medium size problems so far.

Quoting Papadimitriou [35]: "*The PPAD-completeness of Nash suggests that any approach to finding Nash equilibria that aspires to be efficient [...] should explicitly take advantage of computationally beneficial special properties of the game in hand*". Hence to make our algorithm efficient for larger size problems, one could exploit possible sparsity and regularities often present in the data (which will be the case if the game is the normal form of an extensive form game). Indeed specific SDP relaxations for minimization problems that exploit sparsity efficiently have been provided in Kojima et al. [49] and their convergence has been proved in [25] under some condition on the sparsity pattern.

2. Notation and preliminary results

2.1. **Notation and definitions.** Let $\mathbb{R}[x]$ be the ring of real polynomials in the variables $x = (x_1, \ldots, x_n)$ and let $(X^\alpha)_{\alpha\in\mathbb{N}}$ be its canonical basis of monomials. Denote by $\Sigma[x] \subset \mathbb{R}[x]$ the subset (cone) of polynomials that are sums of squares (s.o.s.), and by $\mathbb{R}[x]_d$ the space of polynomials of degree at most d. Finally let $\|x\|$ denote the Euclidean norm of $x \in \mathbb{R}^n$.

With $\mathbf{y} =: (y_\alpha) \subset \mathbb{R}$ being a sequence indexed in the canonical monomial basis (X^α), let $L_\mathbf{y} : \mathbb{R}[x] \to \mathbb{R}$ be the linear functional

$$f \,(= \sum_{\alpha\in\mathbb{N}^n} f_\alpha\, x^\alpha) \longmapsto \sum_{\alpha\in\mathbb{N}^n} f_\alpha\, y_\alpha, \quad f \in \mathbb{R}[x].$$

Moment matrix. Given $\mathbf{y} = (y_\alpha) \subset \mathbb{R}$, the *moment* matrix $M_d(\mathbf{y})$ of order d associated with $\mathbf{y}$, has its rows and columns indexed by (x^α) and its (α,β)-entry is

defined by:

$$M_d(\mathbf{y})(\alpha,\beta) \,:=\, L_{\mathbf{y}}(x^{\alpha+\beta}) \,=\, y_{\alpha+\beta}, \qquad |\alpha|,\,|\beta| \,\leq d.$$

Localizing matrix. Similarly, given $\mathbf{y} = (y_\alpha) \subset \mathbb{R}$ and $\theta \in \mathbb{R}[x]\,(= \sum_\gamma \theta_\gamma x^\gamma)$, the *localizing* matrix $M_d(\theta,\mathbf{y})$ of order d associated with $\mathbf{y}$ and θ, has its rows and columns indexed by (x^α) and its (α,β)-entry is defined by:

$$M_d(\theta,\mathbf{y})(\alpha,\beta) := L_{\mathbf{y}}(x^{\alpha+\beta}\theta(x)) = \sum_\gamma \theta_\gamma y_{\gamma+\alpha+\beta}, \qquad |\alpha|,|\beta| \,\leq d.$$

One says that $\mathbf{y} = (y_\alpha) \subset \mathbb{R}$ has a *representing* measure supported on $\mathbf{K}$ if there is some finite Borel measure μ on $\mathbf{K}$ such that

$$y_\alpha \,=\, \int_{\mathbf{K}} x^\alpha \, d\mu(x), \qquad \forall\, \alpha \in \mathbb{N}^n.$$

For later use, write

$$M_d(\mathbf{y}) \;=\; \sum_{\alpha\in\mathbb{N}^n} y_\alpha\, B_\alpha \tag{2.1}$$

$$M_d(\theta,\mathbf{y}) \;=\; \sum_{\alpha\in\mathbb{N}^n} y_\alpha\, B^\theta_\alpha, \tag{2.2}$$

for real symmetric matrices $(B_\alpha, B^\theta_\alpha)$ of appropriate dimensions. Note that the above two summations contain only finitely many terms.

Definition 2.1 (Putinar's property)**.** *Let $(g_j)_{j=1}^m \subset \mathbb{R}[x]$. A basic closed semi algebraic set $\mathbf{K} := \{x \in \mathbb{R}^n : g_j(x) \geq 0, : j = 1,\ldots,m\}$ satisfies Putinar's property if there exists $u \in \mathbb{R}[x]$ such that $\{x : u(x) \geq 0\}$ is compact and*

$$u \,=\, \sigma_0 + \sum_{j=1}^m \sigma_j\, g_j \tag{2.3}$$

for some s.o.s. polynomials $(\sigma_j)_{j=0}^m \subset \Sigma[x]$. Equivalently, for some $M > 0$ the quadratic polynomial $x \mapsto M - \|x\|^2$ has Putinar's representation (2.3).

Obviously Putinar's property implies compactness of $\mathbf{K}$. However, notice that Putinar's property is not geometric but algebraic as it is related to the representation of $\mathbf{K}$ by the defining polynomials (g_j)'s. Putinar's property holds if e.g. the level set $\{x : g_j(x) \geq 0\}$ is compact for some j, or if all g_j are affine and $\mathbf{K}$ is compact (in which case it is a polytope). In case it is not satisfied and if for some known $M > 0$, $\|x\|^2 \leq M$ whenever $x \in \mathbf{K}$, then it suffices to add the redundant quadratic constraint $g_{m+1}(x) := M - \|x\|^2 \geq 0$ to the definition of $\mathbf{K}$. The importance of Putinar's property stems from the following result:

Theorem 2.2 (Putinar [38])**.** *Let $(g_j)_{j=1}^m \subset \mathbb{R}[x]$ and assume that*

$$\mathbf{K} := \{\, x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1,\ldots,m\}$$

satisfies Putinar's property.

(a) Let $f \in \mathbb{R}[x]$ be strictly positive on $\mathbf{K}$. Then f can be written as u in (2.3).

(b) Let $\mathbf{y} = (y_\alpha)$. Then $\mathbf{y}$ has a representing measure on $\mathbf{K}$ if and only if

$$M_d(\mathbf{y}) \succeq 0, \quad M_d(g_j,\mathbf{y}) \succeq 0, \qquad j = 1,\ldots,m; \quad d = 0,1,\ldots \tag{2.4}$$

We also have:

Lemma 2.3. *Let $\mathbf{K} \subset \mathbb{R}^n$ be compact and let p, q continuous with $q > 0$ on $\mathbf{K}$. Let $M(\mathbf{K})$ be the set of finite Borel measures on $\mathbf{K}$ and let $P(\mathbf{K}) \subset M(\mathbf{K})$ be its subset of probability measures on $\mathbf{K}$. Then*

$$\min_{\mu \in P(\mathbf{K})} \frac{\int_{\mathbf{K}} p\, d\mu}{\int_{\mathbf{K}} q\, d\mu} = \min_{\varphi \in M(\mathbf{K})} \{ \int_{\mathbf{K}} p\, d\varphi : \int_{\mathbf{K}} q\, d\varphi = 1 \} \tag{2.5}$$

$$= \min_{\mu \in P(\mathbf{K})} \int_{\mathbf{K}} \frac{p}{q}\, d\mu = \min_{x \in \mathbf{K}} \frac{p(x)}{q(x)} \tag{2.6}$$

Proof. Let $\rho^* := \min_x \{p(x)/q(x) : x \in \mathbf{K}\}$. As $q > 0$ on $\mathbf{K}$,

$$\frac{\int_{\mathbf{K}} p\, d\mu}{\int_{\mathbf{K}} q\, d\mu} = \frac{\int_{\mathbf{K}} (p/q)\, q\, d\mu}{\int_{\mathbf{K}} q\, d\mu} \geq \rho^*.$$

Hence if $\mu \in P(\mathbf{K})$ then $\int_{\mathbf{K}} (p/q) d\mu \geq \rho^* \int_{\mathbf{K}} d\mu = \rho^*$. On the other hand, with $x^* \in \mathbf{K}$ a global minimizer of p/q on $\mathbf{K}$, let $\mu := \delta_{x^*} \in P(\mathbf{K})$ be the Dirac measure at $x = x^*$. Then $\int_{\mathbf{K}} p d\mu / \int_{\mathbf{K}} q d\mu = p(x^*)/q(x^*) = \int_{\mathbf{K}} (p/q) d\mu = \rho^*$, and therefore

$$\min_{\mu \in P(\mathbf{K})} \frac{\int_{\mathbf{K}} p d\mu}{\int_{\mathbf{K}} q d\mu} = \min_{\mu \in P(\mathbf{K})} \int_{\mathbf{K}} \frac{p}{q} d\mu = \min_{x \in \mathbf{K}} : \frac{p(x)}{q(x)} = \rho^*.$$

Next, for every $\varphi \in M(\mathbf{K})$ with $\int_{\mathbf{K}} q d\varphi = 1$, $\int_{\mathbf{K}} p\, d\varphi \geq \int_{\mathbf{K}} \rho^*\, q\, d\varphi = \rho^*$, and so $\min_{\varphi \in M(\mathbf{K})} \{ \int_{\mathbf{K}} p\, d\varphi : \int_{\mathbf{K}} q\, d\varphi = 1 \} \geq \rho^*$. Finally taking $\varphi := q(x^*)^{-1} \delta_{x^*}$ yields $\int_{\mathbf{K}} q d\varphi = 1$ and $\int_{\mathbf{K}} p\, d\varphi = p(x^*)/q(x^*) = \rho^*$.

