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On alpha stable distribution of wind driven water surface wave slope

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Abstract

We propose a new formulation of the probability distribution function of wind driven water surface slope with an α-stable distribution probability. The mathematical formulation of the probability distribution function is given under an integral formulation. Application to represent the probability of time slope data from laboratory experiments is carried out with satisfactory results. We compare also the α-stable model of the water surface slopes with the Gram-Charlier development and the non-Gaussian model of Liu et al[15]. Discussions and conclusions are conducted on the basis of the data fit results and the model analysis comparison.

Keywords: *Wind driven water surface wave slopes. Probability density function. Non-Gaussian distribution. Alpha-stable distribution.*

During the last three decades, remote sensing of ocean surface has been extensively investigated to study physical processes on the sea surface and air sea interaction mechanism. One of the most important quantities for which depends the ocean remote sensing process is the surface wave slope probability distribution that is known to be non-Gausssian. In the past the general way to describe such probability distribution was to use an expansion technique of the Gaussian law called as Gram-Charlier series. In the present paper, we propose a new formulation of the surface wave slope using alpha stable probability distribution. Application to data from laboratory experiments and study comparison of the alpha stable slope model with existing models are carried out. Our results show the potentiality of the alpha stable laws to describe the non-Gaussian variability of water surface wave slope.

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I. INTRODUCTION:

A knowledge of the sea surface slope distribution and its related statistical quantities is essential to understanding the study of the sea surface from remote sensing process. The use of radar backscatter to characterize the sea surface has been widely investigated as well as from the experiments than from the theoretical approach. At nearly normal radar beam incidence, the importance of the slope distribution is clearly exhibited by the simplest model of cross section radar backscatter based on geometrical optics approach formulated by Valenzuela[27] as:

$$\sigma_0(\theta) = \pi sec^4\theta f(\zeta_x, \zeta_y)|R(0)|^2 \tag{1}$$

where θ is the radar incidence angle, $|R(0)|^2$ is the Fresnel reflection coefficient for normal incidence and $f(\zeta_x, \zeta_y)$ is the probability distribution function (PDF) of the sea surface slope at specular point (ζ_x, ζ_y). Determination of the surface slope distribution has been carried out in a large part with semi-empirical models derived from data observations. The commonly used approach for such purpose was the truncated Gram-Charlier series under the assumptions of the near-Gaussian case and the weakly nonlinear interaction proposed by Longuet-Higgins[16]. The Gram-Charlier series was attempted to include two additional factors: the peakedness and the skewness that are respectively due to second and third order wave-wave interactions. Despite the multiple adaptations made on the Gram-Charlier model, it is now well-known that this approach fails to work in the range of large slope or large surface elevation. In other word, this model is not suitable to represent the reality when large events have higher occurrence. To improve the results from Gram-Charlier series, several works were made by different authors. Among the existing models, those of Liu et al[15], differs from its formulation and its mathematical nature. Indeed, Liu model does not appeal to an expansion technique and is well suitable to represent extreme events. However, this model is originally a symmetrical law including the Gaussian probability and the Cauchy law as limit cases. Owing to the symmetry property, the Liu model is not able to reproduce the skewness effect. Liu extended empirically the model for this purpose.

In the present paper we propose to use the α-stable law to describe the PDF of surface slope. The stable law depends on four parameters: the index of stability, the skewness, the scale and the location parameters that allow one to represent a probability law of random variations including asymmetry, peakedness and peak shift. Also, the α-stable law includes

as limit cases the Gaussian probability and the Cauchy law. Generally speaking, stable law belongs to the family of heavy tails probability laws that is especially well adapted to take in account the large events of random variations. Such strong random variations can be found in fields or laboratory observations of the water surface roughness (Onorato et al[20]).

Evidences from in situ and laboratory experiments on non-Gaussian heavy tails distribution of wind driven water surface waves and their slope may be found in many articles on the literature. The first argument on the non-Gaussian behavior of wind driven water surface waves and their slopes concerns the existence of asymmetry property of wind wavefield that can be quantified by higher order spectral analysis (Leykin et al[13],Joelson et al[10]). Another hallmark of this non-Gaussian behavior is based on the power law observed in the spectrum of wind driven water surface waves. We cite here some works in this sense. Experiment observations was conducted by Huang[8] on time series of sea surface that suggest a random dynamic as a spectral power law associated to fractal description. Mellen and al[17] performed the study of random nonlinear dynamics of laboratory wind waves by the Lévy index and the co-dimension parameter that quantify the multifractal property of the water surface. As pointed out by Solow [24], power law behaviour can arise solely from random distributions having heavy upper tail. A fundamental consequence is that random rough surfaces with slowly decaying power spectral density can have infinite slope variance (Warnick et al[28]). Such case of slope behavior is clearly beyond the Gaussianity assumption and is only reached by the family of heavy tail probability density model to which belongs the α-stable law. Heavy tail distributions of wind driven water surface slope obtained from experiment observations can be found in different works (see for example Cox and Munk [5], Lui et al[15], Chapron et al[2]).

Also direct simulation methods such as a Monte-Carlo simulation based on nonlinear hydrodynamical model are able to reproduce pseudo-random samples that present similar aspects. Recently, Soriano et al[25] studied the effect of this non-Gaussian character on the radar Doppler spectrum of the random surface at grazing angle and concluded on the importance of the non-Gaussian character. Despite these results and the fact that stable laws are discovered since several decades, there are only few applications concerning the scattering on random rough surface with α-stable law. In particular, we cite here the work of Guérin[7], in which, he gave an extension of the expression (1) for the α-stable surface roughness.

The present study, in our opinion, would be the first attempt to apply the stable probability law on random water surface slope representation. The paper is organized as follow: in §2, we recall the main properties of the α-stable law. In §3, the application of stable law to fit instantaneous slope of wind driven water surface is presented. The §4 is devoted to the comparison of the stable PDF with the Gram-Charlier model and the Liu model. In §5, we present our conclusions on the results of the study.

