



HAL
open science

Prime Fano threefolds and integrable systems

Atanas Iliev, Laurent Manivel

► **To cite this version:**

Atanas Iliev, Laurent Manivel. Prime Fano threefolds and integrable systems. *Mathematische Annalen*, 2007, 339 (4), pp.937-955. hal-00330601

HAL Id: hal-00330601

<https://hal.science/hal-00330601>

Submitted on 15 Oct 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Prime Fano threefolds and integrable systems

A. Iliev, L. Manivel

Abstract

For a general K3 surface S of genus g , with $2 \leq g \leq 10$, we prove that the intermediate Jacobians of the family of prime Fano threefolds of genus g containing S as a hyperplane section, form generically an algebraic completely integrable Hamiltonian system.

1 Introduction

Consider a complete family \mathcal{M} of Calabi-Yau threefolds. Following a remarkable result of Donagi and Markman (see [DM1] or [DM2], Theorem 7.7), the relative intermediate Jacobian $\mathcal{J} \rightarrow \tilde{\mathcal{M}}$ over the corresponding family of gauged Calabi-Yau threefolds (a \mathbb{C}^* -bundle over \mathcal{M}), is an *algebraically completely integrable Hamiltonian system* (ACIHS for short), or equivalently a Lagrangian fibration. The base $\tilde{\mathcal{M}}$ of such an ACIHS is known to be endowed with a *special Kähler structure*, which plays an important role in mirror symmetry.

Our goal is to extend this result to families of *log Calabi-Yau varieties with a fixed boundary*. More specifically, we consider complete families of pairs $X \supset S$, with X a prime Fano threefold and S a divisor such that $K_X + S = 0$ (the gauge disappears); the boundary S is itself a fixed K3 surface. In this situation we prove that the family of intermediate Jacobians $J(X)$, turns out to be generically a Lagrangian family of abelian varieties. This is summarized in Theorem 11, which is the main result of our paper:

For a general K3 surface S of genus g , with $2 \leq g \leq 10$, the relative intermediate Jacobian is generically an ACIHS over a complete family of prime Fano threefolds containing S .

Our route towards this result is quite indirect, and in particular much less direct than the one followed by Donagi and Markman in the Calabi-Yau setting. Instead we use a very nice method for constructing ACIHS, introduced by Markman. Consider a smooth complex projective symplectic variety M , and let $Z \subset M$ be a smooth Lagrangian subvariety. Deformations of Z in M are unobstructed, and any smooth deformation of Z remains a Lagrangian subvariety of M . *Then the relative Picard bundle $\text{Pic}^o \rightarrow \mathcal{H}$ is an ACIHS over the base \mathcal{H} of the smooth deformations of Z in M , see 8.1-8.2 in [DM2].*

For example, any smooth curve C of genus g in a K3 surface S is automatically a Lagrangian subvariety of S . In this case the smooth Lagrangian deformations of C in S are the same as the smooth deformations of C in the complete linear system $|C| = \mathbb{P}(H^0(S, C)) \simeq \mathbb{P}^g$. This gives rise to the Beauville integrable system – the relative Jacobian \mathcal{J} over the open subset $\mathcal{U} \subset |C|$ of smooth deformations of C in $|C|$, see [Be2]. Recently Sawon and Yoshioka, by using the

¹ MSC 14J28, 14J45, 14K30, 37J99

² The 1-st author is partially supported by grant MI1503/2005 of the Bulgarian Foundation for Scientific Research

technique of Fourier-Mukai functors, managed to show that for any $k \geq 2$, and for a general K3 surface S with a primitive polarization of degree $d = k^2(2g - 2)$, the punctual Hilbert scheme $Hilb_g S$ is locally isomorphic to a compactified relative Jacobian $\tilde{\mathcal{J}} \rightarrow \mathbb{P}^g$, thus proving the existence of a Lagrangian fibration on $Hilb_g S$ over \mathbb{P}^g , see [Saw].

