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NUMERICAL RANGE AND QUASI-SECTORIAL CONTRACTIONS

YURY ARLINSKIĬ AND VALENTIN ZAGREBNOV

ABSTRACT. We apply a method developed by one of the authors, see [1], to localize the numerical range of *quasi-sectorial* contractions semigroups. Our main theorem establishes a relation between the numerical range of quasi-sectorial contraction semigroups $\{\exp(-tS)\}_{t \geq 0}$, and the maximal sectorial generators S . We also give a new prove of the rate $O(1/n)$ for the operator-norm Euler formula approximation: $\exp(-tS) = \lim_{n \rightarrow \infty} (I + tS/n)^{-n}$, $t \geq 0$, for this class of semigroups.

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1. INTRODUCTION

In what follows the Banach algebra of all *bounded* linear operators on a complex Hilbert space \mathfrak{H} is denoted by $\mathcal{L}(\mathfrak{H})$. We denote by $I_{\mathfrak{H}}$ the identity operator in a Hilbert space \mathfrak{H} . The *domain*, the *range*, and the *null-space* of a linear operator T are denoted by $\text{dom } T$, $\text{ran } T$, and $\text{ker } T$, respectively. For $T \in \mathcal{L}(\mathfrak{H})$ the operators $\text{Re } T = (T + T^*)/2$ and $\text{Im } T = (T - T^*)/2i$ are said to be the *real* and the *imaginary* parts of T .

1.1. Numerical range and the Kato mapping theorem. Let \mathfrak{H} be a complex separable Hilbert space and let A be an (*unbounded*) linear operator in \mathfrak{H} with domain $\text{dom } A$.

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Definition 1.1. *The set of complex numbers*

$$W(A) := \{(Au, u) \in \mathbb{C} : u \in \text{dom } A, \|u\| = 1\}$$

is called the numerical range of A , or its field of values.

According to the Hausdorff-Toeplitz theorem, the numerical range is a convex set. We recall also the following properties of the numerical range (see e.g. [14]).

Proposition 1.2. *Let A be a closed operator in \mathfrak{H} . Then*

- (a) *for any complex number $z \notin \overline{W(A)}$ holds $\ker(A - zI_{\mathfrak{H}}) = \{0\}$ and $\text{ran}(A - zI_{\mathfrak{H}})$ is closed. Moreover, the defect*

$$\text{def}(A - zI_{\mathfrak{H}}) := \dim(\mathfrak{H} \ominus \text{ran}(A - zI_{\mathfrak{H}}))$$

is constant in each connected component of $\mathbb{C} \setminus \overline{W(A)}$.

- (b) *If $z \in \mathbb{C} \setminus \overline{W(A)}$ then*

$$\|(A - zI_{\mathfrak{H}})^{-1}f\| \leq \frac{1}{\text{dist}(z, W(A))} \|f\|, \quad f \in \text{ran}(A - zI_{\mathfrak{H}}).$$

- (c) *If $\text{dom } A$ is dense in \mathfrak{H} and $W(A) \neq \mathbb{C}$, then A is closable.*

Corollary 1.3. *For a bounded operator $A \in \mathcal{L}(\mathfrak{H})$ the spectrum $\sigma(A)$ is a subset of $\overline{W(A)}$.*

For unbounded operator A the relation between spectrum and numerical range is more complicated. We would like to warn, that it may very well happen that $\sigma(A)$ is not contained in $\overline{W(A)}$, but for a closed operator A the essential spectrum $\sigma_{\text{ess}}(A)$ is always a subset of $\overline{W(A)}$. The condition $\text{def}(A - zI) = 0$, $z \notin \overline{W(A)}$ in Proposition 1.2 serves to ensure that for those unbounded operators one gets

$$(1.1) \quad \sigma(T) \subset \overline{W(A)},$$

i.e., the same conclusion as in Corollary 1.3 for bounded operators.

In the sequel we need the following numerical range mapping theorem due to Kato [13].

Proposition 1.4. [13]. *Let $f(z)$ be a rational function on \mathbb{C} , with $f(\infty) = \infty$. Let for some compact and convex set $E' \subset \mathbb{C}$ the inverse function $f^{-1} : E' \rightarrow E \supseteq K$, where K is a convex kernel of E , i.e., is a subset of E such that E is star-shaped with respect to any $z \in K$. If A is bounded operator with $W(A) \subseteq K$, then $W(f(A)) \subseteq E'$.*

Notice that for a convex set E the maximal convex kernel $K = E$.

1.2. Sectorial operators and quasi-sectorial contractions.

Definition 1.5. [14]. *Let S be a linear operator in a Hilbert space \mathfrak{H} . If $\text{Re}(Su, u) \geq 0$ for all $u \in \text{dom } S$, then S is called accretive.*

So, the operator S is accretive if and only if its numerical range is contained in the closed right-half plane of the complex plane. An accretive operator S is called *maximal accretive* (*m-accretive*) if one of the equivalent conditions is satisfied:

- the operator S has no accretive extensions in \mathfrak{H} ;
- the resolvent set $\rho(S)$ is nonempty;
- the operator S is densely defined and closed, and S^* is accretive operator.

The resolvent set $\rho(S)$ of *m-accretive* operator contains the open left half plane and

$$\|(S - zI_{\mathfrak{H}})^{-1}\| \leq \frac{1}{|\text{Re } z|}, \quad \text{Re } z < 0.$$

It is well known [14] that if S is m -accretive operator, then the one-parameter semigroup

$$T(t) = \exp(-tS), \quad t \geq 0$$

is contractive. Conversely, if the family $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup of bounded operators in a Hilbert space \mathfrak{H} , with $T(0) = I_{\mathfrak{H}}$ (C_0 -semigroup) and $T(t)$ is a contraction for each t , then the generator S of $T(t)$:

$$Su := \lim_{t \rightarrow +0} \frac{(I_{\mathfrak{H}} - T(t))u}{t}, \quad u \in \text{dom } S,$$

where domain is defined by condition:

$$\text{dom } S = \left\{ u \in \mathfrak{H} : \lim_{t \rightarrow +0} \frac{(I_{\mathfrak{H}} - T(t))u}{t} \text{ exists} \right\},$$

is an m -accretive operator in \mathfrak{H} . Then the Euler formula approximation:

$$(1.2) \quad T(t) = s - \lim_{n \rightarrow \infty} \left(I_{\mathfrak{H}} + \frac{t}{n} S \right)^{-n}, \quad t \geq 0$$

holds in the strong operator topology, see e.g. [14].

Definition 1.6. [14]. Let $\alpha \in [0, \pi/2)$ and let

$$\mathcal{S}(\alpha) := \{z \in \mathbb{C} : |\arg z| \leq \alpha\}$$

be a sector on the complex plane \mathbb{C} with the vertex at the origin and the semi-angle α .

A linear operator S in a Hilbert space \mathfrak{H} is called sectorial with vertex at $z = 0$ and the semi-angle α if $W(S) \subseteq \mathcal{S}(\alpha)$.

If S is m -accretive and sectorial with vertex at $z = 0$ and the semi-angle α then it is called m -sectorial with vertex at the origin and with the semi-angle α . For short we call these operators m - α -sectorial. The resolvent set of m - α -sectorial operator S contains the set $\mathbb{C} \setminus \mathcal{S}(\alpha)$ and

$$\|(S - zI_{\mathfrak{H}})^{-1}\| \leq \frac{1}{\text{dist}(z, \mathcal{S}(\alpha))}, \quad z \in \mathbb{C} \setminus \mathcal{S}(\alpha).$$

It is well-known [14] that

- a C_0 -semigroup $T(t) = \exp(-tS)$, $t \geq 0$ has *contractive* and *holomorphic* continuation into the sector $\mathcal{S}(\pi/2 - \alpha)$ if and only if the generator S is m - α -sectorial operator,
- the *sesquilinear* form: (Su, v) , $u, v \in \text{dom } S$ is closable.