II. STABLE PROBABILITY DISTRIBUTION

The stable laws were introduced by Paul Lévy [14] during his investigations of the behavior of the sums of independent identically distributed random variables. Then the theory on stable laws was developed by many authors and laid out in many books and articles. We recall here some basic results without proof. A stable distribution law is determined by four parameters: an index of stability α, a location parameter μ, a skewness parameter β and a scale parameter γ. The most common formulation of such law is given by the characteristic function which represents the inverse Fourier transform of the probability distribution function. A random variable X is said to have a stable distribution (noted as X $\sim S_\alpha(\beta, \gamma, \mu)$) if its characteristic function satisfies

$$\Phi_X(t) = \exp\left(i\mu t - \gamma^\alpha |t|^\alpha \left[1 - i\beta \ \mathrm{sign}(t) \ \mathrm{W}(\alpha, t)\right]\right) \tag{2}$$

where

$$\mathrm{W}(\alpha, t) = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi}\log\ |t| & \text{if } \alpha = 1 \end{cases}$$

and

$$\mathrm{sign}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ \text{-1} & \text{if } t < 0 \end{cases}$$

The stability index also called as the characteristic exponent or the tail index satisfies $0 < \alpha \leq 2$ and the skewness parameter $-1 \leq \beta \leq 1$. The location parameter μ is defined only when the stability index is $\alpha > 1$ ($\mu \in \mathbf{R}$). In this case, the first absolute moment is given by $< |X| >= \frac{2\gamma}{\pi}\Gamma(1 - 1/\alpha)$ where $\Gamma(.)$ is the gamma function. If $\beta = 0$, then the distribution is symmetric around μ. The scale parameter also called as dispersion parameter has a real positive value $\gamma > 0$ that determines the width of the distribution law. The location

parameter μ describes the shift of the peak. The tail index α determines the rate at which the tails of the distribution taper off. When $\alpha < 2$, the variance of the random variable X is infinite and the tails exhibit a power-law behavior. More precisely, using the central limit theorem, it can be shown that:

$$\begin{cases} \lim_{t\to\infty} t^{\alpha}\Pr(X > \quad t) = C_{\alpha}(1+\beta)\gamma^{\alpha} \\ \lim_{t\to\infty} t^{\alpha}\Pr(X < -t) = C_{\alpha}(1-\beta)\gamma^{\alpha} \end{cases} \tag{3}$$

where $C_{\alpha} = \frac{1}{\pi}\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})$. From (3), it yields that in general, the pth order moment of a stable random variable is finite if and if only $p < \alpha$.

Probability density function

Excepted from three special cases, the stable distribution do not have closed form expression of its PDF. This lack of closed form formulas for the PDF has a negative consequence on the popularity of the stable distribution. By applying Fourier inversion to (2), an integral form of the PDF is obtained as:

$$f_{\alpha,\beta,\gamma,\mu}(x) = \begin{cases} \frac{1}{\pi} \quad \int_0^{\infty} \exp(-\gamma^{\alpha}z^{\alpha}) \cos\left[(\mathrm{x}-\mu)z + \beta z^{\alpha} \tan(\frac{\alpha\pi}{2})\right] dz \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha \neq 1 \\ \frac{1}{\pi} \quad \int_0^{\infty} \exp(-\gamma z) \cos\left[(\mathrm{x}-\mu)z + \beta z\frac{\pi}{2} \log(|z|)\right] dz \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha = 1 \end{cases} \tag{4}$$

Expression (4) can be analytically solved only for the following exceptions cases:

-for $\alpha = 2$ corresponding to the Gaussian distribution,

$$f_2(x) = \frac{1}{\sqrt{2\pi}\gamma}\exp\left\{-\frac{(x-\mu)^2}{2\gamma^2}\right\} \tag{5}$$

in this case, the skewness parameter is irrelevant. The variable X $\sim S_2(0,\gamma,\mu)$ reduces to the Gaussian random variable X $\sim \mathcal{N}(\mu, \sqrt{2}\gamma)$

-for $\alpha = 1$ and $\beta = 0$ which leads to the Cauchy distribution

$$f_1(x) = \frac{1}{\pi}\frac{\gamma}{\gamma^2 + (x-\mu)^2} \tag{6}$$

The Cauchy random variable writes as X $\sim S_1(0,\gamma,\mu)$

-for $\alpha = 0.5$ and $\beta = 1$ giving the Lévy distribution

$$f_{1/2}(x) = \sqrt{\frac{\gamma}{2\pi}} \quad \frac{\exp\left\{-\frac{\gamma}{2x}\right\}}{x^{3/2}} \tag{7}$$

For the general cases, numerical approximations of (4) was widely used by several authors to represent stable PDF. Different methods exist for this purpose with various level of accuracy. In this paper, we adopt the direct integration method developed by Nolan[19] and Weron[30] using an other formulation of the expression (4) called as the Zolotarev's formulas [31]. The main interest of the Zolotarev's formulas is that instead of expression (4), they do not include infinite integral and then are well adapted to numerical computations. In which follow, we give the Zolotarev's formulas of the PDF of a stable random variable $\mathrm{X} \sim S_\alpha(\beta, \gamma, \mu)$. First, we consider the case of a random variable $\mathrm{X}_0 \sim S_\alpha(\beta, 1, 0)$. By setting $\zeta = -\beta \tan\frac{\pi\alpha}{2}$, the PDF of X_0 can be expressed as:

if $\alpha \neq 1$

$$f_{\alpha,\beta,1,0}(x) = \begin{cases} \dfrac{\alpha(x-\zeta)^{\frac{1}{\alpha-1}}}{\pi|\alpha-1|} \displaystyle\int_{-\zeta}^{\frac{\pi}{2}} \mathrm{V}(\theta,\alpha,\beta) \exp\left\{-(x-\zeta)^{\frac{\alpha}{\alpha-1}} \mathrm{V}(\theta,\alpha,\beta)\right\} d\theta, & \text{if } x > \zeta \\ \dfrac{\Gamma(1+\frac{1}{\alpha}) \cos(\xi)}{\pi}, & \text{if } x = \zeta \\ f_{\alpha,-\beta,1,0}(-x), & \text{if } x < \zeta \end{cases} \tag{8}$$

if $\alpha = 1$

$$f_{1,\beta,1,0}(x) = \begin{cases} \frac{1}{2|\beta|} \exp(-\frac{\pi x}{2\beta}) \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{V}(\theta,1,\beta) \exp(-\exp(-\frac{\pi x}{2\beta}) \mathrm{V}(\theta,1,\beta)) d\theta, & \text{if } \beta \neq 0 \\ \dfrac{1}{\pi(1+x^2)}, & \text{if } \beta = 0 \end{cases}$$

where $\xi = \frac{1}{\alpha} \arctan(-\zeta)$ and

$$\mathrm{V}(\theta,\alpha,\beta) = \begin{cases} (\cos\alpha\xi)^{\frac{1}{\alpha-1}} \left(\dfrac{\cos\theta}{\sin\alpha(\xi+\theta)}\right)^{\frac{\alpha}{\alpha-1}} \dfrac{\cos(\alpha\xi+(\alpha-1)\theta)}{\cos\theta}, & \alpha \neq 1 \\ \dfrac{2}{\pi}\left(\dfrac{\frac{\pi}{2}+\beta\theta}{\cos\theta}\right) \exp\left\{\dfrac{1}{\beta}(\dfrac{\pi}{2}+\beta\theta)\tan\theta\right\}, & \alpha = 1 \end{cases}$$

From the probability distribution given by (8), one can easily build the PDF of the random variable X with the help of the stability property, $\mathrm{X} = \gamma\mathrm{X}_0 + \mu \sim S_\alpha(\beta, \gamma, \mu)$.