The Lagrangian fibrations of Beauville and Sawon are fibered generically by Jacobians of curves on K3 surfaces, and one may ask how to find relevant examples of Lagrangian fibrations arising by Markman's construction from Lagrangian subvarieties of higher dimensional projective symplectic manifolds. In [IM2] we start from the result of Beauville and Donagi that the Fano variety $F(W)$ of lines in a cubic fourfold W is a four-dimensional symplectic variety, and show that the cubic fivefolds containing W give rise to Lagrangian surfaces in $F(W)$. This yields two new examples of algebraic completely integrable systems, one fibered by intermediate Jacobians of cubic fivefolds, and the other by intermediate Jacobians of prime Fano threefolds of genus four. These two new ACIHS are in some sense opposite to the cubic integrable system of Donagi and Markman, given by the relative intermediate Jacobian of smooth cubic threefolds in W , and to its border integrable system by Jacobians of genus 4 curves, see Example 8.22 in [DM2].

In the context of the present paper we consider, instead of the Beauville-Donagi variety, the Hilbert powers $Hilb_d S$ of a smooth K3 surface S contained in a prime Fano threefold. Recall that a prime Fano threefold is a smooth complex threefold X for which the anticanonical divisor $H = -K_X$ is the ample generator of $Pic(X)$ over \mathbb{Z} . Then the number $g = -K_X^3/2 + 1$ is always a positive integer, called the genus of $X = X_{2g-2}$. Prime Fano threefolds exist only for $2 \leq g \leq 12, g \neq 11$, see [IP]. A general member of the anticanonical linear system $|-K_X|$ of a prime Fano threefold $X = X_{2g-2}$, is a K3 surface $S = S_{2g-2}$ of genus g . As known by Mukai, the converse is also true: the general K3 surface of genus g , for $2 \leq g \leq 12$ and $g \neq 11$, can be embedded as an anticanonical divisor in a prime Fano threefold of genus g , see [Muk1] and also [Be4] for a more precise statement. Keeping this in mind, in §2 we show that various families of curves on the prime Fano threefolds $X \supset S$ produce in a natural way Lagrangian subvarieties of the Hilbert schemes $Hilb_d S$, for any $d \geq 1$. Theorem 1, our first main observation, reads:

Let S be a K3 surface in a prime Fano threefold X of genus g . Then for any integer $d \geq 1$, any generically reduced component \mathcal{H} of the Hilbert scheme of smooth curves of degree d on X , is mapped by the intersection map j with S to a Lagrangian subvariety $j(\mathcal{H})$ of $Hilb_d S$.

In §3 we study in detail the family $F(X)$ of conics on a prime Fano threefold $X = X_{2g-2}$. Together with the lines, the conics are one of the two best studied families of curves on prime Fano threefolds (see Ch.4 of [IP] for a survey of the basic results on lines and conics on Fano threefolds). In §3.1 we collect the most important such results in our perspective, beginning with the fact that for X general, the scheme $F(X)$ is a smooth irreducible surface, *the Fano surface* of X .

By Theorem 1, for the general $X \supset S$ the intersection image $j(F(X))$ is a Lagrangian surface in the symplectic fourfold $Hilb_2 S$. By Proposition 4, for any g as above the intersection map $j : F(X) \rightarrow Hilb_2 S$ is an immersion, which is injective for $g \geq 7$. For $g \leq 6$ the map j is almost an isomorphism from $F(X)$ onto its image, except for a finite number of multiple points (presumably simple double points). Then Markman's construction produces an ACIHS whose fibers are Picard varieties of Fano surfaces.

As the referee noticed, a similar map $j : F(X) \rightarrow Hilb_2 S$, for S a K3 surface of degree 8, already appears in [HT]. In their case the Fano threefold X has index two, being the intersection

of two quadrics, and $F(X)$ is the Fano surface of lines on X . This surface is abelian, and varying X yields an abelian fibration $\text{Hilb}_2 S \rightarrow \mathbb{P}\mathcal{I}_S(2)^* \simeq \mathbb{P}^2$.