Denote by $\mathfrak{S}[\cdot, \cdot]$ the closure of the form (Su, v) and by $\mathcal{D}[\mathfrak{S}]$ its domain. By the *first representation theorem* [14] the operator S is *uniquely* associated with the form $\mathfrak{S}[\cdot, \cdot]$ in the following sense:

$$(Su, v) = \mathfrak{S}[u, v] \quad \text{for all } u \in \text{dom } S \quad \text{and for all } v \in \mathcal{D}[\mathfrak{S}].$$

The important relation between the one-parameter semigroups $\{T(t) := \exp(-tS)\}_{t \geq 0}$ generated by the m - α -sectorial operators S and the corresponding closed sesquilinear forms is established in [3]:

$$u \in \mathcal{D}[\mathfrak{S}] = \left\{ u \in \mathfrak{H} : \frac{d}{dt}(T(t)u, u) \Big|_{t=+0} \text{ exists} \right\},$$

and one has that

$$(1.3) \quad \frac{d}{dt}(T(t)u, u) \Big|_{t=+0} = -\mathfrak{S}[u, u], \quad u \in \mathcal{D}[\mathfrak{S}].$$

Notice that in this case of the Euler approximation (1.2) converges to the semigroup in the *operator-norm* topology [7], [8], [19].

Definition 1.7. [8]. For any $\alpha \in [0, \pi/2)$ we define in the complex plane \mathbb{C} a closed domain:

$$(1.4) \quad D_\alpha := \{z \in \mathbb{C} : |z| \leq \sin \alpha\} \cup \{z \in \mathbb{C} : |\arg(1 - z)| \leq \alpha \text{ and } |z - 1| \leq \cos \alpha\}.$$

This is a convex subset of the unit disc $\mathbb{D} = D_{\alpha=\pi/2}$, with the "angle" (in contrast to the "tangent") touching of the boundary $\partial\mathbb{D}$ at the only one point $z = 1$. It is evident that $D_\alpha \subset D_{\beta>\alpha}$.

Definition 1.8. [8]. A contraction C on the Hilbert space \mathfrak{H} is called *quasi-sectorial with semi-angle* $\alpha \in [0, \pi/2)$, if its numerical range $W(C) \subseteq D_\alpha$.

It is evident that if operator C is a *quasi-sectorial* contraction, then $I - C$ is an *m-sectorial* operator with vertex $z = 0$ and semi-angle α . The limits $\alpha = 0$ and $\alpha = \pi/2$ correspond, respectively, to non-negative (i.e. *self-adjoint*) and to some *general* contraction.

Remark 1.9. [8]. Notice that the resolvent family $\{(I_{\mathfrak{H}} + tS)^{-1}\}_{t \geq 0}$ of the *m- α -sectorial* operator S , gives the first non-trivial example of a *quasi-sectorial* contractions, if one considers the semi-angles $\alpha \in [0, \pi/3)$. Below (see Section 2) we show that it can be extended to $\alpha \in [0, \pi/2)$.

The definition of quasi-sectorial contractions was motivated in [8] by the lifting of the Trotter-Kato product formula and the Chernoff theory of semigroup approximation [12] to the operator-norm topology.

Namely, using properties of quasi-sectorial contractions established in [8] for the case of *m- α -sectorial* generator, it was proved that there is *operator-norm* convergence of the Euler semigroup approximation (1.2) and of the Trotter product formula, see [7], [8], [16], [6], [19]. Theorem 2.1 from [8] states that the operators $T(t) = \exp(-tS)$ are quasi-sectorial contractions with $W(T(t)) \subset D_\alpha$ for all $t \geq 0$ and $\alpha \in [0, \pi/2)$. Here operator S stands for *m- α -sectorial* generator. As it is indicated in [18], the proof in [8] has a flaw. In [18] it was corrected, but only for the semi-angles $\alpha \in [0, \pi/4)$.

Our main Theorem 3.4 establishes a quite accurate relation between *m- α -sectorial* generators and the numerical range of the corresponding one-parameter contraction semigroups. It improves the recent result [18] from $\alpha \in [0, \pi/4)$ to $\alpha \in [0, \pi/2)$. To this end we use in the next section some nontrivial results due to [1]-[4], concerning a class of operator contractions and of semigroups on the complex plane. Besides that we use recent results related to the generalizations of the famous von Neumann inequality [15], obtained in [5], [9], [10], [11], in order to give a new prove that in fact the Euler formula (1.2) converges in the operator-norm with the rate $O(1/n)$.

2. OPERATORS OF THE CLASS $C_{\mathfrak{H}}(\alpha)$

A fundamental for us will be the class of contractions introduced for the first time in [1] and studied in [1]-[2]:

Definition 2.1. [1]. Let $\alpha \in (0, \pi/2)$. We say that a bounded operator $T \in \mathcal{L}(\mathfrak{H})$ belongs to the class $C_{\mathfrak{H}}(\alpha)$ if

$$(2.1) \quad \|T \sin \alpha \pm iI_{\mathfrak{H}} \cos \alpha\| \leq 1.$$

It is clear, that the class $C_{\mathfrak{H}}(\alpha)$ is a convex and closed (with respect to the strong operator topology) set, which is intersection of two closed operator balls corresponding to \pm . Moreover, by virtue of (2.1) one immediately concludes that:

$$T \in C_{\mathfrak{H}}(\alpha) \iff -T \in C_{\mathfrak{H}}(\alpha) \iff T^* \in C_{\mathfrak{H}}(\alpha),$$

and that condition (2.1) is equivalent to the following criterium:

$$(2.2) \quad \tan \alpha (\|f\|^2 - \|Tf\|^2) \geq 2|\operatorname{Im}(Tf, f)|, \quad f \in \mathfrak{H}.$$

This inequality implies that the operator T is a contraction. Together with Definition 1.1, it also proves that $T \in C_{\mathfrak{H}}(\alpha)$ is equivalent to the statement that $(I - T^*)(I + T)$ is a bounded m - α -sectorial operator.

According to (2.2) it is natural to identify $C_{\mathfrak{H}}(\alpha = 0)$ with the set of all *self-adjoint* contractions whereas $C_{\mathfrak{H}}(\alpha = \pi/2)$ is the set of general contractions on \mathfrak{H} .

Now following [4] we define for $\alpha \in (0, \pi/2)$ the family of subsets of the complex plane \mathbb{C} :

$$(2.3) \quad C(\alpha) = \{z \in \mathbb{C} : |z \sin \alpha \pm i \cos \alpha| \leq 1\} = \{z \in \mathbb{C} : (1 - |z|^2) \tan \alpha \geq 2|\operatorname{Im} z|\},$$

cf. definitions (2.1), (2.2). Then, similar to (2.1), each set $C(\alpha)$ is the intersection of *two* closed disks of the complex plane which is contained in the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. From (2.3) it is clear that $C(\alpha = 0) = [-1, 1]$.

Remark 2.2. *By virtue of definitions (2.1), (2.2) and (2.3) one gets that $T \in C_{\mathfrak{H}}(\alpha)$ implies for the numerical range: $W(T) \subseteq C(\alpha)$. Then one has:*

$$(2.4) \quad \|\operatorname{Im} T\| \leq \tan(\alpha/2).$$

Notice that $C(\alpha) \setminus D_{\alpha} \neq \emptyset$, i.e. in general the operator T is not quasi-sectorial.

