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Joseph M. Landsberg, Laurent Manivel

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# THE SEXTONIONS AND $E_{7\frac{1}{2}}$

J. M. LANDSBERG<sup>1</sup>, L. MANIVEL

**ABSTRACT.** We fill in the “hole” in the exceptional series of Lie algebras that was observed by Cvitanovic, Deligne, Cohen and deMan. More precisely, we show that the intermediate Lie algebra between  $\mathfrak{e}_7$  and  $\mathfrak{e}_8$  satisfies some of the decomposition and dimension formulas of the exceptional simple Lie algebras. A key role is played by the *sextonions*, a six dimensional algebra between the quaternions and octonions. Using the sextonions, we show similar results hold for the rows of an expanded Freudenthal magic chart. We also obtain new interpretations of the adjoint variety of the exceptional group  $G_2$ .

## 1. INTRODUCTION

In [11, 8, 24] remarkable dimension formulas for the exceptional series of complex simple Lie algebras were established, parametrizing the series by the dual Coxeter number in [11, 8] and using the dimensions of composition algebras in [24]. Cohen and deMan observed that all parameter values giving rise to integer outputs in all the formulas of [11, 8] were already accounted for with essentially one exception, which, were it the dimension of a composition algebra, would be of dimension six and sit between the quaternions and octonions to produce a Lie algebra sitting between  $\mathfrak{e}_7$  and  $\mathfrak{e}_8$ . B. Westbury brought this to our attention and pointed out that were this the case, one would gain an entire new row of Freudenthal’s magic chart. We later learned that this algebra, which we call the *sextionion algebra*, had been observed earlier as a curiosity [19, 20].

In this paper we discuss the sextonions and the extra row of the magic chart it gives rise to. Along the way, we discuss intermediate Lie algebras in general and their homogeneous varieties, in particular the exceptional Lie algebra  $\mathfrak{e}_{7\frac{1}{2}}$  defined by the triality construction of [24] applied to the sextonions. This Lie algebra is intermediate between  $\mathfrak{e}_7$  and  $\mathfrak{e}_8$ . Of course it is not simple but remarkably, shares most of the properties of the simple exceptional Lie algebras discovered by Vogel and Deligne. More generally, many of the dimension formulas of [11, 8, 24] are satisfied by the intermediate Lie algebras and some of the decomposition formulas of [25] hold as well.

Let  $\bar{\mathfrak{g}}$  be a complex simple Lie algebra equipped with its adjoint (5-step) grading induced by the highest root  $\tilde{\alpha}$ :

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{-2} \oplus \bar{\mathfrak{g}}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \bar{\mathfrak{g}}_2.$$

Here  $\bar{\mathfrak{g}}_2 \simeq \mathbb{C}$  is the root space of  $\tilde{\alpha}$  and  $\bar{\mathfrak{g}}_0$  is reductive with a one-dimensional center (except in type  $A$  where the center is two dimensional). Let  $\mathfrak{h} = [\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}_0]$  be its semi-simple part.

Introduce the *intermediate Lie algebra*

$$\mathfrak{g} = \mathfrak{h} \oplus \bar{\mathfrak{g}}_1 \oplus \bar{\mathfrak{g}}_2.$$

Intermediate Lie algebras (sometimes in the forms  $\mathfrak{g}' = \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \bar{\mathfrak{g}}_2$ ,  $\mathfrak{g}'' = \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1$ ,  $\mathfrak{g}''' = \mathfrak{h} \oplus \bar{\mathfrak{g}}_1$ ) have appeared in [31, 29, 15, 16]. Shtepin used them to help decompose  $\bar{\mathfrak{g}}$ -modules as  $\mathfrak{h}$ -modules in a multiplicity free way to make Gelfand-Tsetlin bases. Gelfand and Zelevinsky used them to make representation models for the classical groups (and we hope the varieties discussed here might lead to similar models for the exceptional groups, or even that the triality model will give rise to uniform representation models for all simple Lie groups). Proctor made

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a detailed study of certain representations of the *odd symplectic Lie algebras* and proved a Weyl dimension formula for these.

**Overview.** In §2 we define and discuss the *adjoint varieties* of the intermediate Lie algebras. In §3 we give geometric interpretations of the adjoint variety of the exceptional group  $G_2$ , in particular we show that it parametrizes sextonionic subalgebras of the octonions. We also give a new description of the variety of quaternionic subalgebras of the octonions. In §4 we review the triality construction of Freudenthal's magic square and show how it applies to the sextonions. In §5, we discuss highest weight modules of intermediate Lie algebras, showing how to decompose the Cartan powers of the adjoint representation as an  $\mathfrak{h}$  module. We also remark that some of Vogel's universal decomposition formulas hold. In §6 we show that some of the more refined decomposition formulas of [25] hold for the rows of the extended magic square. Corresponding dimension formulas are stated and proven in §7. Finally in §8 we describe the geometry of closures of the orbits of highest weight vectors  $\mathbb{P}(\overline{G.v}) \subset \mathbb{P}V$  inside the preferred representations described in §6. In particular we get a new (slightly singular) Severi variety that we study in detail.

*Notation:* We use the ordering of roots as in [4]. Unless otherwise specified, all groups  $G$  associated to a Lie algebra  $\mathfrak{g}$  are the adjoint groups.

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## 2. ADJOINT VARIETIES

Let  $\mathfrak{h}$  be a complex simple Lie algebra. The *adjoint variety*  $X_H^{ad} \subset \mathbb{P}\mathfrak{h}$  is the closed  $H$ -orbit in  $\mathbb{P}\mathfrak{h}$ , where  $H$  denotes the adjoint group of  $\mathfrak{h}$ . The adjoint variety parametrizes the highest root spaces: given a line  $\ell$  in  $\mathfrak{h}$  which corresponds to a point of  $X_H^{ad}$ , we can choose a Cartan subalgebra of  $\mathfrak{h}$  and a set of positive roots, such that  $\ell = \mathfrak{h}_{\tilde{\alpha}}$ , the root space of the highest root  $\tilde{\alpha}$ . The goal of this section is to define a cousin of the adjoint variety for intermediate Lie algebras, which we will also call the adjoint variety.

Up to the center, the intermediate Lie algebra  $\mathfrak{g}$  coincides with the parabolic subalgebra of  $\overline{\mathfrak{g}}$  which stabilizes the line  $\overline{\mathfrak{g}}_{\tilde{\alpha}} \in X_G^{ad}$ . Its reductive part  $\mathfrak{h} = [\overline{\mathfrak{g}}_0, \overline{\mathfrak{g}}_0]$  is simple exactly when the support of the adjoint representation is an end of the Dynkin diagram of  $\mathfrak{g}$ . When  $\mathfrak{g} \simeq \mathfrak{sl}_{n+1}$  is of type  $A$ , then  $\mathfrak{h} = \mathfrak{gl}_{n-1}$  and we define the adjoint variety as the closed  $PGL_{n-1}$ -orbit  $\mathbb{F}(1, n-2) \subset \mathbb{P}\mathfrak{gl}_{n-1}$ . When  $\mathfrak{g} \simeq \mathfrak{so}_m$  is of type  $B$  or  $D$ , then  $\mathfrak{h} \simeq \mathfrak{sl}_2 \times \mathfrak{so}_{m-4}$  is not simple and it is not clear how to define the adjoint variety of  $\mathfrak{h}$ . In fact we can take either the adjoint variety of  $\mathfrak{sl}_2$ , a plane conic, or the adjoint variety  $G_Q(2, m-4)$  of  $\mathfrak{so}_{m-4}$ . Notwithstanding this difficulty for the  $\mathfrak{so}_m$ -case, we make the following:

**Definition 2.1.** *Let  $\mathfrak{g}$  be an intermediate Lie algebra. The adjoint variety  $X_G^{ad} \subset \mathbb{P}\mathfrak{g}$  of  $\mathfrak{g}$  is the closure of the  $G$ -orbit of a highest weight line of  $\mathfrak{h}$ .*

Recall that the  $\mathfrak{h}$ -module  $\overline{\mathfrak{g}}_1$  has very nice properties. It is *simple* (except in type  $A$ , for which we have two simple modules exchanged by an outer automorphism), and *minuscule*. The Lie bracket on  $\mathfrak{g}$  induces an invariant symplectic form  $\omega \in \Lambda^2 \overline{\mathfrak{g}}_1^*$  (canonically defined only up to scale), so in particular  $\mathfrak{h} \subset S^2 \overline{\mathfrak{g}}_1^*$ . In fact  $\mathfrak{h}$  generates the ideal of the closed orbit  $H/Q \subset \mathbb{P}\overline{\mathfrak{g}}_1$  which is Legendrian (in type  $A$  it is the union of two disjoint linear spaces, each of which is Legendrian). See, e.g., [22] for proofs of the above assertions.

Given  $x \in \mathfrak{h}$ , let  $q^x \subset S^2 \overline{\mathfrak{g}}_1^*$  denote the quadratic form it determines, defined by

$$q^x(v, w) = \frac{1}{2}\omega(x.v, w) = \frac{1}{2}\omega(x.w, v).$$

The linear span  $\langle q^x \rangle \subset \overline{\mathfrak{g}}_1$  of this quadratic form is the image of the endomorphism  $L_x$  of  $\overline{\mathfrak{g}}_1$  given by the action of  $x$ . Since  $q^x(v, w)$  only depends on  $x.v$  and  $x.w$ , we have an induced quadratic form on  $\langle q^x \rangle$  defined by  $q_x(v) = q^x(u, u)$  when  $v = xu$ .

**Proposition 2.2.** *We have the following description of  $X_G^{ad}$ :*

$$X_G^{ad} = \overline{\{[x, v, q_x(v)] \mid x \in \hat{X}_H^{ad}, v \in \langle \hat{q}^x \rangle\}} \subset \mathbb{P}\mathfrak{g} = \mathbb{P}(\mathfrak{h} \oplus \bar{\mathfrak{g}}_1 \oplus \bar{\mathfrak{g}}_2).$$

*Its affine tangent spaces at  $p = [x, 0, 0] \in X_H^{ad}$  and  $p_0 = [0, 0, 1]$  are*

$$\hat{T}_p X_G^{ad} = \hat{T}_x X_H^{ad} \oplus \langle q^x \rangle, \quad \hat{T}_{p_0} X_G^{ad} = \bar{\mathfrak{g}}_1 \oplus \bar{\mathfrak{g}}_2.$$

*Proof.* Given  $x \in \hat{X}_H^{ad} \subset \hat{X}_G^{ad}$ , consider the action of  $\exp(-u)$  on  $x$ , for  $u \in \bar{\mathfrak{g}}_1$ :

$$\exp(-u)x = x + [x, u] + \frac{1}{2}[u, [u, x]] = x + x.u + q^x(u).$$

The first claim follows, and the description of the tangent space at  $p = [x, 0, 0]$  is clear.

Since  $p_0$  is killed by  $\mathfrak{g}$ ,  $\hat{T}_{p_0} X_G^{ad}$  must be a  $\mathfrak{g}$ -submodule of  $\bar{\mathfrak{g}}$ , contained in  $\bar{\mathfrak{g}}_1 \oplus \bar{\mathfrak{g}}_2$  since a linear action cannot move a vector two steps in a grading. Since  $\bar{\mathfrak{g}}_1$  is irreducible (including in type  $A$  if we take into account the  $\mathbb{Z}_2$ -action), and since the affine tangent space cannot be reduced to  $\bar{\mathfrak{g}}_2$ , there must be equality.  $\square$

Recall that the complex simple Lie algebras can be parametrized by their *Vogel parameters*  $\alpha, \beta, \gamma$  (roughly the Casimir eigenvalues of the nontrivial components of their symmetric square, see [34, 27]). We normalize  $\alpha = -2$ , so that  $t = \alpha + \beta + \gamma = \check{h}$  is the dual Coxeter number. We distinguish  $\beta$  from  $\gamma$  as in [27].

**Lemma 2.3.** *For  $x \in X_H^{ad}$ , the dimension of  $\langle q^x \rangle$  is equal to  $\beta$ .*

*Proof.* Let  $x$  be the root space  $\mathfrak{g}_{\tilde{\alpha}}$  defined by a maximal root  $\tilde{\alpha}$  of  $\mathfrak{h}$ . Then the image of  $L_x$  is the direct sum of the root spaces  $\mathfrak{g}_{\mu+\tilde{\alpha}}$ , for  $\mu$  a root of  $\bar{\mathfrak{g}}_1$  such that  $\mu + \tilde{\alpha}$  is again a root. By [27], Corollary 5.2, there exists exactly  $\beta$  such roots.  $\square$

**Corollary 2.4.** *The dimensions of the adjoint varieties of  $\mathfrak{h}, \mathfrak{g}$  and  $\bar{\mathfrak{g}}$  are related as follows:*

$$\begin{aligned} \dim X_H^{ad} &= 2\check{h} - 3 - 2\beta, \\ \dim X_G^{ad} &= 2\check{h} - 3 - \beta, \\ \dim X_{\bar{G}}^{ad} &= 2\check{h} - 3 \end{aligned}$$

*unless  $\bar{\mathfrak{g}} = \mathfrak{g}_2$ , in which case the formula holds with 2 instead of  $\beta$ .*

*Proof.* The third equality was observed in [21]. That the first and second lines differ by  $\beta$  follows immediately from the fact that  $\dim \langle q^x \rangle = \beta$ , and the description we gave of the tangent space to  $X_G^{ad}$  at  $[x, 0, 0]$ , which is a generic point. Finally, the additional dimensions of the tangent space to  $X_{\bar{G}}^{ad}$  at that point arise from the action of  $\bar{\mathfrak{g}}_{-1}$ , which is symmetric with the action of  $\bar{\mathfrak{g}}_1$  and thus contributes the same value  $\beta$ .  $\square$

**Corollary 2.5.** *The adjoint variety of an intermediate Lie algebra  $X_G^{ad}$  is smooth if and only if  $\beta(\bar{\mathfrak{g}}) = 1$ . In general, the maximal excess dimension of its Zariski tangent spaces is  $\beta(\bar{\mathfrak{g}}) - 1$ .*

*Proof.* By semi-continuity, the excess dimension of the Zariski tangent space must be maximal at the point  $p_0$  of  $X_G^{ad}$ . We have calculated that the dimension of the tangent space at that point is  $2\check{h} - 2$  while  $\dim X_G^{ad} = 2\check{h} - 3 - \beta$ .  $\square$

*Example 1.* If  $\bar{\mathfrak{g}} = \mathfrak{sp}_{2n+2}$ , the adjoint variety is  $v_2(\mathbb{P}^{2n+1})$ . If  $\ell$  is a point of this variety, i.e., a line in  $\mathbb{C}^{2n+2}$ , the Lie algebra  $\mathfrak{h}$  may be identified with  $\mathfrak{sp}(V)$ , for  $V \simeq \ell^\perp/\ell$ , a vector space of dimension  $2n$  endowed with the restriction of the original symplectic form, which is again symplectic. Its adjoint variety is  $v_2(\mathbb{P}V)$ . The intermediate Lie algebra

$$\mathfrak{g} = \mathfrak{sp}(V) \oplus V \oplus \mathbb{C}$$

is an *odd symplectic Lie algebra* (see [29]), and the corresponding adjoint variety is  $v_2(\mathbb{P}\ell^\perp)$ . We thus get a smooth variety with only two  $G$ -orbits, the point  $\ell$  and its complement.

*Example 2.* If  $\bar{\mathfrak{g}} = \mathfrak{sl}_{n+1}$ , the adjoint variety is  $\mathbb{F}_{1,n}$ . A point of this variety is a pair  $(\ell_0, H_0)$ , with  $\ell_0$  a line, and  $H_0$  a hyperplane containing  $\ell_0$ . The Lie algebra  $\mathfrak{h}$  may be identified with  $\mathfrak{gl}(V)$ , where  $V = H_0/\ell_0$  once we have chosen a decomposition of  $\mathbb{C}^{n+1}$  as  $\ell_0 \oplus V \oplus \ell_1$ , with  $\ell_0 \oplus V = H_0$ . Its adjoint variety is the set of pairs  $(\ell \subset H)$ , with  $\ell$  a line and  $H$  a hyperplane in  $V$ , defined by a linear form that we extend by zero on  $\ell_0 \oplus \ell_1$ . The intermediate Lie algebra is

$$\mathfrak{g} = \mathfrak{sl}(V) \oplus V \oplus V^* \oplus \mathbb{C},$$

whose adjoint variety is

$$\{(\ell, H) \in \mathbb{F}_{1,n}, \quad \ell \subset H_0, \ell_0 \subset H\}.$$

This variety has four  $G$ -orbits, and a unique singular point  $(\ell_0, H_0)$ , which is a simple quadratic singularity.

Let  $\theta : S^2\bar{\mathfrak{g}}_1 \rightarrow \mathfrak{h}$  denote the projection map which is dual to the natural inclusion  $\mathfrak{h} \subset S^2\bar{\mathfrak{g}}_1^*$ . The duality on  $\mathfrak{h}$  is taken with respect to the restriction of the Killing form  $K$  of  $\bar{\mathfrak{g}}$ . Explicitly,

$$(1) \quad K(\theta(u), y) = \omega(yu, u) \quad \text{for } u \in \bar{\mathfrak{g}}_1, y \in \mathfrak{h}.$$

**Proposition 2.6.** *Normalize the Killing form so that  $(\alpha_0, \alpha_0) = 2$ .*

*Let  $x \in \hat{X}_H^{ad}$ ,  $y \in \mathfrak{g}_0$  and  $u \in \bar{\mathfrak{g}}_1$ . Then*

$$(2) \quad xyxu = K(x, y)xu,$$

$$(3) \quad \theta(xu) = q^x(u, u)x.$$

*Proof.* By homogeneity, we may suppose that  $x = X_{\tilde{\alpha}}$  belongs to the root space  $\mathfrak{g}_{\tilde{\alpha}}$ . The identity (1) being linear in  $y$  and  $u$ , we can let  $u = X_{\theta}$  for some root  $\theta \in \Phi_1$ . If  $y$  belongs to the Cartan subalgebra, then  $xu$  is an eigenvector of  $y$ , thus  $xyxu$  is a multiple of  $x^2u$ , hence zero. Since  $K(x, y)$  is also zero, we are done.

