HAL
open science

Numerical Range and Quasi-Sectorial Contractions

Yury Arlinskii, Valentin A. Zagrebnov

To cite this version:

Yury Arlinskii, Valentin A. Zagrebnov. Numerical Range and Quasi-Sectorial Contractions. Journal of Mathematical Analysis and Applications, 2010, 366 (1), pp.33-43. 10.1016/j.jmaa.2010.01.036 . hal-00331492

HAL Id: hal-00331492

https://hal.science/hal-00331492

Submitted on 16 Oct 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Draft-fin (Oct.16,2008)

NUMERICAL RANGE AND QUASI-SECTORIAL CONTRACTIONS

YURY ARLINSKIĬ AND VALENTIN ZAGREBNOV

Abstract. We apply a method developed by one of the authors, see [1], to localize the numerical range of *quasi-sectorial* contractions semigroups. Our main theorem establishes a relation between the numerical range of quasi-sectorial contraction semigroups $\{\exp(-tS)\}_{t\geq 0}$, and the maximal sectorial generators S. We also give a new prove of the rate $O(1/n)$ for the operator-norm Euler formula approximation: $\exp(-tS) = \lim_{n\to\infty}(I + tS/n)^{-n}$, $t \geq 0$, for this class of semigroups.

Contents

1. Introduction 1
1.1. Numerical range and the Kato mapping theorem 1
1.2. Sectorial operators and quasi-sectorial contractions 2
2. Operators of the class $C_{\mathfrak{H}}(\alpha)$ 4
3. Numerical range for contractive holomorphic semigroups 7
4. Operator-norm convergence of the Euler formula 11
4.1. A brief history 11
4.2. The von Neumann inequality and its generalizations 12
4.3. Proof of the Theorem 4.1 13
5. Conclusion 14
References 15

1. Introduction

In what follows the Banach algebra of all *bounded* linear operators on a complex Hilbert space $\mathfrak{H}$ is denoted by $\mathcal{L}(\mathfrak{H})$. We denote by $I_{\mathfrak{H}}$ the identity operator in a Hilbert space $\mathfrak{H}$. The *domain*, the *range*, and the *null-space* of a linear operator T are denoted by $\operatorname{dom} T$, $\operatorname{ran} T$, and $\ker T$, respectively. For $T \in \mathcal{L}(\mathfrak{H})$ the operators $\operatorname{Re} T = (T + T^*)/2$ and $\operatorname{Im} T = (T - T^*)/2i$ are said to be the *real* and the *imaginary* parts of T.

1.1. Numerical range and the Kato mapping theorem.

Let $\mathfrak{H}$ be a complex separable Hilbert space and let A be an (*unbounded*) linear operator in $\mathfrak{H}$ with domain $\operatorname{dom} A$.

1991 *Mathematics Subject Classification.* 47A55, 47D03, 81Q10.
Key words and phrases. Operator numerical range; maximal sectorial generators; quasi-sectorial contractions; semigroups on the complex plane.

Definition 1.1. *The set of complex numbers*

$$W(A) := \{(Au, u) \in \mathbb{C} : \ u \in \operatorname{dom} A, \ ||u|| = 1\}$$

is called the numerical range of A, or its field of values.

According to the Hausdorff-Toeplitz theorem, the numerical range is a convex set. We recall also the following properties of the numerical range (see e.g. [14]).

Proposition 1.2. *Let A be a closed operator in $\mathfrak{H}$. Then*

(a) *for any complex number $z \notin \overline{W(A)}$ holds $\ker(A - zI_{\mathfrak{H}}) = \{0\}$ and $\operatorname{ran}(A - zI_{\mathfrak{H}})$ is closed. Moreover, the defect*

$$\operatorname{def}(A - zI_{\mathfrak{H}}) := \dim(\mathfrak{H} \ominus \operatorname{ran}(A - zI_{\mathfrak{H}}))$$

is constant in each connected component of $\mathbb{C} \setminus \overline{W(A)}$.

(b) *If $z \in \mathbb{C} \setminus \overline{W(A)}$ then*

$$||(A - zI_{\mathfrak{H}})^{-1} f|| \leq \frac{1}{\operatorname{dist}(z, W(A))} \, ||f||, \ f \in \operatorname{ran}(A - zI_{\mathfrak{H}}).$$

(c) *If $\operatorname{dom} A$ is dense in $\mathfrak{H}$ and $W(A) \neq \mathbb{C}$, then A is closable.*

Corollary 1.3. *For a bounded operator $A \in \mathcal{L}(\mathfrak{H})$ the spectrum $\sigma(A)$ is a subset of $\overline{W(A)}$.*

For unbounded operator A the relation between spectrum and numerical range is more complicated. We would like to warn, that it may very well happen that $\sigma(A)$ is not contained in $\overline{W(A)}$, but for a closed operator A the essential spectrum $\sigma_{ess}(A)$ is always a subset of $\overline{W(A)}$. The condition $\operatorname{def}(A - zI) = 0, \ z \notin \overline{W(A)}$ in Proposition 1.2 serves to ensure that for those unbounded operators one gets

$$\sigma(T) \subset \overline{W(A)} \ , \tag{1.1}$$

i.e., the same conclusion as in Corollary 1.3 for bounded operators.

In the sequel we need the following numerical range mapping theorem due to Kato [13].

Proposition 1.4. [13]. *Let $f(z)$ be a rational function on $\mathbb{C}$, with $f(\infty) = \infty$. Let for some compact and convex set $E' \subset \mathbb{C}$ the inverse function $f^{-1} : E' \to E \supseteq K$, where K is a convex kernel of E, i.e., is a subset of E such that E is star-shaped with respect to any $z \in K$. If A is bounded operator with $W(A) \subseteq K$, then $W(f(A)) \subseteq E'$.*

Notice that for a convex set E the *maximal* convex kernel $K = E$.

1.2. **Sectorial operators and quasi-sectorial contractions.**

Definition 1.5. [14]. *Let S be a linear operator in a Hilbert space $\mathfrak{H}$. If $\operatorname{Re}(Su, u) \geq 0$ for all $u \in \operatorname{dom} S$, then S is called accretive.*

So, the operator S is accretive if and only if its numerical range is contained in the closed right-half plane of the complex plane. An accretive operator S is called *maximal* accretive (m-accretive) if one of the equivalent conditions is satisfied:

- the operator S has no accretive extensions in $\mathfrak{H}$;
- the resolvent set $\rho(S)$ is nonempty;
- the operator S is densely defined and closed, and S^* is accretive operator.

The resolvent set $\rho(S)$ of m-accretive operator contains the open left half plane and

$$||(S - zI_{\mathfrak{H}})^{-1}|| \leq \frac{1}{|\operatorname{Re} z|} \ , \ \operatorname{Re} z < 0.$$

It is well known [14] that if S is m-accretive operator, then the one-parameter semigroup

$$T(t) = \exp(-tS),\ t \geq 0$$

is contractive. Conversely, if the family $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded operators in a Hilbert space $\mathfrak{H}$, with $T(0) = I_{\mathfrak{H}}$ (C_0-semigroup) and $T(t)$ is a contraction for each t, then the generator S of $T(t)$:

$$Su := \lim_{t\to +0} \frac{(I_{\mathfrak{H}} - T(t))u}{t}\ ,\ u \in \operatorname{dom} S\ ,$$

where domain is defined by condition:

$$\operatorname{dom} S = \left\{ u \in \mathfrak{H} : \lim_{t\to +0} \frac{(I_{\mathfrak{H}} - T(t))u}{t} \quad \text{exists} \right\}\ ,$$

is an m-accretive operator in $\mathfrak{H}$. Then the Euler formula approximation:

$$T(t) = s - \lim_{n\to\infty} \left(I_{\mathfrak{H}} + \frac{t}{n} S \right)^{-n}\ ,\ t \geq 0 \tag{1.2}$$

holds in the strong operator topology, see e.g. [14].