Besides, the operator class $C_{\mathfrak{H}}(\alpha)$ has several interesting properties. In particular the following one [1]:

Proposition 2.3. [1]. *Let T_1 and T_2 belong to the class $C_{\mathfrak{H}}(\alpha)$. Then the operator*

$$(T_1 T_2 + T_2 T_1)/2$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$. In particular, if T_1 and T_2 are two commuting operators from the class $C_{\mathfrak{H}}(\alpha)$, then the product $T = T_1 T_2$ also belongs to $C_{\mathfrak{H}}(\alpha)$.

Remark 2.4. *Notice that the set $C(\alpha)$ inherits a similar property: for each $\alpha \in (0, \pi/2)$ it forms a multiplicative semigroup of complex numbers in the plane \mathbb{C} . Detailed properties of these semigroups have been studied in [4].*

Now we are in position to establish a connection between set of contractions $C_{\mathfrak{H}}(\alpha)$ and m - α -sectorial operators.

Proposition 2.5. [1]. *If S is m - α -sectorial operator, then*

$$T = (I_{\mathfrak{H}} - S)(I_{\mathfrak{H}} + S)^{-1}$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$ and conversely, if $T \in C_{\mathfrak{H}}(\alpha)$, and $\ker(I_{\mathfrak{H}} + T) = \{0\}$, then $S = (I_{\mathfrak{H}} - T)(I_{\mathfrak{H}} + T)^{-1}$ is m - α -sectorial operator.

Now let S be m - α -sectorial operator and let $\lambda > 0$. Then λS is also m - α -sectorial. By Proposition 2.5 the operator $U(\lambda) = (I_{\mathfrak{H}} - \lambda S)(I_{\mathfrak{H}} + \lambda S)^{-1}$ belongs to the class $C_{\mathfrak{H}}(\alpha)$. Let us put

$$(2.5) \quad F(\lambda) := (I_{\mathfrak{H}} + \lambda S)^{-1} = \frac{1}{2}(U(\lambda) + I_{\mathfrak{H}}).$$

Since the operators $U(\lambda)$ and $I_{\mathfrak{H}}$ belong to the class $C_{\mathfrak{H}}(\alpha)$ and since it is a convex set, the operator $F(\lambda)$ is also in the class $C_{\mathfrak{H}}(\alpha)$. Hence, by Remark 2.2 one obtains $W(F(\lambda)) \subseteq C(\alpha)$.

Remark 2.6. *Following the arguments of [8],[18] we can localize the numerical range $W(F(\lambda))$ even more accurate, cf. Remark 1.9. Since for any $u \in \mathfrak{H}$ we have:*

$$(u, F(\lambda)u) = (v_\lambda, v_\lambda) + \lambda(Sv_\lambda, v_\lambda) \in \mathcal{S}(\alpha) ,$$

where $v_\lambda := F(\lambda)u$, it follows that

$$W(F(\lambda)) \subseteq \mathcal{S}(\alpha) \cap C(\alpha) ,$$

for $\lambda > 0$ and $\alpha \in [0, \pi/2)$. Moreover, since $U(\lambda) = 2F(\lambda) - I_{\mathfrak{H}} \in C_{\mathfrak{H}}(\alpha)$, see (2.5), we find that

$$(2.6) \quad \|(F(\lambda) - I_{\mathfrak{H}}/2) \sin \alpha \pm iI_{\mathfrak{H}}(\cos \alpha)/2\| \leq 1/2 .$$

For the numerical range this implies:

$$(2.7) \quad W(F(\lambda)) \subseteq L(\alpha) := \{\zeta \in \mathbb{C} : |(\zeta - 1/2) \sin \alpha \pm (i \cos \alpha)/2| \leq 1/2\} , \quad \text{for } \lambda > 0 .$$

Notice that $L(\alpha) \subset \mathcal{S}(\alpha) \cap C(\alpha)$, see Figure 1.

Now, we follow essentially the line of reasoning of [1] to establish the one-to-one correspondence between m - α -sectorial generators and contraction semigroups of the class $C_{\mathfrak{H}}(\alpha)$, cf. Remark 2.2 about relation to quasi-sectorial contractions.

Theorem 2.7. *If S is m - α -sectorial operator in a Hilbert space \mathfrak{H} , then the corresponding semigroup $T(t) = \exp(-tS) \in C_{\mathfrak{H}}(\alpha)$, for all $t \geq 0$. Conversely, let $T(t) = \exp(-tS)$ for $t \geq 0$ be a C_0 -semigroup of contractions on a Hilbert space \mathfrak{H} . If $T(t) \in C_{\mathfrak{H}}(\alpha)$ for non-negatives t in neighborhood of $t = 0$, then the generator S is an m - α -sectorial operator.*

Proof. Let S be a m - α -sectorial operator and let $\lambda \geq 0$. By (2.5) the operator $F(\lambda) = (I_{\mathfrak{H}} + \lambda S)^{-1}$ belongs to the class $C_{\mathfrak{H}}(\alpha)$. Therefore, by Proposition 2.3 for each $t \geq 0$ and any natural number n the operator

$$F^n \left(\frac{t}{n} \right) = \left(I_{\mathfrak{H}} + \frac{t}{n} S \right)^{-n}$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$. Taking in account that the set $C_{\mathfrak{H}}(\alpha)$ is closed with respect to the strong operator topology, from the Euler formula (1.2) we get that strong limit $T(t) = \exp(-tS)$ also belongs to the class $C_{\mathfrak{H}}(\alpha)$.

Now suppose that semigroup $T(t) = \exp(-tS) \in C_{\mathfrak{H}}(\alpha)$ for $t \in [0, \delta)$, where $\delta > 0$. Define operator family:

$$B_{\pm}(t) := T(t) \sin \alpha \pm i \cos \alpha I_{\mathfrak{H}}, \quad t \geq 0.$$

Since $B_{\pm}(0) = (\sin \alpha \pm i \cos \alpha) I_{\mathfrak{H}}$ and $T(t) \in C_{\mathfrak{H}}(\alpha)$ for $t \in [0, \delta)$, we get

$$\|B_{\pm}(t)f\|^2 \leq \|f\|^2 = \|B_{\pm}(0)\|f\|^2, \quad t \in [0, \delta), \quad f \in \mathfrak{H}.$$

Since

$$\|B_{\pm}(t)f\|^2 = \sin^2 \alpha \|T(t)f\|^2 + \cos^2 \alpha \|f\|^2 \pm 2 \sin \alpha \cos \alpha \operatorname{Im} (T(t)f, f), \quad f \in \mathfrak{H},$$

for all $f \in \operatorname{dom} S$ we have:

$$\frac{d}{dt} (\|B_{\pm}(t)f\|^2) \Big|_{t=+0} = -2 \sin^2 \alpha \operatorname{Re} (Sf, f) \pm 2 \sin \alpha \cos \alpha \operatorname{Im} (Sf, f) \leq 0.$$

Thus, $W(S) \subseteq \mathcal{S}(\alpha)$. But since operator S is m -accretive, it is m - α -sectorial [14]. \square

3. NUMERICAL RANGE FOR CONTRACTIVE HOLOMORPHIC SEMIGROUPS

From Theorem 2.7 it follows, in particular, that for m - α -sectorial generator S the numerical range of the corresponding contraction semigroup

$$W(\exp(-tS)) \subseteq C(\alpha) \quad \text{for all } t \geq 0.$$

But as we warranted in Remark 2.2 it does not imply that this semigroup is quasi-sectorial contraction. It was discovered in [8] that the conformal mapping $z \mapsto z^2$, together with the Kato numerical range theorem (Proposition 1.4) play a special rôle in the theory of quasi-sectorial contractions.