Parameters estimation

The first step on modelling data with a stable law requires the estimation of the four parameters that determine the distribution properties. The estimation process suffers from the

lack of known closed form of the PDF. As a consequence, standard methods based on maximum likelihood principle would not be efficient with regards to the stable PDF dependance. However, there are other numerical ways that do not depend directly on the PDF knowledge to evaluate stable parameters like the sample quantile methods or also the sample characteristic function methods. A comparative analysis and discussions about these methods may be found in Borak and al[1]. Among these existing methods of estimation, we will use the sample characteristic function developed by Koutrovelis[12] and performed by Kogon and Williams[11]. The method is based on regression type on the log-characteristic function. Indeed, the logarithm of the real part and the imaginary part of the log-characteristic function are linear and then, give rise to a regression model with the data.

$$\begin{aligned} \log\,(-\mathrm{Re}[\Psi(t)]) &= \alpha\,\log|t| + \alpha\,\log\gamma \\ \mathrm{Im}[\Psi(t)] &= -\mu t - \beta\,\mathrm{sign}(t)\,\mathrm{W}(\alpha, t) \end{aligned} \tag{9}$$

where Ψ is the log-characteristic function of X and W(.,.) is defined as in expression (2). A more complete description of the method and an extended study may be found in [11].

In the following section, the stable PDF model will be applied on slope data from laboratory experiments on wind driven surface water waves.

III. MEASUREMENTS AND DATA ANALYSIS

Experiments

The data of interest issue from experiments made in the wind wave facility of the IRPHE-IOA Laboratory. Basically, the facility consists of a water tank of 40 m length, 3 m width and 1 m depth. The facility can be described as a combination of a wind tunnel with a wind-wave tank. A schematic representation of the air-sea interaction simulation facility is given in Fig.1. The facility is equipped on a submerged wavemaker that allows to produce mechanical waves from capillary ripples to breaking waves. However in the present work, we focus only on wind wave fields. By means of an axial fan, air flow with velocities ranging from 0 m/s to about 15 m/s generates wind wave fields. Different types of surface waves can be produced by the action of the air flow. It is to be stressed that the facility includes particular devices intended to control various parasitical effects which could coexist with the physical process under consideration. In particular, a permeable wave absorber is disposed

at the upwind end of the water tank to prevent the wave reflection. Within the range of wave frequency of interest in the experiments on surface waves, the reflection coefficient is estimated to be insignificant. As far as wind generated waves are concerned, a specific design has been adopted for a smooth joining of the air flow and the water surface. The design prevents air flow-separation at the facility entrance test section. This insures a natural development of the turbulent boundary layer over the water surface. In addition, later quays disposed in the water tank favor the three-dimensional evolution of the wind waves. Studies with more details about there specific devices are given in a number of publications (see e.g. Coantic et al [3]). We concentrate here on the time evolution of the water surface deflection level at a given position. Measurements of water surface height η(t) were performed by means of two capacitance wave gauges separated by a 5cm along wind distance. These probes using a capacitance wire gauge of 0.3 mm outer diameter were set up on a carriage moving along the wave tank. Experiments were carried out under various conditions, whose main parameters are the wind velocity and the distance between the gauge and the facility entrance section (hereafter denoted fetch). Such distance determines the length action of the air flow over the water surface and is well known to be of crucial importance on the rate growth of surface waves and on its nonlinearity. The time derivative dη/dt of the water surface heights were obtained by use of analog derivators.

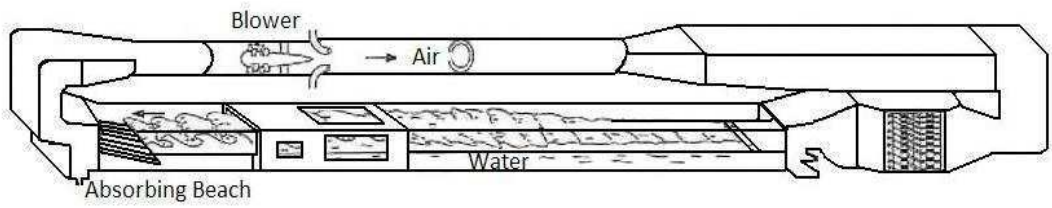

FIG. 1: A schematic representation of the large air-sea interactions facility.

The corresponding processed series are made of n = 36000 samples gathered at a continuous rate of 200 Hz. The water surface wavefields of interest here are known to have temporal as well as spatial dynamics in which dispersion, nonlinearity, dissipation (by the kinematic viscosity or by breaking) and wind action are involved. The results presented herein are limited to some aspects of the temporal evolution at the given space locations.

Case	Fetch (m)	U $_{10}$ (m/s)	α	β	γ	μ
I	2	6	1.7219	-0.0816	0.5434	0.0056
II-1	6	6	1.8688	-1.0000	0.6384	-0.0869
II-2	13	3	1.9822	-1.0000	0.6972	-0.0117
III	26	13	1.9178	-0.9488	0.6481	-0.0051

TABLE I: Estimation of stable parameters $(\alpha, \beta, \gamma, \mu)$ from the data series.

Thus, instantaneous slope s(t) on the upwind direction of the water surface waves could be derived as s(t)=-1/c dη/dt where c corresponds to the phase velocity of the dominant wave frequency. This slope measuring method was tested by comparison with direct measurement performed by a laser slope gauge (see Jhane and Riemer [9]). A good agreement was found particularly for capillary and gravity waves at moderate wind speed.

Data analysis

In this section, we estimate the PDF of time water surface slopes at different wind speed for fetch varying from 2m to 26m. Note that wind speed is estimated at 10m height above the surface(U_{10}). Among the large possibilities of combinations from wind speed and fetch values, we present here some cases that are indicative of the typical behavior of water surface slope. In which follows, we focus on slope behavior of three cases: capillary waves, capillary gravity waves at moderate fetch and low wind speed and gravity waves at large fetch and high wind speed that are typical of fully developed sea surface. In order to compare the different runs of study, the data series are normalized with its rms and mean values. The results of the fit are summarized in Tab.I.

The first line of Tab.I corresponds to the case I of capillary wind wave slopes at moderate wind speed and at a very short fetch. In this case, we find a stability index of 1.72 that is indicative of a strong non-Gaussian character. A very weak value of the skewness parameter is also found traducing a lack of asymmetry on the waveform. The scale parameter is at a relative low value with regard to the normalized Gaussian standard deviation($\sim S_2(0, 1/\sqrt{2}, 0)$). This would be related to a nonlinear effect involved on the generation of capillary waves by wind. The characteristics of the capillary wave slopes are shown in Fig.2. The Fig.2.a depicts a sample of the capillary wave profile and its slope while the corresponding probability

distribution is shown in the part (b) of the figure with a normalized Gaussian law as a reference. At larger fetch value corresponding to case II, the stability index grows. The scale