Definition 1.6. [14]. *Let $\alpha \in [0, \pi/2)$ and let*

$$\mathcal{S}(\alpha) := \{z \in \mathbb{C} : |\arg z| \leq \alpha\}$$

be a sector on the complex plane $\mathbb{C}$ with the vertex at the origin and the semi-angle α.

A linear operator S in a Hilbert space $\mathfrak{H}$ is called sectorial with vertex at $z = 0$ and the semi-angle α if $W(S) \subseteq \mathcal{S}(\alpha)$.

If S is m-accretive and sectorial with vertex at $z = 0$ and the semi-angle α then it is called m-sectorial with vertex at the origin and with the semi-angle α. For short we call these operators m-α-sectorial. The resolvent set of m-α-sectorial operator S contains the set $\mathbb{C} \setminus \mathcal{S}(\alpha)$ and

$$||(S - zI_{\mathfrak{H}})^{-1}|| \leq \frac{1}{\operatorname{dist}(z, \mathcal{S}(\alpha))}\ ,\ z \in \mathbb{C} \setminus \mathcal{S}(\alpha).$$

It is well-known [14] that

- a C_0-semigroup $T(t) = \exp(-tS), t \geq 0$ has *contractive* and *holomorphic* continuation into the sector $\mathcal{S}(\pi/2 - \alpha)$ if and only if the generator S is m-α-sectorial operator,
- the *sesquilinear* form: (Su, v) , $u, v \in \operatorname{dom} S$ is closable.

Denote by $\mathfrak{S}[\cdot,\cdot]$ the closure of the form (Su, v) and by $\mathcal{D}[\mathfrak{S}]$ its domain. By the *first representation theorem* [14] the operator S is *uniquely* associated with the form $\mathfrak{S}[\cdot,\cdot]$ in the following sense:

$$(Su, v) = \mathfrak{S}[u, v] \quad \text{for all} \quad u \in \operatorname{dom} S \quad \text{and for all} \quad v \in \mathcal{D}[\mathfrak{S}].$$

The important relation between the one-parameter semigroups $\{T(t) := \exp(-tS)\}_{t\geq 0}$ generated by the m-α-sectorial operators S and the corresponding closed sesquilinear forms is established in [3]:

$$u \in \mathcal{D}[\mathfrak{S}] = \left\{ u \in \mathfrak{H} : \frac{d}{dt}(T(t)u, u)\Big|_{t=+0} \text{ exits} \right\},$$

and one has that

$$\frac{d}{dt}(T(t)u, u)\Big|_{t=+0} = -\mathfrak{S}[u, u]\ ,\ u \in \mathcal{D}[\mathfrak{S}]\ . \tag{1.3}$$

Notice that in this case of the Euler approximation (1.2) converges to the semigroup in the *operator-norm* topology [7], [8], [19].

Definition 1.7. [8]. *For any $\alpha \in [0, \pi/2)$ we define in the complex plane $\mathbb{C}$ a closed domain:*

$$D_\alpha := \{z \in \mathbb{C} : |z| \leq \sin\alpha\} \cup \{z \in \mathbb{C} : |\arg(1-z)| \leq \alpha \text{ and } |z-1| \leq \cos\alpha\} \ . \tag{1.4}$$

This is a convex subset of the unit disc $\mathbb{D} = D_{\alpha=\pi/2}$, with the "angle" (in contrast to the "tangent") touching of the boundary $\partial\mathbb{D}$ at the only one point $z = 1$. It is evident that $D_\alpha \subset D_{\beta>\alpha}$.

Definition 1.8. [8]. *A contraction C on the Hilbert space $\mathfrak{H}$ is called quasi-sectorial with semi-angle $\alpha \in [0, \pi/2)$, if its numerical range $W(C) \subseteq D_\alpha$.*

It is evident that if operator C is a *quasi-sectorial* contraction, then $I - C$ is an *m-sectorial* operator with vertex $z = 0$ and semi-angle α. The limits $\alpha = 0$ and $\alpha = \pi/2$ correspond, respectively, to non-negative (i.e. *self-adjoint*) and to some *general* contraction.

Remark 1.9. [8]. *Notice that the resolvent family $\{(I_\mathfrak{H} + t\,S)^{-1}\}_{t\geq 0}$ of the m-α-sectorial operator S, gives the first non-trivial example of a quasi-sectorial contractions, if one considers the semi-angles $\alpha \in [0, \pi/3)$. Below (see Section 2) we show that it can be extended to $\alpha \in [0, \pi/2)$.*

The definition of quasi-sectorial contractions was motivated in [8] by the lifting of the Trotter-Kato product formula and the Chernoff theory of semigroup approximation [12] to the operator-norm topology.

Namely, using properties of quasi-sectorial contractions established in [8] for the case of m-α-sectorial generator, it was proved that there is *operator-norm* convergence of the Euler semigroup approximation (1.2) and of the Trotter product formula, see [7], [8], [16], [6], [19]. Theorem 2.1 from [8] states that the operators $T(t) = \exp(-tS)$ are quasi-sectorial contractions with $W(T(t)) \subset D_\alpha$ for all $t \geq 0$ and $\alpha \in [0, \pi/2)$. Here operator S stands for m-α-sectorial generator. As it is indicated in [18], the proof in [8] has a flaw. In [18] it was corrected, but only for the semi-angles $\alpha \in [0, \pi/4]$.

Our main Theorem 3.4 establishes a quite accurate relation between m-α-sectorial generators and the numerical range of the corresponding one-parameter contraction semigroups. It improves the recent result [18] from $\alpha \in [0, \pi/4]$ to $\alpha \in [0, \pi/2)$. To this end we use in the next section some nontrivial results due to [1]-[4], concerning a class of operator contractions and of semigroups on the complex plane. Besides that we use recent results related to the generalizations of the famous von Neumann inequality [15], obtained in [5], [9], [10], [11], in order to give a new prove that in fact the Euler formula (1.2) converges in the operator-norm with the rate $O(1/n)$.

2. Operators of the class $C_\mathfrak{H}(\alpha)$

A fundamental for us will be the class of contractions introduced for the first time in [1] and studied in [1]-[2]:

Definition 2.1. [1]. *Let $\alpha \in (0, \pi/2)$. We say that a bounded operator $T \in \mathcal{L}(\mathfrak{H})$ belongs to the class $C_\mathfrak{H}(\alpha)$ if*

$$||T\,\sin\alpha \pm iI_\mathfrak{H}\cos\alpha|| \leq 1. \tag{2.1}$$

It is clear, that the class $C_\mathfrak{H}(\alpha)$ is a convex and closed (with respect to the strong operator topology) set, which is intersection of two closed operator balls corresponding to $\pm$. Moreover, by virtue of (2.1) one immediately concludes that:

$$T \in C_\mathfrak{H}(\alpha) \iff -T \in C_\mathfrak{H}(\alpha) \iff T^* \in C_\mathfrak{H}(\alpha),$$

and that condition (2.1) is equivalent to the following criterium:

$$\tan\alpha\,(||f||^2 - ||Tf||^2) \geq 2|\mathrm{Im}\,(Tf,f)|,\ f \in \mathfrak{H}. \tag{2.2}$$

This inequality implies that the operator T is a contraction. Together with Definition 1.1, it also proves that $T \in C_{\mathfrak{H}}(\alpha)$ is equivalent to the statement that $(I-T^*)(I+T)$ is a bounded m-α-sectorial operator.

According to (2.2) it is natural to identify $C_{\mathfrak{H}}(\alpha = 0)$ with the set of all *self-adjoint* contractions whereas $C_{\mathfrak{H}}(\alpha = \pi/2)$ is the set of general contractions on $\mathfrak{H}$.