Definition 3.1. Let $\alpha \in [0, \pi/2)$. We define a domain:

$$(3.1) \quad \Omega(\alpha) := \{z^2 : z \in C(\alpha)\}.$$

So, if $f(z) = z^2$, then $\Omega(\alpha) = f(C(\alpha))$. Since (see Remark 2.4) $C(\alpha)$ is a *multiplicative semigroup*, we obtain that $\Omega(\alpha) \subseteq C(\alpha)$, and that the subset $\Omega(\alpha)$ is in turn a multiplicative semigroup [4].

From (2.3) and Proposition 2.3 it follows then that for any $\alpha \in (0, \pi/2)$ the set (3.1) has representation:

$$(3.2) \quad \Omega(\alpha) = \{z \in \mathbb{C} : |\sqrt{z} \sin \alpha \pm i \cos \alpha| \leq 1\} = \{z \in \mathbb{C} : 2|\operatorname{Im} \sqrt{z}| \leq (1 - |z|) \tan \alpha\},$$

with the limiting cases: $\Omega(\alpha = 0) = [0, 1]$ and $\Omega(\alpha = \pi/2) = \overline{\mathbb{D}}$.

Lemma 3.2. The set $\Omega(\alpha)$ (3.1) is convex and $\Omega(\alpha) \subseteq D_\alpha$.

Proof. Let $C_+(\alpha) := \{z \in C(\alpha) : \operatorname{Im} z \geq 0\}$. Then clearly, $\Omega(\alpha) = f(C_+(\alpha))$, where $f(z) = z^2$. Denote

$$\Gamma(\alpha) = \partial C_+(\alpha) \setminus (-1, 1).$$

Then $\partial\Omega(\alpha) = f(\Gamma(\alpha))$. Since

$$\Gamma(\alpha) = \left\{ z : z = \frac{e^{it} - i \cos \alpha}{\sin \alpha}, t \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha \right] \right\},$$

the boundary $\partial\Omega(\alpha)$ can be parameterized as follows:

$$(3.3) \quad \partial\Omega(\alpha) = \left\{ z = \zeta(t) = \frac{(e^{it} - i \cos \alpha)^2}{\sin^2 \alpha}, t \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha \right] \right\}.$$

Put $x = x(t) := \operatorname{Re} \zeta(t)$, $y = y(t) := \operatorname{Im} \zeta(t)$. Then, since

$$\frac{d^2 y}{dx^2} = \frac{y''(t)x'(t) - y'(t)x''(t)}{(x'(t))^3},$$

we obtain

$$\frac{d^2 y}{dx^2} = -\frac{\operatorname{Im}(\zeta'(t)\overline{\zeta''(t)})}{(\operatorname{Re} \zeta'(t))^3}.$$

Further, by explicit calculations we get:

$$\begin{aligned} \zeta'(t) &= \frac{2ie^{it}(e^{it} - i \cos \alpha)}{\sin^2 \alpha}, \quad \zeta''(t) = \frac{2ie^{it}(2ie^{it} + \cos \alpha)}{\sin^2 \alpha}, \\ -\operatorname{Im}(\zeta'(t)\overline{\zeta''(t)}) &= \frac{4(2 + \cos^2 \alpha - 3 \cos \alpha \sin t)}{\sin^4 \alpha}, \quad \operatorname{Re} \zeta'(t) = \frac{2 \cos t(\cos \alpha - 2 \sin t)}{\sin^2 \alpha}. \end{aligned}$$

For $t \in [\pi/2 - \alpha, \pi/2 + \alpha]$, by the estimates

$$\cos^2 \alpha - 3 \cos \alpha \sin t + 2 \geq 2 \cos^2 \alpha - 3 \cos \alpha + 2 = (2 - \cos \alpha)(1 - \cos \alpha) > 0,$$

we obtain $-\operatorname{Im}(\zeta'(t)\overline{\zeta''(t)}) > 0$.

Since $\cos \alpha - 2 \sin t \leq -\cos \alpha < 0$, then for $t \in [\pi/2 - \alpha, \pi/2 + \alpha]$, we get that

$$\begin{aligned} \frac{d^2 y}{dx^2} < 0 & \quad \text{for } t \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2} \right), \\ \frac{d^2 y}{dx^2} > 0 & \quad \text{for } t \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \alpha \right], \end{aligned}$$

which implies that the set $\Omega(\alpha)$ is convex.

Since the mapping: $z \mapsto z^2$ is conformal and $\Omega(\alpha)$ is convex, the proof of the second part of the lemma follows from the estimate

$$(3.4) \quad -(\tan \alpha/2)^2 \leq \operatorname{Re} \zeta(t),$$

where $(\tan \alpha/2)^2 \leq \sin \alpha$ for $\alpha \in (0, \pi/2)$, see (1.4) and (3.3). \square

Notice that in view of relation:

$$y(t) = \frac{\sin 2t - 2 \cos \alpha \cos t}{\sin^2 \alpha},$$

one obtains

$$(3.5) \quad \max_{z \in \Omega(\alpha)} |\operatorname{Im} z| = \frac{\sin 2\gamma - 2 \cos \alpha \sin \gamma}{\sin^2 \alpha},$$

where

$$(3.6) \quad \sin \gamma = \frac{\cos \alpha + \sqrt{\cos^2 \alpha + 8}}{4}, \quad \gamma \in (0, \pi/2).$$

Since $\Omega(\alpha) \subset C(\alpha)$, the number in the right-hand side of (3.5) is less than $\tan(\alpha/2)$.

Lemma 3.3. *Let $\alpha \in [0, \pi/2)$ and let $T \in C_{\mathfrak{S}}(\alpha)$. Then $W(T^{2n}) \subseteq \Omega(\alpha)$ for all natural numbers n .*

Proof. We apply the Kato numerical range mapping theorem (Proposition 1.4) for $f(z) = z^2$ with $E' = \Omega(\alpha)$. Then by Definition 3.1 we have $E := f^{-1}(\Omega(\alpha)) = C(\alpha)$. Since $C(\alpha)$ is a convex set, its maximal convex kernel K coincides with $C(\alpha)$. By virtue of Proposition 2.3 from $T \in C_{\mathfrak{S}}(\alpha)$ it follows that for all natural n one gets $T^n \in C_{\mathfrak{S}}(\alpha)$. Then by Remark 2.2 we obtain $W(T^n) \subseteq C(\alpha) = K$. Applying now the Kato mapping theorem for $f(z) = z^2$ we obtain $W(T^{2n}) \subseteq \Omega(\alpha)$. \square

Now we are in position to prove the main theorem of the present paper.

Theorem 3.4. (1) *Let S be m - α -sectorial operator. Then*

$$(3.7) \quad W(\exp(-tS)) \subseteq \Omega(\alpha), \quad t \geq 0.$$

In particular, $\{\exp(-tS)\}_{t \geq 0}$ is the quasi-sectorial contraction semigroup.