In §4.1 we discuss the less known, but also general fact that the Abel-Jacobi map from the Albanese variety $\text{Alb } F(X)$ of $F(X)$ to the intermediate Jacobian $J(X)$ of X is an isomorphism, at least for the general prime Fano threefold X of any genus g , see Theorem 9 (this also holds for the cubic threefold, and it is an important ingredient in the famous paper of Clemens and Griffiths [CG]). In particular the Picard and the Albanese variety are canonically isomorphic, and our main result follows.

A related construction, as the referee points out, appears in [Be3], where Beauville considers the family of cubic threefolds containing a given K3 surface S of degree six as an anticanonical divisor. Using ideas of Tyurin, he constructs an ACIHS on (a torsor over) the relative intermediate Jacobian of an open part of this family. Note that again, a cubic threefold has index two.

Very recently, using different methods, D. Markushevich extended our Theorem 11 to more general families of Fano threefolds [Mar].

2 Lagrangian images of families of curves on Fano threefolds

Recall from [Muk1] and [Be4] that for any $2 \leq g \leq 12, g \neq 11$, the general K3 surface $S = S_{2g-2}$ of genus g can be represented as an anticanonical divisor in a Fano threefold $X = X_{2g-2}$ of genus g . For $g \geq 3$, the anticanonical bundle $-K_X$ is very ample, so S is just a hyperplane section in the anticanonical embedding of X .

Theorem 1 *Let $X = X_{2g-2}$ be a Fano threefold of genus g , $2 \leq g \leq 12, g \neq 11$, and let S be a general member of the anticanonical system $|-K_X|$. Let \mathcal{H} be an irreducible component of the Hilbert scheme of curves on X , generically reduced, and whose general member is a smooth curve of anticanonical degree d . Consider the intersection map*

$$j : \mathcal{H} \rightarrow \text{Hilb}_d S, \quad j : C \mapsto C \cap S.$$

Then the image $j(\mathcal{H}) \subset \text{Hilb}_d S$ is a Lagrangian subvariety of $\text{Hilb}_d S$.

Proof. Let the curve $C \subset X$ be a general element of \mathcal{H} . Since the anticanonical divisor is base-point free, and $S \in |-K_X|$ is also general, the intersection $C \cap S = \{x_1, \dots, x_d\}$ is a reduced zero-scheme of length d . Consider the differential

$$d_C j : T_C \mathcal{H} = H^0(N_{C/X}) \longrightarrow T_{C \cap S} \text{Hilb}_d S = T_{x_1} S \oplus \dots \oplus T_{x_d} S.$$

The Hilbert scheme $\text{Hilb}_d S$ has a symplectic structure

$$\wedge^2 T_{C \cap S} \text{Hilb}_d S \longrightarrow \wedge^2 T_{x_1} S \oplus \dots \oplus \wedge^2 T_{x_d} S \xrightarrow{\sigma_{x_1}, \dots, \sigma_{x_d}} \mathbb{C} \oplus \dots \oplus \mathbb{C} \xrightarrow{+} \mathbb{C}$$

defined by the choice of a non zero section $\sigma \in H^0(S, K_S) = H^0(S, \Omega_S^2) \simeq \mathbb{C}$.

Denote by j_x the skew-symmetric form defined by the composition

$$\wedge^2 H^0(N_{C/X}) \rightarrow \wedge^2 T_x S \xrightarrow{\sigma_x} \mathbb{C}.$$