Another way to see why this is true is throughout the following argument. The function $\mu \to \frac{\int_{\mathbf{K}} p\, d\mu}{\int_{\mathbf{K}} q\, d\mu}$ is quasi-concave (and also quasi-convex) so that the optimal value of the minimization problem is achieved on the boundary. □

3. Minimizing the max of finitely many rational functions

Let $\mathbf{K} \subset \mathbb{R}^n$ be the basic semi-algebraic set

$$\mathbf{K} := \{x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, \ldots, p\} \tag{3.1}$$

for some polynomials $(g_j) \subset \mathbb{R}[x]$, and let $f_i = p_i/q_i$ be rational functions, $i = 0, 1, \ldots, m$, with $p_i, q_i \in \mathbb{R}[x]$. We assume that:

- $\mathbf{K}$ satisfies Putinar's property (see Definition 2.1) and,
- $q_i > 0$ on $\mathbf{K}$ for every $i = 0, \ldots, m$.

Consider the following problem denoted by **MRF**:

$$\mathbf{MRF}: \quad \rho := \min_x \{ f_0(x) + \max_{i=1,\ldots,m} f_i(x) : x \in \mathbf{K} \}, \tag{3.2}$$

or, equivalently,

$$\mathbf{MRF}: \quad \rho = \min_{x,z} \{ f_0(x) + z : z \geq f_i(x),\ i = 1, \ldots, m;\ x \in \mathbf{K} \}. \tag{3.3}$$

With $\mathbf{K} \subset \mathbb{R}^n$ as in (3.1), let $\widehat{\mathbf{K}} \subset \mathbb{R}^{n+1}$ be the basic semi algebraic set

$$\widehat{\mathbf{K}} := \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : x \in \mathbf{K};\ z\, q_i(x) - p_i(x) \geq 0, i = 1, \ldots, m\} \tag{3.4}$$

and consider the new infinite-dimensional optimization problem

$$\mathcal{P}: \quad \hat{\rho} := \min_{\mu} \{ \int_{\mathbf{K}} (p_0 + z\, q_0)\, d\mu : \int_{\mathbf{K}} q_0\, d\mu = 1, \quad \mu \in M(\widehat{\mathbf{K}}) \} \tag{3.5}$$

where $M(\widehat{\mathbf{K}})$ is the set of finite Borel measures on $\widehat{\mathbf{K}}$. Problem (3.5) is a particular instance of the *Generalized Problem of Moments* for which a general methodology (based on a hierarchy of semidefinite relaxations) has been described in [26]. To make the paper self-contained we explain below how to apply this methodology in the above specific context.

Proposition 3.1. *Let $\mathbf{K} \subset \mathbb{R}^n$ be as in (3.1). Then $\rho = \hat{\rho}$.*

Proof. The following upper and lower bounds

$$\underline{z} := \min_{i=1,\ldots,m} \min_{x\in\mathbf{K}} f_i(x); \quad \overline{z} := \max_{i=1,\ldots,m} \max_{x\in\mathbf{K}} f_i(x),$$

are both well-defined since $\mathbf{K}$ is compact and $q_i > 0$ on $\mathbf{K}$ for every $i = 1, \ldots, m$. Including the additional constraint $\underline{z} \leq z \leq \overline{z}$ in the definition of $\widehat{\mathbf{K}}$ makes it compact without changing the value of ρ. Next observe that

$$\rho = \min_{(x,z)} \left\{ \frac{p_0(x) + zq_0(x)}{q_0(x)} \;:\; (x,z) \in \widehat{\mathbf{K}} \right\}. \tag{3.6}$$

Applying Lemma 2.3 with $\widehat{\mathbf{K}}$ in lieu of $\mathbf{K}$, and with $(x,z) \mapsto p(x,z) := p_0(x)+zq_0(x)$ and $(x,z) \mapsto q(x,z) := q_0(x)$, yields the desired result. □

Remark 3.2. (a) When $m = 0$, that is when one wishes to minimize the rational function f_0 on $\mathbf{K}$, then $\mathcal{P}$ reads $\min_\mu\{\int_\mathbf{K} f_0 d\mu \;:\; \int_\mathbf{K} q_0 d\mu = 1;\; \mu \in M(\mathbf{K})\}$. Using the dual problem $\mathcal{P}^* \;:\; \max\{z \;:\; p_0(x) - zq_0(x) \geq 0 \quad \forall x \in \mathbf{K}\}$, Jibetean and DeKlerk [19] proposed to approximate the optimal value by solving a hierarchy of semidefinite relaxations. The case $m \geq 2$ is a nontrivial extension of the case $m = 0$ because one now wishes to minimize on $\mathbf{K}$ a function which is *not* a rational function. However, by adding an extra variable z, one obtains the moment problem (3.5), which indeed is the same as minimizing the rational function $(x,z) \mapsto \frac{p_0(x)+z\,q_0(x)}{q_0(x)}$ on a domain $\widehat{\mathbf{K}} \subset \mathbb{R}^{n+1}$. And the dual of $\mathcal{P}$ now reads $\max\{\rho : p_0(x) + zq_0(x) - \rho q_0(x) \geq 0 \quad \forall (x,z) \in \widehat{\mathbf{K}}\}$. Hence if $\widehat{\mathbf{K}}$ is compact and satisfies Putinar's property one may use the hierarchy of semidefinite relaxations defined in [19] and adapted to this specific context; see also [26].

(b) The case $m = 1$ (i.e., when one wants to minimize the sum of rational functions $f_0 + f_1$ on $\mathbf{K}$) is also interesting. One way is to reduce to the same denominator $q_0 \times q_1$ and minimize the rational function $(p_0q_1 + p_1q_0)/q_0q_1$. But then the first SDP relaxation of [26, 19] would have to consider polynomials of degree at least $d := \max[\deg p_0 + \deg q_1, \deg p_1 + \deg q_0, \deg q_0 + \deg q_1]$, which may be very penalizing when q_0 and q_1 have large degree (and sometimes it may even be impossible!). Indeed this SDP relaxation has as many as $O(n^d)$ variables and a linear matrix inequality of size $O(n^{\lceil d/2 \rceil})$. In contrast, by proceeding as above in introducing the additional variable z, one now minimizes the rational function $(p_0 + zq_0)/q_0$ which may be highly preferable since the first relaxation only considers polynomials of degree bounded by $\max[\deg p_0, 1 + \deg q_0]$ (but now in $n + 1$ variables). For instance, if $\deg p_i = \deg q_i = v$, $i = 1, 2$, then one has $O(n^{v+1})$ variables instead of $O(n^{2v})$ variables in the former approach.

We next describe how to solve the **MRF** problem via a hierarchy of semidefinite relaxations.

SDP relaxations for solving the MRF problem. As $\mathbf{K}$ is compact and $q_i > 0$ on $\mathbf{K}$, for all i, let

$$M_1 := \max_{i=1,\ldots,m} \left\{ \frac{\max\{|p_i(x)|, x \in \mathbf{K}\}}{\min\{q_i(x), x \in \mathbf{K}\}} \right\}, \tag{3.7}$$

and

$$M_2 := \min_{i=1,\ldots,m} \left\{ \frac{\min\{p_i(x), x \in \mathbf{K}\}}{\max\{q_i(x), x \in \mathbf{K}\}} \right\}. \tag{3.8}$$

Redefine the set $\widehat{\mathbf{K}}$ to be

$$\widehat{\mathbf{K}} := \{(x,z) \in \mathbb{R}^n \times \mathbb{R} : h_j(x,z) \geq 0, \quad j = 1, \ldots p+m+1\} \tag{3.9}$$

with

$$\left\{ \begin{array}{lll} (x,z) \mapsto & h_j(x,z) := g_j(x) & j = 1,\ldots,p \\ (x,z) \mapsto & h_j(x,z) := z\, q_j(x) - p_j(x) & j = p+1,\ldots,p+m \\ (x,z) \mapsto & h_j(x,z) := (M_1 - z)(z - M_2) & j = m+p+1 \end{array} \right. . \tag{3.10}$$

Lemma 3.3. *Let $\mathbf{K} \subset \mathbb{R}^n$ satisfy Putinar's property. Then the set $\widehat{\mathbf{K}} \subset \mathbb{R}^{n+1}$ defined in (3.9) satisfies Putinar's property.*

Proof. Since $\mathbf{K}$ satisfies Putinar's property, equivalently, the quadratic polynomial $x \mapsto u(x) := M - \|x\|^2$ can be written in the form (2.3), i.e., $u(x) = \sigma_0(x) + \sum_{j=1}^p \sigma_j(x) g_j(x)$ for some s.o.s. polynomials $(\sigma_j) \subset \Sigma[x]$. Next, consider the quadratic polynomial

$$(x,z) \mapsto w(x,z) \;=\; M - \|x\|^2 + (M_1 - z)(z - M_2).$$

Obviously, its level set $\{x : w(x,z) \geq 0\} \subset \mathbb{R}^{n+1}$ is compact and moreover, w can be written in the form

$$\begin{aligned} w(x,z) &= \sigma_0(x) + \sum_{j=1}^p \sigma_j(x)\, g_j(x) + (M_1 - z)(z - M_2) \\ &= \sigma_0'(x,z) + \sum_{j=1}^{m+p+1} \sigma_j'(x,z)\, h_j(x,z) \end{aligned}$$

for appropriate s.o.s. polynomials $(\sigma_j') \subset \Sigma[x,z]$. Therefore $\widehat{\mathbf{K}}$ satisfies Putinar's property in Definition 2.1, the desired result. □

We are now in position to define the hierarchy of semidefinite relaxations for solving the **MRF** problem. Let $\mathbf{y} = (y_\alpha)$ be a real sequence indexed in the monomial basis $(x^\beta z^k)$ of $\mathbb{R}[x,z]$ (with $\alpha = (\beta, k) \in \mathbb{N}^n \times \mathbb{N}$).