(2) *The inverse is also true. Let $\{T(t) := \exp(-tS)\}_{t \geq 0}$ be a C_0 -semigroup on a Hilbert space \mathfrak{H} . If in some neighborhood of $t = 0$ the numerical range: $W(T(t \geq 0)) \subseteq \Omega(\alpha)$ for some $\alpha \in [0, \pi/2)$, then the generator S is an m - α -sectorial operator.*

Proof. (1) By Theorem 2.7 the operators $T(t) = \exp(-tS)$ belong to the class $C_{\mathfrak{S}}(\alpha)$ for all $t \geq 0$, but $T(t) = T^2(t/2)$, then by Lemma 3.3 we obtain $W(T(t)) \subseteq \Omega(\alpha)$. Since Lemma 3.2 implies $\Omega(\alpha) \subset D_\alpha$, the operators $T(t)$ are quasi-sectorial contractions.

(2) Fix $u \in \operatorname{dom} S$, $\|u\| = 1$. For $t \in [0, \delta]$ we define two functions:

$$G_{\pm}(t) := |\sin \alpha \sqrt{(T(t)u, u)} \pm i \cos \alpha|^2.$$

Then $G_{\pm}(0) = 1$. The condition $W(T(t)) \subseteq \Omega(\alpha)$ yields (see (3.2)) that $G_{\pm}(t) \leq G_{\pm}(0)$, $t \in [0, \delta]$, which implies:

$$\left. \frac{d}{dt} (G_{\pm}(t)) \right|_{t=+0} \leq 0 .$$

Therefore we get that

$$(3.8) \quad \lim_{t \rightarrow +0} \frac{1 - G_{\pm}(t)}{t} \geq 0 .$$

Further we use the following identities:

$$\begin{aligned} 1 - G_{\pm}(t) &= \sin^2 \alpha (1 - |(T(t)u, u)|) \mp 2 \sin \alpha \cos \alpha \operatorname{Im} \sqrt{(T(t)u, u)} = \\ &= \frac{\sin^2 \alpha \left(1 - (T(t)u, u) + (T(t)u, u)(1 - \overline{(T(t)u, u)}) \right)}{1 + |(T(t)u, u)|} \mp 2 \sin \alpha \cos \alpha \operatorname{Im} \sqrt{(T(t)u, u)}, \\ \frac{1 - G_{\pm}(t)}{t} &= \frac{\sin^2 \alpha \left(1 - (T(t)u, u) + (T(t)u, u)(1 - \overline{(T(t)u, u)}) \right)}{t(1 + |(T(t)u, u)|)} \mp \\ &\mp 2 \sin \alpha \cos \alpha \operatorname{Im} \left(\frac{(T(t)u, u) - 1}{t(\sqrt{(T(t)u, u)} + 1)} \right) . \end{aligned}$$

Since by (1.3) one has:

$$\lim_{t \rightarrow +0} \frac{1 - (T(t)u, u)}{t} = (Su, u) ,$$

and by (3.8) we get

$$\lim_{t \rightarrow +0} \frac{1 - G_{\pm}(t)}{t} = \sin^2 \alpha \operatorname{Re} (Su, u) \pm \sin \alpha \cos \alpha \operatorname{Im} (Su, u) \geq 0 .$$

The last estimate implies that $W(S) \subseteq \mathcal{S}(\alpha)$, i.e., S is m - α -sectorial operator. \square

Thus, the two equivalent conditions: $\exp(-tS) \in C_{\mathfrak{H}}(\alpha)$ (see Theorem 2.7) and $W(\exp(-tS)) \subseteq \Omega(\alpha)$ *completely* characterize m - α -sectorial operators S . Since (3.7) yields $W(\exp(-tS)) \subseteq D_{\alpha}$, the statement in Theorem 3.4(1) is extension of Theorem 2.1 (proven in [18] for $\alpha \in [0, \pi/4)$) to the whole class of the m - α -sectorial generators: $\alpha \in [0, \pi/2)$.

Notice that taking into account the multiplicative semigroup property of the set $\Omega(\alpha)$ and Theorem 3.4 we get by consequence the following inequalities:

$$\left| \sin \alpha \left(\prod_{k=1}^n (\exp(-t_k S_k) u_k, u_k) \right)^{1/2} \pm i \cos \alpha \right| \leq 1$$

for arbitrary m - α -sectorial operators S_1, \dots, S_n , any non-negative numbers t_1, \dots, t_n , and arbitrary normalized vectors u_1, \dots, u_n from \mathfrak{H} . By semi-group property: $\exp(-t_1 S) \exp(-t_2 S) = \exp(-(t_1 + t_2)S)$. Hence we also get

$$|\sin \alpha \sqrt{(\exp(-t_1 S)u, \exp(-t_2 S^*)u)} \pm i \cos \alpha| \leq 1$$

for any $t_1, t_2 \geq 0$ and any $u \in \mathfrak{H}$, $\|u\| = 1$.

Remark that since $\operatorname{Re} z \geq -\tan^2(\alpha/2)$ for all $z \in \Omega(\alpha)$ and $W(\exp(-tS)) \subseteq \Omega(\alpha)$, we obtain:

$$\operatorname{Re} \exp(-tS) \geq -\tan^2(\alpha/2) I_{\mathfrak{H}} , \quad t \geq 0 .$$

The inclusion $W(\exp(-tS)) \subseteq \Omega(\alpha)$ also implies (see (3.5)) that

$$\|\operatorname{Im} \exp(-tS)\| \leq \frac{\sin 2\gamma - 2 \cos \alpha \sin \gamma}{\sin^2 \alpha} < \tan(\alpha/2) , \quad t \geq 0 ,$$

where γ satisfies (3.6). By virtue of relation $\operatorname{Re} T^2 = (\operatorname{Re} T)^2 - (\operatorname{Im} T)^2$ (valid for any bounded operator T) and since one has that $\operatorname{Re} \exp(-tS) = (\operatorname{Re} \exp(-tS/2))^2 - (\operatorname{Im} \exp(-tS/2))^2$, we obtain:

$$\operatorname{Re} \exp(-tS) \geq - \left(\frac{\sin 2\gamma - 2 \cos \alpha \sin \gamma}{\sin^2 \alpha} \right)^2 I_{\mathfrak{H}} > - \tan^2(\alpha/2) I_{\mathfrak{H}}, \quad t \geq 0.$$

Now it is useful to define also the following subset of $C(\alpha)$:

$$(3.9) \quad \begin{aligned} Q(\alpha) &:= \{z \in C(\alpha) : |z \sin \alpha - \cos \alpha| \leq 1\} = \\ &= \{z \in \mathbb{C} : 2|\operatorname{Im} z| \leq (1 - |z|^2) \tan \alpha\} \cap \{z \in \mathbb{C} : -2\operatorname{Re} z \leq (1 - |z|^2) \tan \alpha\}. \end{aligned}$$

So, $Q(\alpha)$ is the intersection of *three* closed disks, cf. (2.3).