Now suppose that  $y = X_{\sigma}$  is a root vector in  $\mathfrak{h}$ . Then  $K(x, y) \neq 0$  if and only if  $\sigma = -\tilde{\alpha}$ . Recall that  $\bar{\mathfrak{g}}_1$  is a minuscule  $\mathfrak{h}$ -module, so that a root of  $\bar{\mathfrak{g}}_1$  is of the form  $\gamma = \omega_0 + \chi$  with  $-1 \leq \chi(H_{\tau}) \leq 1$  for every root  $\tau$  of  $\mathfrak{h}$ . Moreover, for  $u = X_{\gamma}$ ,  $X_{\tau}u \neq 0$  implies that  $\chi(H_{\tau}) = -1$ . This implies that if  $xyxu$  is nonzero,

$$-1 = \chi(H_{\tilde{\alpha}}) = (\chi + \tilde{\alpha})(H_{\sigma}) = (\chi + \tilde{\alpha} + \sigma)(H_{\tilde{\alpha}}),$$

hence  $\sigma(H_{\tilde{\alpha}}) = -\tilde{\alpha}(H_{\tilde{\alpha}}) = -2$ . But this is possible only if  $\sigma = -\tilde{\alpha}$ , in which case  $[y, x] = tH_{\tilde{\alpha}}$  for some scalar  $t \neq 0$ , and

$$xyxu = x[y, x]u = t\gamma(H_{\tilde{\alpha}})xu = t\chi(H_{\tilde{\alpha}})xu = -txu.$$

With our normalization,  $2t = tK(H_{\tilde{\alpha}}, H_{\tilde{\alpha}}) = K(H_{\tilde{\alpha}}, [y, x]) = K([H_{\tilde{\alpha}}, y], x) = -2K(y, x)$ , thus  $K(x, y) = -t$  and finally,  $xyxu = K(x, y)u$ , which is what we wanted to prove.

The second identity is an immediate consequence: from the equation defining  $\theta$ , we get

$$K(y, \theta(v)) = \omega(v, yv) \quad \forall y \in \mathfrak{h}, \forall v \in \bar{\mathfrak{g}}_1.$$

For  $x \in \hat{X}_H^{ad}$ , we get using (1),

$$\begin{aligned} K(y, \theta(xu)) &= \omega(yxu, xu) = -\omega(xyxu, u) \\ &= -K(y, x)\omega(xu, u) = -K(y, q^x(u, u)x), \end{aligned}$$

as claimed. The proof is complete.  $\square$

Recall that the cone over the closed  $H$ -orbit in  $\mathbb{P}\bar{\mathfrak{g}}_1$  is the set of vectors  $v \in \bar{\mathfrak{g}}_1$  such that  $\theta(v) = 0$ , a space of quadratic equations parametrized by  $\mathfrak{h}$  (see [22]).

The following fact was observed case by case in [27]:

**Corollary 2.7.** *The adjoint variety parametrizes a family of  $(\beta - 2)$ -dimensional quadrics on the closed  $H$ -orbit in  $\mathbb{P}\bar{\mathfrak{g}}_1$ .*

*Proof.* For each  $x \in X_H^{ad}$ , the linear space  $\langle q^x \rangle \subset \mathbb{P}\mathfrak{g}_1$  has dimension  $\beta$ , and the identity  $\theta(xu) = q^x(u)x = q_x(xu)x$  shows that on  $\langle q^x \rangle$ , the condition  $\theta(v) = 0$  reduces to a single quadratic condition. This means that the closed orbit cuts  $\langle q^x \rangle$  along a quadric hypersurface.  $\square$

In the case where  $\mathfrak{g} = \mathfrak{so}_m$ , we have two families of maximal quadrics in the adjoint variety  $G_Q(2, m)$ , of dimension 4 and  $m - 4$ . We can either choose  $\beta = 4$  and  $\gamma = m - 4$ , which corresponds to the component  $G_Q(2, m - 4)$  of  $X_H^{ad}$ , or  $\beta = m - 4$  and  $\gamma = 4$ , which corresponds to the one-dimensional component  $v_2(\mathbb{P}^1)$ .

### 3. THE SEXTONIONS

Consider the adjoint variety  $X_{G_2}^{ad} \subset \mathbb{P}\mathfrak{g}_2$ . Since  $G_2 \subset SO(7, \underline{q})$ , we have  $X_{G_2}^{ad} \subset G_{\underline{q}}(2, Im\mathbb{O})$ , the Grassmannian of  $\underline{q}$ -isotropic planes in  $\mathbb{C}^7 = Im\mathbb{O}$ . So the adjoint variety is a  $G_2$ -invariant set of isotropic planes. Here  $\underline{q}$  is the restriction of the quadratic form  $q(a, b) = Re(a\bar{b})$  on  $\mathbb{O}$  to  $Im\mathbb{O}$ , where  $Re(x) := \frac{1}{2}(x + \bar{x})$  in  $\mathbb{O}$ .

We will say a plane  $E \in G(2, Im\mathbb{O})$  is *null* if for all  $u, v \in E$ ,  $uv = 0$ .

**Theorem 3.1.** *The adjoint variety  $X_{G_2}^{ad}$ , the closed  $G_2$ -orbit in  $\mathbb{P}\mathfrak{g}_2$ , parametrizes:*

- (1) *null-planes in  $\mathbb{O}$ ;*
- (2) *rank two derivations of  $\mathbb{O}$ , up to scalars;*
- (3) *six-dimensional subalgebras of  $\mathbb{O}$ .*

*Proof.* The correspondence between these three objects is as follows: if  $U$  is a null-plane in  $\mathbb{O}$ , the orthogonal space  $V$  is a six-dimensional subalgebra, and there is, up to scale, a unique skew-symmetric endomorphism of  $\mathbb{O}$  whose image is  $U$  and kernel is  $V$ .

We first claim that  $X_{G_2}^{ad} = G(2, Im\mathbb{O}) \cap \mathbb{P}\mathfrak{g}_2$ . Note that this intersection is highly non-transverse (of codimension 5 in  $G(2, 7)$  instead of the expected 7), although the set theoretic intersection is a smooth variety. We therefore use a direct geometric description in terms of the associative form  $\phi \in \Lambda^3 Im\mathbb{O}^*$  defined by

$$\phi(x, y, z) = Re[(xy)z - (zy)x].$$

Bryant showed that the stabilizer of  $\phi$  is exactly the group  $G_2$ , see [18]. Note that since  $\dim \Lambda^3 Im\mathbb{O} = \dim \mathfrak{gl}_7 - \dim \mathfrak{g}_2$ , the  $GL_7$ -orbit of  $\phi$  in  $\Lambda^3 Im\mathbb{O}$  is a dense open subset.

On the Grassmannian  $G(2, Im\mathbb{O})$ , we have a tautological rank two vector bundle  $T$ , and a quotient bundle  $Q$  of rank 5. Consider the homogeneous vector bundle  $E = Q^* \otimes \Lambda^2 T^*$ , of rank 5. By the Borel-Weil theorem, the space of global sections of this vector bundle is  $\Gamma(G(2, Im\mathbb{O}), E) = \Lambda^3(Im\mathbb{O})^*$ . We can therefore interpret  $\phi$  as a generic section  $\sigma$  of the vector bundle  $E$ , which is globally generated, being irreducible as a homogeneous vector bundle. By Bertini, the zero-locus of  $\sigma$  is, if not empty, a smooth codimension 5 subvariety of  $G(2, Im\mathbb{O})$ , hence a 5-dimensional smooth variety,  $G_2$ -invariant since  $\phi$  is  $G_2$ -invariant. But the adjoint variety  $X_{G_2}^{ad}$  is the  $G_2$ -orbit of minimal dimension, and this dimension is five. So  $X_{G_2}^{ad}$  must be equal to the zero-locus of  $\sigma$ .

What is this zero-locus explicitly? If we choose a basis  $u_1, u_2$  of a plane  $U$  in  $Im\mathbb{O}$ , the linear form  $\phi(u_1, u_2, \bullet)$  is a linear form on  $Im\mathbb{O}$  (which descends to a linear form on  $Q = Im\mathbb{O}/U$ ), and  $\sigma$  vanishes at  $U$  if and only if this linear form is zero. But for  $z \in Im\mathbb{O}$ ,

$$\begin{aligned} \phi(u_1, u_2, z) &= Re[(u_1 z)u_2 - (u_2 z)u_1] \\ &= q((u_1 z)u_2, 1) - q((u_2 z)u_1, 1) \\ &= -q(u_1 z, u_2) + q(u_2 z, u_1) \\ &= q(z, u_1 u_2) - q(u_2, u_1 z) \\ &= 2q(z, u_1 u_2). \end{aligned}$$

This is zero for all  $u$  if and only if  $u_1 u_2 = r1$  for some scalar  $r$ . But multiplying by  $u_1$  on the left, we get  $-q(u_1)u_2 = ru_1$ , thus  $r = 0$ . We conclude that the zero locus of  $\sigma$  is exactly the

set of null-planes in  $Im\mathbb{O}$ . (In particular, it is not empty! Note also that a null-plane must be  $q$ -isotropic.) This proves our first claim.

Let  $d$  be a rank two derivation of  $\mathbb{O}$ . Since  $d$  has rank two and is skew-symmetric, we can find two independant vectors  $u_1$  and  $u_2$  such that  $d(z) = q(u_1, z)u_2 - q(u_2, z)u_1$ . Since  $d(1) = 0$ , the plane  $U$  generated by  $u_1$  and  $u_2$  is contained in  $Im\mathbb{O}$ . Since  $d$  is a derivation, its kernel  $V = U^\perp$  is a subalgebra of  $\mathbb{O}$ , containing the unit element. For  $v, v' \in V$  and  $u \in U$ , we get  $0 = q(u, vv') = q(\bar{v}u, v')$ , hence  $V.U \subset U$ . This implies that  $U$  must be  $q$ -isotropic, since the right multiplication by a non-isotropic element is invertible. For  $u \in U$  nonzero, consider the right multiplication operator  $R_u : \mathbb{O} \rightarrow \mathbb{O}$ . Then  $R_u(\mathbb{O})$  is a four dimensional  $q$ -isotropic subspace of  $\mathbb{O}$ . Since  $V$  has codimension 2 in  $\mathbb{O}$ ,  $V.U$  has codimension at most two in  $R_u(\mathbb{O})$ , and since it is contained in  $U$  we must have  $R_u(V) = U$ . If  $u' \in U$ , this means that we can find  $v \in V$  such that  $u' = vu$ . But then  $u'u = (vu)u = -q(u)v = 0$ . We conclude that  $U$  is a null-plane. Thus the projectivization of the space of rank two derivations of  $\mathbb{O}$ ,  $G(2, Im\mathbb{O}) \cap \mathbb{P}\mathfrak{g}_2$ , which is non-empty because  $\dim G(2, Im\mathbb{O}) = 10$  and  $\mathbb{P}\mathfrak{g}_2$  has codimension 7 in  $\mathbb{P}\mathfrak{so}_7$ , can be identified with a subvariety of  $X_{G_2}^{ad}$ . Being  $G_2$ -invariant, it must be equal to the adjoint variety. This proves our second claim.

Our third claim follows. On the one hand, the orthogonal space to a null-plane  $U$ , being equal to the kernel of a rank-two derivation, is a six-dimensional subalgebra of  $\mathbb{O}$ . Conversely, we have just proved that the orthogonal to such a subalgebra is a null plane.  $\square$

What is the structure of a six dimensional subalgebra  $S = U^\perp$  of  $\mathbb{O}$ ? To understand it, consider another null plane  $U_-$ , transverse to  $S$ , and let  $S_- := U_-^\perp$ .

**Lemma 3.2.**  $H = S \cap S_-$  is a quaternionic subalgebra.

*Proof.* Being the intersection of two subalgebras,  $H$  is a subalgebra, and contains 1. The hypothesis that  $U_-$  be transverse to  $S$  is equivalent to the fact that  $H$  is tranverse to  $U$  in  $S$ . In particular, the norm restricts to a nondegenerate quadratic form on  $H$ , which must therefore be a quaternionic subalgebra, i.e., isomorphic to  $\mathbb{H}$ .  $\square$

**Lemma 3.3.** The right action of  $H$  on  $U$  identifies  $H$  with  $\mathfrak{gl}(U)$ .

The scalar product on  $\mathbb{O}$  identifies  $U_-$  with  $U^*$ .

We can be even more precise and explicitly describe the octonionic multiplication in terms of the decomposition

$$\mathbb{O} = \mathfrak{gl}(U) \oplus U \oplus U^*.$$

An explicit computation shows that his multiplication is given by the formula

$$(4) \quad (X, u, u^*)(Y, v, v^*) = (XY - 2u \otimes v^* - 2(v \otimes u^*)^0, X^0v + Yu, X^tv^* + Y^*u^*),$$

and that the norm is

$$q(X, u, u^*) = \det(X) + 2\langle u, u^* \rangle.$$

Here  $X^0 = \text{trace}(X)I - X$ , so that the map  $X \mapsto X^0$  is the reflection in the hyperplane perpendicular to the identity. Moreover,  $(X^0)^t$  is the cofactor matrix of  $X$ , as  $XX^0 = (\det X)I$ . Also, note that  $(XY)^0 = Y^0X^0$ .

Restricting to  $S = \mathfrak{gl}(U) \oplus U$ , we get the multiplication law

$$(5) \quad (X, u)(Y, v) = (XY, X^0v + Yu),$$

while the norm  $q(X, u) = \det(X)$  becomes degenerate, with kernel  $U$ .

**Lemma 3.4.** The decomposition  $H^\perp = U \oplus U_-$  into the direct sum of two null-planes, is unique.

*Proof.* Use the multiplication law (4) on  $H^\perp$ .  $\square$

Lemma 3.4 provides an interesting way to parametrize the set of quaternionic subalgebras of  $\mathbb{O}$ . First note that the Schubert condition  $U^\perp \cap U_- = 0$  defines a  $G_2$ -invariant divisor  $D$  in

the linear system  $|\mathcal{O}(1,1)|$  on  $X_{G_2}^{ad} \times X_{G_2}^{ad} \subset \mathbb{P}\mathfrak{g}_2 \times \mathbb{P}\mathfrak{g}_2$ . This divisor descends to a very ample divisor in  $Sym^2 X_{G_2}^{ad}$ , which we still denote by  $D$ . Note that  $D$  contains the diagonal, which is the singular locus of  $Sym^2 X_{G_2}^{ad}$ .

**Proposition 3.5.** *Let  $\mathcal{H}$  denote the set of quaternionic subalgebras of  $\mathbb{O}$ .*

*We have the  $G_2$ -equivariant identifications*

$$\mathcal{H} \simeq G_2/GL_2 \rtimes \mathbb{Z}_2 \simeq Sym^2 X_{G_2}^{ad} - D.$$

*Proof.* The fact that  $\mathcal{H}$  is  $G_2$ -homogeneous is well-known, see e.g. the proof of Theorem 1.27, page 57 of [30]. We prove that the stabilizer  $K$  of a quaternionic subalgebra  $H$  is isomorphic to  $GL_2$ . This stabilizer also preserves  $H^\perp$ , hence, by the lemma, the pair  $U, U_-$ . The subgroup  $K^0$  of  $K$  preserving  $U$ , is therefore either equal to  $K$ , or a normal subgroup of index two.

For  $m \in GL(U)$ , the endomorphism  $\rho_m$  of  $\mathbb{O}$  defined by

$$\rho_m(X, u, u^*) = (Ad(m)X, mu, {}^t m^{-1}u^*)$$

is easily checked to be an algebra automorphism, and the map  $m \mapsto \rho_m$  defines an isomorphism of  $GL(U)$  with  $K^0$ .

Now let  $\sigma \in K - K^0$ . Since the restriction of  $\sigma$  to  $H$  is an algebra automorphism,

$$\sigma(X, u, u^*) = (Ad(s)X, P(u^*), Q(u))$$

for some  $s \in GL(U)$ , and some invertible operators  $P : U^* \rightarrow U$  and  $Q : U \rightarrow U^*$ . Composing with a element of  $K^0$ , we may suppose that  $Ad(s) = 1$  and  $P \circ Q = \varepsilon I$ , with  $\varepsilon = \pm 1$ . For  $u \in U$  and  $u^* \in U^*$ , the condition that  $\sigma(u.u^*) = \sigma(u).\sigma(u^*)$  gives  $\varepsilon = 1$  and  $\langle Qu, u \rangle = 0$ . Thus  $Q$  is a skew-symmetric endomorphism from  $U$  to  $U^*$ , and we check that this is sufficient to ensure that  $\sigma$  is an automorphism. We conclude that  $K^0 \neq K = K^0 \rtimes \mathbb{Z}_2$ .