We shall see first that $j_{x_1} + \cdots + j_{x_d} = 0$, i.e. that $j(\mathcal{H})$ is an isotropic subvariety of $Hilb_d S$ at the point $j(C) = x_1 + \cdots + x_d$ with respect to the symplectic structure on $Hilb_d S$. For this, we observe that since $K_X(S)|_S = K_S$ and $H^1(X, K_X) = 0$, σ lifts uniquely to a section of $K_X(S)$, a meromorphic 3-form on X with poles on S . Since $K_{X|C} = K_C \otimes (\det N_{C/X})^*$, we get by restriction to C a section $\tilde{\sigma}$ of $K_C(C \cap S) \otimes (\det N_{C/X})^*$. Let α, β be global sections of $N_{C/X}$. Then $\alpha \wedge \beta$ is a global section of $\det N_{C/X}$, and its product with $\tilde{\sigma}$ gives a section $(\sigma, \alpha \wedge \beta)$ of $K_C(C \cap S)$, a meromorphic form on C with poles at x_1, \dots, x_d . Now, for any $x \in S$ the following identity takes place

$$(*) \quad j_x(\alpha, \beta) = \text{res}_x(\sigma, \alpha \wedge \beta).$$

To verify (*), choose local coordinates u, v, w on X around x , such that u, v are local equations of C and w a local equation of S . Our generator σ of K_S can be written locally as $f du \wedge dv$ for f some regular function on S , and it is the residue of the form $\tilde{\sigma}$ on X which is given locally by

$$\tilde{\sigma} = F du \wedge dv \wedge \frac{dw}{w}, \quad F|_S = f.$$

Since $\partial/\partial u$ and $\partial/\partial v$ are local generators of $N_{C/X}$, we write locally $\alpha \wedge \beta$ as $a \partial/\partial u \wedge \partial/\partial v$. The result of the contraction with the restriction of $\tilde{\sigma}$ to C is $(\sigma, \alpha \wedge \beta) = F|_C a \frac{dw}{w}$, whose residue at x is $\text{res}_x(\sigma, \alpha \wedge \beta) = F(x)a(x) = f(x)a(x)$. On the other hand, $j_x(\alpha, \beta)$ is obtained by evaluating σ_x on $a(x)\partial/\partial u \wedge \partial/\partial v$ considered as a vector of $\wedge^2 T_x S$. This gives $j_x(\alpha, \beta) = f(x)a(x)$, as claimed.

Next, by (*) and the residue theorem,

$$j_{x_1}(\alpha, \beta) + \cdots + j_{x_d}(\alpha, \beta) = \text{res}_{x_1}(\sigma, \alpha \wedge \beta) + \cdots + \text{res}_{x_d}(\sigma, \alpha \wedge \beta) = 0,$$

which proves that $j(\mathcal{H})$ is an isotropic subvariety of $Hilb_d S$ at $x_1 + \cdots + x_d$.

Since $x_1 + \cdots + x_d$ as above is a general point of $j(\mathcal{H})$, then $j(\mathcal{H})$ is an isotropic subvariety of $Hilb_d S$; and in order to prove that $j(\mathcal{H})$ is a Lagrangian subvariety of $Hilb_d S$ it remains to see that $j(\mathcal{H})$ has dimension d .

For this, consider the differential of j at a general member C of the family \mathcal{H} .

Let I_C and $N = N_{C/X}$ denote the ideal sheaf and the normal sheaf of C in X . Since the general $C \in \mathcal{H}$ is smooth, in particular C is a locally complete intersection in X , then $N_{C/X}$ is locally free. Let

$$T_C \mathcal{H} = H^0(N) = \text{Hom}(I_C/(I_C)^2, \mathcal{O}_C) \xrightarrow{d_C j} T_{C \cap S} Hilb_d S = \text{Hom}(I_{C \cap S}^S/(I_{C \cap S}^S)^2, \mathcal{O}_{C \cap S})$$

be the natural map deduced from the identity $I_{C \cap S}^S = (I_C + I_S)/I_S$ for the ideal sheaf of $C \cap S$ in S , and let

$$T_{C \cap S} Hilb_d S = \text{Hom}(I_{C \cap S}^S/(I_{C \cap S}^S)^2, \mathcal{O}_{C \cap S}) \xrightarrow{\theta} \text{Hom}(I_C/(I_C)^2, \mathcal{O}_{C \cap S}) = N \otimes_{\mathcal{O}_X} \mathcal{O}_{C \cap S}$$