Proposition 3.5. *The set $Q(\alpha)$ is a multiplicative semigroup on \mathbb{C} .*

Proof. First we note that $Q(\alpha)$ contains the set $C_+(\alpha) := \{z \in C(\alpha) : \operatorname{Re} z \geq 0\}$. Moreover, since $\operatorname{Re} iz = -\operatorname{Im} z$, by (3.9) we obtain also that the set

$$B(\alpha) := \{z \in C(\alpha) : iz \in C(\alpha)\},$$

has a non-empty intersection with $Q(\alpha)$, such that

$$(3.10) \quad B(\alpha) \cap Q(\alpha) = B(\alpha) \quad \text{and} \quad Q(\alpha) \setminus B(\alpha) \subset C_+(\alpha).$$

In [4] it is shown that the set $B(\alpha)$ is the *ideal* of the multiplicative semigroup $C(\alpha)$, i.e., $z\xi \in B(\alpha)$ for all $z \in B(\alpha)$ and all $\xi \in C(\alpha)$. Let $z, \xi \in Q(\alpha) \setminus B(\alpha)$. Since $B(\alpha)$ has the properties (3.10), then

$$\arg z, \arg \xi \in (-\pi/4, \pi/4).$$

Finally, since $z\xi \in C(\alpha)$ and $\arg(z\xi) \in (-\pi/2, \pi/2)$, we obtain $z\xi \in Q(\alpha)$. \square

Notice that in particular Proposition 3.5 yields:

$$z \in Q(\alpha) \Rightarrow z^n \in Q(\alpha).$$

Moreover, if $\varphi_n(z) = z^n$, then the set $\varphi_n(Q(\alpha))$ is also a multiplicative semigroup and

$$\varphi_n(Q(\alpha)) \subset Q(\alpha).$$

Since by (2.3) and (3.9) one gets that

$$C(\alpha) = Q(\alpha) \cup \{-Q(\alpha)\},$$

where $\{-Q(\alpha)\} := \{z : -z \in Q(\alpha)\}$, by (3.1) and (3.10) we have

$$\Omega(\alpha) = f(Q(\alpha))$$

when $f(z) = z^2$. Thus, we obtain:

$$(3.11) \quad \Omega(\alpha) \subset Q(\alpha).$$

Remark 3.6. *Another way to check (3.11) is the following argument. Let $\xi = z^2$, where $z \in C(\alpha)$. Then $\xi \in \Omega(\alpha)$, see (3.1). Since $\operatorname{Re} \xi = (\operatorname{Re} z)^2 - (\operatorname{Im} z)^2$ and $(\operatorname{Im} z)^2 \leq |\operatorname{Im} z|$, we also get*

$$\begin{aligned} -2\operatorname{Re} \xi &= -2(\operatorname{Re} z)^2 + 2(\operatorname{Im} z)^2 \leq 2|\operatorname{Im} z| \leq \tan \alpha(1 - |z|^2) = \\ &= \tan \alpha(1 - |\xi|) \leq \tan \alpha(1 - |\xi|^2), \end{aligned}$$

which means by (3.9) that $\Omega(\alpha) \subset Q(\alpha)$.

Remark 3.7. Let D_α be the set (1.4) introduced in [8]. From definition (3.9) one deduces that $Q(\alpha) \subset D_\alpha$, as well as that D_α is not a subset of $C(\alpha)$, i.e., $D_\alpha \cap C(\alpha) \neq D_\alpha$. In addition, by virtue of (2.7) and (3.2) we obtain: $L(\alpha) \subset \Omega(\alpha)$, where $L(\alpha)$ is defined by (2.7). So, for all $\alpha \in [0, \pi/2)$ and $F(\lambda) = (I_{\mathfrak{H}} + \lambda S)^{-1}$, $\lambda > 0$ we have the following inclusions (see Figure 1):

$$\begin{aligned} L(\alpha) &\subset \Omega(\alpha) \subset Q(\alpha) \subset D_\alpha, \\ W(F(\lambda)) &\subseteq L(\alpha), \\ W(F^n(\lambda)) &\subseteq C(\alpha) \quad \text{and} \quad W(F^{2n}(\lambda)) \subseteq \Omega(\alpha) \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

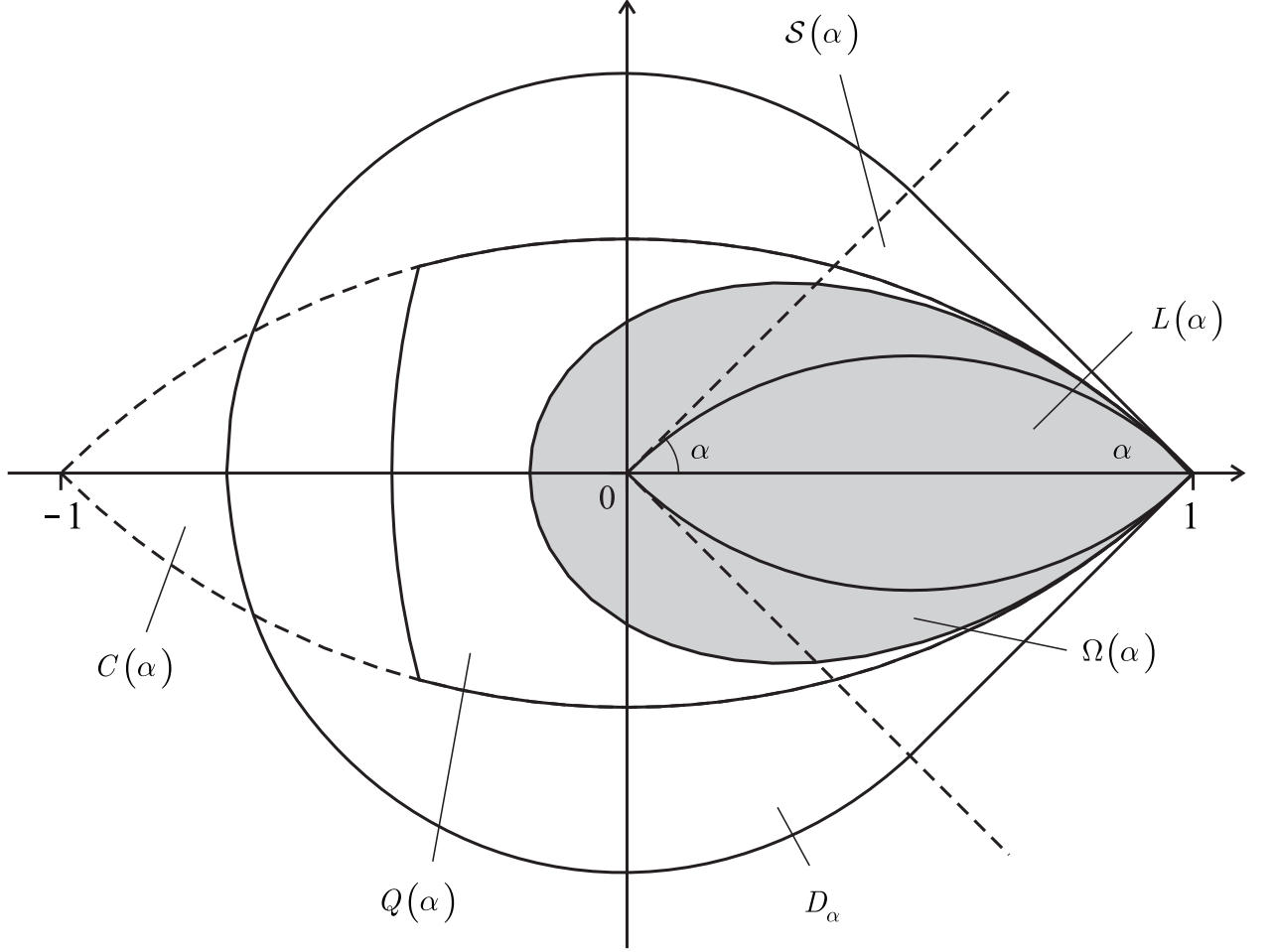


Figure 1

4. OPERATOR-NORM CONVERGENCE OF THE EULER FORMULA

4.1. A brief history. Theorem 5.1 in [8] states that

$$(4.1) \quad \|(I_{\mathfrak{H}} + tS/n)^{-n} - \exp(-tS)\| = O(\ln n/n)$$

for m - α -sectorial boundedly invertible operator S in \mathfrak{H} and for all $t \in S_{\pi/2-\alpha}$. The proof of (4.1) is essentially based on two estimates for a quasi-sectorial contraction C , i.e. $W(C) \subseteq D_\alpha$, also established in [8]. The first is

$$(4.2) \quad \|C^n - C^{n+1}\| \leq \frac{K}{n+1}$$

for all $n \in \mathbb{N}$ and for some $K > 0$ depending on α , and the second one is

$$(4.3) \quad \|C^n - \exp(n(C - I_{\mathfrak{H}}))\| = O(1/n^{1/3}).$$

Then application of (4.2) and (4.3) for the particular case of the operator $C = (I_{\mathfrak{H}} + tS/n)^{-1}$ leads to (4.1).