Finally, the decomposition  $H^\perp = U \oplus U_-$ , being unique, defines an injective  $G_2$ -equivariant morphism from  $\mathcal{H}$  to  $Sym^2 X_{G_2}^{ad}$ . Since  $U^\perp = H \oplus U$  and  $U_-^\perp = H \oplus U_-$ , we have that  $U^\perp \cap U_- = U_-^\perp \cap U = 0$ , so that the image of  $\mathcal{H}$  is contained in the complement of the divisor  $D$ . In fact there is equality, by Lemma 3.2.  $\square$

**Proposition 3.6.** *The automorphism group of  $S$  fits into an exact sequence*

$$1 \longrightarrow R \longrightarrow Aut(S) \longrightarrow GL(U) \longrightarrow 1,$$

*where the radical  $R$  is a four-dimensional vector space considered with its natural abelian group structure. The induced action of  $GL(U)$  on  $R$  identifies  $R$  with  $S^3 U \otimes (\det U)^{-1}$ .*

*Proof.* Recall that the multiplication on  $S = H \oplus U \simeq \mathfrak{gl}(U) \oplus U$  is given by

$$(X, u)(Y, v) = (XY, X^0 v + Yu).$$

An automorphism  $\rho$  of  $S$  will preserve the norm, hence the kernel  $U$  of this quadratic form. We can therefore write

$$\rho(X, u) = (\rho_2(X), \sigma(X) + \rho_1(u)),$$

where  $\rho_1 \in GL(U)$ ,  $\sigma \in Hom(\mathfrak{gl}(U), U)$  and  $\rho_2 \in Aut(H)$ . In particular, we can find  $r \in GL(U)$  such that  $\rho_2 = Ad(r)$ . A straightforward computation shows that  $\rho$  is an automorphism of  $S$  if and only if the following conditions hold:

$$(6) \quad \sigma(XY) = \rho_2(X)^0 \sigma(Y) + \rho_2(Y) \sigma(X) \quad \forall X, Y \in \mathfrak{gl}(U),$$

$$(7) \quad \rho_1(Yu) = \rho_2(Y) \rho_1(u) \quad \forall Y \in \mathfrak{gl}(U), u \in U,$$

$$(8) \quad \rho_1(X^0 v) = \rho_2(X)^0 \rho_1(v) \quad \forall X \in \mathfrak{gl}(U), v \in U.$$

The second condition yields  $\rho_1 Y = r Y r^{-1} \rho_1$  for all  $Y \in \mathfrak{gl}(U)$ , so  $r^{-1} \rho_1$  is a homothety and  $\rho_2 = Ad(\rho_1)$ . Since  $r^0 = (\det r) r^{-1}$ , the third condition follows. Since the first equation is certainly verified by  $\sigma = 0$ , the map  $\rho \mapsto \rho_1$  defines a surjective morphism from  $Aut(S)$  to  $GL(U)$ . Note that this surjection is split, since  $GL(U)$  can be identified with the subgroup



$Aut_H(S)$  of automorphisms of  $S$  preserving  $H$ . Consider the kernel of this extension, i.e., the normal subgroup of  $Aut(S)$  consisting in morphisms of type

$$\rho(X, u) = (X, \sigma(X) + u),$$

where  $\sigma \in Hom(\mathfrak{gl}(U), U)$  is subject to the condition that

$$\sigma(XY) = X^0\sigma(Y) + Y\sigma(X) \quad \forall X, Y \in \mathfrak{gl}(U).$$

Letting  $Y = I$ , we see that  $\sigma(I) = 0$ . For  $Y = X$  we get  $\sigma(X^2) = \text{trace}(X)\sigma(X)$ , but since  $X^2 = \text{trace}(X)X - \det(X)I$ , this follows from  $\sigma(I) = 0$ . So the symmetric part of the condition is fulfilled, and we are left with the skew-symmetric part,

$$\sigma([X, Y]) = -2(X\sigma(Y) - Y\sigma(X)) \quad \forall X, Y \in \mathfrak{sl}(U).$$

An explicit computation shows that this defines a four dimensional subspace  $R$  of  $Hom(\mathfrak{gl}(U), U)$  (take the standard basis  $X, Y, H$  of  $\mathfrak{sl}(U)$  and check that  $\sigma$  is uniquely defined by the choice of  $\sigma(X)$  and  $\sigma(Y)$ , which is arbitrary). The conjugation action of  $\rho \in GL(U) = Aut_H(S)$  is by  $\rho(\sigma) = \rho^{-1} \circ \sigma \circ Ad(\rho)$ . We finally choose a two dimensional torus in  $GL(U)$  and compute the weights of this action, and they are those of  $S^3U \otimes (\det U)^{-1}$ . This concludes the proof.  $\square$

**Proposition 3.7.** *Let  $Aut_S(\mathbb{O})$  denote the group of automorphisms of  $\mathbb{O}$  preserving  $S$ . The restriction map  $Aut_S(\mathbb{O}) \rightarrow Aut(S)$  is surjective with one dimensional kernel.*

Note that  $Aut_S(\mathbb{O})$  is a maximal parabolic subgroup of  $Aut(\mathbb{O}) = G_2$ , since the adjoint variety  $X_{G_2}^{ad} = Aut(\mathbb{O})/Aut_S(\mathbb{O})$ .

*Proof.* We begin with a technical lemma. For  $\sigma \in R \subset Hom(\mathfrak{gl}(U), U)$ , let  $\sigma^\dagger \in Hom(U^*, \mathfrak{gl}(U))$  denote its transpose with respect to the norm on  $\mathbb{O}$ . Since the polarisation of the determinant is the symmetric bilinear form  $\det(X, Y) = \text{trace}(X)\text{trace}(Y) - \text{trace}(XY)$ , this means that

$$\frac{1}{2}(\text{trace}(Y)\text{trace}(\sigma^\dagger(u^*)) - \text{trace}(Y\sigma^\dagger(u^*))) = \langle \sigma(Y), u^* \rangle \quad \forall u^* \in U^*, Y \in \mathfrak{gl}(U).$$

Since  $\sigma(I) = 0$ , we can characterize  $\sigma^\dagger$  as the unique morphism from  $U^*$  to  $\mathfrak{sl}(U)$  such that

$$\text{trace}(Y\sigma^\dagger(u^*)) = -2\langle \sigma(Y), u^* \rangle \quad \forall u^* \in U^*, Y \in \mathfrak{sl}(U).$$

**Lemma 3.8.** *For all  $\sigma \in R$ , we have the identities*

$$(9) \quad \langle \sigma(u \otimes v^*), w^* \rangle = \langle \sigma(u \otimes w^*), v^* \rangle \quad \forall u \in U, v^*, w^* \in U^*$$

$$(10) \quad \sigma^\dagger(v^*)u = -2\sigma(u \otimes v^*) \quad \forall u \in U, v^* \in U^*$$

$$(11) \quad \sigma^\dagger(X^*v^*) = \sigma^\dagger(v^*)X + 2\sigma(X) \otimes v^* \quad \forall v^* \in U^*, X \in \mathfrak{gl}(U).$$

*Proof of the lemma.* Let  $X = u \otimes v^*$  and  $Y = u \otimes w^*$ . Then  $XY = \langle u, v^* \rangle Y$  and  $X^0 = \langle u, v^* \rangle I - u \otimes v^*$ . The identity  $\sigma(XY) = X^0\sigma(Y) + Y\sigma(X)$  gives

$$\langle u, v^* \rangle \sigma(u \otimes v^*) = (\langle u, v^* \rangle I - u \otimes v^*)\sigma(u \otimes w^*) + (u \otimes w^*)\sigma(u \otimes v^*),$$

and the first identity follows. We deduce that for all  $w^* \in U^*$ ,

$$\begin{aligned} \langle \sigma^\dagger(v^*)u, w^* \rangle &= \text{trace}(\sigma^\dagger(v^*), u \otimes w^*) \\ &= -2\langle \sigma(u \otimes w^*), v^* \rangle \\ &= -2\langle \sigma(u \otimes v^*), w^* \rangle. \end{aligned}$$

This gives the second identity. Finally, for all  $Y \in \mathfrak{gl}(U)$ , we have

$$\begin{aligned} \text{trace}(Y\sigma^\dagger(X^*v^*)) &= -2\langle \sigma(Y), X^*v^* \rangle \\ &= -2\langle X^0\sigma(Y), v^* \rangle \\ &= -2\langle \sigma(XY), v^* \rangle + 2\langle Y\sigma(X), v^* \rangle \\ &= \text{trace}(XY\sigma^\dagger(v^*)) + 2\text{trace} Y(s(X) \otimes v^*), \end{aligned}$$

and this implies the last identity.  $\square$

**Lemma 3.9.** *For all  $\sigma \in R$ , the map  $d_\sigma \in \text{End}(\mathbb{O})$  defined by*

$$d_\sigma(X, u, u^*) = (-\sigma^\dagger(u^*), \sigma(X), 0), \quad X \in \mathfrak{gl}(U), u \in U, u^* \in U^*,$$

*is a derivation of  $\mathbb{O}$ .*

*Proof.* Easy verification with the formulas of the previous lemma.  $\square$

**Lemma 3.10.** *For all  $\rho \in \mathfrak{gl}(U)$ , the map  $d_\rho \in \text{End}(\mathbb{O})$  defined by*

$$d_\rho(X, u, u^*) = (ad(\rho)X, \rho(u), -\rho^t(u^*)), \quad X \in \mathfrak{gl}(U), u \in U, u^* \in U^*,$$

*is a derivation of  $\mathbb{O}$ .*

*Proof.* Straightforward.  $\square$

We can now complete the proof of Proposition 3.7. The differential at the identity of the map  $\text{Aut}_S(\mathbb{O}) \rightarrow \text{Aut}(S)$  is the natural restriction map  $\text{Der}_S(\mathbb{O}) \rightarrow \text{Der}(S)$ . The two previous lemmas imply that this map is surjective. The map  $\text{Aut}_S(\mathbb{O}) \rightarrow \text{Aut}(S)$  is therefore surjective as well.

Consider some automorphism  $\gamma$  of  $\mathbb{O}$  acting trivially on  $S$ . The simple fact that it preserves the norm implies that

$$\gamma(X, u, u^*) = (X, u + \delta(u^*), u^*)$$

for some skew-symmetric map  $\delta : U^* \rightarrow U$ . Up to scale, there is only one such skew-symmetric map. Moreover, being a rank two skew-symmetric map with a null plane for image, it must be a derivation. We conclude that the kernel of the restriction map  $\text{Aut}_S(\mathbb{O}) \rightarrow \text{Aut}(S)$  is the additive group of automorphisms of the form  $I + d$ ,  $d$  a rank two derivation with image in  $U$ .  $\square$

**Definition 3.11.** *In what follows, we fix a six dimensional subalgebra  $S$  of  $\mathbb{O}$ , denote it by  $\mathbb{S}$  and call it the sextonion algebra. Recall formula (5), which gives a model of the sextonion algebra over an arbitrary field, for example over the real numbers :  $\mathbb{S} \simeq \mathfrak{gl}(U) \oplus U$  for some two-dimensional vector space  $U$ , and the product is given by the simple formula*

$$(X, u)(Y, v) = (XY, X^0v + Yu),$$

where  $X^0 = \text{trace}(X)I - X$ . We get a six-dimensional alternative algebra, with zero divisors.

#### 4. REVIEW OF THE TRIALITY AND $r$ -ALITY CONSTRUCTIONS

For  $\mathbb{A}$  a composition algebra, define the *triality group*

$$T(\mathbb{A}) = \{\theta = (\theta_1, \theta_2, \theta_3) \in SO(\mathbb{A})^3 \mid \theta_3(xy) = \theta_1(x)\theta_2(y) \forall x, y \in \mathbb{A}\}.$$

There are three natural actions of  $T(\mathbb{A})$  on  $\mathbb{A}$  corresponding to its three projections on  $SO(\mathbb{A})$ , and we denote these representations by  $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ . See [24] for more details. We let  $\mathfrak{t}(\mathbb{A})$  denote the corresponding Lie algebra.

Now let  $\mathbb{A}$  and  $\mathbb{B}$  be two composition algebras. Then

$$\mathfrak{g}(\mathbb{A}, \mathbb{B}) = \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B}) \oplus (\mathbb{A}_1 \otimes \mathbb{B}_1) \oplus (\mathbb{A}_2 \otimes \mathbb{B}_2) \oplus (\mathbb{A}_3 \otimes \mathbb{B}_3)$$

is naturally a semi-simple Lie algebra when  $\mathbb{A}, \mathbb{B}$  are among  $0, \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ .

The triality Lie algebras can be generalized to  $r$ -ality for all  $r$  to recover the generalized Freudenthal chart (see [23]). For  $r > 3$  we have

$$\mathfrak{t}_r(\mathbb{R}) = 0, \quad \mathfrak{t}_r(\mathbb{C}) = \mathbb{C}^{\oplus(r-1)}, \quad \mathfrak{t}_r(\mathbb{H}) = \mathfrak{sl}_2^{\times r} \quad \mathfrak{t}_r(\mathbb{S}) = \mathfrak{sl}_2^{\times r} \oplus \mathbb{C}^{2(r-1)}$$

and

$$\mathfrak{g}_r(\mathbb{A}, \mathbb{B}) = \mathfrak{t}_r(\mathbb{A}) \times \mathfrak{t}_r(\mathbb{B}) \oplus \bigoplus_{1 \leq i < j \leq r} \mathbb{A}_{ij} \otimes \mathbb{B}_{ij}.$$

The algebras  $\mathfrak{g}_r(\mathbb{A}, \mathbb{B})$  are all semi-simple when  $\mathbb{A}, \mathbb{B}$  are among  $0, \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , and moreover, all simple Lie algebras except  $\mathfrak{g}_2$  arise by this construction ( $\mathfrak{g}_2$  can be recovered by supplementing this list with the derivation algebras).

The goal of this section is to show that this construction works with the sextonions, which is not a complexified composition algebra since its natural quadratic form is degenerate. Nevertheless, the definitions of the triality group and algebra make sense.

**Proposition 4.1.** *The triality algebra  $\mathfrak{t}(\mathbb{S}) = \text{Der}(\mathbb{S}) \oplus \text{Im}(\mathbb{S})^{\oplus 2}$ . Its dimension is 18.*

*Proof.* Same proof as in Barton & Sudbery, [2].  $\square$

There is no natural inclusion of  $\mathfrak{t}(\mathbb{S})$  in  $\mathfrak{t}(\mathbb{O})$ , but the subalgebra  $\mathfrak{t}_{\mathbb{S}}(\mathbb{O}) \subset \mathfrak{t}(\mathbb{O})$  of triples  $\theta \in \mathfrak{so}(\mathbb{O})$  such that  $\theta_i(\mathbb{S}) \subset \mathbb{S}$  for  $i = 1, 2, 3$ , is a kind of substitute for  $\mathfrak{t}(\mathbb{S})$ .

**Corollary 4.2.** *The natural morphism  $\mathfrak{t}_{\mathbb{S}}(\mathbb{O}) \rightarrow \mathfrak{t}(\mathbb{S})$  is surjective with one dimensional kernel.*

This allows one to determine the structure of  $\mathfrak{t}(\mathbb{S})$ . Indeed,  $\mathfrak{t}_{\mathbb{S}}(\mathbb{O})$  is the subalgebra of  $\mathfrak{t}(\mathbb{O})$  preserving  $\mathbb{S}$ . This is the same as preserving its orthogonal, which is a null plane  $U$ . The Grassmannian of isotropic planes in  $\mathbb{O}$  is homogeneous under the action of  $T(\mathbb{O}) = \text{Spin}_8$ ; in fact, it is the adjoint variety of  $\text{Spin}_8$ . The stabilizer of an isotropic two plane, for example the stabilizer of  $U$ , is therefore a maximal parabolic subgroup, which can also be defined as the stabilizer of a highest root space. Recall that the choice of a highest root space in  $\bar{\mathfrak{g}} = \mathfrak{so}_8$  induces a 5-grading  $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{-2} \oplus \bar{\mathfrak{g}}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \bar{\mathfrak{g}}_2$ . The stabilizer of the highest root space  $\bar{\mathfrak{g}}_2$  is  $\bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \bar{\mathfrak{g}}_2 = \mathfrak{t}_{\mathbb{S}}(\mathbb{O})$ . We let  $\mathfrak{t}^*(\mathbb{S}) = [\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}_0] \oplus \bar{\mathfrak{g}}_1 \oplus \bar{\mathfrak{g}}_2$ , the intermediate subalgebra of  $\bar{\mathfrak{g}} = \mathfrak{so}_8$ .

Using [26], section 3.4, we can prove:

**Proposition 4.3.** *If  $A, B, C, D$  are 2-dimensional vector spaces, we have identifications*

$$\begin{aligned} \mathfrak{t}(\mathbb{O}) &= \mathfrak{sl}(A) \times \mathfrak{sl}(B) \times \mathfrak{sl}(C) \times \mathfrak{sl}(D) \quad \oplus \quad A \otimes B \otimes C \otimes D \\ \cup &\quad \cup \quad \cup \\ \mathfrak{t}^*(\mathbb{S}) &= \mathfrak{sl}(A) \times \mathfrak{sl}(B) \times \mathfrak{sl}(C) \oplus \mathbb{C} \quad \oplus \quad A \otimes B \otimes C \\ \cup &\quad \cup \\ \mathfrak{t}(\mathbb{H}) &= \mathfrak{sl}(A) \times \mathfrak{sl}(B) \times \mathfrak{sl}(C) \end{aligned}$$

The triality algebra  $\mathfrak{t}(\mathbb{S})$  is then also identified with  $\mathfrak{sl}(A) \times \mathfrak{sl}(B) \times \mathfrak{sl}(C) \oplus \mathbb{C} \oplus A \otimes B \otimes C$ , but the Lie algebra structure is not exactly the same as that of  $\mathfrak{t}^*(\mathbb{S})$ . (In the notation of the introduction,  $\mathfrak{t}(\mathbb{S})$  corresponds to the intermediate algebra  $\mathfrak{g}''$ .)