Let $h_0(x,z) := p_0(x) + z q_0(x)$, and let $v_j := \lceil (\deg h_j)/2 \rceil$ for every $j = 0, \ldots, m+p+1$. For $r \geq r_0 := \max_{j=0,\ldots,p+m+1} v_j$, introduce the hierarchy of semidefinite programs:

$$\mathbf{Q}_r : \left\{ \begin{array}{lll} \min\limits_{\mathbf{y}} & L_{\mathbf{y}}(h_0) & \\ \text{s.t.} & M_r(\mathbf{y}) & \succeq 0 \\ & M_{r-v_j}(h_j, \mathbf{y}) & \succeq 0, \quad j = 1,\ldots,m+p+1 \\ & L_y(q_0) & = 1, \end{array} \right. \tag{3.11}$$

with optimal value denoted $\inf \mathbf{Q}_r$ (and $\min \mathbf{Q}_r$ if the infimum is attained).

Theorem 3.4. *Let* $\mathbf{K} \subset \mathbb{R}^n$ *(compact) be as in (3.1). Let* $\mathbf{Q}_r$ *be the semidefinite program (3.11) with* $(h_j) \subset \mathbb{R}[x,z]$ *and* M_1, M_2 *defined in (3.10) and (3.7)-(3.8) respectively. Then:*

(a) $\inf \mathbf{Q}_r \uparrow \rho$ *as* $r \to \infty$.

(b) Let $\mathbf{y}^r$ *be an optimal solution of the SDP relaxation* $\mathbf{Q}_r$ *in (3.11). If*

$$\operatorname{rank} M_r(\mathbf{y}^r) \;=\; \operatorname{rank} M_{r-r_0}(\mathbf{y}^r) \;=\; t \tag{3.12}$$

then $\min \mathbf{Q}_r = \rho$ *and one may extract* t *points* $(x^*(k))_{k=1}^t \subset \mathbf{K}$*, all global minimizers of the* **MRF** *problem.*

(c) Let $\mathbf{y}^r$ *be a nearly optimal solution of the SDP relaxation (3.11) (with say* $\inf \mathbf{Q}_r \leq L_{\mathbf{y}^r} \leq \inf \mathbf{Q}_r + 1/r$*. If (3.6) has a unique global minimizer* $x^* \in \mathbf{K}$ *then the vector of first-order moments* $(L_{\mathbf{y}^r}(x_1), \ldots, L_{\mathbf{y}^r}(x_n))$ *converges to* x^* *as* $r \to \infty$.

Proof. As already mentioned in Remark 3.2, convergence of the dual of the semidefinite relaxations (3.11) was first proved in Jibetean and de Klerk [19] for minimizing a rational function on a basic compact semi-algebraic set (in our context, for minimizing the rational function $(x,z) \mapsto (p_0(x) + zq_0(x))/q_0(x)$ on the set $\widehat{\mathbf{K}} \subset \mathbb{R}^{n+1}$). See also [26, §4.1] and [28, Theor. 3.2]. In particular to get (b) see [28, Theor. 3.4]. The proof of (c) is easily adapted from Schweighofer [43]. □

Remark 3.5. Hence, by Theorem 3.4(b), when finite convergence occurs one may extract $t := \operatorname{rank} M_r(\mathbf{y})$ global minimizers. On the other hand, a *generic* **MRF** problem has a unique global minimizer $x^* \in \mathbf{K}$ and in this case, even when the convergence is only asymptotic, one may still obtain an approximation of x^* (as closely as desired) from the vector of first-order moments $(L_{\mathbf{y}^r}(x_1), \ldots, L_{\mathbf{y}^r}(x_n))$. For instance, one way to have a unique global minimizer is to ϵ-perturb the objective function of the **MRF** problem by some randomly generated polynomial of a sufficiently large degree, or to slightly perturb the coefficients of the data (h_i, g_j) of the **MRF** problem.

To solve (3.11) one may use e.g. the Matlab based public software GloptiPoly 3 [15] dedicated to solve the generalized problem of moments described in [26]. It is an extension of GloptiPoly [14] previously dedicated to solve polynomial optimization problems. A procedure for extracting optimal solutions is implemented in Gloptipoly when the rank condition (3.12) is satisfied[5]. For more details the interested reader is referred to [15] and `www.laas.fr/~henrion/software/`.

Remark 3.6. If g_j is affine for every $j = 1, \ldots, p$ and if p_j is affine and $q_j \equiv 1$ for every $j = 0, \ldots, m$, then h_j is affine for every $j = 0, \ldots, m$. One may also replace the single quadratic constraint $h_{m+p+1}(x,z) = (M_1 - z)(z - M_2) \geq 0$ with the two equivalent linear constraints $h_{m+p+1}(x,z) = M_1 - z \geq 0$ and $h_{m+p+2}(x,z) = z - M_2 \geq 0$. In this case, it suffices to solve the single semidefinite relaxation $\mathbf{Q}_1$,

[5]In fact GloptiPoly 3 extracts *all* solutions because most SDP solvers that one may call in GloptiPoly 3 (e.g. SeDuMi) use primal-dual interior points methods with the self-dual embedding technique which find an optimal solution in the relative interior of the set of optimal solutions; see [27, §4.4.1, p. 663]. In the present context of (3.11) this means that at an optimal solution $\mathbf{y}^*$, the moment matrix $M_r(\mathbf{y}^*)$ has maximum rank and its rank corresponds to the number of solutions.

which is in fact a linear program. Indeed, for $r = 1$, $\mathbf{y} = (y_0, (x, z), Y)$ and

$$M_1(\mathbf{y}) = \left[\begin{array}{c|c} y_0 & (x \quad z) \\ \hline \begin{pmatrix} x \\ z \end{pmatrix} & Y \end{array} \right].$$

Then (3.11) reads

$$\mathbf{Q}_1 : \left\{ \begin{array}{lll} \min\limits_{\mathbf{y}} & h_0(x) & \\ \text{s.t.} & M_1(\mathbf{y}) & \succeq 0 \\ & h_j(x,z) & \geq 0, \quad j = 1, \ldots, m+p+2 \\ & y_0 & = 1. \end{array} \right. .$$

As $v_j = 1$ for every j, $M_{1-1}(h_j, \mathbf{y}) \succeq 0 \Leftrightarrow M_0(h_j, \mathbf{y}) = L_{\mathbf{y}}(h_j) = h_j(x,z) \geq 0$, a linear constraint. Hence the constraint $M_1(\mathbf{y}) \succeq 0$ can be discarded as given any (x, z) one may always find Y such that $M_1(\mathbf{y}) \succeq 0$. Therefore, (3.11) is a linear program. This is fortunate for finite zero-sum games applications since computing the value is equivalent to minimizing a maximum of finitely many linear functions (and it is already known that it can be solved by Linear Programming).

4. Applications to finite games

In this section we show that several solution concepts of static and dynamic finite games reduce to solving the **MRF** problem (3.2). Those are just examples and one expects that such a reduction also holds in a much larger class of games (when they are described by finitely many scalars).