Now following [4] we define for $\alpha \in (0, \pi/2)$ the family of subsets of the complex plane $\mathbb{C}$:

$$C(\alpha) = \{z \in \mathbb{C} : |z\sin\alpha \pm i\cos\alpha| \leq 1\} = \left\{z \in \mathbb{C} : (1-|z|^2)\tan\alpha \geq 2|\mathrm{Im}\,z|\right\}, \tag{2.3}$$

cf. definitions (2.1), (2.2). Then, similar to (2.1), each set $C(\alpha)$ is the intersection of *two* closed disks of the complex plane which is contained in the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. From (2.3) it is clear that $C(\alpha = 0) = [-1, 1]$.

Remark 2.2. *By virtue of definitions (2.1), (2.2) and (2.3) one gets that $T \in C_{\mathfrak{H}}(\alpha)$ implies for the numerical range: $W(T) \subseteq C(\alpha)$. Then one has:*

$$||\mathrm{Im}\,T|| \leq \tan(\alpha/2)\ . \tag{2.4}$$

Notice that $C(\alpha) \setminus D_\alpha \neq \emptyset$, i.e. in general the operator T is not quasi-sectorial.

Besides, the operator class $C_{\mathfrak{H}}(\alpha)$ has several interesting properties. In particular the following one [1]:

Proposition 2.3. [1]. *Let T_1 and T_2 belong to the class $C_{\mathfrak{H}}(\alpha)$. Then the operator*

$$(T_1T_2 + T_2T_1)/2$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$. In particular, if T_1 and T_2 are two commuting operators from the class $C_{\mathfrak{H}}(\alpha)$, then the product $T = T_1\,T_2$ also belongs to $C_{\mathfrak{H}}(\alpha)$.

Remark 2.4. *Notice that the set $C(\alpha)$ inherits a similar property: for each $\alpha \in (0, \pi/2)$ it forms a multiplicative semigroup of complex numbers in the plane $\mathbb{C}$. Detailed properties of these semigroups have been studied in* [4].

Now we are in position to establish a connection between set of contractions $C_{\mathfrak{H}}(\alpha)$ and m-α-sectorial operators.

Proposition 2.5. [1]. *If S is m-α-sectorial operator, then*

$$T = (I_{\mathfrak{H}} - S)(I_{\mathfrak{H}} + S)^{-1}$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$ and conversely, if $T \in C_{\mathfrak{H}}(\alpha)$, and $\ker(I_{\mathfrak{H}} + T) = \{0\}$, then $S = (I_{\mathfrak{H}} - T)(I_{\mathfrak{H}} + T)^{-1}$ is m-α-sectorial operator.

Now let S be m-α-sectorial operator and let $\lambda > 0$. Then λS is also m-α-sectorial. By Proposition 2.5 the operator $U(\lambda) = (I_{\mathfrak{H}} - \lambda S)(I_{\mathfrak{H}} + \lambda S)^{-1}$ belongs to the class $C_{\mathfrak{H}}(\alpha)$. Let us put

$$F(\lambda) := (I_{\mathfrak{H}} + \lambda S)^{-1} = \frac{1}{2}(U(\lambda) + I_{\mathfrak{H}}). \tag{2.5}$$

Since the operators $U(\lambda)$ and $I_{\mathfrak{H}}$ belong to the class $C_{\mathfrak{H}}(\alpha)$ and since it is a convex set, the operator $F(\lambda)$ is also in the class $C_{\mathfrak{H}}(\alpha)$. Hence, by Remark 2.2 one obtains $W(F(\lambda)) \subseteq C(\alpha)$.

Remark 2.6. *Following the arguments of* [8],[18] *we can localize the numerical range* $W(F(\lambda))$ *even more accurate, cf. Remark 1.9. Since for any* $u \in \mathfrak{H}$ *we have:*

$$(u, F(\lambda)u) = (v_\lambda, v_\lambda) + \lambda(Sv_\lambda, v_\lambda) \in \mathcal{S}(\alpha) \ ,$$

where $v_\lambda := F(\lambda)u$*, it follows that*

$$W(F(\lambda)) \subseteq \mathcal{S}(\alpha) \cap C(\alpha) \ ,$$

for $\lambda > 0$ *and* $\alpha \in [0, \pi/2)$*. Moreover, since* $U(\lambda) = 2F(\lambda) - I_{\mathfrak{H}} \in C_{\mathfrak{H}}(\alpha)$*, see (2.5), we find that*

$$\|(F(\lambda) - I_{\mathfrak{H}}/2)\sin\alpha \pm iI_{\mathfrak{H}}(\cos\alpha)/2\| \leq 1/2 \ . \tag{2.6}$$

For the numerical range this implies:

$$W(F(\lambda)) \subseteq L(\alpha) := \{\zeta \in \mathbb{C} : |(\zeta - 1/2)\sin\alpha \pm (i\cos\alpha)/2| \leq 1/2\} \ , \quad \textit{for} \quad \lambda > 0 \ . \tag{2.7}$$

Notice that $L(\alpha) \subset \mathcal{S}(\alpha) \cap C(\alpha)$*, see Figure 1.*

Now, we follow essentially the line of reasoning of [1] to establish the one-to-one correspondence between m-α-sectorial generators and contraction semigroups of the class $C_{\mathfrak{H}}(\alpha)$, cf. Remark 2.2 about relation to quasi-sectorial contractions.

Theorem 2.7. *If* S *is* m-α*-sectorial operator in a Hilbert space* $\mathfrak{H}$*, then the corresponding semigroup* $T(t) = \exp(-tS) \in C_{\mathfrak{H}}(\alpha)$*, for all* $t \geq 0$*. Conversely, let* $T(t) = \exp(-tS)$ *for* $t \geq 0$ *be a* C_0*-semigroup of contractions on a Hilbert space* $\mathfrak{H}$*. If* $T(t) \in C_{\mathfrak{H}}(\alpha)$ *for non-negatives* t *in neighborhood of* $t = 0$*, then the generator* S *is an* m-α*-sectorial operator.*

Proof. Let S be a m-α-sectorial operator and let $\lambda \geq 0$. By (2.5) the operator $F(\lambda) = (I_{\mathfrak{H}} + \lambda S)^{-1}$ belongs to the class $C_{\mathfrak{H}}(\alpha)$. Therefore, by Proposition 2.3 for each $t \geq 0$ and any natural number n the operator

$$F^n\left(\frac{t}{n}\right) = \left(I_{\mathfrak{H}} + \frac{t}{n}S\right)^{-n}$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$. Taking in account that the set $C_{\mathfrak{H}}(\alpha)$ is closed with respect to the strong operator topology, from the Euler formula (1.2) we get that strong limit $T(t) = \exp(-tS)$ also belongs to the class $C_{\mathfrak{H}}(\alpha)$.

Now suppose that semigroup $T(t) = \exp(-tS) \in C_{\mathfrak{H}}(\alpha)$ for $t \in [0, \delta)$, where $\delta > 0$. Define operator family:

$$B_\pm(t) := T(t)\ \sin\alpha \pm i\cos\alpha I_{\mathfrak{H}},\ t \geq 0.$$

Since $B_\pm(0) = (\sin\alpha \pm i\cos\alpha)I_{\mathfrak{H}}$ and $T(t) \in C_{\mathfrak{H}}(\alpha)$ for $t \in [0, \delta)$, we get

$$||B_\pm(t)f||^2 \leq ||f||^2 = ||B_\pm(0)||f||^2,\ t \in [0, \delta),\ f \in \mathfrak{H}.$$

Since

$$||B_\pm(t)f||^2 = \sin^2\alpha||T(t)f||^2 + \cos^2\alpha||f||^2 \pm 2\sin\alpha\ \cos\alpha\,\mathrm{Im}\,(T(t)f, f),\ f \in \mathfrak{H},$$

for all $f \in \mathrm{dom}\, S$ we have:

$$\frac{d}{dt}\left(||B_\pm(t)f||^2\right)\Big|_{t=+0} = -2\sin^2\alpha\,\mathrm{Re}\,(Sf, f) \pm 2\sin\alpha\ \cos\alpha\,\mathrm{Im}\,(Sf, f) \leq 0.$$

Thus, $W(S) \subseteq \mathcal{S}(\alpha)$. But since operator S is m-accretive, it is m-α-sectorial [14]. □

3. Numerical range for contractive holomorphic semigroups

From Theorem 2.7 it follows, in particular, that for m-α-sectorial generator S the numerical range of the corresponding contraction semigroup

$$W(\exp(-tS)) \subseteq C(\alpha) \quad \text{for all} \quad t \geq 0 \ .$$

But as we warranted in Remark 2.2 it does not imply that this semigroup is quasi-sectorial contraction. It was discovered in [8] that the conformal mapping : $z \mapsto z^2$, together with the Kato numerical range theorem (Proposition 1.4) play a special rôle in the theory of quasi-sectorial contractions.