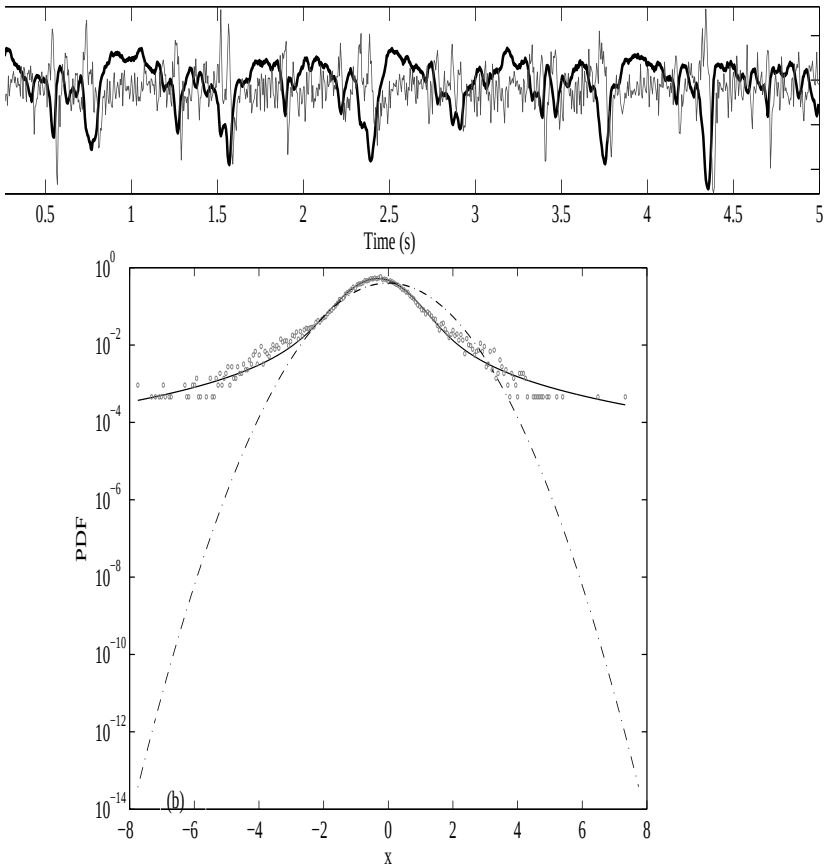

FIG. 2: Fetch=2m. U_{10} = 6m/s. Part(a): Thick line: Normalized sample of surface wave. Thin line: Normalized instantaneous slope sample. Part(b): 'o': PDF of slope experiment data. Solid line: α stabe PDF. Dashed-dotted line: Normalized Gaussian PDF

parameter also increases. A striking feature is that the skewness parameter reaches to its negative limit value. This is in good agreement with time evolution of the slope presented in Fig3.a. Observing also that the corresponding surface waves are dominated by a Stokes component, we suggest the possible rôle of the Stokes effect in the asymmetric behavior of the slope and in the limit value obtained for the skewness parameter in the present case. However, in the general case, relation between asymmetry in the surface and in the slope variation is not easy to establish especially for a random wavefield. The Stokes effect yields to a waveform of more steeper crests increasing locally slopes and more flatter troughs. Results of run at fetch of 6m and wind speed of 6m/s (case II-1) are shown in Fig.3. The

asymmetry property is clearly visible on the slope samples (Fig.3.a) and is well represented by the PDF model (Fig.3.b). Modulational instabilities begin to occur in the wavefield owing to the nonlinear wave-wave interaction process called as the Benjamin-Feir instabilities. The sample of the wave signal presents wave trains formation that are reminiscent of these instabilities effects (see Fig.3.a). Such nonlinear effects are known in this case to dominate the effects of randomness. This would explain the reduction of slope magnitude and also the increase of the stability index to the near-Gaussian situation. By increasing the fetch value but taking a low wind speed (case II-2), the rate of input wind energy will be weaker than the rate of the self-nonlinearity of the wavefield. Thus, one can more emphasize the wave modulation effect on the slope distribution and on its PDF. The corresponding PDF is shown in Fig.4.b with a very similar form to the normalized Gaussian except for the negative tail that is indicative of the presence of Stokes effect. Finally, at high wind speed and large fetch corresponding to the case III, one can find surprisingly that values of the index stability but also the skewness and the scale parameter reduce reaching again to a clearly non-Gaussian situation. In agreement with these values, from the Fig.5.a, it may be found that the magnitude of the wave slopes grows locally at the crests and the wave signal behaves strongly asymmetrical. From the physical background, it was shown by Trulsen and Dysthe[26] that such situation may be explained from theoretical model based on a modified nonlinear Schrödinger equation that allows one to predict the stabilization of the Stokes effects by the strong wind such that modulational instabilities are suppressed. The PDF of the case III are given in Fig.5.b in which it may be found that the stable PDF model reproduce well the distribution of the negative slope values and also the skewness of the slopes. However, at large positive values, the model fails to represent the data distribution. No clear interpretation was found for this difference but we believe that additional physical process such as breaking wave would be involved.

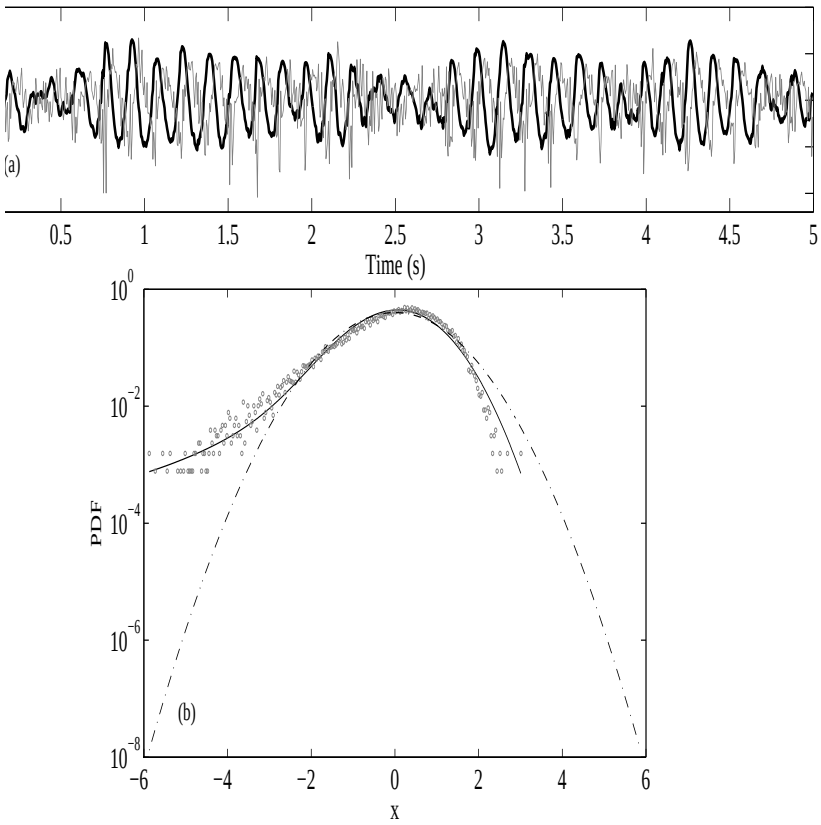

FIG. 3: Fetch=6m. U_{10} = 6m/s. Part(a): Thick line: Normalized sample of surface wave. Thin line: Normalized instantaneous slope sample. Part(b): 'o': PDF of slope experiment data. Solid line: α stabe PDF. Dashed-dotted line: Normalized Gaussian PDF