4.1. **Standard static games.** A finite game is a tuple $(N, \{S^i\}_{i=1,\ldots,N}, \{g^i\}_{i=1,\ldots,N})$ where $N \in \mathbb{N}$ is the set of players, S^i is the finite set of pure strategies of player i and $g^i : \mathbf{S} \to \mathbb{R}$ is the payoff function of player i, where $\mathbf{S} := S^1 \times \ldots \times S^N$. The set

$$\Delta^i = \left\{ \left(p^i(s^i)\right)_{s^i \in S^i} : \quad p^i(s^i) \geq 0, \sum_{s^i \in S^i} p^i(s^i) = 1 \right\}$$

of probability distributions over S^i is called the set of mixed strategies of player i. Notice that Δ^i is a compact basic semi-algebraic set. If each player j chooses the mixed strategy $p^j(\cdot)$, the vector denoted $p = \left(p^1, \ldots, p^N\right) \in \mathbf{\Delta} := \Delta^1 \times \ldots \times \Delta^N$ is called a *profile* of mixed strategies and the expected payoff of a player i is

$$g^i(p) = \sum_{s=(s^1,\ldots,s^N) \in S} p^1(s^1) \times \ldots \times p^i(s^i) \times \ldots \times p^N(s^N) g^i(s).$$

This is nothing but the multi-linear extension of g^i. For a player i, and a profile p, let p^{-i} be the profile of the other players except i: that is $p^{-i} = (p^1, \ldots, p^{i-1}, p^{i+1}, \ldots, p^N)$. Let $\mathbf{S}^{-i} = S^1 \times \ldots \times S^{i-1} \times S^{i+1} \times \ldots \times S^N$ and define

$$g^i(s^i, p^{-i}) = \sum_{s^{-i} \in S^{-i}} p^1(s^1) \times \ldots \times p^{i-1}(s^{i-1}) \times p^{i+1}(s^{i+1}) \times \ldots \times p^N(s^N) g^i(s),$$

where $s^{-i} := (s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^N) \in S^{-i}$.

A profile p_0 is a Nash [33] equilibrium if and only for all $i = 1, ..., N$ and all $s^i \in S^i$, $g^i(p_0) \geq g^i(s^i, p_0^{-i})$ or equivalently if:

$$p_0 \in \arg\min_{p \in \mathbf{\Delta}} \left\{ \max_{i=1,\ldots,N} \max_{s^i \in S^i} \left\{ g^i(s^i, p_0^{-i}) - g^i(p_0) \right\} \right\}. \tag{4.1}$$

Since each finite game admits at least one Nash equilibrium [33], the optimal value of the min-max problem (4.1) is zero. Notice that (4.1) is a particular instance of the **MRF** problem (3.2) (with a set $\mathbf{K} = \Delta$ that satisfies Putinar's property and with $q_i = 1$ for every $i = 0, \ldots, m$), and so Theorem 3.4 applies. Finally, observe that the number m of polynomials in the inner double "max" of (4.1) (or, equivalently, m in (3.2)) is just $m = \sum_{i=1}^n |S^i|$, i.e., m is just the total number of all possible actions.

Hence by solving the hierarchy of SDP relaxations (3.11), one can approximate the value of the min-max problem as closely as desired. In addition, if (3.12) is satisfied at some relaxation $\mathbf{Q}_r$, then one may extract all the Nash equilibria of the game.

If there is a unique equilibrium p^* then by Theorem 3.4(c), one may obtain a solution arbitrary close to p^* and so obtain an ϵ-equilibrium in finite time. Since game problems are not generic **MRF** problems, they have potentially several equilibria which are all global minimizers of the associated **MRF** problem. Also, perturbing the data of a finite game still leads to a non generic associated **MRF** problem with possibly multiple solutions. However, as in Remark 3.5, one could perturb the **MRF** problem associated with the original game problem to obtain (generically) an ϵ-perturbed **MRF** problem with a unique optimal solution. Notice that the ϵ-perturbed **MRF** is not necessarily coming from a finite game. Doing so, by Theorem 3.4(c), one obtains a sequence that converges asymptotically (and sometimes in finitely many steps) to an ϵ-equilibrium of the game problem. Recently, Lipton, Markakis and Mehta [31] provided an algorithm that computes an ϵ-equilibrium in less than exponential time but still not polynomial (namely $n^{\frac{logn}{\epsilon^2}}$ where n is the total number of strategies). This promising result yields Papadimitriou [35] to argue that "*finding a mixed Nash equilibrium is PPAD-complete raises some interesting questions regarding the usefulness of Nash equilibrium, and helps focus our interest in alternative notions (most interesting among them the approximate Nash equilibrium)*".

Example 4.1. Consider the simple illustrative example of a 2×2 game with data

	s_1^2	s_2^1
s_1^1	(a, c)	$(0, 0)$
s_2^1	$(0, 0)$	(b, d)

for some scalars (a, b, c, d). Denote $x_1 \in [0, 1]$ the probability for player 1 of playing s_1^1 and $x_2 \in [0, 1]$ the probability for player 2 of playing s_1^2. Then one must solve

$$\min_{x_1, x_2 \in [0,1]} \max \begin{cases} ax_1 - ax_1x_2 - b(1-x_1)(1-x_2) \\ b(1-x_2) - ax_1x_2 - b(1-x_1)(1-x_2) \\ cx_1 - cx_1x_2 - d(1-x_1)(1-x_2) \\ d(1-x_1) - cx_1x_2 - d(1-x_1)(1-x_2) \end{cases}.$$

We have solved the hierarchy of semidefinite programs (3.11) with GloptiPoly 3 [15]. For instance, the moment matrix $M_1(\mathbf{y})$ of the first SDP relaxation $\mathbf{Q}_1$ reads

$$M_1(\mathbf{y}) = \begin{bmatrix} y_0 & y_{100} & y_{010} & y_{001} \\ y_{100} & y_{200} & y_{010} & y_{001} \\ y_{010} & y_{110} & y_{020} & y_{011} \\ y_{001} & y_{101} & y_{011} & y_{002} \end{bmatrix},$$

and $\mathbf{Q}_1$ reads

$$\mathbf{Q}_1 : \left\{ \begin{array}{ll} \min\limits_{\mathbf{y}} & y_{001} \\ \text{s.t.} & M_1(\mathbf{y}) \succeq 0 \\ & y_{001} - ay_{100} + ay_{110} + b(y_0 - y_{100} - y_{010} + y_{110}) \geq 0 \\ & y_{001} - by_0 + by_{010} + ay_{110} + b(y_0 - y_{100} - y_{010} + y_{110}) \geq 0 \\ & y_{001} - cy_{100} + cy_{110} + d(y_0 - y_{100} - y_{010} + y_{110}) \geq 0 \\ & y_{00} - dy_0 + dy_{100} + cy_{110} + d(y_0 - y_{100} - y_{010} + y_{110}) \geq 0 \\ & y_{100} - y_{200} \geq 0;\ y_{010} - y_{020} \geq 0;\ 9 - y_{002} \geq 0 \\ & y_0 = 1 \end{array} \right. .$$

With $(a, b, c, d) = (0.05, 0.82, 0.56, 0.76)$, solving $\mathbf{Q}_3$ yields the optimal value $3.93.10^{-11}$ and the three optimal solutions $(0, 0)$, $(1, 1)$ and $(0.57575, 0.94253)$. With randomly generated $a, b, c, d \in [0, 1]$ we also obtained a very good approximation of the global optimum 0 and 3 optimal solutions in most cases with $r = 3$ (i.e. with moments or order 6 only) and sometimes $r = 4$.

We have also solved 2-player non-zero-sum $p \times q$ games with randomly generated reward matrices $A, B \in \mathbb{R}^{p\times q}$ and $p, q \leq 5$. We could solve $(5, 2)$ and $(4, q)$ (with $q \leq 3$) games exactly with the 4th (sometimes 3rd) SDP relaxation, i.e. $\inf \mathbf{Q}_4 = O(10^{-10}) \approx 0$ and one extracts an optimal solution [6]. However, the size is relatively large and one is close to the limit of present public SDP solvers like SeDuMi. Indeed, for a 2-player $(5, 2)$ or $(4, 3)$ game, $\mathbf{Q}_3$ has 923 variables and $M_3(\mathbf{y}) \in \mathbb{R}^{84\times 84}$, whereas $\mathbf{Q}_4$ has 3002 variables and $M_4(\mathbf{y}) \in \mathbb{R}^{210\times 210}$. For a $(4, 4)$ game $\mathbf{Q}_3$ has 1715 variables and $M_3(\mathbf{y}) \in \mathbb{R}^{120\times 120}$ and $\mathbf{Q}_3$ is still solvable, whereas $\mathbf{Q}_4$ has 6434 variables and $M_4(\mathbf{y}) \in \mathbb{R}^{330\times 330}$.

Finally we have also solved randomly generated instances of 3-player non-zero sum games with $(2, 2, 2)$ actions and $(3, 3, 2)$ actions. In all $(2, 2, 2)$ cases the 4th relaxation $\mathbf{Q}_4$ provided the optimal value and the rank-test (3.12) was passed (hence allowing to extract global minimizers). For the $(3, 3, 2)$ games, the third relaxation $\mathbf{Q}_3$ was enough in 30% of cases and the fourth relaxation $\mathbf{Q}_4$ in 80% of cases.