Definition 3.1. *Let $\alpha \in [0, \pi/2)$. We define a domain:*

$$\Omega(\alpha) := \left\{ z^2 : z \in C(\alpha) \right\} . \tag{3.1}$$

So, if $f(z) = z^2$, then $\Omega(\alpha) = f(C(\alpha))$. Since (see Remark 2.4) $C(\alpha)$ is a *multiplicative semigroup*, we obtain that $\Omega(\alpha) \subseteq C(\alpha)$, and that the subset $\Omega(\alpha)$ is in turn a multiplicative semigroup [4].

From (2.3) and Proposition 2.3 it follows then that for any $\alpha \in (0, \pi/2)$ the set (3.1) has representation:

$$\Omega(\alpha) = \left\{ z \in \mathbb{C} : |\sqrt{z} \sin\alpha \pm i \cos\alpha| \leq 1 \right\} = \left\{ z \in \mathbb{C} : 2|\mathrm{Im}\sqrt{z}| \leq (1 - |z|) \tan\alpha \right\} , \tag{3.2}$$

with the limiting cases: $\Omega(\alpha = 0) = [0, 1]$ and $\Omega(\alpha = \pi/2) = \overline{\mathbb{D}}$.

Lemma 3.2. *The set $\Omega(\alpha)$ (3.1) is convex and $\Omega(\alpha) \subseteq D_\alpha$.*

Proof. Let $C_+(\alpha) := \{z \in C(\alpha) : \mathrm{Im}\, z \geq 0\}$. Then clearly, $\Omega(\alpha) = f(C_+(\alpha))$, where $f(z) = z^2$. Denote

$$\Gamma(\alpha) = \partial C_+(\alpha) \setminus (-1, 1).$$

Then $\partial\Omega(\alpha) = f(\Gamma(\alpha))$. Since

$$\Gamma(\alpha) = \left\{ z : z = \frac{e^{it} - i\cos\alpha}{\sin\alpha},\ t \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha\right] \right\},$$

the boundary $\partial\Omega(\alpha)$ can be parameterized as follows:

$$\partial\Omega(\alpha) = \left\{ z = \zeta(t) = \frac{(e^{it} - i\cos\alpha)^2}{\sin^2\alpha},\ t \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha\right] \right\}. \tag{3.3}$$

Put $x = x(t) := \mathrm{Re}\,\zeta(t)$, $y = y(t) := \mathrm{Im}\,\zeta(t)$. Then, since

$$\frac{d^2y}{dx^2} = \frac{y''(t)x'(t) - y'(t)x''(t)}{(x'(t))^3} ,$$

we obtain

$$\frac{d^2y}{dx^2} = -\frac{\mathrm{Im}\,(\zeta'(t)\overline{\zeta''(t)})}{(\mathrm{Re}\,\zeta'(t))^3} .$$

Further, by explicit calculations we get:

$$\zeta'(t) = \frac{2ie^{it}(e^{it} - i\cos\alpha)}{\sin^2\alpha} \quad , \quad \zeta''(t) = \frac{2ie^{it}(2ie^{it} + \cos\alpha)}{\sin^2\alpha} ,$$

$$-\mathrm{Im}\,(\zeta'(t)\overline{\zeta''(t)}) = \frac{4(2 + \cos^2\alpha - 3\cos\alpha\sin t)}{\sin^4\alpha} \quad , \quad \mathrm{Re}\,\zeta'(t) = \frac{2\cos t(\cos\alpha - 2\sin t)}{\sin^2\alpha} .$$

For $t \in [\pi/2 - \alpha, \pi/2 + \alpha]$, by the estimates

$$\cos^2\alpha - 3\cos\alpha\sin t + 2 \geq 2\cos^2 - 3\cos\alpha + 2 = (2 - \cos\alpha)(1 - \cos\alpha) > 0,$$

we obtain $-\mathrm{Im}\,(\zeta'(t)\overline{\zeta''(t)}) > 0$.

Since $\cos\alpha - 2\sin t \leq -\cos\alpha < 0$, then for $t \in [\pi/2-\alpha, \pi/2+\alpha]$, we get that

$$\frac{d^2y}{dx^2} < 0 \quad \text{for} \quad t \in \left[\frac{\pi}{2}-\alpha, \frac{\pi}{2}\right) \ ,$$
$$\frac{d^2y}{dx^2} > 0 \quad \text{for} \quad t \in \left(\frac{\pi}{2}, \frac{\pi}{2}+\alpha\right] \ ,$$

which implies that the set $\Omega(\alpha)$ is convex.

Since the mapping: $z \mapsto z^2$ is conformal and $\Omega(\alpha)$ is convex, the proof of the second part of the lemma follows from the estimate

$$-(\tan\alpha/2)^2 \leq \operatorname{Re}\zeta(t) \ , \tag{3.4}$$

where $(\tan\alpha/2)^2 \leq \sin\alpha$ for $\alpha \in (0, \pi/2)$, see (1.4) and (3.3). □

Notice that in view of relation:

$$y(t) = \frac{\sin 2t - 2\cos\alpha\cos t}{\sin^2\alpha} \ ,$$

one obtains

$$\max_{z\in\Omega(\alpha)} |\operatorname{Im} z| = \frac{\sin 2\gamma - 2\cos\alpha\sin\gamma}{\sin^2\alpha} \ , \tag{3.5}$$

where

$$\sin\gamma = \frac{\cos\alpha + \sqrt{\cos^2\alpha + 8}}{4} \ , \quad \gamma \in (0, \pi/2) \ . \tag{3.6}$$

Since $\Omega(\alpha) \subset C(\alpha)$, the number in the right-hand side of (3.5) is less than $\tan(\alpha/2)$.

Lemma 3.3. *Let $\alpha \in [0, \pi/2)$ and let $T \in C_{\mathfrak{H}}(\alpha)$. Then $W(T^{2n}) \subseteq \Omega(\alpha)$ for all natural numbers n.*

Proof. We apply the Kato numerical range mapping theorem (Proposition 1.4) for $f(z) = z^2$ with $E' = \Omega(\alpha)$. Then by Definition 3.1 we have $E := f^{-1}(\Omega(\alpha)) = C(\alpha)$. Since $C(\alpha)$ is a convex set, its maximal convex kernel K *coincides* with $C(\alpha)$. By virtue of Proposition 2.3 from $T \in C_{\mathfrak{H}}(\alpha)$ it follows that for all natural n one gets $T^n \in C_{\mathfrak{H}}(\alpha)$. Then by Remark 2.2 we obtain $W(T^n) \subseteq C(\alpha) = K$. Applying now the Kato mapping theorem for $f(z) = z^2$ we obtain $W(T^{2n}) \subseteq \Omega(\alpha)$. □

Now we are in position to proof the main theorem of the present paper.

Theorem 3.4. (1) *Let S be m-α-sectorial operator. Then*

$$W(\exp(-tS)) \subseteq \Omega(\alpha) \ , \ t \geq 0. \tag{3.7}$$

In particular, $\{\exp(-tS)\}_{t\geq 0}$ is the quasi-sectorial contraction semigroup.