We include the sextonions in the triality construction by letting

$$\begin{aligned} \mathfrak{g}(\mathbb{A}, \mathbb{S})^+ &= \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}_{\mathbb{S}}(\mathbb{O}) \oplus (\mathbb{A}_1 \otimes \mathbb{S}_1) \oplus (\mathbb{A}_2 \otimes \mathbb{S}_2) \oplus (\mathbb{A}_3 \otimes \mathbb{S}_3), \\ \mathfrak{g}(\mathbb{A}, \mathbb{S}) &= \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}^*(\mathbb{S}) \oplus (\mathbb{A}_1 \otimes \mathbb{S}_1) \oplus (\mathbb{A}_2 \otimes \mathbb{S}_2) \oplus (\mathbb{A}_3 \otimes \mathbb{S}_3). \end{aligned}$$

We can also define  $\mathfrak{g}(\mathbb{S}, \mathbb{S})$  by replacing  $\mathbb{A}$  with  $\mathbb{S}$  and  $\mathfrak{t}(\mathbb{A})$  with  $\mathfrak{t}^*(\mathbb{S})$  in this formula. Since  $\mathfrak{t}^*(\mathbb{S})$  is a subalgebra of  $\mathfrak{t}(\mathbb{O})$ ,  $\mathfrak{g}(\mathbb{A}, \mathbb{S})$  is defined as a subvector space of  $\mathfrak{g}(\mathbb{A}, \mathbb{O})$ .

**Proposition 4.4.**  *$\mathfrak{g}(\mathbb{A}, \mathbb{S})$  is a Lie subalgebra of  $\mathfrak{g}(\mathbb{A}, \mathbb{O})$ .*

*Proof.* From the definition of the Lie bracket of  $\mathfrak{g}(\mathbb{A}, \mathbb{O})$  given in [24], we see that we just need to check that the maps  $\Psi_i : \Lambda^2 \mathbb{O}_i \rightarrow \mathfrak{t}(\mathbb{O})$  take  $\Lambda^2 \mathbb{S}_i \subset \Lambda^2 \mathbb{O}_i$  inside  $\mathfrak{t}^*(\mathbb{S}) \subset \mathfrak{t}(\mathbb{O})$ . This is clear for  $\Psi_1$ , since the image of  $\Psi_1(s, s')$  is just the plane generated by  $s$  and  $s'$ . This is also clear for  $\Psi_2$  and  $\Psi_3$ :  $\Psi_2(s, s')$  and  $\Psi_3(s, s')$  are defined in terms of left and right multiplication by  $s$  or  $s'$ , so that the subalgebra  $\mathbb{S}$  is preserved when  $s$  and  $s'$  belong to it.  $\square$

**Proposition 4.5.** *For  $\mathbb{A} \neq \mathbb{S}$ ,  $\mathfrak{g}(\mathbb{A}, \mathbb{S})$  is the intermediate subalgebra of the simple Lie algebra  $\mathfrak{g}(\mathbb{A}, \mathbb{O})$ . It is a maximal parabolic subalgebra minus the one dimensional center, and its semi-simple part is equal to the simple Lie algebra  $\mathfrak{g}(\mathbb{A}, \mathbb{H})$ .*

*Proof.* We saw in [24] that a Cartan subalgebra of  $\mathfrak{g}(\mathbb{A}, \mathbb{O})$  is given by the product of two Cartan subalgebras in  $\mathfrak{t}(\mathbb{A})$  and  $\mathfrak{t}(\mathbb{O})$ . Moreover, once we have chosen a set of positive roots for  $\mathfrak{t}(\mathbb{O}) = \mathfrak{so}_8$ , its highest root can be chosen as a highest root for  $\mathfrak{g}(\mathbb{A}, \mathbb{O})$ . Since  $\mathfrak{so}_8 = \Lambda^2 \mathbb{O}$ , we can identify the highest root line with an isotropic two plane in  $\mathbb{O}$ , which we can choose to be

the null-plane  $\mathbb{H}^\perp$ . It is then straightforward to check that the stabilizer of the highest root line in the adjoint representation  $\mathfrak{g}(\mathbb{A}, \mathbb{O})$  is exactly  $\mathfrak{g}(\mathbb{A}, \mathbb{S})^+$ , and our first claim follows. The second claim is a simple exercise. Note that  $\mathfrak{g}(\mathbb{A}, \mathbb{H})$  is embedded in  $\mathfrak{g}(\mathbb{A}, \mathbb{O})$  through the natural embedding of  $\mathfrak{t}(\mathbb{H}) \simeq \mathfrak{t}_{\mathbb{H}}(\mathbb{O}) \subset \mathfrak{t}(\mathbb{O})$ , as explained in [26], section 3.6.  $\square$

**Definition 4.6.** We denote by  $\mathfrak{e}_{7\frac{1}{2}}$  the algebra  $\mathfrak{g}(\mathbb{S}, \mathbb{O})$ , which is intermediate between the exceptional algebras  $\mathfrak{e}_7 = \mathfrak{g}(\mathbb{H}, \mathbb{O})$  and  $\mathfrak{e}_8 = \mathfrak{g}(\mathbb{O}, \mathbb{O})$ .

Although we have no direct proof, we observe that, for  $a, b = 0, 1, 2, 4, 6, 8$ :

$$(12) \quad \dim \text{Der}(\mathbb{A}) = \frac{4(a-1)(a-2)}{a+4}$$

$$(13) \quad \dim \mathfrak{t}(\mathbb{A}) = \frac{6a(a-1)}{a+4}$$

$$(14) \quad \dim \mathfrak{g}(\mathbb{A}, \mathbb{B}) = \frac{3(4a+ab+4b-4)(2a+ab+2b)}{(a+4)(4+b)}$$

Here are the resulting algebras  $\mathfrak{g}(\mathbb{A}, \mathbb{B})$  giving rise to an expanded magic chart. The first row is the dimension of  $\mathbb{A}$ . The first column contains the derivation algebras:

$-2/3$	0	1	2	4	6	8
0	0	$A_1$	$A_2$	$C_3$	$C_3.H_{14}$	$F_4$
0	$T_2$	$A_2$	$2A_2$	$A_5$	$A_5.H_{20}$	$E_6$
$A_1$	$3A_1$	$C_3$	$A_5$	$D_6$	$D_6.H_{32}$	$E_7$
$A_1.H_4$	$(3A_1).H_8$	$C_3.H_{14}$	$A_5.H_{20}$	$D_6.H_{32}$	$D_6.H_{32}.H_{44}$	$E_7.H_{56}$
$G_2$	$D_4$	$F_4$	$E_6$	$E_7$	$E_7.H_{56}$	$E_8$

The convention here is that a Lie algebra  $G.H_{2n}$  means that the Lie algebra of type  $G$  has a representation  $V$  of dimension  $2n$  which admits an invariant symplectic form  $\omega$ . Then  $G$  acts on the Heisenberg algebra of  $(V, \omega)$  and  $G.H_{2n}$  denotes the semi-direct product. These algebras are not reductive and the Heisenberg algebra is the radical.

There is another series of Lie algebras, the Barton-Sudbery intermediate Lie algebras of [2]. These are called intermediate because they are intermediate between the derivation algebras and the triality algebras. This gives the following table:

0	0	0
0	$T_1$	$T_2$
$A_1$	$2A_1$	$3A_1$
$A_1.H_4$	$2A_1.H_6$	$3A_1.H_8$
$G_2$	$B_3$	$D_4$

## 5. UNIVERSAL DECOMPOSITIONS

Let  $\mathfrak{g}$  be an intermediate Lie algebra and write  $V = \overline{\mathfrak{g}}_1$ .

**5.1. Decomposition of  $\mathfrak{g} \otimes \mathfrak{g}$ .** In order to decompose  $S^2\mathfrak{g}$ ,  $\Lambda^2\mathfrak{g}$ , we need to understand the decomposition of  $\mathfrak{h} \otimes V$ . This turns out to be uniform:

**Proposition 5.1.** Let  $\mathfrak{g} = \mathfrak{h} \oplus V \oplus \mathbb{C}$  be an intermediate Lie algebra. Then

$$\mathfrak{h} \otimes V = \mathfrak{h}V \oplus V \oplus (\mathfrak{h}V)_{\text{Ad}}$$

where  $(\mathfrak{h}V)_{Aad}$  is as follows:

$$\begin{array}{ccc} \mathfrak{h} & V & (\mathfrak{h}V)_{Aad} \\ \mathfrak{sl}_2 \times \mathfrak{so}_n & \mathbb{C}^2 \otimes \mathbb{C}^n = W \otimes V_{\omega_1} & W \otimes (V_{\omega_1} \oplus V_{\omega_3}) \\ \mathfrak{sl}_n & \mathbb{C}^n \oplus \mathbb{C}^{n*} = V_{\omega_1} \oplus V_{\omega_{n-1}} & V_{\omega_1 + \omega_{n-2}} \oplus V_{\omega_2 + \omega_{n-1}} \\ \mathfrak{c}_n & V_{\omega_1} & V_{\omega_1 + \omega_2} \end{array}$$

and from [25] we recall for the subexceptional series:

	$A_1$	$A_1^{\oplus 3}$	$C_3$	$A_5$	$D_6$	$E_7$
$V$	[3]	[1, 1, 1]	[0, 0, 1]	[0, 0, 1, 0, 0]	[0, 0, 0, 0, 0, 1]	[0, 0, 0, 0, 0, 0, 1]
$\mathfrak{h}$	[2]	[2, 0, 0]	[2, 0, 0]	[1, 0, 0, 0, 1]	[0, 1, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0, 0]
$(\mathfrak{h}V)_{Aad}$	[1]	[1, 1, 1] $\otimes \rho$	[1, 1, 0]	[1, 1, 0, 0, 0]	[1, 0, 0, 0, 1, 0]	[0, 1, 0, 0, 0, 0, 0]

In the column corresponding to  $A_1^{\oplus 3}$ ,  $\rho$  denotes the two-dimensional irreducible representation of  $\Gamma = \mathfrak{S}_3$ .

Given two modules  $V, W$ , the module  $(VW)_{Aad}$  is defined and discussed in [25], section 2.3. In particular, in most cases it may be determined by pictorial methods using Dynkin diagrams. In the case  $W \subseteq I_2(X) \subset S^2V^*$ , where  $X$  is the closed  $G$  orbit in  $\mathbb{P}V$ , then  $(VW)_{Aad}$  is a space of linear syzygies among the quadrics in  $W$ .

Recall the universal decomposition formulas of Vogel  $\Lambda^2\mathfrak{h} = \mathfrak{h} \oplus \mathfrak{h}_2$ ,  $S^2\mathfrak{h} = \mathfrak{h}^2 \oplus \mathfrak{h}_Q \oplus \mathfrak{h}_{Q'} \oplus \mathbb{C}$  [34]. We obtain uniform decompositions of

$$\begin{aligned} S^2\mathfrak{g} &= S^2\mathfrak{h} \oplus S^2V \oplus \mathfrak{h} \otimes V \oplus \mathfrak{h} \oplus V \oplus \mathbb{C}, \\ \Lambda^2\mathfrak{g} &= \Lambda^2\mathfrak{h} \oplus \Lambda^2V \oplus \mathfrak{h} \otimes V \oplus \mathfrak{h} \oplus V. \end{aligned}$$

Here, Vogel's decompositions work if we take

$$\begin{aligned} \mathfrak{g}_2 &= \mathfrak{h}_2 \oplus V_2 \oplus \mathfrak{h}V \oplus (\mathfrak{h}V)_{Aad} \oplus \mathfrak{h} \oplus V, \\ \mathfrak{g}^2 &= \mathfrak{h}^2 \oplus \mathfrak{h}V \oplus V^2 \oplus \mathfrak{h} \oplus V \oplus \mathbb{C}, \\ \mathfrak{g}_Q &= \mathfrak{h}_Q \oplus (\mathfrak{h}V)_{Aad} \oplus V \oplus \mathfrak{h} \end{aligned}$$

(the last equation assumes we are in the case  $\mathfrak{h}_{Q'} = 0$ ). It would be interesting to determine to what extent the Cartan powers of  $\mathfrak{g}_Q$  satisfy the dimension formulas of [27].

**5.2. Cartan powers of  $\mathfrak{g}$ .** One can check that the formulas above really define  $\mathfrak{g}$ -submodules of  $\mathfrak{g} \otimes \mathfrak{g}$ . For example,  $\mathfrak{g}_2$  is the  $\mathfrak{g}$ -submodule of  $\Lambda^2\mathfrak{g}$  generated by  $\mathfrak{h}_2$ .

In general, given an irreducible finite dimensional  $\bar{\mathfrak{g}}$ -module  $W_\lambda$  with highest weight line  $\ell_\lambda$ , we can define a highest weight  $\mathfrak{g}$ -module module  $V_\lambda$  by taking  $V_\lambda = U(\mathfrak{g})\ell_\lambda$ . Note that this is the same as taking  $V_\lambda = U(\bar{\mathfrak{g}}_1)W'_\lambda = S^\bullet(\bar{\mathfrak{g}}_1)W'_\lambda$ , where  $W'_\lambda$  is the  $\mathfrak{h}$ -module  $U(\mathfrak{h}).\ell_\lambda$ . As in [31], where the case of classical intermediate algebras was studied, weights  $\lambda, \mu$  of  $\bar{\mathfrak{g}}$  will give rise to the same  $\mathfrak{g}$  module if and only if they project to the same weight in the weight lattice of  $\mathfrak{h}$  (considered as a subspace of the weight lattice of  $\bar{\mathfrak{g}}$ ).

In general we have no effective way of computing  $V_\lambda$  from  $W_\lambda$  but we do have the following special case:

**Proposition 5.2.** *Suppose that the highest weights of  $V$  and  $\mathfrak{h}$  are linearly independant. Then, as an  $\mathfrak{h}$ -module,*

$$\mathfrak{g}^{(k)} = \bigoplus_{p+q \leq k} \mathfrak{h}^{(p)} V^{(q)}.$$

*Proof.* As a subspace of  $S^k \mathfrak{g}$ , the Cartan power  $\mathfrak{g}^{(k)}$  is generated by the powers  $x^k$  of the highest weight vectors of  $\mathfrak{h}$ , and their images by successive applications of vectors in  $V = \mathfrak{g}_1$ . For  $v, w \in V$ , we have

$$\begin{aligned} \text{ad}(v)x^k &= kx^{k-1}(xv), \\ \text{ad}(w)\text{ad}(v)x^k &= k(k-1)x^{k-2}(xv)(xw) + kx^{k-1}\omega(xv, w). \end{aligned}$$

First observe that the last expression is symmetric in  $v$  and  $w$ , so that the action of  $V$  induces an action of  $\text{Sym}(V)$ . Second, the last term is a multiple of  $x^{k-1}$ , and that kind of terms generate  $\mathfrak{g}^{(k-1)}\mathfrak{g}_2$ . By induction on  $k$ , we are reduced to proving that the  $\mathfrak{h}$ -module spanned by tensors of the form  $x^{k-q}(xv)^q$ , for  $x$  a highest weight vector of  $\mathfrak{g}_0$  and  $v \in V$ , is a copy of  $\mathfrak{h}^{(k-q)}V^{(q)}$ . Since it follows from the hypothesis that the weights of these modules, as  $k$  and  $q$  vary, are distinct, Schur's lemma will imply our claim.

We first prove that we can suppose that  $xv$  is a highest weight vector in  $V$ . To see this, recall that the image of  $x$  in  $\mathbb{P}V$  is the linear space denoted  $\langle q^x \rangle$  in §2, and  $\langle q^x \rangle \cap X_H^{\text{ad}}$  is a smooth quadric hypersurface in  $\langle q^x \rangle$  whose equation is

$$0 = q^x(xu, xv) = \omega(u, xv) = \omega(v, xu).$$

The first expression shows that this quantity does not depend on  $v$ , but only on  $xv$ , and the second one shows that it does not depend on  $u$ , but only on  $xu$ . Let  $V_x = x \cdot V \subset V$  denote the deprojectivization of  $\langle q^x \rangle$ . We get  $S^q V_x = S^{(q)}V_x \oplus q^x S^{(q-2)}V_x \oplus \dots$ . Since  $q^x$  is given by expressions of type  $\omega(v, xu)$ , it must be considered as belonging to  $\mathfrak{g}_2$ , and we remain with  $S^{(q)}V_x$  only. By definition, this space is generated by  $q$ -th powers of vectors that belong to the quadric hypersurface  $q^x = 0$ , hence also to the cone over the closed  $G_0$ -orbit in  $\mathbb{P}V$ . This proves our claim that we can suppose  $xv$  to be a highest weight vector.

The stabilizers  $\mathfrak{g}_0^x$  and  $\mathfrak{g}_0^{xv}$  are two parabolic subalgebras of  $\mathfrak{g}_0$ . Their intersection must therefore contain a Cartan subalgebra, and we can choose a Borel subalgebra containing this Cartan subalgebra and contained in  $\mathfrak{g}_0^x$ . In other words, we may suppose that  $x = x_{\tilde{\alpha}}$  is a highest root vector of  $\mathfrak{g}_0$ , while  $xv$  is a weight vector of  $V$ . Of course we can also suppose that  $v$  itself is a weight vector, say of weight  $\mu$ , so that the weight of  $xv$  is  $\mu + \tilde{\alpha}$ .