Another important concept in game theory is the min-max payoff $\underline{v}_i$ which plays an important role in repeated games (Aumann and Shapley [2]):

$$\underline{v}_i = \min_{p^{-i}\in \mathbf{\Delta}^{-i}} \max_{s^i \in S^i} g^i(s^i, p^{-i})$$

where $\mathbf{\Delta}^{-i} = \Delta^1 \times ... \times \Delta^{i-1} \times \Delta^{i+1} \times ... \times \Delta^N$. This is again a particular instance of the **MRF** problem (3.2). Hence, it seems more difficult to compute the approximate min-max strategies compared to approximate Nash equilibrium strategies because we do not know in advance the value of $\underline{v}_i$ while we know that the min-max value associated to the Nash problem is always zero. This is not surprising: in theory, computing a Nash-equilibrim is PPAD-complete [35] while computing the min-max

[6] In general, it is not known which relaxation suffices to solve the min-max problem. Also, as already mentioned, GloptiPoly 3 extracts *all* solutions because most SDP solvers that one may call in GloptiPoly 3 (e.g. SeDuMi) use primal-dual interior points methods with the self-dual embedding technique which find an optimal solution in the relative interior of the set of optimal solutions. This is explained in [27, §4.4.1, p. 663]

is NP-hard [3]. In the case of two players, the function $g^i(s^i, p^{-i})$ is linear in p^{-i}. By remark 3.6 it suffices to solve the first relaxation $\mathbf{Q}_1$, a linear program.

Remark 4.2. The Nash equilibrium problem may be reduced to solving a system of polynomial equations (see e.g. [8]). In the same spirit, an alternative for the Nash-equilibrium problem (but not for the **MRF** problem in general) is to apply the moment approach described in Lasserre et al. [27] for finding *real* roots of polynomial equations. If there are finitely many Nash equilibria then its convergence is *finite* and in contrast with the algebraic methods [8, 30, 47] mentioned above, it provides *all* real solutions without computing all complex roots.

4.2. **Loomis games.** Loomis [32] extended the min-max theorem to zero-sum games with a rational fraction. His model may be extended to N-player games as follows. Our extension is justified by the next section.

Associated with each player $i \in N$ are two functions $g^i : \mathbf{S} \to \mathbb{R}$ and $f^i : \mathbf{S} \to \mathbb{R}$ where $f^i > 0$ and $\mathbf{S} := S^1 \times ... \times S^N$. With same notation as in the last section, let their multi-linear extension to $\mathbf{\Delta}$ still be denoted by g^i and f^i. That is, for $p \in \mathbf{\Delta}$, let:

$$g^i(p) = \sum_{s=(s^1,...,s^N)\in S} p^1(s^1) \times ... \times p^i(s^i) \times ... \times p^N(s^N) g^i(s).$$

and similarly for f^i.

Definition 4.3. *A Loomis game is defined as follows. The strategy set of player i is Δ^i and if the profile $p \in \mathbf{\Delta}$ is chosen, his payoff function is $h^i(p) = \frac{g^i(p)}{f^i(p)}$.*

One can show the following new lemma[7].

Lemma 4.4. *A Loomis game admits a Nash equilibrium[8].*

Proof. Note that each payoff function is quasi-concave in p^i (and also quasi-convex so that it is a quasi-linear function). Actually, if $h^i(p_1^i, p^{-i}) \geq \alpha$ and $h^i(p_2^i, p^{-i}) \geq \alpha$ then for any $\delta \in [0, 1]$,

$$g^i(\delta p_1^i + (1-\delta)p_2^i, p^{-i}) \geq f^i(\delta p_1^i + (1-\delta)p_2^i, p^{-i})\alpha,$$

so that $h^i(\delta p_1^i + (1-\delta)p_2^i, p^{-i}) \geq \alpha$. One can now apply Glicksberg's [11] theorem because the strategy sets are compact, convex, and the payoff functions are quasi-concave and continuous. □

Corollary 4.5. *A profile $p_0 \in \mathbf{\Delta}$ is a Nash equilibrium of a Loomis game if and only if*

$$p_0 \in \arg\min_{p\in\Delta} \left\{ \max_{i=1,...,N} \max_{s^i\in S^i} \left\{ h^i(s^i, p^{-i}) - h^i(p) \right\} \right\}. \tag{4.2}$$

[7]As far as we know, non-zero sum Loomis games are not considered in the literature. This model could be of interest in situations where there are populations with many players. A mixed strategy for a population describes the proportion of players in the population that uses some pure action. $g^i(p)$ is the non-normalized payoff of population i and $f^i(p)$ may be interpreted as the value of money for population i so that $h^i(p) = \frac{g^i(p)}{f^i(p)}$ is the normalized payoff of population i.

[8]Clearly, the lemma and its proof still hold in infinite games where the sets S^i are convex-compact-metric and the functions f^i and g^i are continuous. The summation in the multi-linear extension should be replaced by an integral.

Proof. Clearly, $p_0 \in \mathbf{\Delta}$ is an equilibrium of the Loomis game if and only if

$$p_0 \in \arg\min_{p\in\Delta} \left\{ \max_{i=1,\ldots,N} \max_{\widetilde{p}^i\in\Delta^i} \left\{ \frac{g(\widetilde{p}^i, p^{-i})}{f^i(\widetilde{p}^i, p^{-i})} - \frac{g^i(p)}{f^i(p)} \right\} \right\}.$$

Using the quasi-linearity of the payoffs or Lemma 2.3, one deduces that:

$$\max_{\widetilde{p}^i\in\Delta^i} \frac{g^i(\widetilde{p}^i, p^{-i})}{f^i(\widetilde{p}^i, p^{-i})} = \max_{s^i\in S^i} \frac{g^i(s^i, p^{-i})}{f^i(s^i, p^{-i})}$$

which is the desired result. □

Again, the min-max optimization problem (4.2) is a particular instance of the **MRF** problem (3.2) and so can be solved via the hierarchy of semidefinite relaxations (3.11). Notice that in (4.2) one has to minimize the supremum of rational functions (in contrast to the supremum of polynomials in (4.1)).

4.3. **Finite absorbing games.** This subclass of stochastic games has been introduced by Kohlberg [21]. The following formulas are established in [23]. It shows that absorbing games could be reduced to Loomis games. An N-player finite absorbing game is defined as follows.

As above, there are N finite sets $(S^1, ..., S^N)$. There are two functions $g^i : \mathbf{S} \to \mathbb{R}$ and $f^i : \mathbf{S} \to \mathbb{R}$ associated to each player $i \in \{1, ..., N\}$ and a probability transition function $q : \mathbf{S} \to [0, 1]$.

The game is played in discrete time as follows. At each stage $t = 1, 2, ...$, if the game has not been absorbed before that day, each player i chooses (simultaneously) at random an action $s^i_t \in S^i$. If the profile $s_t = (s^1_t, ..., s^N_t)$ is chosen, then:

(i) the payoff of player i is $g^i(s_t)$ at stage t.

(ii) with probability $1-q(s_t)$ the game is terminated (absorbed) and each player i gets at every stage $s > t$ the payoff $f^i(s_t)$, and

(iii) with probability $q(s_t)$ the game continues (the situation is repeated at stage $t + 1$).

Consider the λ-discounted game $G(\lambda)$ $(0 < \lambda < 1)$. If the payoff of player i at stage t is $r^i(t)$ then its λ-discounted payoff in the game is $\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} r^i(t)$. Hence, a player is optimizing his expected λ-discounted payoff.

Let $\widetilde{g}^i = g^i \times q$ and $\widetilde{f}^i = f^i \times (1-q)$ and extend $\widetilde{g}^i$, $\widetilde{f}^i$ and q multilinearly to $\mathbf{\Delta}$ (as above in Nash and Loomis games).

A profile $p \in \Delta$ is a stationary equilibrium of the absorbing game if (1) each player i plays iid at each stage t the mixed strategy p^i until the game is absorbed and (2) this is optimal for him in the discounted absorbing game if the other players do not deviate.

Lemma 4.6. *A profile $p_0 \in \Delta$ is a stationary equilibrium of the absorbing game if and only if it is a Nash equilibrium of the Loomis game with payoff functions $p \to \frac{\lambda\widetilde{g}^i(p)+(1-\lambda)\widetilde{f}^i(p)}{\lambda q(p)+(1-q(p))}$, $i = 1, ..., N$.*

Proof. See Laraki [23]. □

As shown in [23], the min-max of a discounted absorbing game satisfies:

$$\underline{v}_i = \min_{p^{-i}\in\mathbf{\Delta}^{-i}} \max_{s^i\in S^i} \frac{\lambda\widetilde{g}^i(s^i, p^{-i}) + (1-\lambda)\widetilde{f}^i(s^i, p^{-i})}{\lambda q(s^i, p^{-i}) + (1 - q(s^i, p^{-i}))}.$$

Hence solving a finite absorbing game is equivalent to solving a Loomis game (hence a particular instance of the **MRF** problem (3.2)) which again can be solved via the hierarchy of semidefinite relaxations (3.11). Again one has to minimize the supremum of rational functions.

5. Zero-sum polynomial games

Let $\mathbf{K}_1 \subset \mathbb{R}^{n_1}$ and $\mathbf{K}_2 \subset \mathbb{R}^{n_2}$ be two basic and closed semi-algebraic sets (not necessarily with same dimension):

$$\mathbf{K}_1 := \{x \in \mathbb{R}^{n_1} : g_j(x) \geq 0, \quad j = 1, \ldots, m_1\} \tag{5.1}$$

$$\mathbf{K}_2 := \{x \in \mathbb{R}^{n_2} : h_k(x) \geq 0, \quad k = 1, \ldots, m_2\} \tag{5.2}$$

for some polynomials $(g_j) \subset \mathbb{R}[x_1, \ldots x_{n_1}]$ and $(h_k) \subset \mathbb{R}[x_1, \ldots x_{n_2}]$.