(2) *The inverse is also true. Let $\{T(t) := \exp(-tS)\}_{t\geq 0}$ be a C_0-semigroup on a Hilbert space $\mathfrak{H}$. If in some neighborhood of $t = 0$ the numerical range: $W(T(t \geq 0)) \subseteq \Omega(\alpha)$ for some $\alpha \in [0, \pi/2)$, then the generator S is an m-α-sectorial operator.*

Proof. (1) By Theorem 2.7 the operators $T(t) = \exp(-tS)$ belong to the class $C_{\mathfrak{H}}(\alpha)$ for all $t \geq 0$, but $T(t) = T^2(t/2)$, then by Lemma 3.3 we obtain $W(T(t)) \subseteq \Omega(\alpha)$. Since Lemma 3.2 implies $\Omega(\alpha) \subset D_\alpha$, the operators $T(t)$ are quasi-sectorial contractions.

(2) Fix $u \in \operatorname{dom} S$, $||u|| = 1$. For $t \in [0, \delta]$ we define two functions:

$$G_\pm(t) := |\sin\alpha\sqrt{(T(t)u, u)} \pm i\cos\alpha|^2 \ .$$

Then $G_\pm(0) = 1$. The condition $W(T(t)) \subseteq \Omega(\alpha)$ yields (see (3.2)) that $G_\pm(t) \leq G_\pm(0)$, $t \in [0, \delta]$, which implies:

$$\frac{d}{dt}\left(G_\pm(t)\right)\Big|_{t=+0} \leq 0 \ .$$

Therefore we get that

$$\lim_{t\to+0} \frac{1 - G_\pm(t)}{t} \geq 0. \tag{3.8}$$

Further we use the following identities:

$$1 - G_\pm(t) = \sin^2\alpha(1 - |(T(t)u, u)|) \mp 2\sin\alpha\,\cos\alpha\,\mathrm{Im}\,\sqrt{(T(t)u, u)} =$$
$$= \frac{\sin^2\alpha\left(1 - (T(t)u, u) + (T(t)u, u)(1 - \overline{(T(t)u, u)})\right)}{1 + |(T(t)u, u|} \mp 2\sin\alpha\,\cos\alpha\,\mathrm{Im}\,\sqrt{(T(t)u, u)},$$

$$\frac{1 - G_\pm(t)}{t} = \frac{\sin^2\alpha\left(1 - (T(t)u, u) + (T(t)u, u)(1 - \overline{(T(t)u, u)})\right)}{t(1 + |(T(t)u, u|)} \mp$$
$$\mp 2\sin\alpha\,\cos\alpha\,\mathrm{Im}\left(\frac{(T(t)u, u) - 1}{t(\sqrt{(T(t)u, u)} + 1)}\right) \ .$$

Since by (1.3) one has:

$$\lim_{t\to+0} \frac{1 - (T(t)u, u)}{t} = (Su, u) \ ,$$

and by (3.8) we get

$$\lim_{t\to+0} \frac{1 - G_\pm(t)}{t} = \sin^2\alpha\,\mathrm{Re}\,(Su, u) \pm \sin\alpha\,\cos\alpha\,\mathrm{Im}\,(Su, u) \geq 0 \ .$$

The last estimate implies that $W(S) \subseteq \mathcal{S}(\alpha)$, i.e., S is m-α-sectorial operator. □

Thus, the two equivalent conditions: $\exp(-tS) \in C_{\mathfrak{H}}(\alpha)$ (see Theorem 2.7) and $W(\exp(-tS)) \subseteq \Omega(\alpha)$ *completely* characterize m-α-sectorial operators S. Since (3.7) yields $W(\exp(-tS)) \subset D_\alpha$, the statement in Theorem 3.4(1) is extension of Theorem 2.1 (proven in [18] for $\alpha \in [0, \pi/4)$) to the whole class of the m-α-sectorial generators: $\alpha \in [0, \pi/2)$.

Notice that taking into account the multiplicative semigroup property of the set $\Omega(\alpha)$ and Theorem 3.4 we get by consequence the following inequalities:

$$\left|\sin\alpha\left(\prod_{k=1}^{n}(\exp(-t_kS_k)u_k, u_k))\right)^{1/2} \pm i\cos\alpha\right| \leq 1$$

for arbitrary m-α-sectorial operators $S_1, \ldots S_n$, any non-negative numbers $t_1, \ldots, t_n$, and arbitrary normalized vectors $u_1, \ldots, u_n$ from $\mathfrak{H}$. By semi-group property:
$\exp(-t_1S)\exp(-t_2S) = \exp(-(t_1 + t_2)S)$. Hence we also get

$$|\sin\alpha\sqrt{(\exp(-t_1S)u, \exp(-t_2S^*)u)} \pm i\cos\alpha| \leq 1$$

for any $t_1, t_2 \geq 0$ and any $u \in \mathfrak{H}$, $||u|| = 1$.

Remark that since $\mathrm{Re}\,z \geq -\tan^2(\alpha/2)$ for all $z \in \Omega(\alpha)$ and $W(\exp(-tS)) \subseteq \Omega(\alpha)$, we obtain:

$$\mathrm{Re}\,\exp(-tS) \geq -\tan^2(\alpha/2)\ I_{\mathfrak{H}}\ ,\ t \geq 0\ .$$

The inclusion $W(\exp(-tS)) \subseteq \Omega(\alpha)$ also implies (see (3.5)) that

$$||\mathrm{Im}\,\exp(-tS)|| \leq \frac{\sin 2\gamma - 2\cos\alpha\sin\gamma}{\sin^2\alpha} < \tan(\alpha/2)\ ,\ t \geq 0\ ,$$

where γ satisfies (3.6). By virtue of relation $\mathrm{Re}\, T^2 = (\mathrm{Re}\, T)^2 - (\mathrm{Im}\, T)^2$ (valid for any bounded operator T) and since one has that $\mathrm{Re}\, \exp(-tS) = (\mathrm{Re}\, \exp(-tS/2))^2 - (\mathrm{Im}\, \exp(-tS/2))^2$, we obtain:

$$\mathrm{Re}\, \exp(-tS) \geq -\left(\frac{\sin 2\gamma - 2\cos\alpha \sin\gamma}{\sin^2\alpha}\right)^2 I_{\mathfrak{H}} \ > -\tan^2(\alpha/2)\ I_{\mathfrak{H}}\ ,\ t \geq 0\ .$$

Now it is useful to define also the following subset of $C(\alpha)$:

$$\begin{aligned} &Q(\alpha) := \{z \in C(\alpha) : |z \sin\alpha - \cos\alpha| \leq 1\} = \\ &= \{z \in \mathbb{C} : 2|\mathrm{Im}\, z| \leq (1-|z|^2)\tan\alpha\} \cap \{z \in \mathbb{C} : -2\mathrm{Re}\, z \leq (1-|z|^2)\tan\alpha\}\ . \end{aligned} \tag{3.9}$$

So, $Q(\alpha)$ is the intersection of *three* closed disks, cf. (2.3).