Now we use the fact that  $V$  is a minuscule  $\mathfrak{g}$ -module. In particular,  $\mu(H_{\tilde{\alpha}})$  and  $(\mu + \tilde{\alpha})(H_{\tilde{\alpha}}) = \mu(H_{\tilde{\alpha}}) + 2$  belong to  $\{-1, 0, +1\}$ , hence  $\mu(H_{\tilde{\alpha}}) = -1$ . For simplicity, suppose that  $\mathfrak{g}_0$  is not of type  $A$ , so that the highest root is a multiple of a fundamental weight  $\omega_{\gamma}$ . Then  $(\mu + \tilde{\alpha})(H_{\gamma}) > \mu(H_{\gamma}) \geq -1$ , so  $(\mu + \tilde{\alpha})(H_{\gamma}) \geq 0$ . If  $(\mu + \tilde{\alpha})(H_{\beta}) \geq 0$  for every simple root  $\beta$ , then  $\mu + \tilde{\alpha}$  is a dominant weight, hence the highest weight of  $V$ . If  $(\mu + \tilde{\alpha})(H_{\beta}) < 0$  for some simple root  $\beta$ , necessarily distinct from  $\gamma$ , then the corresponding reflection stabilizes  $\tilde{\alpha}$ , but changes  $\mu + \tilde{\alpha}$  into the greater root  $\mu + \tilde{\alpha} + \beta$ . By induction, we may therefore suppose that the weight  $\mu + \tilde{\alpha}$  of  $xv$  is the highest weight of  $V$ . Then  $x^{k-q}(xv)^q$  is a highest weight vector of the Cartan product  $\mathfrak{g}_0^{(k-q)}\mathfrak{g}_1^{(q)}$ , and we are done.  $\square$

*Remark.* The only case of rank greater than two, for which the hypothesis of Proposition 5.2 does not hold, is when  $\mathfrak{g} = \mathfrak{sp}_{2n+1}$  is an odd symplectic Lie algebra. The highest weight of  $\mathfrak{h} = \mathfrak{sp}_{2n}$  is  $2\omega_1$ , twice the weight of  $V = \mathbb{C}^{2n}$ . The Proposition does not hold in that case, but it is easy to see that as an  $\mathfrak{sp}_{2n}$ -module

$$\mathfrak{sp}_{2n+1}^{(k)} = S^{2k}\mathbb{C}^{2n} \oplus S^{2k-1}\mathbb{C}^{2n} \oplus \dots \oplus \mathbb{C}^{2n} \oplus \mathbb{C} \simeq S^{2k}(\mathbb{C}^{2n} \oplus \mathbb{C}).$$

*Remark.* Consider the intermediate Lie algebra of  $\mathfrak{sl}_{n+2}$ , that we denote by

$$\tilde{\mathfrak{sl}}_{n+1} = \mathfrak{sl}_n \oplus \mathbb{C}^n \oplus (\mathbb{C}^n)^* \oplus \mathbb{C}.$$

By the previous theorem, the decomposition of its Cartan powers into  $\mathfrak{sl}_n$ -modules is

$$\tilde{\mathfrak{sl}}_{n+1}^{(k)} = \bigoplus_{p+q+r \leq k} V_{(p+q)\omega_1 + (p+r)\omega_{n-1}}.$$

This is exactly the formula for the restriction of the  $\mathfrak{sl}_{n+1}$ -module  $\mathfrak{sl}_{n+1}^{(k)}$  to  $\mathfrak{sl}_n$  given by the usual branching rule. We therefore have two different Lie algebras,  $\mathfrak{sl}_{n+1}$  and  $\tilde{\mathfrak{sl}}_{n+1}$ , not only with the same dimension, but such that in any degree, their Cartan powers have the same dimensions.

## 6. DECOMPOSITION FORMULAS IN THE MAGIC CHART

The sextonions allow one to add a new column to Freudenthal's magic square. We know that for each row of the original square, there are a few preferred representations, leading to nice dimension and decomposition formulas for some of their plethysms (see [25]). In this section we address the problem of extending these results to the sextonionic case. What the preferred representations should be is easy to imagine: take a preferred representation  $V_{\mathbb{O}}$  from the octonionic column; it contains a preferred representation  $V_{\mathbb{H}}$  from the quaternionic column, and the sextonionic representation  $V_{\mathbb{S}}$  is simply the  $\mathfrak{g}$ -submodule of  $V_{\mathbb{O}}$  generated by  $V_{\mathbb{H}}$ , where  $\mathfrak{g}$  is the intermediate Lie algebra.

We adopt the notation  $V_0 = \mathbb{C} \oplus \bar{\mathfrak{g}}_1$ . In several of the modules below  $V_0$  will replace the trivial representation in the decomposition formulas. This makes sense when the trivial representation corresponds to the copy of  $\mathbb{C}$  in  $\bar{\mathfrak{g}}_0$ .

**6.1. First row.** Here we have one distinguished representation, call it  $V = \mathcal{J}_3(\mathbb{S})_0$ . It is the complement of the symplectic form in  $\Lambda^2 \mathbb{C}^6 \oplus \mathbb{C}^{6*} = \Lambda^2(\mathbb{C}^6 + \mathbb{C})$ .

As graded  $\mathfrak{sp}_6$ -modules, we have

$$\begin{aligned} (15) \quad & V = V_{\omega_2} \oplus V_{\omega_1} \\ (16) \quad & \mathfrak{g} = V_{2\omega_1} \oplus V_{\omega_3} \oplus \mathbb{C} \\ (17) \quad & V_2 = V_{\omega_1+\omega_3} \oplus (V_{\omega_1+\omega_2} \oplus V_{\omega_1}) \oplus V_{\omega_2} \end{aligned}$$

We have the following decomposition formulas, which agree with those in [25]:

$$\begin{aligned} (18) \quad & S^2 V = V^2 \oplus V \oplus V_0 \\ (19) \quad & \Lambda^2 V = \mathfrak{g} \oplus V_2. \end{aligned}$$

**6.2. Second row.** Here, we have two dual distinguished representations, call one of them  $V = \mathcal{J}_3(\mathbb{S}) = \Lambda^2(\mathbb{C}^6 \oplus \mathbb{C})$ . As graded  $\mathfrak{sl}_6$ -modules, we have

$$\begin{aligned} (20) \quad & V = V_{\omega_2} \oplus V_{\omega_5} \\ (21) \quad & V^* = V_{\omega_4} \oplus V_{\omega_1} \\ (22) \quad & \mathfrak{g} = V_{\omega_1+\omega_5} \oplus V_{\omega_3} \oplus \mathbb{C} \\ (23) \quad & V_0 = \mathbb{C} \oplus V_{\omega_3} \end{aligned}$$

The  $\mathfrak{sl}_6$ -module  $V$  is exceptional in the sense of [3] and its symmetric algebra behaves the same as the rest of the Severi series, namely

$$S^d V = \bigoplus_{i+2j+3k=d} V^{(i)}(V^*)^{(j)}(V_0)^{(k)}$$

where we take  $\mathfrak{sl}_6$ -Cartan products in the factors.

**6.3. Third row.** There are three distinguished representations, which we call  $V = \mathcal{Z}_2(\mathbb{S}), V_2, \mathfrak{g}$ . As graded  $\mathfrak{so}_{12}$ -modules they are

$$\begin{aligned} (24) \quad & V = V_{\omega_6} \oplus V_{\omega_1} \\ (25) \quad & \mathfrak{g} = V_{\omega_2} \oplus V_{\omega_5} \oplus \mathbb{C} \\ (26) \quad & V_2 = V_{\omega_4} \oplus (V_{\omega_1+\omega_6} \oplus V_{\omega_5}) \oplus V_{\omega_2} \\ (27) \quad & V_0 = \mathbb{C} \oplus V_{\omega_5} \end{aligned}$$

Here again,  $V$  is exceptional in the sense of [3] and its symmetric algebra behaves the same as the rest of the subexceptional series, namely

$$S^d V = \bigoplus_{i+2j+3k+4l+4m=d} V^{(i+k)} \mathfrak{g}^{(j)} V_0^{(l)} V_2^{(m)}$$

Here some care must be taken in interpreting the formula. In Brion's list there are 14 generators of the symmetric algebra which do not coincide with the generators we use. The critical difference is that the product  $\mathfrak{g}V$ , is not the Cartan product as  $\mathfrak{sl}_6$ -modules, but instead

$$\mathfrak{g}V = V_{\omega_2+\omega_6} \oplus (V_{\omega_5+\omega_6} \oplus V_{\omega_3}) \oplus V_{\omega_5}$$

where note that the  $V_{\omega_3}$  would not appear in the  $\mathfrak{sl}_6$  Cartan product. All other products coincide with the Cartan product in  $\mathfrak{sl}_6$ . Thus the interpretation of the algebra structure is different.

The justification for  $\mathfrak{g}V$  is as follows. In  $V_{\omega_2} \otimes V_{\omega_6}$ , the submodule  $V_{\omega_2+\omega_6}$  is generated by tensors of the form  $P \otimes S$  with  $P \in G_Q(2, 12)$ ,  $S \in \mathbb{S}_6 \subset G_Q(6, 12)$  an isotropic 6-plane, where  $P \subset S$ . We have a map  $V_{\omega_5} \otimes V_{\omega_6} \rightarrow V_{\omega_1}$ , which may be seen geometrically as follows. Let  $S' \in \mathbb{S}'_6 \subset \mathbb{P}V_{\omega_5}$  be a 6-plane in the other family. Generically  $S \cap S'$  is a point of the quadric. This geometric intersection extends to a linear map  $V_{\omega_5} \otimes V_{\omega_6} \rightarrow V_{\omega_1}$ . The action of  $V_{\omega_5}$  thus produces tensors of the form  $v \otimes P$  with no incidence condition on  $v$  and  $P$ , in particular a projection to  $V_{\omega_3}$  by wedging them together.

## 7. DIMENSION FORMULAS

We have the following generalizations of the theorems in [24]:

**Theorem 7.1.** *Let  $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{g}_2, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_{7\frac{1}{2}} = \mathfrak{e}_7 \oplus V_{\omega_7} \oplus \mathbb{C}, \mathfrak{e}_8$ , with respectively  $a = -4/3, -1, -2/3, 0, 1, 2, 4, 6, 8$ . Then for all  $k \geq 0$ ,*

$$\dim \mathfrak{g}^{(k)} = \frac{3a + 2k + 5}{3a + 5} \frac{\binom{k+2a+3}{k} \binom{k+\frac{5a}{2}+3}{k} \binom{k+3a+4}{k}}{\binom{k+\frac{a}{2}+1}{k} \binom{k+a+1}{k}}.$$

**Theorem 7.2.** *Let  $V$  be the distinguished module, of dimension  $6a + 8$ , of a Lie algebra  $\mathfrak{g}$  in the subexceptional series, with  $a = -\frac{2}{3}, 0, 1, 2, 4, 6, 8$ . Then*

$$\begin{aligned} \dim \mathfrak{g}^{(k)} &= \frac{2k+2a+1}{2a+1} \frac{\binom{k+\frac{3a}{2}-1}{k} \binom{k+\frac{3a}{2}+1}{k} \binom{k+2a}{k}}{\binom{k+\frac{a}{2}-1}{k} \binom{k+\frac{a}{2}+1}{k}}, \\ \dim V^{(k)} &= \frac{a+k+1}{a+1} \frac{\binom{k+2a+1}{k} \binom{k+\frac{3a}{2}+1}{k}}{\binom{k+\frac{a}{2}}{k}}, \\ \dim V_2^{(k)} &= \frac{(4k+3a+2)}{(k+1)(3a+2)} \frac{\binom{k+a}{k} \binom{k+a+1}{k} \binom{k+\frac{3a}{2}-1}{k} \binom{k+\frac{3a}{2}}{k} \binom{2k+2a+1}{2k}}{\binom{k+\frac{a}{2}-1}{k} \binom{k+\frac{a}{2}}{k} \binom{2k+a}{2k}}. \end{aligned}$$

**Theorem 7.3.** *Let  $V$  be the distinguished module in the Severi series, with  $a = -\frac{2}{3}, 0, 1, 2, 4, 6, 8$ . Then*

$$\dim V^{(k)} = \frac{(2k+a)(k+a)}{a^2} \frac{\binom{k+a-1}{k} \binom{k+\frac{3a}{2}-1}{k}}{\binom{k+\frac{a}{2}}{k}}.$$

Unfortunately our proofs are just case by case applications of the Weyl dimension formulas, plus the decomposition formulas from Proposition 5.2 and the previous section. Even then we obtain in each case a polynomial  $P(k)$  of the correct degree, but that is not obviously the same polynomial as obtained above. To check we used Maple to test that the two polynomials agree on  $\deg P + 1$  points and therefore must be equal.

Here are outlines of the proofs:



**Severi case.** The triality formula for  $a = 6$  predicts

$$\dim V^{(k)} = \frac{(2k+6)(k+6)}{36} \frac{\binom{k+5}{k} \binom{k+8}{k}}{\binom{k+3}{k}}$$

which, as a function of  $k$  is a polynomial of degree 12. We compare with the Weyl dimension formula applied to the  $\mathfrak{sl}_6$ -module

$$V^k = (V_{\omega_2} \oplus V_{\omega_5})^{(k)} = \sum_{i=0}^k V_{(k-i)\omega_2 + i\omega_5}$$

which also gives a polynomial of degree 12 in  $k$ . First note that for all positive roots  $\alpha$  we have  $(\omega_2, \alpha)$  and  $(\omega_5, \alpha)$  either 0 or 1. Separate the positive roots of  $\mathfrak{sl}_6$  into four groups accordingly:  $\Delta_{0,0}, \Delta_{1,0}, \Delta_{0,1}, \Delta_{1,1}$  where the first subscript is  $(\omega_2, \alpha)$  and the second is  $(\omega_5, \alpha)$ .  $\Delta_{0,0}$  has four elements, three of which have  $(\rho, \alpha) = 1$  and one with  $(\rho, \alpha) = 2$ .  $\Delta_{1,0}$  has three elements, with  $(\rho, \alpha) = 1, 2, 3$ .  $\Delta_{0,1}$  has six elements, with  $(\rho, \alpha) = 1, 2, 2, 3, 3, 4$ .  $\Delta_{1,1}$  has two elements, with  $(\rho, \alpha) = 4, 5$ . Thus the numerator in the WDF becomes

$$2(k+5)(k+4) \left[ \sum_{i=0}^k (k-i+2)^2 (k-i+3)^2 (k-i+1)(k-i+4)(i+3)(i+2)(i+1) \right]$$

Dividing by the denominator, and considering, e.g., the  $i = [k/2]$  term), we obtain another polynomial that is a sum of  $k+1$  terms of degree 11 in  $k$ , but these terms collapse by using formulas for  $\sum_k i^k$  to give a polynomial of degree twelve. One then easily checks they agree for the first 13 values of  $k$  so they must be equal.

**Exceptional row.** For  $\mathfrak{g}^k$  in the exceptional row the dimension of the relevant  $\mathfrak{e}_7$  modules are as follows:

$$\dim V_{i\omega_1 + j\omega_7} = \frac{(j+5)(2i/17 + j/17 + 1) \binom{j+9}{9} \binom{11+i}{11} \binom{8+i}{8} \binom{16+i+j}{16} \binom{13+i+j}{13}}{5 \binom{3+i}{3} \binom{8+i+j}{8} \binom{5+i+j}{5}}$$

For  $\dim \mathfrak{g}^{(k)}$ , one takes the sum over  $i+j \leq k$  and compares it with the triality formula. Both are polynomials of degree 45 in  $k$  but they are not obviously equal so we evaluated them both at 45 points (plus zero) to check their equality.

**Subexceptional row.** We calculate as above. The relevant dimensions of the  $\mathfrak{so}_{12}$ -modules that need to be summed over are respectively

$$\begin{aligned} \dim V_{i\omega_5 + j\omega_2} &= \frac{(2i+j+9)(j+3) \binom{i+5}{5} \binom{i+4}{4} \binom{i+j+8}{8} \binom{i+j+7}{7} \binom{j+5}{5}}{27(i+1) \binom{i+j+3}{3} \binom{i+j+4}{4}} \\ \dim V_{a\omega_4 + b(\omega_1 + \omega_6) + c\omega_5 + d\omega_2} &= \frac{1}{158018273280000} (1+b)^2 (2+b+d)(3+b+d)(4+a+b+d)^2 (5+a+b+c+d)^2 (1+d)(2+d) \\ &\quad \times (3+a+d)(4+a+c+d)(2+a)(3+a+c)(1+a)(2+a+c)(1+c)(9+2a+2b+c+2d) \\ &\quad \times (8+2a+2b+c+d)(7+2a+2b+c+d)(6+a+2b+c+d)(5+a+2b+d)(7+2a+b+c+d) \\ &\quad \times (6+2a+b+c+d)(5+2a+b+c)(4+a+b+c)(3+a+b)(3+a+b+c)(2+a+b) \\ \dim V_{a\omega_6 + b\omega_1} &= \frac{1}{548674560000} (1+b)(2+b)(3+b)(4+b)(5+b)(9+a+b)(8+a+b)(7+a+b) \\ &\quad \times (6+a+b)(5+a+b)(7+a)(6+a)(5+a)^2 (4+a)^2 (3+a)^2 (2+a)(1+a). \end{aligned}$$

*Remark.* This raises an obvious question. To what extent are the dimension formulas proved in [24, 27], valid for intermediate Lie algebras? In particular, in [27] we gave a general dimension formula for the Cartan powers of a simple Lie algebra  $\mathfrak{g}$  in terms of its Vogel's parameters  $\alpha, \beta, \gamma$ . Theorem 7.1 is the specialization of that formula to the exceptional series, and extends to the intermediate Lie algebra  $\mathfrak{e}_{7\frac{1}{2}}$  with Vogel's parameters  $\alpha = -2, \beta = 10, \gamma = 16$ .