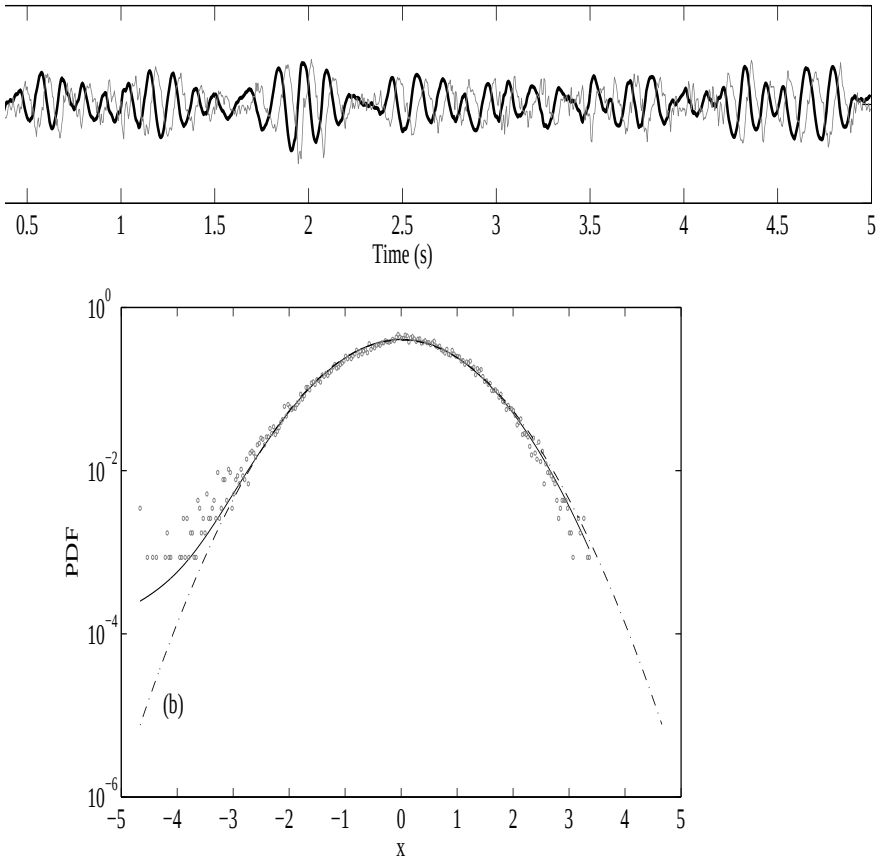

FIG. 4: Fetch=13m. U_{10} = 3m/s. Part(a): Thick line: Normalized sample of surface wave. Thin line: Normalized instantaneous slope sample. Part(b): 'o': PDF of slope experiment data. Solid line: α stabe PDF. Dashed-dotted line: Normalized Gaussian PDF

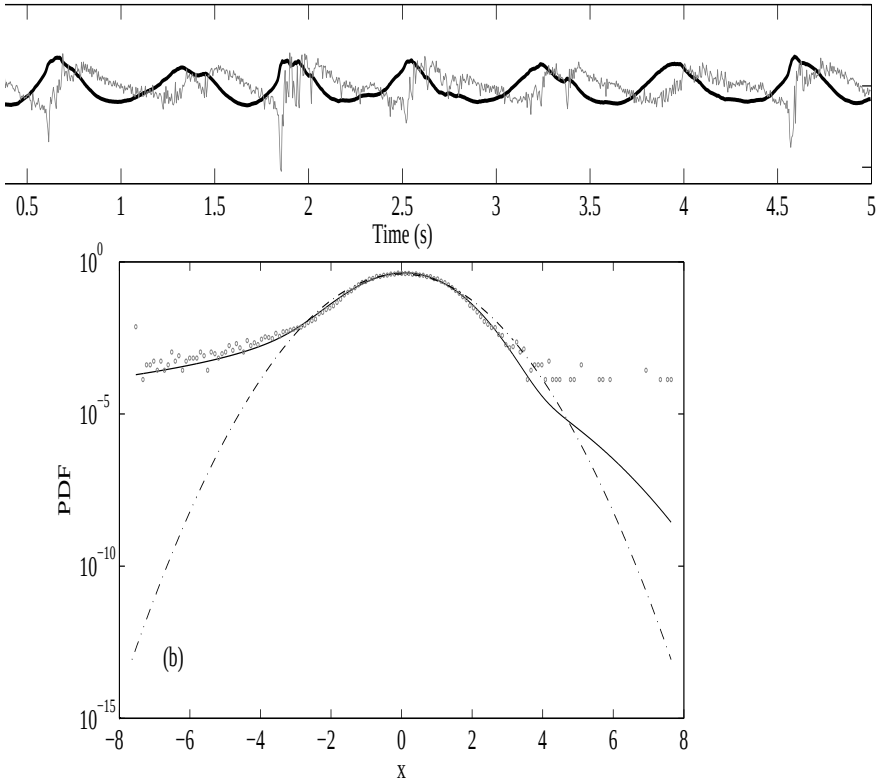

FIG. 5: Fetch=26m. U_{10} = 13m/s. Part(a): Thick line: Normalized sample of surface wave. Thin line: Normalized instantaneous slope sample. Part(b): 'o': PDF of slope experiment data. Solid line: α stabe PDF. Dashed-dotted line: Normalized Gaussian PDF

IV. COMPARISON WITH OTHER MODELS OF SLOPE DISTRIBUTION

In this section, we compare the properties of the stable law with the Gram Charlier and the Liu models. The Gram-Charlier model based on an expansion technique called as Edgworth series was widely used in the past and permitted to develop a large part of the understanding of the non-Gaussian behavior of the sea surface. However, the limit of this model was earlier found by several authors. Many efforts were conducted to develop alternative ways from the statistical expansion technique. The Liu model was built in this context to improve the statistical representation of the sea surface slopes.