Let $P(\mathbf{K}_i)$ be the set of Borel probability measures on $\mathbf{K}_i$, $i = 1, 2$, and consider the following min-max problem:

$$\mathbf{P}: \quad J^* = \min_{\mu \in P(\mathbf{K}_1)} \max_{\nu \in P(\mathbf{K}_2)} \int_{\mathbf{K}_2} \int_{\mathbf{K}_1} p(x, z)\, d\mu(x)\, d\nu(z) \tag{5.3}$$

for some polynomial $p \in \mathbb{R}[x, z]$.

If $\mathbf{K}_1$ and $\mathbf{K}_2$ are compact, it is well-known that

$$\begin{aligned} J^* &= \min_{\mu \in P(\mathbf{K}_1)} \max_{\nu \in P(\mathbf{K}_2)} \int_{\mathbf{K}_2} \int_{\mathbf{K}_1} p(x, z)\, d\mu(x)\, d\nu(z) \\ &= \max_{\nu \in P(\mathbf{K}_2)} \min_{\mu \in P(\mathbf{K}_1)} \int_{\mathbf{K}_1} \int_{\mathbf{K}_2} p(x, z)\, d\nu(z)\, d\mu(x), \end{aligned} \tag{5.4}$$

that is, there exist $\mu^* \in P(\mathbf{K}_1)$ and $\nu^* \in P(\mathbf{K}_2)$ such that:

$$J^* = \int_{\mathbf{K}_2} \int_{\mathbf{K}_1} p(x, z)\, d\mu^*(x)\, d\nu^*(z). \tag{5.5}$$

The probability measures μ^* and ν^* are the optimal strategies of players 1 and 2 respectively.

For the univariate case $n = 1$, Parrilo [37] showed that J^* is the optimal value of a *single* semidefinite program, namely the semidefinite program (7) in [37, p. 2858], and mentioned how to extract optimal strategies since there exist optimal strategies (μ^*, ν^*) with finite support. In [37] the author mentions that extension to the multivariate case is possible. We provide below such an extension which, in view of the proof of its validity given below, is non trivial. The price to pay for this extension is to replace a single semidefinite program with a hierarchy of semidefinite programs of increasing size. But contrary to the polynomial optimization case in e.g. [24], proving convergence of this hierarchy is more delicate because one has (simultaneously in the same SDP) moment matrices of increasing size and an s.o.s.-representation of some polynomial in Putinar's form (2.3) with increasing degree bounds for the s.o.s. weights. In particular, the convergence is not monotone anymore. When we do $n = 1$ in this extension, one retrieves the original result of Parrilo [37], i.e., the first semidefinite program in the hierarchy (5.9) reduces to (7) in [37, p. 2858] and provides us with the exact desired value.

Semidefinite relaxations for solving P. With $p \in \mathbb{R}[x,z]$ as in (3.2), write

$$\begin{aligned} p(x,z) &= \sum_{\alpha\in\mathbb{N}^{n_2}} p_\alpha(x)\, z^\alpha \qquad \text{with} \\ p_\alpha(x) &= \sum_{\beta\in\mathbb{N}^{n_1}} p_{\alpha\beta}\, x^\beta, \qquad |\alpha| \,\leq\, d_z \end{aligned} \tag{5.6}$$

where d_z is the total degree of p when seen as polynomial in $\mathbb{R}[z]$. So, let $p_{\alpha\beta} := 0$ for every $\beta \in \mathbb{N}^{n_1}$ whenever $|\alpha| > d_z$.

Let $r_j := \lceil \deg g_j/2 \rceil$, for every $j = 1,\ldots,m_1$, and consider the following semidefinite program:

$$\left\{\begin{array}{ll} \min\limits_{\mathbf{y},\lambda,Z^k} & \lambda \\ \text{s.t.} & \lambda\, I_{\alpha=0} - \sum\limits_{\beta\in\mathbb{N}^{n_1}} p_{\alpha\beta}\, y_\beta \,=\, \langle Z^0, B_\alpha\rangle + \sum\limits_{k=1}^{m_2} \langle Z^k, B_\alpha^{h_k}\rangle, \quad |\alpha| \leq 2d \\ & \\ & M_d(\mathbf{y}) \,\succeq\, 0 \\ & M_{d-r_j}(g_j,\mathbf{y}) \,\succeq\, 0, \quad j = 1,\ldots,m_1 \\ & y_0 \,=\, 1 \\ & \\ & Z^k \,\succeq\, 0, \quad k = 0,1,\ldots m_2, \end{array}\right. \tag{5.7}$$

where $\mathbf{y}$ is a finite sequence indexed in the canonical basis (x^α) of $\mathbb{R}[x]_{2d}$. Denote by λ_d^* its optimal value. In fact, with $h_0 \equiv 1$ and $p_\mathbf{y} \in \mathbb{R}[z]$ defined by:

$$z \,\mapsto\, p_\mathbf{y}(z) \,:=\, \sum_{\alpha\in\mathbb{N}^{n_2}} \left(\sum_{\beta\in\mathbb{N}^{n_1}} p_{\alpha\beta}\, y_\beta \right) z^\alpha, \tag{5.8}$$

the semidefinite program (5.7) has the equivalent formulation:

$$\left\{\begin{array}{ll} \min\limits_{\mathbf{y},\lambda,\sigma_k} & \lambda \\ \text{s.t.} & \lambda \,-\, p_\mathbf{y}(\cdot) \,=\, \sum\limits_{k=0}^{m_2} \sigma_k\, h_k \\ & \\ & M_d(\mathbf{y}) \,\succeq\, 0 \\ & M_{d-r_j}(g_j,\mathbf{y}) \,\succeq\, 0, \quad j = 1,\ldots,m_1 \\ & y_0 \,=\, 1 \\ & \\ & \sigma_k \in \Sigma[z];: \deg\sigma_k + \deg h_k \leq 2d, \quad k = 0,1,\ldots,m_2, \end{array}\right. \tag{5.9}$$

where the first constraint should be understood as an equality of polynomials. Observe that for any admissible solution $(\mathbf{y},\lambda)$ and $p_\mathbf{y}$ as in (5.8),

$$\lambda \,\geq\, \max_z \{ p_\mathbf{y}(z) \,:\, z \in \mathbf{K}_2 \}. \tag{5.10}$$

Similarly, with p as in (3.2), write

$$\begin{aligned} p(x,z) &= \sum_{\alpha\in\mathbb{N}^{n_1}} \hat{p}_\alpha(z)\, x^\alpha \qquad \text{with} \\ \hat{p}_\alpha(z) &= \sum_{\beta\in\mathbb{N}^{n_2}} \hat{p}_{\alpha\beta}\, z^\beta, \qquad |\alpha| \,\leq\, d_x \end{aligned} \tag{5.11}$$

where d_x is the total degree of p when seen as polynomial in $\mathbb{R}[x]$. So, let $\hat{p}_{\alpha\beta} := 0$ for every $\beta \in \mathbb{N}^{n_2}$ whenever $|\alpha| > d_x$.

Let $l_k := \lceil \deg h_k/2 \rceil$, for every $k = 1,\ldots,m_2$, and with

$$x \,\mapsto\, \hat{p}_{\mathbf{y}}(x) \,:=\, \sum_{\alpha\in\mathbb{N}^{n_1}} \left(\sum_{\beta\in\mathbb{N}^{n_2}} \hat{p}_{\alpha\beta}\, y_\beta \right) x^\alpha, \tag{5.12}$$

consider the following semidefinite program (with $g_0 \equiv 1$):

$$\left\{ \begin{array}{ll} \max\limits_{\mathbf{y},\gamma,\sigma_j} & \gamma \\ \text{s.t.} & \hat{p}_{\mathbf{y}}(\cdot) - \gamma \,=\, \sum\limits_{j=0}^{m_1} \sigma_j\, g_j \\ & \\ & M_d(\mathbf{y}) \,\succeq\, 0 \\ & M_{d-l_k}(h_k,\mathbf{y}) \,\succeq\, 0, \quad k = 1,\ldots,m_2 \\ & y_0 \,=\, 1 \\ & \\ & \sigma_j \in \Sigma[x];\ \deg\sigma_j + \deg g_j \leq 2d, \quad j = 0,1,\ldots,m_1. \end{array} \right. \tag{5.13}$$

where $\mathbf{y}$ is a finite sequence indexed in the canonical basis (z^α) of $\mathbb{R}[z]_{2d}$. Denote by γ_d^* its optimal value. In fact, (5.13) is the dual of the semidefinite program (5.7).