Also, the remark we made at the end of section 5 shows that the formula for  $\dim \mathfrak{g}^{(k)}$  holds for  $\tilde{\mathfrak{sl}}_n$  with the same parameters  $\alpha = -2$ ,  $\beta = 2$ ,  $\gamma = n$  as for  $\mathfrak{sl}_n$ . Another interesting case is the intermediate Lie algebra of  $\mathfrak{sp}_{2n+2}$ , the odd symplectic algebra

$$\mathfrak{sp}_{2n+1} = \mathfrak{sp}_{2n} \oplus \mathbb{C}^{2n} \oplus \mathbb{C}.$$

We have seen that as an  $\mathfrak{sp}_{2n}$ -module,  $\mathfrak{sp}_{2n+1}^{(k)} \simeq S^{2k}(\mathbb{C}^{2n} \oplus \mathbb{C})$ , which has dimension  $\binom{2n+2k}{2k}$ . Again, that's exactly what our dimension formula predicts for Vogel's parameters  $\alpha = -2$ ,  $\beta = 1$ ,  $\gamma = n + \frac{5}{2}$ .

*Question:* How could one incorporate the intermediate Lie algebras into the formalism of the universal Lie algebra developed by Vogel and Deligne? A first obstacle is that we not longer have an invariant quadratic form, which was a basic ingredient in their categorical constructions.

## 8. SEXTONIONIC GEOMETRY

In this section we study a few projective varieties that can be defined naturally in terms of the sextonions, in the same way as some more familiar varieties are defined in terms of the usual (complexified) composition algebras. In particular, we investigate in some detail the geometry of the projective plane over  $\mathbb{S}$ , which is a singular but close cousin of the famous four Severi varieties  $\mathbb{AP}^2$ , for  $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ . Then we consider the Grassmannian  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ , again a singular variety but which shares the very nice geometric properties of the smooth varieties  $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$  for  $\mathbb{A} = 0, \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$  [23].

**8.1.  $\mathbb{S}$ -lines.** For  $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , an  $\mathbb{A}$ -line is a smooth quadric of dimension  $a$ , and can be described as the image of the Veronese map

$$\nu_2 : \mathbb{P}(\mathbb{A} \oplus \mathbb{A}) \dashrightarrow \mathbb{P}\mathcal{J}_2(\mathbb{A}), \quad \nu_2(x, y) = \begin{pmatrix} x\bar{x} & x\bar{y} \\ y\bar{x} & y\bar{y} \end{pmatrix}.$$

Here  $\mathcal{J}_k(\mathbb{A})$  denotes the algebra of Hermitian matrices of order  $k$  with coefficients in  $\mathbb{A}$ . The image of this map is the quadric defined by the vanishing of the determinant.

All this makes perfect sense for  $\mathbb{A} = \mathbb{S}$ , except that the determinantal quadric in  $\mathbb{P}\mathcal{J}_2(\mathbb{S})$  is not smooth. Indeed,  $\mathcal{J}_2(\mathbb{S}) = \mathcal{J}_2(\mathbb{H}) \oplus \mathcal{A}_2(\mathbb{H}^\perp)$ , where  $\mathcal{A}_2(\mathbb{H}^\perp)$  denotes the (two-dimensional) space of skew-symmetric matrices with coefficients in  $\mathbb{H}^\perp \subset \mathbb{S}$ . If we write a matrix  $M \in \mathcal{J}_2(\mathbb{S})$  as  $M = R + S$ , with  $R \in \mathcal{J}_2(\mathbb{H})$  and  $S \in \mathcal{A}_2(\mathbb{H}^\perp)$ , then  $\det(M) = \det(R)$ . We conclude that:

*An  $\mathbb{S}$ -line  $\mathbb{SP}^1$  is a singular quadric of dimension 6 in  $\mathbb{P}\mathcal{J}_2(\mathbb{S}) \simeq \mathbb{P}^7$ , singular along a line.*

**8.2. The sextonionic plane.** For  $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , the  $\mathbb{A}$ -plane  $\mathbb{AP}^2$  can be defined as the image of the Veronese map

$$\nu_2 : \mathbb{P}(\mathbb{A} \oplus \mathbb{A} \oplus \mathbb{A}) \dashrightarrow \mathbb{P}\mathcal{J}_3(\mathbb{A}), \quad \nu_2(x, y, z) = \begin{pmatrix} x\bar{x} & x\bar{y} & x\bar{z} \\ y\bar{x} & y\bar{y} & y\bar{z} \\ z\bar{x} & z\bar{y} & z\bar{z} \end{pmatrix}.$$

While  $\mathcal{J}_3(\mathbb{S})$  is a Jordan algebra, in fact a Jordan subalgebra of the exceptional simple Jordan algebra  $\mathcal{J}_3(\mathbb{O})$ , it is *not simple*. In fact, we can write  $\mathcal{J}_3(\mathbb{S}) = \mathcal{J}_3(\mathbb{H}) \oplus \mathcal{A}_3(\mathbb{H}^\perp)$ . A computation shows that  $\mathcal{A}_3(\mathbb{H}^\perp)$  is a two-sided Jordan ideal of  $\mathcal{J}_3(\mathbb{S})$ , and its square is obviously zero. Therefore,  $\mathcal{A}_3(\mathbb{H}^\perp)$  is the radical of  $\mathcal{J}_3(\mathbb{S})$ , whose semi-simple part is  $\mathcal{J}_3(\mathbb{H})$ .

**Proposition 8.1.** *The derivation algebra of  $\mathcal{J}_3(\mathbb{S})$  is  $\text{Der } \mathcal{J}_3(\mathbb{S}) \simeq \mathfrak{g}(\mathbb{R}, \mathbb{S})$ .*

The same statement holds for the normed algebras, and the proof of [2] works for  $\mathbb{S}$  without change.

Let  $x, y, z \in \mathbb{H}$  and  $r, s, t \in \mathbb{H}^\perp$ . Then

$$\nu_2(x + r, y + s, z + t) = \begin{pmatrix} x\bar{x} & x\bar{y} & x\bar{z} \\ y\bar{x} & y\bar{y} & y\bar{z} \\ z\bar{x} & z\bar{y} & z\bar{z} \end{pmatrix} + \begin{pmatrix} 0 & r\bar{y} - xs & r\bar{z} - xt \\ s\bar{x} - yr & 0 & s\bar{z} - yt \\ t\bar{x} - zr & t\bar{y} - zs & 0 \end{pmatrix}.$$

The first summand is in  $\mathcal{J}_3(\mathbb{H})$ , and the second in  $\mathcal{A}_3(\mathbb{H}^\perp)$ , since  $\mathbb{H}^\perp$  is a two-sided ideal of  $\mathbb{S}$ .

Now recall that there is a natural identification of  $\mathcal{J}_3(\mathbb{H})$  with  $\Lambda^2\mathbb{C}^6$ , such that the  $\mathbb{H}$ -plane  $\mathbb{H}\mathbb{P}^2$  is identified with the Grassmannian  $G(2, 6) \subset \mathbb{P}\Lambda^2\mathbb{C}^6$ . Let  $W = \mathbb{C}^6$ .

**Proposition 8.2.** *There is a natural identification of  $\mathcal{J}_3(\mathbb{S})$  with  $V = \Lambda^2W \oplus W^*$ , such that  $\mathbb{S}\mathbb{P}^2$  is identified with the closure of the set of pairs  $[\sigma, w] \in \mathbb{P}V$ , where  $\sigma$  belongs to  $G(2, 6)$ , and  $w$  represents a hyperplane of  $W$  containing the plane  $\sigma$ .*

*Proof.* The fact that  $\mathcal{J}_3(\mathbb{S})$  may be identified with  $V = \Lambda^2W \oplus W^*$  was noticed in 6.2. Now  $\mathbb{S}\mathbb{P}^2$  is a subvariety of  $\mathbb{P}V$ , stable under the natural action of the intermediate Lie algebra  $\mathfrak{g} = \mathfrak{sl}_6 \oplus \Lambda^3\mathbb{C}^6 \oplus \mathbb{C}$ . An easy explicit computation shows that it contains the set of pairs  $[\sigma, w]$ , where  $\sigma$  represents a plane contained in the hyperplane defined by  $w$ . But this is a rank-four vector bundle over  $G(2, 6)$ , hence an irreducible variety of dimension 12, hence an open subset of the irreducible variety  $\mathbb{S}\mathbb{P}^2$ . This implies our claim.  $\square$

**Corollary 8.3.** *The variety  $\mathbb{S}\mathbb{P}^2$  is singular along  $\mathbb{P}W^* \simeq \mathbb{P}^5$ .*

This is in agreement with the principle stated in [6], following which the very nice algebraic properties of the normed algebras have their geometric counterpart in the smoothness of the associated projective varieties. For example,  $\mathcal{J}_k(\mathbb{O})$  is no longer a Jordan algebra for  $k \geq 4$ , and every natural definition of the  $\mathbb{O}$ -projective space  $\mathbb{O}\mathbb{P}^{k-1}$  gives a singular variety.

**Corollary 8.4.** *The action of  $PSL_6$  on  $\mathbb{S}\mathbb{P}^2$  has three orbits: the singular locus  $\mathbb{P}W^*$ , the Grassmannian  $G(2, 6) \subset \mathbb{P}\Lambda^2W$ , and their complement. The smooth locus of  $\mathbb{S}\mathbb{P}^2$  is the total space of a rank four homogeneous vector bundle over  $G(2, 6)$ .*

The projective planes  $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$  are the four Severi varieties, the only smooth  $n$ -dimensional varieties  $X \subset \mathbb{P}^m$ , with  $m = \frac{3n}{2} + 2$ , whose secant variety (the determinantal cubic) is not the whole ambient space. The  $\mathbb{S}$ -plane has the same properties, except that it is not smooth, as we have just seen. (Note that,  $\mathbb{S}\mathbb{P}^2$  is not optimal for Zak's theorem on singular varieties with secant defect, see [36], II.2.8, although it is naturally contained in  $J(\mathbb{P}W^*, \mathbb{H}\mathbb{P}^2)$  defined below, which is optimal, and the two varieties have the same secant variety.)

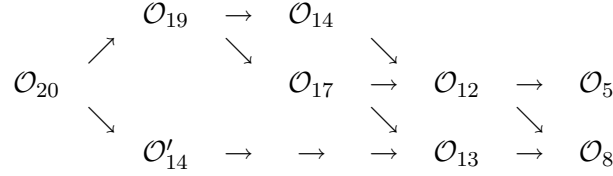
**Proposition 8.5.** *The secant variety of  $\mathbb{S}\mathbb{P}^2$  is the determinantal cubic, a cone over the determinantal cubic in  $\mathbb{P}\mathcal{J}_3(\mathbb{H})$ .*

*Proof.* The secant variety is clearly contained in the determinantal cubic. Equality means that any pair  $(\omega, h)$ , where  $\omega \in \Lambda^2W$  has rank four and  $h$  is a generic linear form, can be written as a sum  $(\alpha, k) + (\beta, l)$ , where  $\alpha, \beta$  have rank two, and  $k$  (respectively  $l$ ) defines a hyperplane containing the plane  $A$  (respectively  $B$ ) defined by  $\alpha$  (respectively  $\beta$ ). This implies that  $k|_B = h|_B$  and  $l|_A = h|_A$ . Conversely, we can choose any decomposition  $\omega = \alpha + \beta$  into a sum of rank two elements, define  $k$  and  $l$  on  $A \oplus B$  by the conditions that  $k|_A = 0$ ,  $k|_B = h|_B$  and  $l|_A = h|_A$ ,  $l|_B = 0$ , and then adjust freely on a complement  $C$  of  $A \oplus B$  so that  $(k + l)|_C = h|_C$ . Then  $h = k + l$ , and we are done.  $\square$

### 8.3. Orbits in $\mathcal{J}_3(\mathbb{S})$ .

**Proposition 8.6.** *The action of  $PSL_6$  on  $\mathbb{P}V$  is prehomogeneous. The open orbit is the complement of the determinantal cubic and the linear subspace  $\mathbb{P}\Lambda^2W \simeq \mathbb{P}^{14}$ . In fact there are exactly nine  $PSL_6$ -orbits in  $\mathbb{P}V$ .*

The orbits are very easy to describe. For a pair  $(\omega, h) \in \Lambda^2 W$ , the rank of  $\omega$  can be 0, 2, 4 or 6, and  $h$  can define a hyperplane containing or not the kernel of  $\omega$ , or be zero. The incidence diagram is as follows, where  $\mathcal{O}_k$  denotes an orbit of dimension  $k$ :



It is more natural to consider the action on  $\mathbb{P}\mathcal{J}_3(\mathbb{S})$  of the automorphism group of  $\mathcal{J}_3(\mathbb{S})$ , or of the group  $PSL(3, \mathbb{S})$  preserving the determinant. We define  $SL(3, \mathbb{S})$  to be the closed subgroup of  $GL(\mathcal{J}_3(\mathbb{S}))$  with Lie algebra  $\mathfrak{g}(\mathbb{C}, \mathbb{S}) \simeq Der(\mathcal{J}_3(\mathbb{S})) \oplus \mathcal{J}_3(\mathbb{S})_0$ , where the space of traceless matrices  $\mathcal{J}_3(\mathbb{S})_0$  acts on  $\mathcal{J}_3(\mathbb{S})$  by multiplication. (Recall that the Lie algebra structure follows from the fact that for any  $x, y \in \mathcal{J}_3(\mathbb{S})$ , the bracket  $D_{x,y} = [M_x, M_y]$  of the multiplication operators by  $x$  and by  $y$ , is a derivation of  $\mathcal{J}_3(\mathbb{S})$ .) In fact,  $\mathfrak{g}(\mathbb{C}, \mathbb{S})$  is our intermediate Lie algebra  $\mathfrak{g}$ .

Clearly,  $G(2, 6)$  is *not* stable under the action of  $PSL(3, \mathbb{S})$ , since otherwise  $\Lambda^2 \mathbb{C}^6$  would be stable under the action of  $\mathfrak{g}(\mathbb{C}, \mathbb{S})$ . We conclude:

**Proposition 8.7.** *The action of  $PSL(3, \mathbb{S})$  on  $\mathbb{S}\mathbb{P}^2$  has only two orbits: the singular locus  $\mathbb{P}W^*$ , and the smooth locus.*

We now examine the  $PSL(3, \mathbb{S})$  orbits in  $\mathbb{P}\mathcal{J}_3(\mathbb{S})$ . Let  $J(\mathbb{P}W^*, \mathbb{H}\mathbb{P}^2)$  denote the cone over  $\mathbb{H}\mathbb{P}^2 = G(2, 6)$ , and note that  $\mathbb{S}\mathbb{P}^2 \subset J(\mathbb{P}W^*, \mathbb{H}\mathbb{P}^2)$ . A point  $p \in J(\mathbb{P}W^*, \mathbb{H}\mathbb{P}^2) \setminus \mathbb{S}\mathbb{P}^2$ , can be represented by a sum

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & r & s \\ -r & 0 & t \\ -s & -t & 0 \end{pmatrix}$$

with  $t \neq 0$ . We prove that the tangent space  $T_p = \mathfrak{g}(\mathbb{S}, \mathbb{C}).p$  to the orbit of this point, has the same dimension as the cone over  $\mathbb{H}\mathbb{P}^2$ , implying that  $J(\mathbb{P}W^*, \mathbb{H}\mathbb{P}^2) \setminus \mathbb{S}\mathbb{P}^2$  is a single  $PSL(3, \mathbb{S})$  orbit.

First note that the action of  $PSL(3, \mathbb{H})$  contributes by the dimension of  $\mathbb{H}\mathbb{P}^2$ . What remains to prove is that  $\mathcal{A}_3(\mathbb{H}^\perp)$  is contained in  $T_p$ . Recall that  $\mathfrak{g}(\mathbb{S}, \mathbb{C}) = Der \mathcal{J}_3(\mathbb{S}) \oplus \mathcal{J}_3(\mathbb{S})_0$ . By the left action of  $\mathcal{A}_3(\mathbb{H}^\perp) \subset \mathcal{J}_3(\mathbb{S})_0$ , we get that

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix} \in T_p \quad \forall a, b \in \mathbb{H}^\perp.$$

Then we use the action of the triality algebra  $\mathfrak{t}(\mathbb{H}) \subset \mathfrak{t}(\mathbb{S}) \subset Der \mathcal{J}_3(\mathbb{S})$ . Recall that  $\mathfrak{t}(\mathbb{H}) \simeq \mathfrak{sl}(A) \times \mathfrak{sl}(B) \times \mathfrak{sl}(C)$ , where  $A, B, C$  have dimension two, and that  $\mathbb{S} \simeq A \otimes B \oplus C$ . In particular,  $t$  being a nonzero vector in  $C \simeq \mathbb{H}^\perp$  can be taken to any vector in  $C$ , so that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix} \in T_p \quad \forall c \in \mathbb{H}^\perp.$$

Thus  $\mathcal{A}_3(\mathbb{H}^\perp) \subset T_p$ , and our claim is proved.