The Gram-Charlier series

The traditional method of interpreting the wind wave slopes has been to fit them to a Gram-Charlier series, which can be derived by assuming that surface waves are weakly nonlinear (Longuet-Higgins[16]). Basically, in this procedure the PDF of the wave slopes s(t) writes as:

$$\phi_{\mathrm{GC}}(s) = \mathrm{G}(s)\left[1 + \frac{\mathrm{C}_3}{6}\,\mathrm{H}_3(s) + \frac{\mathrm{C}_4}{24}\,\mathrm{H}_4(s)\right] \tag{10}$$

where

$$\mathrm{G}(s) = \frac{1}{\sigma\sqrt{2\pi}}\,\exp\left[-\left(\frac{s- < s >}{\sqrt{2}\sigma}\right)^2\right]$$

and σ is the rms of the slope, when H_n is the nth Hermite polynomial. The coefficients C_n can be calculated from the statistical moments of ϕ_{GC}(s) and G(s) as:

$$C_n = \frac{1}{\sigma^n}\int s^n\left[\phi_{\mathrm{GC}}(s) - \mathrm{G}(s)\right]ds$$

Clearly, the coefficients C_3 and C_4 in expression (10) are intended to include the skewness and the kurtosis that are the consequences of nonlinearities in the underlying physical processes. Of interest is that the PDF model becomes non-Gaussian. Note also that expression (10) is not always guaranteed to be positive, and is therefore not a valid probability distribution. The Gram-Charlier series diverges in many cases of interest. It converges only if ϕ_{GC}(s) falls off faster than $\exp(-s^2/4)$ at infinity (Cramér[6]). For water waves applications, the skewness and kurtosis coefficients (C_3 and C_4) are found to be related to physical parameters in particular to the air flow speed value at 10m altitude (noted as the U_{10}). Here we adopt the values of these coefficients used by Cox and Munk[4] and Plant[21] from a range of wind speed varying from 0.72 to 10.2 m/s (Note that [4] uses the wind speed at height of 12.5m

instead of the traditional U_{10}). We fit the Gram-Charlier model with stable law using the Nelder-Mead[18] simplex method. The results of the fit are given in Tab.II. The method is based on the minimization of the euclidian norm of the error $\epsilon = || f_{\alpha,\beta,\gamma,\mu} - \phi_{GC}||$ for 1000 points varying from -5 to +5. From each case, values of the best stable parameters and the error ϵ are shown.

$U_{12.5}$ (m/s)	σ	C_3^{CM}	C_4^{CM}	α	β	γ	μ	ϵ
0.72	0.0005	0.101	0.127	1.951	-0.330	0.695	-0.05	$1e^{-1}$
3.93	0.0098	0.003	0.129	1.957	0.022	0.695	0	$1.9e^{-2}$
4.92	0.0174	-0.080	-0.019	1.980	0.900	0.708	0.040	$8.6e^{-2}$
6.30	0.0170	-0.143	0.101	1.974	0.880	0.697	0.070	$1.5e^{-1}$
8.00	0.0191	-0.156	0.173	1.930	0.260	0.690	0.07	$1.5e^{-1}$
10.2	0.0357	-0.283	0.128	1.910	0.400	0.690	0.13	$2.7e^{-1}$

TABLE II: Estimation of stable parameters $(\alpha, \beta, \gamma, \mu)$ from the Gram-Charlier series used by Cox and Munk (1954) model

As a general remark for all cases, the values of the stability index α are closed to 2 that traduces the near-Gaussian assumption adopted in the Gram-Charlier model. The error ϵ increases with the wind speed values. Rather than a simple difference from the calculation, these error values reveal a more profound departure on the nature of both models. More precisely, at low wind speed, when the occurrence of large values of slope would be rare, a better agreement is found in particular around the peak of the PDF. At high wind speed, the results of the Gram-Charlier series differ clearly from those of the stable law as well as at the tails of the probability than at the middle part of the probability law. The Gram-Charlier model underestimates the extreme events giving a less heavy tails on the probability law (see Fig.6). At the middle part of probability law, it overestimates the effect of the skewness. Note that no clear tendency is found between the stable parameter β and the skewness C_3^{CM}. Finally, the location parameter μ is globally around zero and the scale parameter γ is found at a quasi constant value around $\frac{1}{\sqrt{2}}$ that are directly related to the normalized Gaussian law of the function G(s) included in the expression (10).

The Liu model of slope PDF

A more advanced model is developed by Liu et al[15] to fit the joint probability of the

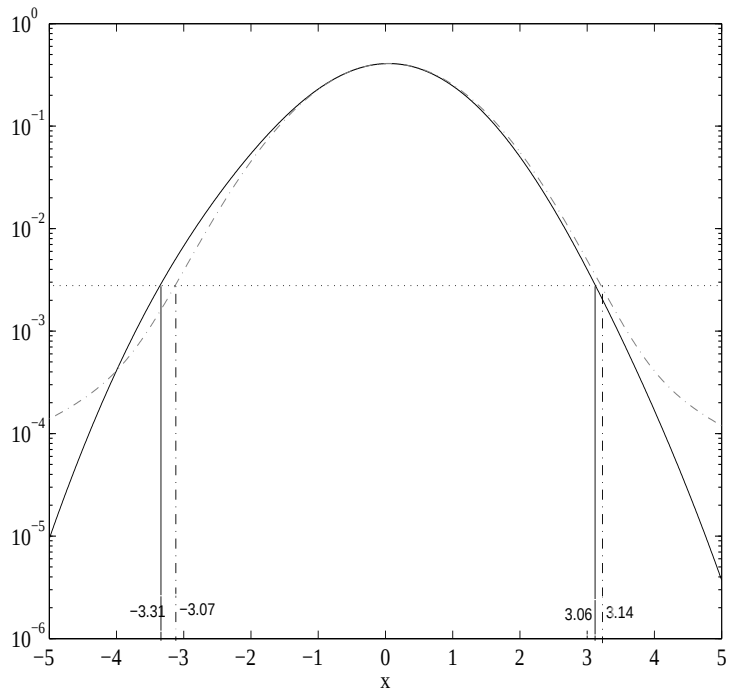

FIG. 6: Results of the fit for $U_{12.5}$ =8m/s. The solid line is the Gram-Charlier PDF. Dashed line represents the stable law of the fit. A fixed value of the probability is drawn with dotted line that allows one to visualize the asymmetry of both laws.

surface wave slopes. Liu model is also based on the weakly nonlinear assumption of the surface [16]. The basic form of Liu slope PDF writes as:

$$f_L^n(x,y) = \frac{n}{2\pi(n-1)}\left[1 + \frac{x^2}{(n-1)} + \frac{y^2}{(n-1)}\right]^{-(n+2)/2} \tag{11}$$

where x,y are normalized slope components and n an integer parameter describing the peakedness of the surface slopes. Note that (11) refers to the case of isotropic sea surface and is not always suitable for anisotropic real surface. An empirical formulation inspired by [4][16] is used by [15] to handle the skewness effect. According to [15], the expression (11) is an improvement of the Gram-Charlier distribution in particular for the range of large slopes. For convenience, here we consider the expression for y=0, i.e. we focus only on upwind direction. As a first remark, the positivity of the expression (11) is obviously guaranteed however it may be found that the expression of $f_L^n(x,0)$ does not represent a valid probability density function for any value of n. Indeed, the integral of this expression is not

equal to unity. We correct this problem by adopting a normalized form of the Liu PDF as:

$$\phi_L^m(x) = \frac{1}{\sqrt{\pi}} \left[1 + \frac{x^2}{(m-1)}\right]^{(-\frac{m}{2}+1)} \frac{\Gamma(\frac{m}{2}-1)\sqrt{\frac{1}{m-1}}}{\Gamma(\frac{m}{2}-\frac{3}{2})} \tag{12}$$