Proposition 3.5. *The set $Q(\alpha)$ is a multiplicative semigroup on $\mathbb{C}$.*

Proof. First we note that $Q(\alpha)$ contains the set $C_+(\alpha) := \{z \in C(\alpha) : \mathrm{Re}\, z \geq 0\}$. Moreover, since $\mathrm{Re}\, iz = -\mathrm{Im}\, z$, by (3.9) we obtain also that the set

$$B(\alpha) := \{z \in C(\alpha) : iz \in C(\alpha)\}\ ,$$

has a non-empty intersection with $Q(\alpha)$, such that

$$B(\alpha) \cap Q(\alpha) = B(\alpha) \ \text{ and } \ Q(\alpha) \setminus B(\alpha) \subset C_+(\alpha)\ . \tag{3.10}$$

In [4] it is shown that the set $B(\alpha)$ is the *ideal* of the multiplicative semigroup $C(\alpha)$, i.e., $z\,\xi \in B(\alpha)$ for all $z \in B(\alpha)$ and all $\xi \in C(\alpha)$. Let $z, \xi \in Q(\alpha) \setminus B(\alpha)$. Since $B(\alpha)$ has the properties (3.10), then

$$\arg z, \arg \xi \in (-\pi/4, \pi/4).$$

Finally, since $z\,\xi \in C(\alpha)$ and $\arg(z\,\xi) \in (-\pi/2, \pi/2)$, we obtain $z\,\xi \in Q(\alpha)$. □

Notice that in particular Proposition 3.5 yields:

$$z \in Q(\alpha) \Rightarrow z^n \in Q(\alpha).$$

Moreover, if $\varphi_n(z) = z^n$, then the set $\varphi_n(Q(\alpha))$ is also a multiplicative semigroup and

$$\varphi_n(Q(\alpha)) \subset Q(\alpha).$$

Since by (2.3) and (3.9) one gets that

$$C(\alpha) = Q(\alpha) \cup \{-Q(\alpha)\},$$

where $\{-Q(\alpha)\} := \{z : -z \in Q(\alpha)\}$, by (3.1) and (3.10) we have

$$\Omega(\alpha) = f(Q(\alpha))$$

when $f(z) = z^2$. Thus, we obtain:

$$\Omega(\alpha) \subset Q(\alpha). \tag{3.11}$$

Remark 3.6. *Another way to check* (3.11) *is the following argument. Let $\xi = z^2$, where $z \in C(\alpha)$. Then $\xi \in \Omega(\alpha)$, see (3.1). Since* $\mathrm{Re}\, \xi = (\mathrm{Re}\, z)^2 - (\mathrm{Im}\, z)^2$ *and* $(\mathrm{Im}\, z)^2 \leq |\mathrm{Im}\, z|$, *we also get*

$$\begin{aligned} -2\mathrm{Re}\, \xi = -2(\mathrm{Re}\, z)^2 + 2(\mathrm{Im}\, z)^2 \leq 2|\mathrm{Im}\, z| \leq \tan\alpha(1-|z|^2) = \\ = \tan\alpha(1-|\xi|) \leq \tan\alpha(1-|\xi|^2)\ , \end{aligned}$$

which means by (3.9) that $\Omega(\alpha) \subset Q(\alpha)$.

Remark 3.7. *Let D_α be the set (1.4) introduced in* [8]. *From definition (3.9) one deduces that $Q(\alpha) \subset D_\alpha$, as well as that D_α is not a subset of $C(\alpha)$, i.e., $D_\alpha \cap C(\alpha) \neq D_\alpha$. In addition, by virtue of (2.7) and (3.2) we obtain: $L(\alpha) \subset \Omega(\alpha)$, where $L(\alpha)$ is defined by (2.7). So, for all $\alpha \in [0, \pi/2)$ and $F(\lambda) = (I_{\mathfrak{H}} + \lambda S)^{-1}$, $\lambda > 0$ we have the following inclusions* (*see Figure 1*):

$$L(\alpha) \subset \Omega(\alpha) \subset Q(\alpha) \subset D_\alpha \ ,$$
$$W(F(\lambda)) \subseteq L(\alpha) \ ,$$
$$W(F^n(\lambda)) \subseteq C(\alpha) \quad \textit{and} \quad W(F^{2n}(\lambda)) \subseteq \Omega(\alpha) \quad \textit{for all} \quad n \in \mathbb{N} \ ,$$

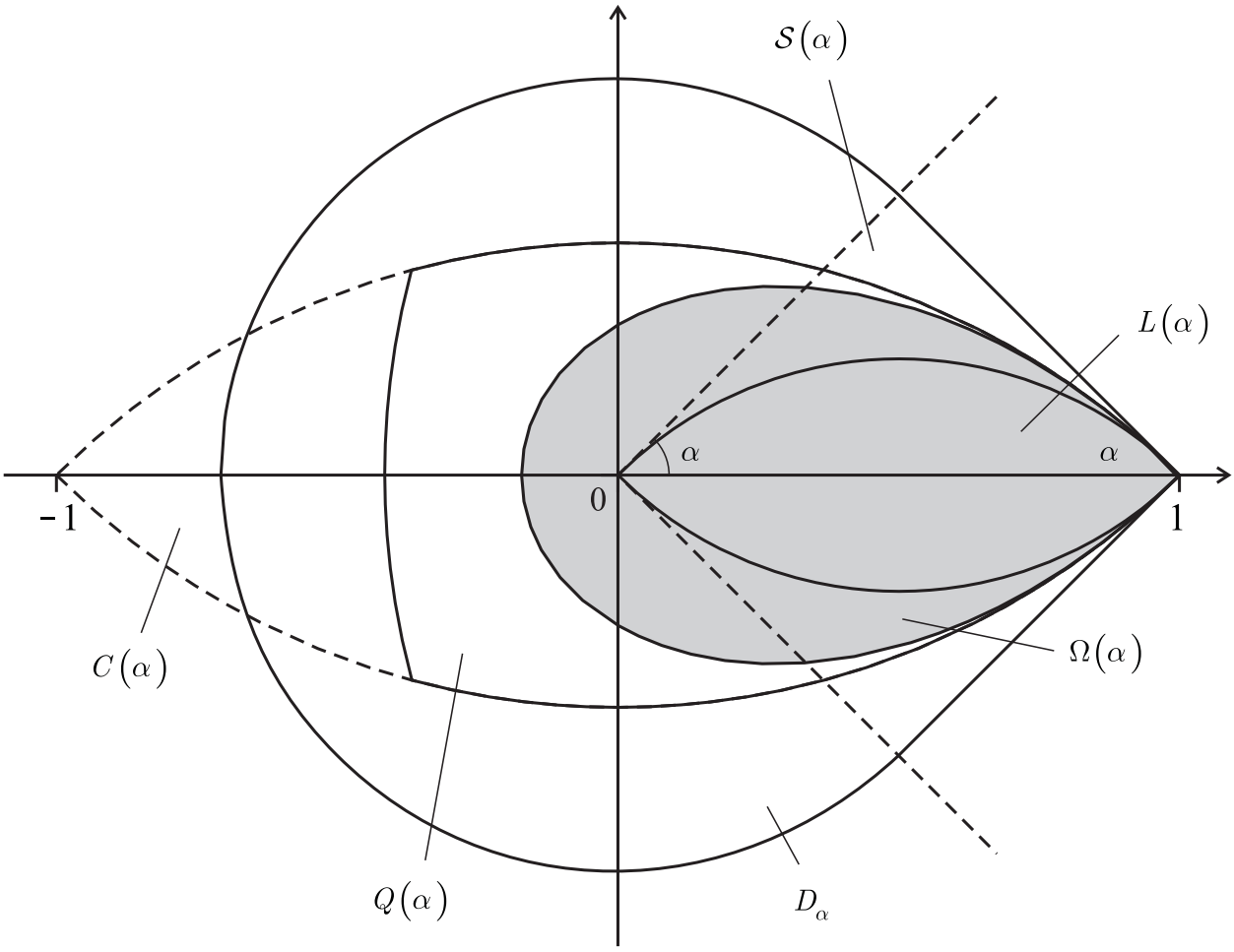

Figure 1

4. Operator-norm convergence of the Euler formula

4.1. **A brief history.** Theorem 5.1 in [8] states that

$$\left\|(I_{\mathfrak{H}} + tS/n)^{-n} - \exp(-tS)\right\| = O\left(\ln n/n\right) \tag{4.1}$$

for m-α-sectorial boundedly invertible operator S in $\mathfrak{H}$ and for all $t \in S_{\pi/2-\alpha}$. The proof of (4.1) is essentially based on two estimates for a quasi-sectorial contraction C, i.e. $W(C) \subseteq D_\alpha$, also established in [8]. The first is

$$\left\|C^n - C^{n+1}\right\| \leq \frac{K}{n+1} \tag{4.2}$$

for all $n \in \mathbb{N}$ and for some $K > 0$ depending on α, and the second one is

$$\left\|C^n - \exp\left(n(C - I_{\mathfrak{H}})\right)\right\| = O\left(1/n^{1/3}\right) \ . \tag{4.3}$$

Then application of (4.2) and (4.3) for the particular case of the operator $C = (I_{\mathfrak{H}} + tS/n)^{-1}$ leads to (4.1).