Now, a point in  $\sigma(\mathbb{S}\mathbb{P}^2) \setminus J(\mathbb{P}W^*, \mathbb{H}\mathbb{P}^2)$ , can be represented by a sum

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & r & s \\ -r & 0 & t \\ -s & -t & 0 \end{pmatrix}.$$

The action of  $\mathcal{A}_3(\mathbb{H}^\perp)$  by multiplication is trivial on the second factor. Since for  $a, b, c \in \mathbb{H}^\perp$ ,

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b/2 \\ -a & 0 & c/2 \\ -b/2 & -c/2 & 0 \end{pmatrix},$$

we see that the orbit of this point must be open in the determinantal hypersurface  $\sigma(\mathbb{SP}^2)$ , independantly of  $r, s, t$ . We conclude:

**Proposition 8.8.** *The orbit closures of  $PSL(3, \mathbb{S})$  in  $\mathbb{PJ}_3(\mathbb{S})$  are*

$$\mathbb{P}^5 = \mathbb{SP}_{sing}^2 \subset \mathbb{SP}^2 \subset J(\mathbb{PW}^*, \mathbb{HP}^2) \subset \sigma(\mathbb{SP}^2) \subset \mathbb{PJ}_3(\mathbb{S}).$$

As in the case of  $\mathbb{AP}^2 \subset \mathbb{PJ}_3(\mathbb{A})$ , with  $\mathbb{A}$  normed, we get a simple chain of orbit closures. The two differences here are that  $\mathbb{SP}^2$  is singular, and a proper subvariety of the cone over  $\mathbb{HP}^2$ .

#### 8.4. Linear spaces in $\mathbb{SP}^2$ .

**Proposition 8.9.** *The open orbit of the adjoint variety  $X^{ad}(\mathbb{S}, \mathbb{C})$  parametrizes a family of  $\mathbb{P}^4$ 's in  $\mathbb{SP}^2$ . This family has dimension 15.*

*Proof.* We prove that if  $x$  belongs to the open orbit in  $X^{ad}(\mathbb{S}, \mathbb{C})$ , its image in  $\mathcal{J}_3(\mathbb{S})$  defines a  $\mathbb{P}^4$  contained in  $\mathbb{SP}^2$ . By homogeneity, we may suppose that  $x$  belongs to the adjoint variety  $\mathbb{F}_{1,5}$  of  $\mathfrak{sl}_6$ , and corresponds to a pair  $(\ell \subset H)$ , for  $\ell$  a line and  $H$  a hyperplane in  $W \simeq \mathbb{C}^6$ . Its action on  $\mathcal{J}_3(\mathbb{S}) = \Lambda^2 W \oplus W^*$  has for image  $\ell \wedge H \oplus H$ , a five dimensional vector space. The projectivization of this vector space is clearly contained in  $\mathbb{SP}^2$ , by Proposition 8.2, because a nonzero vector in  $\ell \wedge H$  defines a two-plane containing  $\ell$  and contained in  $H$ .  $\square$

Unlike the Severi varieties, there are other families of unextendable linear spaces on  $\mathbb{SP}^2$ :

**Proposition 8.10.** *The unextendable linear spaces on  $\mathbb{SP}^2$  are as follows:*

- A 10-dimensional family of  $\mathbb{P}^5$ 's parametrized by a one point compactification of a line bundle over  $G(3, W)$ .
- An irreducible family of  $\mathbb{P}^4$ 's of dimension 15, with an open subset given by the smooth locus of the adjoint variety  $X^{ad}(\mathbb{S}, \mathbb{C})$ ,
- An irreducible family of  $\mathbb{P}^4$ 's of dimension 15, with an open subset given by the total space of the vector bundle  $\Lambda^2 Q^*(1)$ , where  $Q$  denotes the rank four quotient bundle on  $\mathbb{PW}$ .

*Proof.* Let  $P \subset \mathbb{SP}^2$  be an unextendable linear space. Its projection to  $\mathbb{P}\Lambda^2 W$  is a linear space contained in  $G(2, 6)$ , so is either the set of planes containing a line  $\ell$  and contained in a  $k$ -dimensional space  $L$ , or the set of planes contained in a three plane  $M$ .

In the second case, again in an adapted basis,  $P$  must be generated by vectors  $e_1 \wedge e_2 + ze_3^*, e_2 \wedge e_3 + ze_1^*, e_3 \wedge e_1 + ze_2^*, e_4^*, e_5^*, e_6^*$ . We thus get a family of  $\mathbb{P}^5$ 's on  $\mathbb{SP}^2$ , parametrized by a  $\mathbb{C}$ -bundle over the Grassmannian  $G(3, 6)$ . This family becomes complete when we add to it a single point, corresponding to the singular set  $\mathbb{PW} \simeq \mathbb{P}^5$  of  $\mathbb{SP}^2$ .

In the first case, in an adapted basis,  $P$  must be generated by vectors  $e_1 \wedge e_2 + h_2, \dots, e_1 \wedge e_k + h_k, e_{k+1}^*, \dots, e_6^*$ , where  $h_2, \dots, h_{k+2}$  are linear forms such that  $h_i(e_1) = 0$  for all  $i$  and the matrix  $h_i(e_j)$ ,  $2 \leq i, j \leq k+2$ , is skew-symmetric. In particular,  $P$  has affine dimension 5. If  $k = 0$ , we get the singular  $\mathbb{P}^5$ . Note also that  $k \neq 1, 2$ , otherwise  $P$  would be extendable. Thus  $k \geq 3$  and we get a family of  $\mathbb{P}^4$ 's on  $\mathbb{SP}^2$  of dimension  $k(6-k) + (k-1) + \binom{k-1}{2} = k(6-k) + \binom{k}{2}$  (choice of  $L$  plus choice of  $\ell \subset L$  plus choice of  $h$ ). Note that for  $k = 5$ , we recover the 15-dimensional family parametrized by the open orbit of the adjoint variety. But  $k = 6$  gives another family of the same dimension. Note that in that case,  $\ell$  being the line generated by  $e_1$ , the map  $h$  should be seen as a skew-symmetric morphism from  $W/\ell$  to  $\ell^\perp \simeq (W/\ell)^*$ , depending linearly on the vector we choose on  $\ell$ . Thus our family is parametrized by the vector bundle  $\Lambda^2 Q(1)$  on  $\mathbb{PW}$ .

Finally, it is easy to check that the other cases belong to the closure of these two maximal families.  $\square$

**8.5. Point-line geometry.** When  $\mathbb{A}$  is a normed algebra, the  $\mathbb{A}$ -plane is covered by a family of  $\mathbb{A}$ -lines (i.e.  $\mathbb{A}\mathbb{P}^1$ 's) parametrized by  $\mathbb{A}\mathbb{P}^2$  itself. This family of  $\mathbb{A}\mathbb{P}^1$ 's defines a plane projective geometry on  $\mathbb{A}\mathbb{P}^2$ , in the sense that two generic points are joined by a unique line, and two generic lines meet in a unique point. We now show that the same picture holds for  $\mathbb{S}\mathbb{P}^2$ .

The  $\mathbb{A}$ -lines can be described as the entry-loci of the points inside the secant cubic. For the sextonions, we choose a pair  $(\omega, h)$ , where  $\omega \in \Lambda^2 W$  has rank four, and  $h$  is a linear form. Denote by  $P$  the support of  $\omega$ , i.e., the four plane which is the image of the contraction by  $W^*$ . A computation shows that the entry-locus of  $(\omega, h)$  is the intersection of  $\mathbb{S}\mathbb{P}^2$  with the linear space  $\phi_h(\Lambda^2 P) + P^\perp$ , where  $\phi_h = Id_{\Lambda^2 W} + \psi_h$  for an endomorphism  $\psi_h : \Lambda^2 P \rightarrow P^*$  defined by  $h$ , more precisely by the restriction of  $h$  to  $P$ .

To be more explicit, suppose that  $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$ , and let's try to solve the equation  $(\omega, h) = (\alpha, k) + (\beta, l)$ . Around  $\alpha_0 = e_1 \wedge e_2$ ,  $\beta_0 = e_3 \wedge e_4$ , a solution of the equation  $\omega = \alpha + \beta$  can be written

$$\begin{aligned}\alpha &= (1 + sv - ut)^{-1}(e_1 + se_3 + te_4) \wedge (e_2 + ue_3 + ve_4), \\ \beta &= (1 + sv - ut)^{-1}(e_3 - ve_1 + te_2) \wedge (e_4 + ue_1 - se_2).\end{aligned}$$

Then  $h = k + l$ , with  $k|_\alpha = 0$  and  $l|_\beta = 0$ , if  $k(e_1 + se_3 + te_4) = k(e_2 + ue_3 + ve_4) = 0$ ,  $k(e_3 - ve_1 + te_2) = h(e_3 - ve_1 + te_2)$  and  $k(e_4 + ue_1 - se_2) = h(e_4 + ue_1 - se_2)$ . This gives

$$\begin{aligned}(1 + sv - ut)k(e_1) &= (sv - ut)h(e_1) - sh(e_3) - th(e_4), \\ (1 + sv - ut)k(e_2) &= (sv - ut)h(e_2) - uh(e_3) - vh(e_4), \\ (1 + sv - ut)k(e_3) &= h(e_3) - vh(e_1) + th(e_2), \\ (1 + sv - ut)k(e_4) &= h(e_4) + uh(e_1) - sh(e_2).\end{aligned}$$

Letting  $h(e_i) = h_i$ , and completing  $e_1, e_2, e_3, e_4$  into a basis of  $W$ , we deduce that the projective span of  $(\alpha, k)$  is

$$\begin{aligned}[\sum_{1 \leq i < j \leq 4} Z_{ij} e_i \wedge e_j, & h_1 Z_{34} + h_3 Z_{23} + h_4 Z_{24}, h_2 Z_{34} + h_3 Z_{13} - h_4 Z_{14}, \\ & h_3 Z_{12} - h_1 Z_{14} - h_2 Z_{24}, h_4 Z_{12} + h_1 Z_{13} + h_2 Z_{23}, k_5, k_6],\end{aligned}$$

and this is a point of  $\mathbb{S}\mathbb{P}^2$  when  $\sum_{1 \leq i < j \leq 4} Z_{ij} e_i \wedge e_j$  has rank two. We can describe this linear space in a more invariant way as follows. The two form  $\omega$  defines the four-space  $P$ , and the non-zero vector  $\wedge^2 \omega \in \Lambda^4 P$  which allows one to identify  $P$  with its dual. Choose a supplement  $P^\circ$  to  $P^\perp$  in  $W^*$ , so that the composition  $P^\circ \hookrightarrow W^* \rightarrow P^*$  is a natural isomorphism. Then the linear space above is the set of vectors

$$Z + Z \wedge (h \lrcorner \omega) / (\omega \wedge \omega) + Y, \quad Z \in \Lambda^2 P, \quad Y \in P^\perp,$$

where  $h \lrcorner \omega$  belongs to  $P$ , hence  $Z \wedge (h \lrcorner \omega)$  to  $\Lambda^3 P = P^* \otimes \Lambda^4 P$ , so that we obtain after division by  $(\omega \wedge \omega)$  a vector in  $P^*$  that we identify with  $P^\circ \subset W^*$ . The resulting vector is uniquely defined only up to  $P^\perp$ , but the  $Y$  term allows one to ignore that point.

This space is therefore defined only by the tensor  $\omega \wedge \omega + h \lrcorner \omega \in \Lambda^4 W \oplus W$ , where  $\omega \wedge \omega$  is a decomposable tensor in  $\Lambda^4 W \simeq \Lambda^2 W^*$ , and  $h \lrcorner \omega \in W$  is a linear form on  $W^*$  vanishing on the plane defined by  $\omega \wedge \omega$ . We finally get a family of  $\mathbb{S}$ -lines parametrized by the smooth part of the dual plane  $\hat{\mathbb{S}}\mathbb{P}^2$ .

It remains to understand how these  $\mathbb{S}$ -lines degenerate when we approach the singular set of this dual plane. To see this, we compute the entry locus of a generic point of the determinantal hypersurface in  $\mathbb{P}\mathcal{J}_3(\mathbb{S})$ , of the form  $\omega + h$ , with  $\omega \in \Lambda^2 W$  a decomposable tensor and  $h$  linear form which is not identically zero on the plane defined by  $\omega$ . We check that this entry locus only depends on the kernel of the restriction of  $h$  to that plane: precisely, if  $e$  is a generator of that line, it is a smooth 8-dimensional quadric obtained as the intersection of  $\mathbb{S}\mathbb{P}^2$  with the linear space  $e \wedge W \oplus e^\perp$ . Such a smooth quadric is clearly covered by 6-dimensional quadrics singular along a line, which can be obtained as limits of  $\mathbb{S}\mathbb{P}^1$ 's on  $\mathbb{S}\mathbb{P}^2$ . And the family of these smooth quadrics is naturally parametrized by  $\mathbb{P}W$ , the singular set of the dual plane  $\hat{\mathbb{S}}\mathbb{P}^2$ .

Using this explicit description, we easily get:

**Proposition 8.11.** *Two generic  $\mathbb{S}$ -lines on  $\mathbb{S}\mathbb{P}^2$  meet in a unique point. Through two generic points of  $\mathbb{S}\mathbb{P}^2$  passes a unique  $\mathbb{S}$ -line.*

**8.6. The first row.** To pass to the variety  $X = \mathbb{S}\mathbb{P}_0^2 \subset \mathbb{P}V$  of the first row, as with the rest of the series we take a hyperplane section, but now the hyperplane section is no longer generic, as it cuts only the first factor. The variety  $\mathbb{S}\mathbb{P}_0^2$  has a corresponding description where  $G(2, W)$  is replaced by the  $\omega$ -isotropic Grassmanian  $G_\omega(2, W)$ .

**8.7. The Grassmannian  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ .** The varieties from the third line of the geometric Freudenthal square have several interesting interpretations, as Lagrangian Grassmannians of symplectic  $\mathbb{A}$ -subspaces of  $\mathbb{A}^6$ , or cubic curves over the simple Jordan algebras  $\mathcal{J}_3(\mathbb{A})$ , or conformal compactifications of these Jordan algebras. Recall that they are defined as the closures of the images of the maps

$$\nu_3 : \mathcal{J}_3(\mathbb{A}) \rightarrow \mathbb{P}\mathcal{Z}_2(\mathbb{A}), \quad \nu_3(x) = \begin{pmatrix} 1 & x \\ Q(x) & \det(x) \end{pmatrix},$$

where  $Q(x)$  denotes the cofactor matrix of  $x$  (see [23], section 1.2, or [7], section 6). We use the same definition over the sextonions. Our first claim is about the equations of the resulting variety  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ . (For the nondegenerate case, this is Proposition 6.2 in [7], but the proof is not quite correct). The following argument works in general. We begin by exhibiting a set of quadratic equations of  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ , which define it set-theoretically.

**Lemma 8.12.** *The variety  $G_\omega(\mathbb{S}^3, \mathbb{S}^6) \subset \mathbb{P}\mathcal{Z}_2(\mathbb{S})$  is the set of matrices  $\begin{pmatrix} s & x \\ y & t \end{pmatrix}$ , such that*

$$Q(x) = sy, \quad Q(y) = tx, \quad xy = stI.$$

*Proof.* We must prove that such a matrix belongs to the closure of  $\nu_3(\mathcal{J}_3(\mathbb{S}))$ . This is clear if  $s \neq 0$ . Since

$$\nu_3(x^{-1}) = \begin{pmatrix} \det(x) & Q(x) \\ x & 1 \end{pmatrix},$$

this is also true for  $t \neq 0$ . But  $w \in \mathcal{J}_3(\mathbb{S})$  acts on  $\mathcal{Z}_2(\mathbb{S})$  by the translation

$$t_w \begin{pmatrix} s & x \\ y & t \end{pmatrix} = \begin{pmatrix} s & x + sw \\ y + 2Q(x, w) + sQ(w) & t + \text{trace}(yw) + \text{trace}(xQ(w)) + s\det(w) \end{pmatrix},$$

where  $Q(x, w)$  denotes the polarization of  $Q$ . This action of  $\mathcal{J}_3(\mathbb{S})$  preserves our set of quadratic equations, but clearly not the subspace of matrices such that  $t = 0$ . The claim follows.  $\square$

Let  $Sp(6, \mathbb{S})$  denote the closed subgroup of  $GL(\mathcal{Z}_2(\mathbb{S}))$  defined by the Lie algebra  $\mathfrak{g}(\mathbb{S}, \mathbb{H})$ . It contains the group  $Sp(6, \mathbb{H}) = Spin_{12}$ , whose action on  $\mathcal{Z}_2(\mathbb{S})$  leaves invariant the subspace  $\mathcal{Z}_2(\mathbb{H}) \simeq \Delta_+$ . Remember that the closed orbit of  $PSp(6, \mathbb{H})$  in  $\mathbb{P}\mathcal{Z}_2(\mathbb{H})$  is the spinor variety  $\mathbb{S}_+ = G_\omega(\mathbb{H}^3, \mathbb{H}^6)$ .

**Proposition 8.13.** *The variety  $G_\omega(\mathbb{S}^3, \mathbb{S}^6) \subset \mathbb{P}\mathcal{Z}_2(\mathbb{S})$  is the highest weight variety of  $PSp(6, \mathbb{S})$  in  $\mathbb{P}\mathcal{Z}_2(\mathbb{S})$ . It can be interpreted as the subvariety of  $\mathbb{P}(\Delta_+ \oplus U)$ , consisting of the closure of the set of pairs  $(\Sigma, u)$  where  $\Sigma$  defines a maximal isotropic subspace in the family  $\mathbb{S}_+ = G_\omega(\mathbb{H}^3, \mathbb{H}^6)$ , and  $u$  belongs to  $\Sigma$ .*

*Proof.* A stable complement is given by the space of matrices with coefficients in  $\mathbb{H}^\perp$ . This complement has dimension 12 and, since the action is nontrivial, it must coincide with the natural representation of  $Spin_{12}$  on  $U \simeq \mathbb{C}^{12}$ . (This proves the description of  $V$  in 6.3.)