The expression (12) is defined for $m \geq 4$. For m=4, one can easily find that

$$\phi_L^4(x) = \frac{1}{\pi}\left(\frac{\sqrt{3}}{3+x^2}\right) \tag{13}$$

Expression (13) belongs to the Cauchy stable law with X $\sim S_1(0, \sqrt{3}, 0)$. On the other hand, taking the logarithm of (12), the limit for large value of m writes as:

$$\log\, \phi_L^m(x) \simeq_{m\to\infty} -\frac{m-2}{2} \log\left(1 + \frac{x^2}{m-1}\right) \tag{14}$$

which reaches by simple series expansion to the normal distribution of X $\sim \mathcal{N}(0,1)$ or also the stable law with X $\sim S_2(0, \frac{1}{\sqrt{2}}, 0)$. The limit cases of the Liu formulation belong to the symmetrical stable distribution.

For intermediate values, the Liu PDF differs slightly from the stable distribution. In order to illustrate this relation, we compute a numerical adjustment of the stable parameters α and γ to the expression (12) for m=4.5. The adjustment is based on the minimization of the euclidian norm $||\, \mathrm{f}_{\alpha,0,\gamma,0}(x) - \phi_L^m(x)|| \leq \epsilon$ for 3000 points from x=-15 to 15. The results are shown in Fig.7. It may be found that the major difference between the two laws are clearly localized at the medium range values of the probability structure.

By varying the peakedness parameter m, this difference stays quite systematically. Analytically, one can also decompose the modified Liu model to study its property. As an example, taking m=6, the Liu PDF writes as:

$$\phi_L^6(x) = \frac{1}{\pi}\frac{\sqrt{5}}{5+x^2} + \frac{1}{\pi}\frac{\sqrt{5}(5-x^2)}{(5+x^2)^2} \tag{15}$$

where the first term is a Cauchy law $S_1(0, \sqrt{5}, 0)$. As shown from the Fig.8, the second term possess a significant positive values at the middle and the medium parts of the probability law. The magnitude of these positive values are responsible of the height of Liu PDF and the departure from the stable law at the medium range values. The negative values of the second term of (15) explain the similarity of the asymptotic behavior of both laws observed in Fig.7. For higher values of m, the same expansion holds with a Cauchy law $S_1(0, \sqrt{m-1}, 0)$ as a first term but the second term becomes a sum of rational functions whose integral vanishes to ensure the normalization of the PDF.

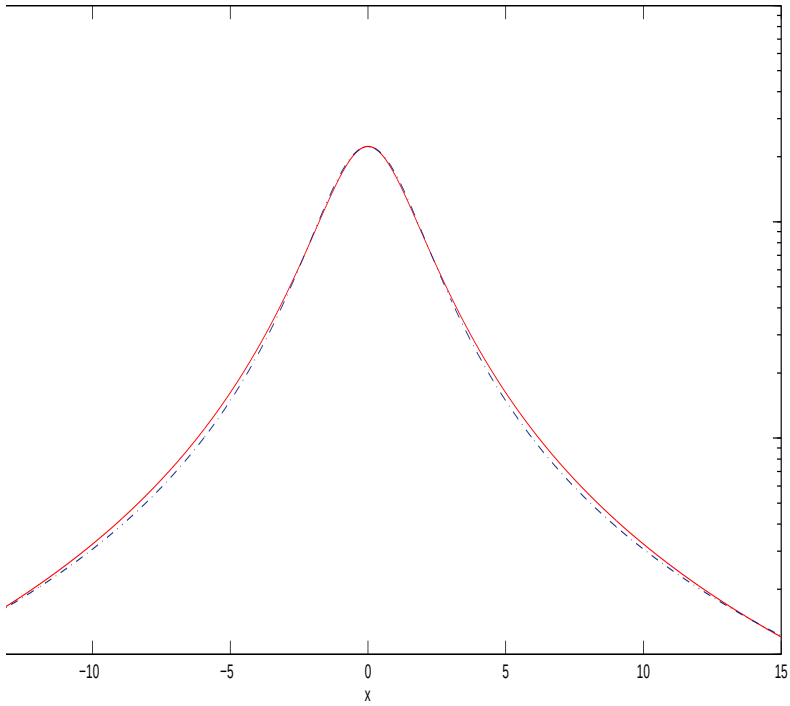

FIG. 7: Semi-log representation of Liu PDF for m=4.5 and the stable law fit with $\alpha = 1.221$ and $\gamma = 1.3384$. The error $\epsilon = 0.066277$. Solid line: Liu PDF model. Dashed line: Stable law.

According to the probability theory, the modified Liu PDF belongs to the family of the Generalized Student or T-distribution whose the expression writes as:

$$\phi_S^m(x) = \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{m}{2}) \quad \Gamma(\frac{1}{2})} \left(\frac{a}{(x^2+a^2)^{m+\frac{1}{2}}} \right) \tag{16}$$

where m is now called as freedom degrees and a is a scale parameter. The characteristic function of the T-distribution is given by:

$$\psi_S^m(t) = \quad 2\frac{|\mathrm{at}|^{\frac{m}{2}} \quad K_{\frac{m}{2}}(|\mathrm{at}|)}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} \tag{17}$$

where K_i is the modified Bessel function. From (2) and (17), one can evaluate the difference between the stable and the T-distribution laws. As said in previous section, the stable laws decay as $|x|^{-\alpha-1}$ with $\alpha < 2$. Here, it may be shown that the T-distribution laws have