Observe that for any admissible solution $(\mathbf{y},\gamma)$ and $\hat{p}_{\mathbf{y}}$ as in (5.12),

$$\gamma \,\leq\, \min_x \{ \hat{p}_{\mathbf{y}}(x) \,:\, x \in \mathbf{K}_1 \}. \tag{5.14}$$

Theorem 5.1. *Let* $\mathbf{P}$ *be the min-max problem defined in (3.2) and assume that both* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *are compact and satisfy Putinar's property (see Definition 2.1). Let* λ_d^* *and* γ_d^* *be the optimal values of the semidefinite program (5.9) and (5.13) respectively. Then* $\lambda_d^* \to J^*$ *and* $\gamma_d^* \to J^*$ *as* $d \to \infty$.

We also have a test to detect whether finite convergence has occurred.

Theorem 5.2. *Let* $\mathbf{P}$ *be the min-max problem defined in (3.2).*

Let λ_d^* *be the optimal value of the semidefinite program (5.9), and suppose that with* $r := \max_{j=1,\ldots,m_1} r_j$*, the condition*

$$\operatorname{rank} M_{d-r}(\mathbf{y}) \,=\, \operatorname{rank} M_d(\mathbf{y}) \quad (=:\, s_1) \tag{5.15}$$

holds at an optimal solution $(\mathbf{y},\lambda,\sigma_k)$ *of (5.9).*

Let γ_t^* *be the optimal value of the semidefinite program (5.13), and suppose that with* $r := \max_{k=1,\ldots,m_2} l_k$*, the condition*

$$\operatorname{rank} M_{t-r}(\mathbf{y}') \,=\, \operatorname{rank} M_t(\mathbf{y}') \quad (=:\, s_2) \tag{5.16}$$

holds at an optimal solution $(\mathbf{y}',\gamma,\sigma_j)$ *of (5.13).*

If $\lambda_d^* = \gamma_t^*$ then $\lambda_g^* = \gamma_t^* = J^*$ and an optimal strategy for player 1 (resp. player 2) is a probability measure μ^* (resp. ν^*) supported on s_1 points of $\mathbf{K}_1$ (resp. s_2 points of $\mathbf{K}_2$).

For a proof the reader is referred to §8.

Remark 5.3. In the univariate case, when $\mathbf{K}_1, \mathbf{K}_2$ are (not necessarily bounded) intervals of the real line, the optimal value $J^* = \lambda_d^*$ (resp. $J^* = \gamma_d^*$) is obtained by solving the single semidefinite program (5.9) (resp. (5.13)) with $d = d_0$, which is equivalent to (7) in Parrilo [37, p. 2858].

6. Zero-sum polynomial absorbing games

As in the previous section, consider two compact basic semi-algebraic sets $\mathbf{K}_1 \subset \mathbb{R}^{n_1}$, $\mathbf{K}_2 \subset \mathbb{R}^{n_2}$ and polynomials g, f and $q : \mathbf{K}_1 \times \mathbf{K}_2 \to [0,1]$. Recall that $P(\mathbf{K}_1)$ (resp. $P(\mathbf{K}_2)$) denotes the set of probability measures on $\mathbf{K}_1$ (resp. $\mathbf{K}_2$). The absorbing game is played in discrete time as follows. At stage $t = 1, 2, ...$ player 1 chooses at random $x_t \in \mathbf{K}_1$ (using some mixed action $\mu_t \in P(\mathbf{K}_1)$) and, simultaneously, Player 2 chooses at random $y_t \in \mathbf{K}_2$ (using some mixed action $\nu_t \in P(\mathbf{K}_2)$).

(i) player 1 receives $g(x_t, y_t)$ at stage t;

(ii) with probability $1 - q(x_t, y_t)$ the game is absorbed and player 1 receives $f(x_t, y_t)$ in all stages $s > t$;

and

(iii) with probability $q(x_t, y_t)$ the game continues (the situation is repeated at step $t+1$).

If the stream of payoffs is $r(t)$, $t = 1, 2, ...$, the λ-discounted-payoff of the game is $\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} r(t)$.

Let $\widetilde{g} = g \times q$ and $\widetilde{f} = f \times (1-q)$ and extend $\widetilde{g}$, $\widetilde{f}$ and q multilinearly to $P(\mathbf{K}_1) \times P(\mathbf{K}_2)$.

Player 1 maximizes the expected discounted-payoff and player 2 minimizes that payoff. Using an extension of the Shapley operator [45] one can deduce that the game has a value $v(\lambda)$ that uniquely satisfies,

$$\begin{aligned} v(\lambda) &= \max_{\mu \in P(\mathbf{K}_1)} \min_{\nu \in P(\mathbf{K}_2)} \int_\Theta \left(\lambda \widetilde{g} + (1-\lambda) v(\lambda) p + (1-\lambda) \widetilde{f} \right) d\mu \otimes \nu \\ &= \min_{\nu \in P(\mathbf{K}_2)} \max_{\mu \in P(\mathbf{K}_1)} \int_\Theta \left(\lambda \widetilde{g} + (1-\lambda) v(\lambda) p + (1-\lambda) \widetilde{f} \right) d\mu \otimes \nu \end{aligned}$$

with $\Theta := \mathbf{K}_1 \times \mathbf{K}_2$. As in the finite case, it may be shown [23] that the problem may be reduced to a zero-sum Loomis game, that is:

$$v(\lambda) = \max_{\mu \in P(\mathbf{K}_1)} \min_{\nu \in P(\mathbf{K}_2)} \frac{\int_\Theta P \, d\mu \otimes \nu}{\int_\Theta Q \, d\mu \otimes \nu} = \min_{\nu \in P(\mathbf{K}_2)} \max_{\mu \in P(\mathbf{K}_1)} \frac{\int_\Theta P \, d\mu \otimes \nu}{\int_\Theta Q \, d\mu \otimes \nu} \tag{6.1}$$

where

$$\begin{aligned} (x,y) \mapsto P(x,y) &:= \lambda \widetilde{g}(x,y) + (1-\lambda) \widetilde{f}(x,y) \\ (x,y) \mapsto Q(x,y) &:= \lambda q(x,y) + 1 - q(x,y) \end{aligned}$$

Or equivalently, as it was originally presented by Loomis [32], $v(\lambda)$ is the unique real t such that

$$\begin{aligned} 0 &= \max_{\mu\in P(\mathbf{K}_1)} \min_{\nu\in P(\mathbf{K}_2)} \left[\int_\Theta (P(x,y) - t\,Q(x,y))\, d\mu(x)d\nu(y)\right] \\ &= \min_{\nu\in P(\mathbf{K}_2)} \min_{\nu\in P(\mathbf{K}_1)} \left[\int_\Theta (P(x,y) - t\,Q(x,y))\, d\mu(x)d\nu(y)\right]. \end{aligned}$$

Actually, the function $s:\mathbb{R}\to\mathbb{R}$ defined by:

$$t \to s(t) := \max_{\mu\in P(\mathbf{K}_1)} \min_{\nu\in P(\mathbf{K}_2)} \left[\int_\Theta (P(x,y) - tQ(x,y))\, d\mu(x)d\nu(y)\right]$$

is continuous, strictly decreasing from $+\infty$ to $-\infty$ as t increases in $(-\infty,+\infty)$.

In the univariate case, if $\mathbf{K}_1$ and $\mathbf{K}_2$ are both real intervals (not necessarily compact), then evaluating $s(t)$ for some fixed t can be done by solving a *single* semidefinite program; see Remark 5.3. Therefore, in this case, one may approximate the optimal value s^* $(= s(t^*))$ of the game by binary search on t and so, the problem can be solved in a polynomial time. This extends Shah and Parrilo [44].

7. Conclusion

We have proposed a common methodology to approximate the optimal value of games in two different contexts. The first algorithm, intended to compute (or approximate) Nash equilibria in mixed strategies for static finite games or dynamic absorbing games, is based on a hierarchy of semidefinite programs to approximate the supremum of finitely many rational functions on a compact basic semi-algebraic set. Actually this latter formulation is also of self-interest in optimization. The second algorithm, intended to approximate the optimal value of polynomial games whose action sets are compact basic semi-algebraic sets, is also based on a hierarchy of semidefinite programs. Not surprisingly, as the latter algorithm comes from a min-max problem over sets of measures, it is a subtle combination of moment and s.o.s. constraints whereas in polynomial optimization it is entirely formulated either with moments (primal formulation) or with s.o.s. (dual formulation). Hence the above methodology illustrates the power of the combined moment-s.o.s. approach. A natural open question arises: how to adapt the second algorithm to compute Nash equilibria of a non-zero-sum polynomial game?

8. Appendix

8.1. **Proof of Theorem 5.1.** We first need the following partial result.