be the map deduced from the restriction $I_C \rightarrow I_{C \cap S}^S$. Since this restriction map is surjective, θ is injective. Moreover, the composition

$$H^0(N) \xrightarrow{\theta \circ d_C j} N \otimes_{\mathcal{O}_X} \mathcal{O}_{C \cap S}$$

is just the evaluation map $ev_{C \cap S}$ on the punctual scheme $C \cap S$. In particular, the kernel of $d_C j$ coincides with the kernel of $ev_{C \cap S}$, which can easily be computed from the exact sequence $0 \rightarrow \mathcal{O}_C(-S) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{C \cap S} \rightarrow 0$, twisted by the locally free sheaf N . We get

$$\text{Ker } d_C j = H^0(N(-S)) = H^0(N \otimes K_{X|C}) = H^0(N^* \otimes K_C) \simeq H^1(N)^*.$$

This implies that the relative dimension of j is $h^1(N)$. Then, if g denotes the arithmetic genus of C , the Riemann-Roch yields

$$\dim j(\mathcal{H}) = h^0(N) - h^1(N) = \chi(N) = \text{deg} N + 2(1 - g) = -K_X \cdot C = d.$$

This concludes the proof. \square

Example. Let X be a prime Fano threefold of genus $g \geq 3$. When $g = 3$ the linear system $| -K_X |$ is either very ample and embeds X in \mathbb{P}^4 as a quartic hypersurface, or it defines a double covering of the smooth 3-dimensional quadric, see Propositions 4.1.11 and 4.1.12 in [IP]. For $g > 3$, the anticanonical bundle is very ample and defines an embedding $X \subset \mathbb{P}^{g+1}$. Consider the family \mathcal{H}_{can} of canonical curves on X , that is, codimension two linear sections of X . Then $\dim \mathcal{H}_{can} = \dim G(2, g+2) = 2g$, while these curves have anticanonical degree $-K_X^3 = 2g - 2 < \dim \mathcal{H}_{can}$. The image of the map

$$j : \mathcal{H}_{can} \longrightarrow \text{Hilb}_{2g-2} S$$

is the family of canonical punctual schemes on S , defined again by codimension two linear sections of S . Its dimension is $\dim j(\mathcal{H}_{can}) = \dim G(2, g+1) = 2g - 2$, as expected, and j is a \mathbb{P}^2 -fibration. Note that a canonical curve C has normal bundle $N = \mathcal{O}_C(1) \oplus \mathcal{O}_C(1) = K_C \oplus K_C$, so clearly $h^1(N) = 2$. The fact that $j(\mathcal{H}_{can})$ is Lagrangian already appears in [Voi], Exemple 3.

The proof of the previous statement also provides a criterion for the map j to be immersive.

Corollary 2 *Let \mathcal{H} be as above, and let C be a curve of the family parameterized by \mathcal{H} . Suppose that C is a locally complete intersection in X , with normal bundle N . Then j is an immersion at C if and only if the evaluation map*

$$ev_{C \cap S} : H^0(N) \longrightarrow N \otimes_{\mathcal{O}_X} \mathcal{O}_{C \cap S}$$

is injective.

In the next section this criterion will be applied to conics on prime Fano threefolds.

3 Fano surfaces of conics on prime Fano threefolds

Let X be a prime Fano threefold of genus g . It is well known that X contains lines, but there is only a one-dimensional family of lines on X , and in particular X is not covered by lines, see §4.2 in [IP]. But X is covered by conics, and the Fano scheme $F(X)$ of conics on X appears as a fundamental object of study, *ibid*.

In this section we focus on $F(X)$, for X a general prime Fano threefold, and we apply to conics the main result of the previous section.