Later the estimate (4.1) was improved in [16] to

$$(4.4) \quad \|C^n - \exp(n(C - I_{\mathfrak{H}}))\| = O\left((\ln n/n)^{1/2}\right),$$

and in [6] to

$$(4.5) \quad \|C^n - \exp(n(C - I_{\mathfrak{H}}))\| = O(1/n^{1/2}).$$

In [16] the following bound is proved for the operator-norm convergence rate:

$$(4.6) \quad \|(I_{\mathfrak{H}} + tS/n)^{-n} - \exp(-tS)\| \leq c/n,$$

where c is a constant depending on the operator S . The bounds (4.3), (4.4), (4.5), and (4.6) are obtained in [16] and [6] by means of the probability theory methods (the *Poisson distribution* and the *Central Limit Theorem*) that improves the estimate (4.3).

Here, using the results obtained in [11], [5], [10], and [9], and generalizations of some von Neumann inequality [15], we prove the theorem, which makes more explicit the right-hand side of the estimate (4.6).

Theorem 4.1. *Let S be an m - α -sectorial operator in a Hilbert space \mathfrak{H} . Then*

$$\|(I_{\mathfrak{H}} + tS/n)^{-n} - \exp(-tS)\| \leq K(\alpha)/(n \cos^2 \alpha), \quad t \geq 0,$$

where

$$(\pi \sin \alpha)/2\alpha \leq K(\alpha) \leq \min\left(2 + 2/\sqrt{3}, (\pi - \alpha)/\alpha\right).$$

4.2. The von Neumann inequality and its generalizations. The spectral sets theory was introduced by von Neumann [15] in order to extend the functional calculus to the case of non-normal operators in Hilbert spaces.

Definition 4.2. [15]. *A set $\sigma \subset \mathbb{C}$ is a spectral set of the operator A in a Hilbert space \mathfrak{H} if it is closed and if for any bounded rational function $u(z)$ on σ one has*

$$\|u(A)\| \leq \sup_{z \in \sigma} |u(z)|.$$

Proposition 4.3. [15]. *A necessary and sufficient condition for one of the domains*

$$|z - a| \leq r, \quad |z - a| \geq r, \quad \operatorname{Re}(az) \geq b$$

to be a spectral set of A in \mathfrak{H} is that

$$\|A - aI_{\mathfrak{H}}\| \leq r, \quad \|(A - aI_{\mathfrak{H}})^{-1}\| \leq \frac{1}{r}, \quad \operatorname{Re}(aT) \geq b.$$

Definition 4.4. *Let \mathcal{D} be an open convex subset of the complex plane ($\mathcal{D} \neq \emptyset$, $\mathcal{D} \neq \mathbb{C}$) and let A be a linear operator in a Hilbert space \mathcal{H} with $W(A) \subset \overline{\mathcal{D}}$. The set \mathcal{D} is called K -spectral set for the operator A , if*

$$\|u(A)\| \leq K \sup_{z \in \mathcal{D}} |u(z)|$$

for all rational functions $u(z)$ without pole in the spectrum of A .

Let $\mathcal{D}_k := \{|z - a_k| < r_k\}$, $k = 1, 2$ be two disks such that $\mathcal{D}_k \cap \mathcal{D}_2 = \{\xi_1, \xi_2\}$. Let

$$\mathfrak{L} := \mathcal{D}_1 \cap \mathcal{D}_2.$$

The set \mathfrak{L} is said to be a *convex lens-shaped domain* [5]. Denote by $2\alpha \in (0, \pi)$ the angle of the lens L at the vertices. The operator $A \in \mathcal{L}(\mathfrak{H})$ is called of the *lenticular \mathfrak{L} -type* [5] if

$$\|A - a_1 I_{\mathfrak{H}}\| \leq r_1 \quad \text{and} \quad \|A - a_2 I_{\mathfrak{H}}\| \leq r_2.$$

The next Proposition is established in [5].

Proposition 4.5. *Let \mathfrak{L} be a convex lens-shaped domain of the complex plane with angle 2α . There exists a best positive constant $K(\alpha)$ such that the inequality*

$$(4.7) \quad \|p(A)\| \leq K(\alpha) \sup_{z \in \mathfrak{L}} |p(z)|,$$

holds for all polynomials $p(z)$, for all operators $A \in \mathcal{L}(\mathfrak{H})$ of \mathfrak{L} -type, and for all Hilbert spaces \mathfrak{H} . The constant $K(\alpha)$ depends only on α and satisfies the inequality

$$\frac{\pi}{2\alpha} \sin \alpha \leq K(\alpha) \leq \min \left(2 + \frac{2}{\sqrt{3}}, \frac{\pi - \alpha}{\alpha} \right).$$

Notice that by the Mergelyan Theorem [17], the inequality (4.7) remains valid if p is holomorphic in \mathfrak{L} and continuous in $\overline{\mathfrak{L}}$.

4.3. Proof of the Theorem 4.1.

Proof. Let

$$\mathcal{D}_1 = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{2} - \frac{i}{2} \cot \alpha \right| < \frac{1}{2 \sin \alpha} \right\},$$

$$\mathcal{D}_2 = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{2} + \frac{i}{2} \cot \alpha \right| < \frac{1}{2 \sin \alpha} \right\}.$$

Then (see (2.7)) $\mathfrak{L} = L(\alpha) \setminus \partial L(\alpha)$, i.e. interior of the set $L(\alpha)$.

Fix $t \geq 0$ and let

$$C = F(t) := (I_{\mathfrak{H}} + tS)^{-1}.$$

Then from (2.6) it follows that the operator C is of \mathfrak{L} -type.

Since

$$F(t/n) = F(t) \{(n-1)F(t)/n + I_{\mathfrak{H}}/n\}^{-1}, \quad n \in \mathbb{N},$$

we get also that

$$F^n(t/n) = C^n \{(n-1)C/n + I_{\mathfrak{H}}/n\}^{-n}.$$

Put

$$h_n(z) := \exp(1 - 1/z) - z^n \{(n-1)z/n + 1/n\}^{-n}, \quad z \in \mathfrak{L}.$$

Because $\operatorname{Re} z > 0$ for all $z \in \mathfrak{L}$, the function $h_n(z)$ is holomorphic in \mathfrak{L} and continuous in $\overline{\mathfrak{L}} = L(\alpha)$. Moreover, since

$$\exp(-tS) = \exp(I_{\mathfrak{H}} - C^{-1}),$$

we obtain that

$$\exp(-tS) - (I_{\mathfrak{H}} + tS/n)^{-n} = h_n(C).$$

The fractional-linear conformal transformation $w \mapsto z = 1/(1+w)$ maps the sector $\mathcal{S}(\alpha)$ onto $L(\alpha)$. So, let

$$g_n(w) := h_n((1+w)^{-1}) = \exp(-w) - (1+w/n)^{-n}, \quad w \in (\mathcal{S}(\alpha) \setminus \partial \mathcal{S}(\alpha)).$$