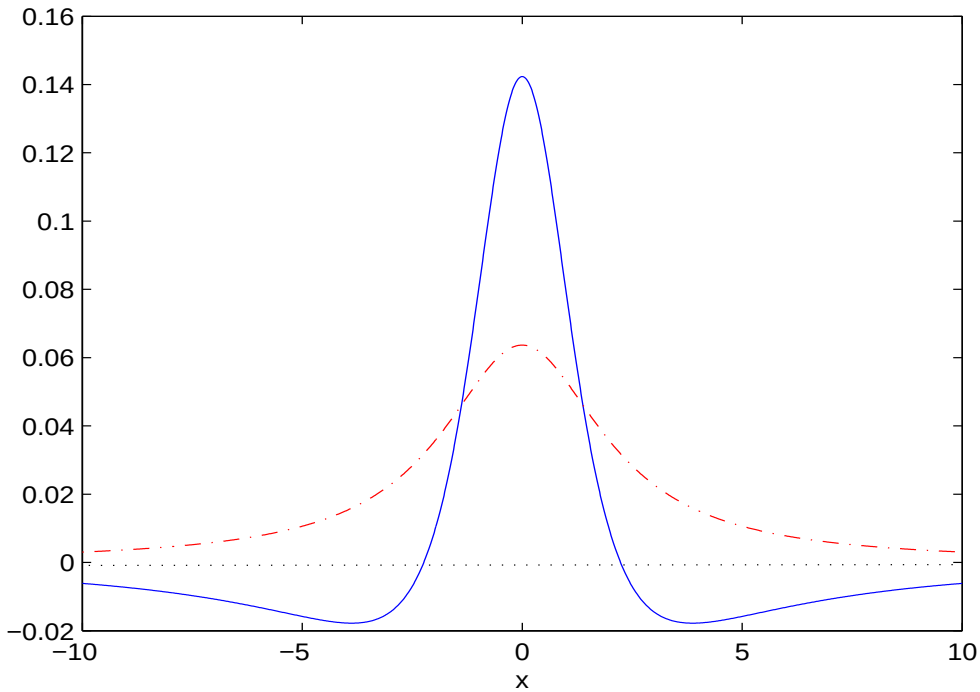

FIG. 8: The corresponding figure of expression (15). The Cauchy law (first term) is on dashed line while solid line represents the second term with negative part. The zero value is drawn with dotted line

the same asymptotic behavior as $|x|^{-m-1}$ for all m>0. Note that as a consequence, the T-distribution laws are more accurate than the stable laws to approximate the near-Gaussian situation. Another significant difference concerns the variance of the laws: for all m>2, the T-distribution laws have finite variance given by $\sigma^2 = \frac{a^2}{m-2}$ while the stable laws do not possess finite variance except only for $\alpha = 2$. From the practical viewpoint, this lack of variance may be of crucial importance when one have to compute statistical values as the mean square slope that is necessary to evaluate the radar backscatter cross section.

Finally, from the expression (16), it appears that the T-distribution do not include a skewness property while the stable laws are naturally able to express the skewness effect. Recalling the importance of the skewness effect on the dynamic of the surface waves, this point may have a significant consequence on the accuracy of the slope distribution model.

V. CONCLUSION

The main purpose of this study was to highlight the possible use of α-stable laws for water surface slopes representation. As said in the introduction of this paper, the fundamental argument on the use of α-stable laws stems first from experimental observations of the main property of the wind driven water surface waves: a possible fractal representation, related to a spectral power law, nonlinear characters of the wind wavefield resulting in an asymmetry and a modulational instability of the surface, and a randomness of the slope, such as these abrupt changes shown here in section 3.

Considering these strongly disordered behaviors that may be found on slope variations, α-stable laws seem à priori to be a potential candidate for this objective. Starting from this idea, the study propose to apply the stable laws to represent the probability distribution of the slopes of wind driven water surface from laboratory experiments. It is found that the stable laws are well adapted to model the random characteristic of the slopes data in different cases of experiment configurations. From the proposed cases, the four stable parameters behave in good agreement with the effects of different physical processes observed and expected in the experiments. In particular, the stability index traduces fairly the non-Gaussianity of the slopes in the same sense as the Stokes and/or the modulational instability effects. Also the skewness parameter reproduce well the asymmetry tendency of the slopes under various situations. However, at extreme experiment condition (large fetch and high wind speed), the stable laws fail to reproduce the behavior of the upper part of the slopes.

From more theoretical consideration, comparisons with the Gram-Charlier and the Liu et al[15] models are carried out. It appears that the stable law differs completely from the Gram-Charlier model as well as in its mathematical structure than from the characteristics of its results. Indeed, one can recall that the Gram-Charlier model at its usual truncation order (third or fourth order) is part of the near-Gaussian approximation models, while the stable law is totally non-Gaussian except only when the stability index reaches to the maximum value.

The comparison with the Liu model shows more similarities. Recalling that Liu model belongs to the T-distribution family, it is well established that both models are part of the so-called "heavy tail probability laws". They are both infinitely divisible probability laws but the T-distribution is not stable. This means that sum of independent identically T-

distributed variables does not follow the same law unlike the stable law that it is always the case.

From mathematical viewpoint such stability property lies to a fundamental characteristic based on the generalization of the central limit theorem (see Samorodnitsky and Taqqu[23]) and traduces the fact that the α-stable laws constitute limit laws (or attractor laws) for all additive random processes. As a special case, we recall that for $\alpha = 2$, from the central limit theorem, the Gaussian law is a limit law for all additive finite variance random processes. This is the profound justification of the popularity of the Gaussian model in many physical domains. In the realm of water surface waves, the random wavefield is usually considered as the sum of infinite number of random components. Owing to the weakness of the interactions between components, the corresponding motion of each component may be regarded as independent (see for example Phillips [22]). Consequently, under the generalisation of the central limit theorem, the choice of stable law is then theoretically justified with a clear mathematical basis.

With regard to the practical viewpoint, the differences between the two laws are more important and have to be stressed here with respect to the study objective. These differences stand on four points concerning the closed form of the PDF, the finite variance value, the skewness representation and the interpretation of the physical processes through the PDF parameters.

As said in previous section, the T-distribution has a closed form of the PDF for all values of its parameters . The stable distribution does not possess a closed form of its PDF except for three special cases. This lack of closed form is apparently an inconvenient property of the stable law. However, as shown in this study, there exists robust techniques to implement the integral form of the stable PDF (see Zolotarev [31], Weron[30], Nolan[19]). Other expansion techniques based on the Bergström series are also available to evaluate the stable PDF.

For physical applications, an appeal to statistical quantities associated with the PDF model would be useful. For example, for the sea surface remote sensing problem, it is known that the mean and the variance of the slopes are of importance. This is historically related to the assumption of near-Gaussian state in the development. Recalling that T-distribution has finite variance for all freedom degree greater than 2. This later appears as a well adapted choice for this purpose. However, the T-distribution and the stable distribution constitute fully non-Gaussian alternative and then fails to belong in the category of quasi-

Gaussian solutions. In our opinion, an extension of the remote sensing or the hydrodynamical relationships in the sense as developed by Guérin[7] is required for a fully non-Gaussian solution.

As far as non-Gaussian random water surface is concerned, the skewness is undoubtedly one of its most important properties. The stable distribution has asymmetric form that allows one to include naturally the skewness. This is not the case of the T-distribution whose no extension to asymmetric case is previewed. Empirical asymmetric formulation has to be found (see Liu et al[15]).

Before ending the paper, we focus on the interpretation of the physical processes through the probability distribution model. As we have shown in this study, the four stable parameters are able to traduce correctly the effects of physical processes as the asymmetry and the kurtosis of the slopes or also its level of non-Gaussianity. Use of α-stable distribution instead of the traditional Gaussian or quasi-Gaussian distributions, combined to hydrodynamical laws will be of great importance on understanding of the physical behavior of the water wave slopes. In this sense, extension of the present work to the space representation of water wave slopes by the use of multivariate α-stable distribution has to be carried out and constitutes our prospective task. An other way to be explored is also the use of truncated stable law that allows one to obtain finite values of statistical moments.

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