Lemma 8.1. *Let $(\mathbf{y}^d)_d$ be a sequence of admissible solutions of the semidefinite program (5.7). Then there exists $\hat{\mathbf{y}}\in\mathbb{R}^\infty$ and a subsequence (d_i) such that $\mathbf{y}^{d_i}\to\hat{\mathbf{y}}$ pointwise as $i\to\infty$, that is,*

$$\lim_{i\to\infty} y_\alpha^{d_i} = \hat{y}_\alpha, \qquad \forall \alpha\in\mathbb{N}^n. \tag{8.1}$$

The proof is omitted because it is exactly along the same lines as the proof of Theorem 3.4 as among the constraints of the feasible set, one has

$$y_0^d = 1, \quad M_d(\mathbf{y}^d)\succeq 0, \quad M_d(g_j,\mathbf{y}^d)\succeq 0,\ j=1,\ldots,m_1.$$

Proof of Theorem 5.1. Let $\mu^* \in P(\mathbf{K}_1), \nu^* \in P(\mathbf{K}_2)$ be optimal strategies of player 1 and player 2 respectively, and let $\mathbf{y}^* = (y^*_\alpha)$ be the sequence of moments of μ^* (well-defined because $\mathbf{K}_1$ is compact). Then

$$\begin{aligned}
J^* &= \max_{\nu\in P(\mathbf{K}_2)} \int_{\mathbf{K}_2} \left(\int_{\mathbf{K}_1} p(x,z) d\mu^*(x) \right) d\nu(z) \\
&= \max_{\nu\in P(\mathbf{K}_2)} \int_{\mathbf{K}_2} \sum_{\alpha\in\mathbb{N}^{n_2}} \left(\sum_{\beta\in\mathbb{N}^{n_1}} p_{\alpha\beta} \int_{\mathbf{K}_1} x^\beta \, d\mu^*(x) \right) z^\alpha d\nu(z) \\
&= \max_{\nu\in P(\mathbf{K}_2)} \int_{\mathbf{K}_2} \sum_{\alpha\in\mathbb{N}^{n_2}} \left(\sum_{\beta\in\mathbb{N}^{n_1}} p_{\alpha\beta} y^*_{\alpha\beta} \right) z^\alpha d\nu(z) \\
&= \max_{\nu\in P(\mathbf{K}_2)} \int_{\mathbf{K}_2} p_{\mathbf{y}^*}(z) \, d\nu(z) \\
&= \max_z \{ p_{\mathbf{y}^*}(z) \,:\, z \in \mathbf{K}_2 \} \\
&= \min_{\lambda,\sigma_k} \{ \lambda : \lambda - p_{\mathbf{y}^*}(\cdot) \,=\, \sigma_0 + \sum_{k=1}^{m_2} \sigma_k \, h_k; \quad (\sigma_j)_{j=0}^{m_2} \subset \Sigma[z] \}
\end{aligned}$$

with $z \mapsto p_{\mathbf{y}^*}(z)$ defined in (5.8). Therefore, with $\epsilon > 0$ fixed arbitrary,

$$J^* - p_{\mathbf{y}^*}(\cdot) + \epsilon \,=\, \sigma_0^\epsilon + \sum_{k=1}^{m_2} \sigma_k^\epsilon \, h_k, \tag{8.2}$$

for some polynomials $(\sigma_k^\epsilon) \subset \Sigma[z]$ of degree at most $2d^1_\epsilon$. So $(\mathbf{y}^*, J^* + \epsilon, \sigma_k^\epsilon)$ is an admissible solution for the semidefinite program (5.9) whenever $d \geq \max_j r_j$ and $d \geq d^1_\epsilon + \max_k l_k$, because

$$2d \geq \deg \sigma_0^\epsilon \,; \quad 2d \geq \deg \sigma_k^\epsilon + \deg h_k, \quad k = 1, \ldots, m_2. \tag{8.3}$$

Therefore,

$$\lambda^*_d \,\leq\, J^* + \epsilon, \qquad \forall d \,\geq\, \tilde{d}^1_\epsilon := \max \left[\max_j r_j, \, d^1_\epsilon + \max_k l_k \right]. \tag{8.4}$$

Now, let $(\mathbf{y}^d, \lambda_d)$ be an admissible solution of the semidefinite program (5.9) with value $\lambda_d \leq \lambda^*_d + 1/d$. By Lemma 8.1, there exists $\hat{\mathbf{y}} \in \mathbb{R}^\infty$ and a subsequence (d_i) such that $\mathbf{y}^{d_i} \to \hat{\mathbf{y}}$ pointwise, that is, (8.1) holds. But then, invoking (8.1) yields

$$M_d(\hat{\mathbf{y}}) \succeq 0 \quad \text{and} \quad M_d(g_j, \hat{\mathbf{y}}) \succeq 0, \quad \forall j = 1, \ldots, m_1; \quad d = 0, 1, \ldots$$

By Theorem 2.2, there exists $\hat{\mu} \in P(\mathbf{K}_1)$ such that

$$\hat{y}_\alpha \,=\, \int_{\mathbf{K}_1} x^\alpha \, d\hat{\mu}, \qquad \forall \alpha \in \mathbb{N}^n.$$

On the other hand,

$$\begin{aligned}
J^* &\leq \max_{\nu\in P(\mathbf{K}_2)} \int_{\mathbf{K}_2} \left(\int_{\mathbf{K}_1} p(x,z) d\hat{\mu}(x) \right) d\nu(z) \\
&= \max_z \{ p_{\hat{\mathbf{y}}}(z) \,:\, z \in \mathbf{K}_2 \} \\
&= \min \{ \lambda : \lambda - p_{\hat{\mathbf{y}}}(\cdot) \,=\, \sigma_0 + \sum_{k=1}^{m_2} \sigma_k \, h_k; \quad (\sigma_j)_{j=0}^{m_2} \subset \Sigma[z] \}
\end{aligned}$$

with

$$z \mapsto p_{\hat{\mathbf{y}}}(z) := \sum_{\alpha\in\mathbb{N}^n} \left(\sum_{\beta\in\mathbb{N}^n} p_{\alpha\beta}\, \hat{y}_\beta \right) z^\alpha.$$

Next, let $\rho := \max_{z\in\mathbf{K}_2} p_{\hat{\mathbf{y}}}(z)$ (hence $\rho \geq J^*$), and consider the polynomial

$$z \mapsto p_{\mathbf{y}^d}(z) := \sum_{\alpha\in\mathbb{N}^n} \left(\sum_{\beta\in\mathbb{N}^n} p_{\alpha\beta}\, y^d_\beta \right) z^\alpha.$$

It has same degree as $p_{\hat{\mathbf{y}}}$, and by (8.1), $\|p_{\hat{\mathbf{y}}}(\cdot) - p_{\mathbf{y}^{d_i}}(\cdot)\| \to 0$ as $i \to \infty$.

Hence, $\max_{z\in\mathbf{K}_2}\, p_{\mathbf{y}^{d_i}}(z) \to \rho$ as $i \to \infty$, and by construction of the semidefinite program (5.9), $\lambda^*_{d_i} \geq \max_{z\in\mathbf{K}_2}\, p_{\mathbf{y}^{d_i}}(z)$.

Therefore, $\lambda^*_{d_i} \geq \rho - \epsilon$ for all sufficiently large i (say $d_i \geq d^2_\epsilon$) and so, $\lambda^*_{d_i} \geq J^* - \epsilon$ for all $d_i \geq d^2_\epsilon$. This combined with $\lambda^*_{d_i} \leq J^* + \epsilon$ for all $d_i \geq \tilde{d}^1_\epsilon$, yields the desired result that $\lim_{i\to\infty} \lambda^*_{d_i} = J^*$ because $\epsilon > 0$ fixed was arbitrary;

Finally, as the converging subsequence (r_i) was arbitrary, we get that the entire sequence (λ^*_d) converges to J^*. The convergence $\gamma^*_d \to J^*$ is proved with similar arguments. □

8.2. **Proof of Theorem 5.2.** By the flat extension theorem of Curto and Fialkow (see e.g. [26]), $\mathbf{y}$ has a representing s_1-atomic probability measure μ^* supported on $\mathbf{K}_1$ and similarly, $\mathbf{y}'$ has a representing s_2-atomic probability measure ν^* supported on $\mathbf{K}_2$. But then from the proof of Theorem 5.1,

$$\begin{aligned}
J^* &\leq \max_{\psi\in P(\mathbf{K}_2)} \int_{\mathbf{K}_2} \left(\int_{\mathbf{K}_1} p(x,z) d\mu^*(x) \right) d\psi(z) \\
&= \max_{\psi\in P(\mathbf{K}_2)} \int_{\mathbf{K}_2} p_{\mathbf{y}}(z) d\psi(z) = \max_z \{\, p_{\mathbf{y}}(z) \,:\, z \in \mathbf{K}_2 \} \leq \lambda^*_d. \\
J^* &\geq \min_{\psi\in P(\mathbf{K}_1)} \int_{\mathbf{K}_1} \left(\int_{\mathbf{K}_2} p(x,z) d\nu^*(z) \right) d\psi(x) \\
&= \min_{\psi\in P(\mathbf{K}_1)} \int_{\mathbf{K}_1} \hat{p}_{\mathbf{y}'}(x) d\psi(x) = \min_x \{\, \hat{p}_{\mathbf{y}'}(x) \,:\, x \in \mathbf{K}_1 \} \geq \gamma^*_t,
\end{aligned}$$

and so as $\lambda^*_d = \gamma^*_t$ one has $J^* = \lambda^*_d = \gamma^*_t$. This in turn implies that μ^* (resp. ν^*) is an optimal strategy for player 1 (resp. player 2). □

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