3.1 Conics on general prime Fano threefolds

First recall that the genus g of a prime Fano threefold X can be any integer between 2 and 12, except 11. For $g = 2$, the anticanonical divisor is base point free and defines X as a double cover of \mathbb{P}^3 ramified along a general sextic hypersurface (a sextic double solid). For $g \geq 3$, $-K_X$ is very ample and if $g \leq 10$, it embeds X as a complete intersection in some homogeneous space Σ , as follows from the work of Mukai (see [Muk1], and also [IP]). For the convenience of the reader we recall below, for each value of g , the nature of Σ , its dimension d , and the multidegree δ of X in Σ .

g	δ	Σ	d
3	4	\mathbb{P}^4	4
4	2, 3	\mathbb{P}^5	5
5	2, 2, 2	\mathbb{P}^6	6
6	2, 1, 1	$G(2, 5)$	6
7	1, 1, 1, 1, 1, 1	$OG(5, 10)$	10
8	1, 1, 1, 1, 1	$G(2, 6)$	8
9	1, 1, 1	$LG(3, 6)$	6
10	1, 1	G_2^{ad}	5

Here $G(r, d)$ (resp. $OG(r, d)$, $LG(r, d)$) is the Grassmannian of r -dimensional subspaces (resp. isotropic subspaces) of a d -dimensional vector space (resp. endowed with a non degenerate symmetric or skew-symmetric bilinear form, where we consider only one connected component in the case of $OG(r, 2r)$). G_2^{ad} denotes the adjoint variety of the exceptional group G_2 , the closed G_2 -orbit in the projectivized Lie algebra $\mathbb{P}\mathfrak{g}_2$.

For any $2 \leq g \leq 12$, $g \neq 11$ the Fano scheme of conics $F(X)$ is known to be a smooth surface, and the normal bundle to a general conic $q \subset X$ is trivial, see [CV] for $g = 2$, [CMW] for $g = 3$, [IM2] for $g = 4$, and Proposition 4.2.5 in [IP] for $g \geq 5$.

For $g = 12$, the Fano surface $F(X) \simeq \mathbb{P}^2$, see Remark 5.2.15 in [IP].

For $g = 10$, $F(X)$ is an Abelian surface. Since we could not find any reference for this fact, we include a proof of the following statement:

Proposition 3 *For X a general Fano threefold of genus 10, the Fano surface $F(X)$ and the intermediate Jacobian $J(X)$ are isomorphic.*

Proof. We first prove that $F(X)$ is an Abelian surface. The idea to prove this is to construct an Abelian fibration on the Hilbert square of a general K3 surface S of genus 10. The existence of this Abelian fibration is known by [HT] (see also [Saw] for a more general treatment of Abelian fibrations on Hilbert powers of K3 surfaces).

Following Mukai, $S = \mathbb{P}_S^{10} \cap G_2^{ad}$ is a codimension three linear section of G_2^{ad} , the adjoint variety of G_2 (see 3.3 for more details). Let $F(G_2^{ad})$ denote the family of conics on G_2^{ad} . We define $\mathcal{F}(S) \subset F(G_2^{ad}) \times \text{Hilb}_2 S$ as the incidence correspondence of pairs (q, ξ) , such that $\xi \subset q$. By Lemma 7, the projection $\mathcal{F}(S) \rightarrow \text{Hilb}_2 S$ is surjective and birational. By Lemma 8, this morphism is bijective, hence an isomorphism. In particular $F(S)$ is smooth.

Now, there is a morphism $p : F(S) \rightarrow \mathbb{P}\mathcal{I}_S(1)^* \simeq \mathbb{P}^2$, mapping (q, ξ) to the linear span of $S \cup q$. This morphism is well-defined since S , being general, does not contain any conic.

By Matsushita's theorem (see [Msh]), the general fiber of p is an abelian surface in $Hilb_2 S$. Since this general fiber can be identified with the family of conics contained in the general Fano threefold $X = G_2^{ad} \cap \mathbb{P}_X^{11}$ containing S as a hyperplane section, we are done. \square

The isomorphism of the Abelian surface $F(X)$ with the intermediate Jacobian of $J(X)$ can be derived on the base of the birational geometry of X as follows.