We have  $S^2(\Delta_+ \oplus U)^* = S^2\Delta_+ \oplus (\Delta_+ \otimes U) \oplus S^2U$ . Using the notation of Bourbaki for the weights of  $\mathfrak{so}_{12}$ , we have  $\Delta_+ = V_{\omega_6}$  and

$$S^2\Delta_+ = V_{2\omega_6} \oplus V_{\omega_2}, \quad \Delta_+ \otimes U = V_{\omega_1 + \omega_6} \oplus V_{\omega_5}, \quad S^2U = V_{2\omega_1} \oplus \mathbb{C}.$$

Now recall Lemma 8.12, and decompose  $\mathcal{Z}_2(\mathbb{S})$  into  $\mathcal{Z}_2(\mathbb{H}) \oplus (\mathbb{S}^\perp \oplus \mathbb{S}^\perp) = \Delta_+ \oplus U$ . We see that  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$  has quadratic equations of different types: those only involving  $\mathcal{Z}_2(\mathbb{H})$  are the

quadratic equations of  $G_\omega(\mathbb{H}^3, \mathbb{H}^6)$ , which gives  $V_{\omega_2}$ . There are also equations of mixed type, i.e. from  $\Delta_+ \otimes U$ .

It follows that  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$  can be defined as a set pairs  $(\Sigma, u)$ , where  $\Sigma \in \mathbb{P}\Delta_+$  defines a maximal isotropic subspace of  $U$  in one of the two families of these, and  $u$  belongs to some subspace of  $U$  defined by  $\Sigma$  in some invariant way. The only possibility is that this space is  $\Sigma$  itself (it cannot be zero since  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$  is certainly not contained in  $G_\omega(\mathbb{H}^3, \mathbb{H}^6)$ , and it cannot be the whole of  $U$  since we do have mixed equations). In particular,  $u$  must be isotropic, and the trivial factor of  $S^2U$  must appear in the space of quadratic equations of  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ .

*Remark.* We conclude that the space of quadratic equations of  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ , as an  $\mathfrak{so}_{12}$ -module, is  $V_{\omega_2} \oplus V_{\omega_5} \oplus \mathbb{C} = \mathfrak{so}_{12} \oplus \Delta_- \oplus \mathbb{C}$ . But this is just the intermediate Lie algebra  $\mathfrak{g} = \mathfrak{g}(\mathbb{S}, \mathbb{H})$ , which is no surprise since on the third line of the magic chart, we have the invariant symplectic form  $\omega$ , which allows to associate to every vector  $x \in \mathfrak{g}$  the quadratic form  $q_x(v) = \omega(v, xv)$ .

**Corollary 8.14.**  *$G_\omega(\mathbb{S}^3, \mathbb{S}^6)$  is singular along the quadric  $\mathbb{Q}^{10} \subset \mathbb{P}U$ . Its smooth locus has two orbits under the action of  $PSO_{12}$ , but is homogeneous under the action of  $PSp(6, \mathbb{S})$ .*

*Proof.* The Zariski tangent space of  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$  at a point  $(0, u) \in \mathbb{Q}^{10} \subset \mathbb{P}U \subset \mathbb{P}(\Delta_+ \oplus U)$  certainly contains the line of  $\Delta_+$  generated by  $\Sigma \in \mathbb{P}\Delta_+$  parametrizing any maximal isotropic subspace of  $U$  containing  $u$ . The linear span of such  $\Sigma$ 's is isomorphic with a half-spin representation of  $Spin(u^\perp/Cu) = Spin_{10}$  - in particular, its dimension is 16. But our Zariski tangent space also contains  $U$ , obviously, and we already get  $16 + 12 = 28$  dimensions, which is more than the dimension, 21, of  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ . This proves the first claim.

The complement of  $\mathbb{Q}^{10}$  is the set of pairs  $(\Sigma, u)$ , where  $\Sigma$  is non zero and parametrizes a maximal isotropic subspace of  $U$  containing  $u$ . The action of  $PSO_{12}$  gives two orbits, one where  $u = 0$  and one where  $u \neq 0$ . But the condition  $u = 0$  is not  $\mathfrak{g}(\mathbb{S}, \mathbb{H})$ -invariant, so the complement of  $\mathbb{Q}^{10}$  is  $PSp(6, \mathbb{S})$ -homogeneous, hence smooth, and exactly equal to the smooth locus of  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ .  $\square$

**Proposition 8.15.** *The orbit closures of  $PSp(6, \mathbb{S})$  in  $\mathbb{P}\mathcal{Z}_2(\mathbb{S})$  are the cones over the four  $PSp(6, \mathbb{H})$ -orbits in  $\mathbb{P}\mathcal{Z}_2(\mathbb{H})$  with vertex  $\mathbb{P}U$ , the variety  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$  and its singular locus  $\mathbb{Q}^{10} \subset \mathbb{P}U$ .*

*Proof.* Consider a point in  $\mathbb{P}\mathcal{Z}_2(\mathbb{H}) \subset \mathbb{P}\mathcal{Z}_2(\mathbb{S})$ , given by some matrix  $m = \begin{pmatrix} s & x \\ y & t \end{pmatrix}$ . We want to understand when the tangent space  $\mathfrak{g}(\mathbb{S}, \mathbb{H}).m$  to the orbit of  $m$  contains  $U \simeq \mathcal{Z}_2(\mathbb{H}^\perp)$ , the subspace of  $\mathcal{Z}_2(\mathbb{S})$  consisting of matrices all of whose coefficients are in  $\mathbb{H}^\perp$  (in particular, the diagonal coefficients must be zero). Note that this condition is certainly  $PSp(6, \mathbb{H})$ -invariant.

The action of  $\mathcal{A}_3(\mathbb{H}^\perp)$  and its dual provide us with the matrices

$$\begin{pmatrix} 0 & su \\ xu & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & yv \\ tv & 0 \end{pmatrix}, \quad u, v \in \mathcal{A}_3(\mathbb{H}^\perp).$$

We can certainly solve the equations  $su + yv = p$ ,  $xu + tv = q$ , as soon as the matrix  $stI - xy$  is invertible. This is the case if  $m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , a point in the complement of the tangent quartic, which is an open  $PSp(6, \mathbb{H})$ -orbit in  $\mathbb{P}\mathcal{Z}_2(\mathbb{H})$ .

This is no longer true if we consider the point  $m = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$  on the tangent hypersurface. But let us move this point by the translation  $t_w$ , to get the point

$$t_w(m) = \begin{pmatrix} 1 & x + w \\ Q(x, w) + Q(w) & \text{trace}(xQ(w)) + \det(w) \end{pmatrix}.$$

Now the matrix we want to be invertible is  $z_w = \text{trace}(xQ(w))I - xQ(w) - xQ(x, w) - wQ(x, w)$ . It is enough to find  $w$  such that the degree two part  $wQ(x, w)$  is invertible. We claim that this is possible as soon as the rank of  $x$  is at least two. Indeed, if we represent  $x$  by some



diagonal matrix with at least two nonzero eigenvalues, and if we also choose  $w$  to be diagonal, a straightforward computation shows that  $wQ(x, w)$  is again diagonal with generically nonzero eigenvalues.

We conclude that for any point in  $\mathbb{P}(\Delta_+ \oplus U)$  of the form  $p = (\Sigma, u)$ , where  $\Sigma$  does not belong to the cone over  $\mathbb{S}_+ = G_\omega(\mathbb{H}^3, \mathbb{H}^6)$ , the  $PSp(6, \mathbb{S})$ -orbit of  $p$  must be the whole cone over the  $PSp(6, \mathbb{H})$ -orbit of  $\Sigma$ , with vertex  $\mathbb{P}U$ .

A similar computation shows that when  $\Sigma$  is a nonzero vector in the cone over  $\mathbb{S}_+$ , there are only two cases up to the  $PSp(6, \mathbb{S})$ -action: either  $u$  does belong to  $\Sigma$ , or not.  $\square$

**Proposition 8.16.** *A point in  $\mathbb{P}\mathcal{Z}_2(\mathbb{S})$ , outside the tangential quartic, belongs to a unique secant to  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ . In particular,  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$  has only one apparent double point.*

*Proof.* We use the same argument as in [7]: A general point  $m = \begin{pmatrix} s & x \\ y & t \end{pmatrix}$  in  $\mathcal{Z}_2(\mathbb{S})$  does not belong to the hyperplane at infinity ( $s = 0$ ). By translation we can then suppose that  $x = 0$ . It is easy to check that if  $y$  is invertible,  $m$  cannot belong to a secant line joining two points of  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ , one of which in the hyperplane at infinity. So we need to solve the equation

$$\begin{pmatrix} s & x \\ y & t \end{pmatrix} = \lambda \begin{pmatrix} 1 & a \\ Q(a) & \det(a) \end{pmatrix} + \mu \begin{pmatrix} 1 & b \\ Q(b) & \det(b) \end{pmatrix}.$$

Using the identities  $Q(Q(a)) = \det(a)a$  and  $aQ(a) = \det(a)I$ , which are valid in  $\mathcal{J}_3(\mathbb{S})$ , one checks that this equation has for unique solution

$$a = \frac{\mu - \lambda Q(y)}{\lambda} \frac{Q(y)}{st} \quad \text{and} \quad b = \frac{\lambda - \mu Q(y)}{\mu} \frac{Q(y)}{st},$$

where the scalars  $\lambda$  and  $\mu$  are uniquely defined by the conditions that

$$\lambda + \mu = s \quad \text{and} \quad \frac{(\lambda - \mu)^2}{\lambda\mu} = \frac{st^2}{\det(y)}. \quad \square$$

*Remark.* The property of having only one apparent double point is equivalent to the fact that the projection of the variety from a general tangent space is birational. For the varieties  $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ , this projection can be interpreted as the map  $Q : \mathbb{P}\mathcal{J}_3(\mathbb{A}) \dashrightarrow \mathbb{P}\mathcal{J}_3(\mathbb{A})$ . This is an involutive birational isomorphism because of the identity  $Q(Q(a)) = \det(a)a$ , and this also holds over the sextonions.

**8.8. The adjoint variety  $X^{ad}(\mathbb{S}, \mathbb{H})$ .** We conclude with a brief sketch of study of this variety. Remember the identification  $\mathfrak{g}(\mathbb{S}, \mathbb{H}) = \mathfrak{so}_{12} \oplus \Delta_- \oplus \mathbb{C}$ . We define  $X^{ad}(\mathbb{S}, \mathbb{H}) \subset \mathbb{P}\mathfrak{g}(\mathbb{S}, \mathbb{H})$  as the closure of the space of triples  $(P, \Sigma, z)$  such that:  $P \in \mathfrak{so}_{12}$  parametrizes a point of the adjoint variety  $X^{ad}(\mathbb{H}, \mathbb{H})$ , i.e., an isotropic plane in  $U = \mathbb{C}^{12}$ ;  $\Sigma$  parametrizes a maximal isotropic space in  $U$  from the family  $\mathbb{S}_-$ , containing  $P$ ;  $z$  is any scalar.

**Proposition 8.17.** *The variety  $X^{ad}(\mathbb{S}, \mathbb{H}) \subset \mathbb{P}\mathfrak{g}(\mathbb{S}, \mathbb{H})$  is the  $PSp(6, \mathbb{S})$ -adjoint variety. Its dimension is 25.*

This is in agreement with the fact that for the third row of Freudenthal's square, the dimension of the adjoint variety is  $4a + 1$  in the nondegenerate case.

**Proposition 8.18.** *The smooth locus of  $X^{ad}(\mathbb{S}, \mathbb{H})$  parametrizes a family of 8-dimensional quadrics on  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ .*

We can also consider the space of lines on  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ . The lines which are not contained in the singular locus form a quasi-homogeneous variety, linearly nondegenerate inside  $\mathbb{P}\Lambda^{(2)}\mathcal{Z}_2(\mathbb{S})$ .

We leave to the reader the problem of showing that points of  $X^{ad}(\mathbb{S}, \mathbb{H})$ , lines in  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ , and points of  $G_\omega(\mathbb{S}^3, \mathbb{S}^6)$ , are the elements – points, lines and planes respectively, of a six-dimensional symplectic geometry.

We also leave to the future the problem of studying the varieties in the sextonionic and expanded octonionic rows. For the octonionic row, we should get four quasi-homogeneous varieties defining a metasymplectic geometry in the sense of Freudenthal.

## REFERENCES

- [1] Baez J., *The octonions*, Bull. Amer. Math. Soc. **39** (2002), no. 2, 145–205.
- [2] Barton C.H., Sudbery A., *Magic squares and matrix models of Lie algebras*, Adv. Math. **180** (2003), no. 2, 596–647.
- [3] Brion M.: *Représentations exceptionnelles des groupes semi-simples*, Ann. scient. Ec. Norm. Sup. **18**, 345–387 (1985).
- [4] Bourbaki N., *Groupes et algèbres de Lie*, Hermann, Paris 1968.
- [5] Chaput P. E., *Severi varieties*, Math. Z. **240** (2002), no. 2, 451–459.
- [6] Chaput P. E., *Scorza varieties and Jordan algebras*, Indagationes Math. **14** (2003), 169–182.
- [7] Clerc J.L., *Special prehomogeneous vector spaces associated to  $F_4, E_6, E_7, E_8$  and simple Jordan algebras of rank 3*, J. Algebra **264** (2003), no. 1, 98–128.
- [8] Cohen A.M., de Man R.: *Computational evidence for Deligne’s conjecture regarding exceptional Lie groups*, C.R.A.S. **322** (1996), 427–432.
- [9] Cvitanović P., *Group theory*, available at [www.nbi.dk/GroupTheory/](http://www.nbi.dk/GroupTheory/), a monograph in preparation (1984–2002).
- [10] Cohen A.M., van Leeuwen M.A., Lissers B., LiE, a package for Lie group computations, CAN, Amsterdam, 1992.
- [11] Deligne P., *La série exceptionnelle des groupes de Lie*, C.R.A.S. **322**, 321–326 (1996).
- [12] Deligne P., de Man R., *The exceptional series of Lie groups*, C.R.A.S. **323**, 577–582 (1996).
- [13] Deligne P., Gross B., *La série exceptionnelle et ses descendants*, C.R.A.S. **335** (2002), no. 11, 877–881.
- [14] Freudenthal H., *Oktaven, Ausnahmegruppen und Oktavengeometrie* (notes from 1951), Geom. Dedicata **19**, 7–63 (1985).
- [15] Gelfand I., Zelevinsky A., *Representation models for classical groups and their higher symmetries*, in The mathematical heritage of Elie Cartan (Lyon, 1984). Astérisque 1985, Numéro Hors Serie, 117–128.
- [16] Gelfand I., Zelevinsky A., *Models of representations of classical groups and their hidden symmetries*, (Russian) Funktsional. Anal. i Prilozhen. **18** (1984), no. 3, 14–31.
- [17] Gross B., Wallach R., *On quaternionic discrete series representations and their continuations*, J. reine angew. Math. **481**, 73–123 (1996).
- [18] Harvey F.R., *Spinors and calibrations*, Perspectives in Math. Vol. 9, Academic Press, 1990.
- [19] Jeurissen R.H., *The automorphism groups of octave algebras*, Doctoral dissertation, University of Utrecht 1970.
- [20] Kleinfeld E., *On extensions of quaternions*, Indian J. Math. **9** (1968) 443–446.
- [21] Knop F., *Ein neuer Zusammenhang zwischen einfachen Gruppen und einfachen Singularitäten*, Invent. Math. **90** (1987), 579–604.
- [22] Landsberg J.M., Manivel L., *Construction and classification of simple Lie algebras via projective geometry*, Selecta Mathematica **8** (2002), 137–159.
- [23] Landsberg J.M., Manivel L., *The projective geometry of Freudenthal’s magic square*, J. Algebra **239**, 477–512 (2001).
- [24] Landsberg J.M., Manivel L., *Triality, exceptional Lie algebras, and Deligne dimension formulas*, Adv. Math. **171** (2002), 59–85.
- [25] Landsberg J.M., Manivel L., *Series of Lie groups*, Michigan Math. Journal **52** (2004), 453–479.
- [26] Landsberg J.M., Manivel L., *Representation theory and projective geometry*, in Algebraic Transformation Groups and Algebraic Varieties, Ed. V.L. Popov, Encyclopaedia of Mathematical Sciences **132**, Springer 2004.
- [27] Landsberg J.M., Manivel L., *A universal dimension formula for complex simple Lie algebras*, arXiv:math.RT/0401296.
- [28] Mukai S., *Simple Lie algebra and Legendre variety*, preprint 1998.
- [29] Proctor I., *Odd symplectic groups*, Invent. Math. **92**, 307–332 (1988).
- [30] Rosenfeld B., *Geometry of Lie groups*, Mathematics and its Applications **393**, Kluwer 1997, xviii+393 pp.
- [31] Shtepin V., *Intermediate Lie algebras and their finite-dimensional representations*, Acad. Sci. Izv. Math. **43** (1994), no. 3, 559–579.
- [32] Tits J., *Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles I. Construction*, Indag. Math. **28**, 223–237 (1966).
- [33] Tits J., *Les groupes de Lie exceptionnels et leur interprétation géométrique*, Bull. Soc. Math. Belg. **8**, 48–81 (1956).

- [34] Vogel P., *The universal Lie algebra*, preprint 1999.
- [35] Westbury B.W., *Sextonions and the magic square*, preprint 2003.
- [36] Zak F., *Tangents and secants of algebraic varieties*, Translations of Math. Monographs **127**, AMS 1993.

Joseph M. Landsberg  
School of Mathematics,  
Georgia Institute of Technology,  
Atlanta, GA 30332-0160  
USA  
E-mail: jml@math.gatech.edu

Laurent Manivel  
Institut Fourier, UMR 5582 du CNRS  
Université Grenoble I, BP 74  
38402 Saint Martin d'Hères cedex  
FRANCE  
E-mail: Laurent.Manivel@ujf-grenoble.fr