Then clearly:

$$\sup_{z \in \mathfrak{L}} |h_n(z)| = \sup_{w \in (\mathcal{S}(\alpha) \setminus \partial \mathcal{S}(\alpha))} |g_n(w)| = \sup_{w \in \partial \mathcal{S}(\alpha)} |g_n(w)|.$$

Since

$$\partial \mathcal{S}(\alpha) = \{x \exp(-i\alpha), x \in \mathbb{R}_+\} \cup \{x \exp(i\alpha), x \in \mathbb{R}_+\},$$

we have to estimate the value of

$$\sup_{x \in \mathbb{R}_+} |g_n(x \exp(\pm i\alpha))|.$$

To this end we use the representation:

$$g_n(x \exp(i\alpha)) = - \int_0^x \frac{d}{ds} \left((1 + se^{i\alpha}/n)^{-n} \exp(-(x-s)e^{i\alpha}) \right) ds ,$$

where

$$\frac{d}{ds} \left((1 + se^{i\alpha}/n)^{-n} \exp(-(x-s)e^{i\alpha}) \right) = \frac{se^{i\alpha}}{n} (1 + se^{i\alpha}/n)^{-(n+1)} \exp(-(x-s)e^{i\alpha}) .$$

By elementary inequalities:

$$\begin{aligned} |1 + se^{i\alpha}/n|^{n+1} &= (1 + s^2/n^2 + 2s \cos \alpha/n)^{(n+1)/2} \geq (1 + (s \cos \alpha)/n)^{n+1} \geq \\ &\leq 1 + ((n+1)s \cos \alpha)/n \geq 1 + s \cos \alpha, \end{aligned}$$

we obtain

$$s |1 + (s \cos \alpha)/n|^{-(n+1)} \leq s/(1 + s \cos \alpha) \leq 1/\cos \alpha .$$

Therefore, we obtain as an upper bound:

$$|g_n(x \exp(i\alpha))| \leq \int_0^x \frac{\exp(-(x-s) \cos \alpha)}{n \cos \alpha} ds \leq \frac{1}{n \cos^2 \alpha} (1 - \exp(-x \cos \alpha)) \leq \frac{1}{n \cos^2 \alpha} .$$

Similarly one obtains the estimate:

$$|g_n(x \exp(-i\alpha))| \leq \frac{1}{n \cos^2 \alpha} .$$

Thus,

$$\sup_{x \in \mathbb{R}_+} |g_n(x \exp(\pm i\alpha))| \leq \frac{1}{n \cos^2 \alpha} .$$

Now by Proposition 4.5 we get

$$(4.8) \quad \|\exp(-tS) - (I_{\mathfrak{H}} + tS/n)^{-n}\| = \|h_n(C)\| \leq \frac{K(\alpha)}{n \cos^2 \alpha} , \quad t \geq 0 ,$$

which completes the proof. \square

5. CONCLUSION

Now several remarks are in order:

(a) Theorem 2.1 from [8] states that for quasi-sectorial contractions one has:

$$(5.1) \quad W(\exp(-tS)) \subseteq D_\alpha, \quad \alpha \in [0, \pi/2),$$

Here S stands for m - α -sectorial generator of contraction. As it is indicated in [18], the proof in [8] has a flaw. In [18] it was corrected but only for the range $\alpha \in [0, \pi/4]$. Because

$$\Omega(\alpha) \subset Q(\alpha) \subset D_\alpha,$$

our main Theorem 3.4 shows that the original claim (5.1) in [8] is indeed correct for all $\alpha \in [0, \pi/2)$.

(b) Since in Theorem 3.4 we proved in fact that $W(\exp(-tS)) \subseteq \Omega(\alpha)$, this means we give in the present papers more precise localization of $W(\exp(-tS))$ than it was claimed in [8].

(c) The inclusion (5.1) plays an important rôle for the operator-norm error estimates of the norm convergence in the Euler formula (1.2), see the recent paper [18], the references quoted there and in particular [7]. In the present paper we give a new proof of the operator-norm convergent Euler formula for the optimal error estimate with explicit indication of its α -dependence (4.8) for m - α -sectorial generators.

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REFERENCES

- [1] Yu.M. Arlinskiĭ. A class of contractions in Hilbert space. *Ukrain. Matematicheskii Zhurnal*, 39 (1987), No.6, 691-696(Russian). English translation in: *Ukrainian Mathematical Journal*, 39 (1987), No.6, 560–564.
- [2] Yu.M. Arlinskiĭ. Characteristic functions of operators of the class $C(\alpha)$. *Izv.Vyssh. Uchebn.Zaved.Mat.* 1991, N2, 13-21(Russian).
- [3] Yu.M. Arlinskiĭ. Closed sectorial forms and one-parameter contraction semigroups, *Matematicheskie Zametki*, 61 (1997), No.5, 643-654 (Russian). English translation in: *Mathematical Notes*, 61 (1997), No. 5, 537–546.
- [4] Yu.M. Arlinskiĭ. On some semigroups on the complex plane. *Semigroup Forum*, 70 (2005), No.3, 329–346.
- [5] B. Beckermann and M. Crouzeix, A lenticular version of a von Neumann inequality, *Archiv der Mathematik*, 86 (2006), 352–355.
- [6] V. Bentkus and V. Paulauskas, Optimal error estimates in operator-norm approximations of semigroups, *Lett. Math. Phys.* 68 (2004) 131–138.
- [7] V. Cachia. Euler’s exponential formula for semigroups. *Semigroup Forum*, 68 (2004), No.1, 1–24.
- [8] V. Cachia and V.A. Zagrebnov, Operator-norm approximation of semigroups by quasi-sectorial contractions, *Journ. Funct. Anal.*, 180 (2001), 176-194.
- [9] M. Crouzeix, Numerical range and functional calculus in Hilbert space. *Journal of Functional Analysis*, vol. 244 (2007), 668-690.
- [10] M. Crouzeix and B. Delyon, Some estimates for analytic functions of the strip or a sectorial operators, *Archiv der Mathematik*, 81, (2003), 559–566.
- [11] B. Delyon and F. Delyon, Generalization of von Neumann’s spectral sets and integral representation of operators, *Bull. Soc. Math. Fr.*, 127 (1999), No. 1, 25–42.
- [12] P.R. Chernoff, *Product formulas, nonlinear semigroups, and addition of unbounded operators*. Memoirs of the American Mathematical Society, No. 140. American Mathematical Society, Providence, R. I., 1974.
- [13] T. Kato Some mapping theorem for the numerical range, *Proc.Japan Acad.* 41 (1966), 652–655.
- [14] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, Heidelberg, 1995.
- [15] J. von Neumann, Eine spectraltheorie für allgemeine operatoren eines unitären raumes, *Math. Nachrichten*, 4 (1951), 251–281.
- [16] V. Paulauskas, On operator-norm approximation of some semigroups by quasi-sectorial operators, *J. Funct. Anal.* 207 (2004) 58–67.
- [17] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, 1987.
- [18] V.A. Zagrebnov. Quasi-Sectorial Contractions, *Journ. Funct. Anal.*, 254 (2008), 2503–2511.
- [19] V.A. Zagrebnov, *Topics in the theory of Gibbs semigroups*. Leuven Notes in Mathematical and Theoretical Physics. Series A: Mathematical Physics, 10. Leuven University Press, Leuven, 2003.

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