Later the estimate (4.1) was improved in [16] to

$$\|C^n - \exp(n(C - I_{\mathfrak{H}}))\| = O\left((\ln n/n)^{1/2}\right), \tag{4.4}$$

and in [6] to

$$\|C^n - \exp(n(C - I_{\mathfrak{H}}))\| = O\left(1/n^{1/2}\right). \tag{4.5}$$

In [16] the following bound is proved for the operator-norm convergence rate:

$$\left\|(I_{\mathfrak{H}} + tS/n)^{-n} - \exp(-tS)\right\| \leq c/n, \tag{4.6}$$

where c is a constant depending on the operator S. The bounds (4.3), (4.4), (4.5), and (4.6) are obtained in [16] and [6] by means of the probability theory methods (the *Poisson distribution* and the *Central Limit Theorem*) that improves the estimate (4.3).

Here, using the results obtained in [11], [5], [10], and [9], and generalizations of some von Neumann inequality [15], we prove the theorem, which makes more explicit the right-hand side of the estimate (4.6).

Theorem 4.1. *Let S be an m-α-sectorial operator in a Hilbert space $\mathfrak{H}$. Then*

$$\left\|(I_{\mathfrak{H}} + tS/n)^{-n} - \exp(-tS)\right\| \leq K(\alpha)/(n\cos^2\alpha)\ ,\ t \geq 0\ ,$$

where

$$(\pi\sin\alpha)/2\alpha \leq K(\alpha) \leq \min\left(2 + 2/\sqrt{3}\ ,\ (\pi - \alpha)/\alpha\right)\ .$$

4.2. **The von Neumann inequality and its generalizations.** The spectral sets theory was introduced by von Neumann [15] in order to extend the functional calculus to the case of non-normal operators in Hilbert spaces.

Definition 4.2. [15]. *A set $\sigma \subset \mathbb{C}$ is a spectral set of the operator A in a Hilbert space $\mathfrak{H}$ if it is closed and if for any bounded rational function $u(z)$ on σ one has*

$$||u(A)|| \leq \sup_{z\in\sigma} |u(z)|.$$

Proposition 4.3. [15]. *A necessary and sufficient condition for one of the domains*

$$|z - a| \leq r,\ |z - a| \geq r,\ \mathrm{Re}\,(az) \geq b$$

to be a spectral set of A in $\mathfrak{H}$ is that

$$||A - aI_{\mathfrak{H}}|| \leq r,\ ||(A - aI_{\mathfrak{H}})^{-1}|| \leq \frac{1}{r},\ \mathrm{Re}\,(aT) \geq b.$$

Definition 4.4. *Let $\mathcal{D}$ be an open convex subset of the complex plane ($\mathcal{D} \neq \emptyset$, $\mathcal{D} \neq \mathbb{C}$) and let A be a linear operator in a Hilbert space $\mathcal{H}$ with $W(A) \subset \overline{\mathcal{D}}$. The set $\mathcal{D}$ is called K-spectral set for the operator A, if*

$$||u(A)|| \leq K \sup_{z\in\mathcal{D}} |u(z)|$$

for all rational functions $u(z)$ without pole in the spectrum of A.

Let $\mathcal{D}_k := \{|z - a_k| < r_k\}$, $k = 1, 2$ be two disks such that $\mathcal{D}_k \cap \mathcal{D}_2 = \{\xi_1, \xi_2\}$. Let

$$\mathfrak{L} := \mathcal{D}_1 \cap \mathcal{D}_2.$$

The set $\mathfrak{L}$ is said to be a *convex lens-shaped* domain [5]. Denote by $2\alpha \in (0, \pi)$ the angle of the lens L at the vertices. The operator $A \in \mathcal{L}(\mathfrak{H})$ is called *of the lenticular $\mathfrak{L}$-type* [5] if

$$||A - a_1 I_{\mathfrak{H}}|| \leq r_1 \quad \text{and} \quad ||A - a_2 I_{\mathfrak{H}}|| \leq r_2.$$

The next Proposition is established in [5].

Proposition 4.5. *Let $\mathfrak{L}$ be a convex lens-shaped domain of the complex plane with angle 2α. There exists a best positive constant $K(\alpha)$ such that the inequality*

$$||p(A)|| \leq K(\alpha) \sup_{z\in\mathfrak{L}} |p(z)|, \tag{4.7}$$

holds for all polynomials $p(z)$, for all operators $A \in \mathcal{L}(\mathfrak{H})$ of $\mathfrak{L}$-type, and for all Hilbert spaces $\mathfrak{H}$. The constant $K(\alpha)$ depends only on α and satisfies the inequality

$$\frac{\pi}{2\alpha}\sin\alpha \leq K(\alpha) \leq \min\left(2+\frac{2}{\sqrt{3}},\ \frac{\pi-\alpha}{\alpha}\right).$$

Notice that by the Mergelyan Theorem [17], the inequality (4.7) remains valid if p is holomorphic in $\mathfrak{L}$ and continuous in $\overline{\mathfrak{L}}$.

4.3. **Proof of the Theorem 4.1.**

Proof. Let

$$\mathcal{D}_1 = \left\{z\in\mathbb{C} : \left|z-\frac{1}{2}-\frac{i}{2}\cot\alpha\right| < \frac{1}{2\sin\alpha}\right\},$$
$$\mathcal{D}_2 = \left\{z\in\mathbb{C} : \left|z-\frac{1}{2}+\frac{i}{2}\cot\alpha\right| < \frac{1}{2\sin\alpha}\right\}.$$

Then (see (2.7)) $\mathfrak{L} = L(\alpha)\setminus\partial L(\alpha)$, i.e. interior of the set $L(\alpha)$.

Fix $t\geq 0$ and let

$$C = F(t) := (I_{\mathfrak{H}} + tS)^{-1}\ .$$

Then from (2.6) it follows that the operator C is of $\mathfrak{L}$-type.

Since

$$F(t/n) = F(t)\{(n-1)F(t)/n + I_{\mathfrak{H}}/n\}^{-1}\ ,\ n\in\mathbb{N}\ ,$$

we get also that

$$F^n(t/n) = C^n\{(n-1)C/n + I_{\mathfrak{H}}/n\}^{-n}\ .$$

Put

$$h_n(z) := \exp(1-1/z) - z^n\{(n-1)z/n + 1/n\}^{-n}\ ,\ z\in\mathfrak{L}\ .$$

Because $\operatorname{Re} z > 0$ for all $z\in\mathfrak{L}$, the function $h_n(z)$ is holomorphic in $\mathfrak{L}$ and continuous in $\overline{\mathfrak{L}} = L(\alpha)$. Moreover, since

$$\exp(-tS) = \exp(I_{\mathfrak{H}} - C^{-1})\ ,$$

we obtain that

$$\exp(-tS) - (I_{\mathfrak{H}} + tS/n)^{-n} = h_n(C)\ .$$

The fractional-linear conformal transformation $w\mapsto z = 1/(1+w)$ maps the sector $\mathcal{S}(\alpha)$ onto $L(\alpha)$. So, let

$$g_n(w) := h_n((1+w)^{-1}) = \exp(-w) - (1+w/n)^{-n}\ ,\ w\in(\mathcal{S}(\alpha)\setminus\partial\mathcal{S}(\alpha))\ .$$

Then clearly:

$$\sup_{z\in\mathfrak{L}}|h_n(z)| = \sup_{w\in(\mathcal{S}(\alpha)\setminus\partial\mathcal{S}(\alpha))}|g_n(w)| = \sup_{w\in\partial\mathcal{S}(\alpha)}|g_n(w)|\ .$$

Since

$$\partial\mathcal{S}(\alpha) = \{x\exp(-i\alpha),\ x\in\mathbb{R}_+\}\cup\{x\exp(i\alpha),\ x\in\mathbb{R}_+\}\ ,$$

we have to estimate the value of

$$\sup_{x\in\mathbb{R}_+}|g_n(x\exp(\pm i\alpha)|\ .$$

To this end we use the representation:

$$g_n(x\exp(i\alpha)) = -\int\limits_0^x \frac{d}{ds}\left((1+se^{i\alpha}/n)^{-n}\,\exp(-(x-s)e^{i\alpha})\right)ds\ ,$$

where

$$\frac{d}{ds}\left((1+se^{i\alpha}/n)^{-n}\,\exp(-(x-s)e^{i\alpha})\right) = \frac{se^{i\alpha}}{n}\,(1+se^{i\alpha}/n)^{-(n+1)}\,\exp(-(x-s)e^{i\alpha})\ .$$

By elementary inequalities:

$$\begin{aligned}&|1+se^{i\alpha}/n|^{n+1} = (1+s^2/n^2+2s\cos\alpha/n)^{(n+1)/2} \geq (1+(s\cos\alpha)/n)^{n+1} \geq \\ &\leq 1+((n+1)s\cos\alpha)/n \geq 1+s\cos\alpha,\end{aligned}$$

we obtain

$$s\ |1+(s\cos\alpha)/n|^{-(n+1)} \leq s/(1+s\cos\alpha) \leq 1/\cos\alpha\ .$$

Therefore, we obtain as an upper bound:

$$|g_n(x\exp(i\alpha))| \leq \int\limits_0^x \frac{\exp(-(x-s)\cos\alpha)}{n\cos\alpha}ds \leq \frac{1}{n\cos^2\alpha}(1-\exp(-x\cos\alpha)) \leq \frac{1}{n\cos^2\alpha}.$$

Similarly one obtains the estimate:

$$|g_n(x\exp(-i\alpha))| \leq \frac{1}{n\cos^2\alpha}.$$

Thus,

$$\sup_{x\in\mathbb{R}_+}|g_n(x\exp(\pm i\alpha)| \leq \frac{1}{n\cos^2\alpha}.$$

Now by Proposition4.5 we get

$$\left\|\exp(-tS)-(I_{\mathfrak{H}}+tS/n)^{-n}\right\| = ||h_n(C)|| \leq \frac{K(\alpha)}{n\cos^2\alpha}\ ,\ t\geq 0\ , \tag{4.8}$$

which completes the proof. □

5. Conclusion

Now several remarks are in order:

(a) Theorem 2.1 from [8] states that for quasi-sectorial contractions one has:

$$W(\exp(-tS)) \subseteq D_\alpha,\ \alpha\in[0,\pi/2), \tag{5.1}$$

Here S stands for m-α-sectorial generator of contraction. As it is indicated in [18], the proof in [8] has a flaw. In [18] it was corrected but only for the range $\alpha\in[0,\pi/4]$. Because

$$\Omega(\alpha)\subset Q(\alpha)\subset D_\alpha,$$

our main Theorem 3.4 shows that the original claim (5.1) in [8] is indeed correct for all $\alpha\in[0,\pi/2)$.

(b) Since in Theorem 3.4 we proved in fact that $W(\exp(-tS))\subseteq\Omega(\alpha)$, this means we give in the present papers more precise localization of $W(\exp(-tS))$ than it was claimed in [8].

(c) The inclusion (5.1) plays an important rôle for the operator-norm error estimates of the norm convergence in the Euler formula (1.2), see the recent paper [18], the references quoted there and in particular [7]. In the present paper we give a new proof of the operator-norm convergent Euler formula for the optimal error estimate with explicit indication of its $\alpha-$dependence (4.8) for m-α-sectorial generators.

Acknowledgements

V.Z. is thankful to Pavel Bleher for useful remarks related to the multiplicative semigroups on the complex plane.

References

[1] Yu.M. Arlinskiĭ. A class of contractions in Hilbert space. Ukrain. Matematicheskii Zhurnal, 39 (1987), No.6, 691-696(Russian). English translation in: Ukrainian Mathematical Journal, 39 (1987), No.6, 560–564.

[2] Yu.M. Arlinskiĭ. Characteristic functions of operators of the class $C(\alpha)$. Izv.Vyssh. Uchebn.Zaved.Mat. 1991, N2, 13-21(Russian).

[3] Yu.M. Arlinskiĭ. Closed sectorial forms and one-parameter contraction semigroups, Matematicheskie Zametki, 61 (1997), No.5, 643-654 (Russian). English translation in: Mathematical Notes, 61 (1997), No. 5, 537–546.

[4] Yu.M. Arlinskiĭ. On some semigroups on the complex plane. Semigroup Forum, 70 (2005), No.3, 329–346.

[5] B. Beckermann and M. Crouzeix, A lenticular version of a von Neumann inequality, Archiv der Mathematik, 86 (2006), 352–355.

[6] V. Bentkus and V. Paulauskas, Optimal error estimates in operator-norm approximations of semigroups, Lett. Math. Phys. 68 (2004) 131–138.

[7] V. Cachia. Euler's exponential formula for semigroups. Semigroup Forum, 68 (2004), No.1, 1–24.

[8] V. Cachia and V.A. Zagrebnov, Operator-norm approximation of semigroups by quasi-sectorial contractions, Journ. Funct. Anal., 180 (2001), 176-194.

[9] M. Crouzeix, Numerical range and functional calculus in Hilbert space. Journal of Functional Analysis, vol. 244 (2007), 668-690.

[10] M. Crouzeix and B. Delyon, Some estimates for analytic functions of the strip or a sectorial operators, Archiv der Mathematik, 81, (2003), 559–566.

[11] B. Delyon and F. Delyon, Generalization of von Neumann's spectral sets and integral representation of operators, Bull. Soc. Math. Fr., 127 (1999), No. 1, 25–42.

[12] P.R. Chernoff, *Product formulas, nonlinear semigroups, and addition of unbounded operators.* Memoirs of the American Mathematical Society, No. 140. American Mathematical Society, Providence, R. I., 1974.

[13] T. Kato Some mapping theorem for the numerical range, Proc.Japan Acad. 41 (1966), 652–655.

[14] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, Heidelberg, 1995.

[15] J. von Neumann, Eine spectraltheorie für allgemeine operatore eines unitären raumes, Math. Nachrichten, 4 (1951), 251–281.

[16] V. Paulauskas, On operator-norm approximation of some semigroups by quasi-sectorial operators, J. Funct. Anal. 207 (2004) 58–67.

[17] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, 1987.

[18] V.A. Zagrebnov. Quasi-Sectorial Contractions, Journ. Funct. Anal., 254 (2008), 2503–2511.

[19] V.A. Zagrebnov, *Topics in the theory of Gibbs semigroups.* Leuven Notes in Mathematical and Theoretical Physics. Series A: Mathematical Physics, 10. Leuven University Press, Leuven, 2003.

Department of Mathematical Analysis, East Ukrainian National University, Kvartal Molodyozhny 20-A, Lugansk 91034, Ukraine

E-mail address: yury_arlinskii@yahoo.com; yma@snu.edu.ua

Université de la Méditerranée and Centre de Physique Théorique - UMR 6207, Luminy - Case 907, Marseille 13288, Cedex 9, France

E-mail address: zagrebnov@cpt.univ-mrs.fr