Let $\ell \subset X$ be a line. By Theorem 4.3.3(vii) of [IP], the double projection π_ℓ from ℓ defines a birational isomorphism from X to a 3-fold quadric Q . The double projection π_ℓ contracts the family of conics intersecting ℓ to a smooth curve $\Gamma \subset Q$ of genus 2 and degree 7; therefore the intermediate Jacobian $J(X)$ is isomorphic to the Jacobian $J(\Gamma)$ of Γ .

By Matsusaka's criterion the Abelian surface $F(X)$ will be isomorphic to $J(\Gamma)$ if we can find on $F(X)$ a curve $C \simeq \Gamma$ with self-intersection number $C^2 = 2$, see [Msk]. We shall show that such a curve is the base $F_\ell \subset F(X)$ of the family of conics on X intersecting ℓ . First, by the preceding $F_\ell \simeq \Gamma$. Next, the base of the family $G(X)$ of lines on X is a smooth irreducible curve (of genus 10), see Theorem 4.2.7 in [IP]. Therefore all the curves F_ℓ , $\ell \in G(X)$ are algebraically equivalent to each other on the surface $F(X)$. So $F_\ell^2 = F_\ell.F_m$ for any other line m on X . If $m \subset X$ is general then the intersection number $F_\ell.F_m$ is the number of conics $q \subset X$ that intersect both ℓ and m . But according to Theorem 4.3.3 (vii) of [IP], the conics $q \in F_\ell$ sweep out a quadratic section D_ℓ of X . So D_ℓ intersects the line m at two points, and therefore $F_\ell.F_m = D_\ell.m = 2$. \square

For $g = 9$, $F(X)$ is a ruled surface with base a smooth plane quartic C_X such that $J(X) \simeq J(C_X)$, see [Ili] and [IR], Corollary 2.7.9.

For $g = 8$, $F(X)$ is isomorphic to the Fano surface of lines on the unique smooth cubic threefold Y_X birational to X , [IM1]. The Fano surfaces of lines on cubic threefolds have been studied in deep by Clemens and Griffiths, [CG].

For $g = 7$, $F(X)$ is isomorphic to the symmetric square $Sym^2 C_X$ of a genus 7 curve C_X , such that $J(X) = J(C_X)$, see [IMr] or [Kuz].

For $g = 6$, the Fano surface $F(X)$ is smooth by [Log], Proposition 0.1.

For $g = 3, 4, 5$, the Fano threefold X is a complete intersection. This allows to describe $F(X)$ as the zero-locus of a section of some globally generated vector bundle; in particular $F(X)$ is smooth for X general, see [CMW] for $g = 3$, [IM2] for $g = 4$, [Wel] and [PB] for $g = 5$.

Finally, the smoothness of $F(X)$ for $g = 2$ is proved by Ceresa and Verra, see Proposition 1.14 in [CV].

We compile some numerical data for the Fano surface $F = F(X)$ of a general prime Fano threefold X of genus g .

g	c_2	c_1^2	p_a	ν	q	F
12	3	9	0	6	0	\mathbb{P}^2
10	0	0	-1	9	2	<i>Abelian</i>
9	-8	-16	-3	12	3	<i>ruled</i>
8	27	45	5	16	5	
7	66	114	14	24	7	<i>Sym²C</i>
6	384	720	91	39	10	
5	1760	3376	427	148	14	
4	11961	23355	2942	318	20	
3	172704	341040	42811	972	30	
2	?	?	?	?	52	

We denote by ν the number of conics through a general point of X . For the value of ν for $g \geq 8$, see the table (2.8.1) in [Tak]. In the cases $g = 7$ and $g = 6$, the numbers $\nu = 24$ and $\nu = 39$ can be computed by the same approach as in [Tak], see e.g. Theorem 4.5.8 in [IP]. The computation for $g = 3$ is done in [CMW], and the case $g = 4$ is in [IM2]. The invariants in genus 5 are in [Wel], except for the computation of ν which can be obtained with the same techniques as for $g = 3, 4$.

Beware that in the case $g = 6$ the Fano surface $F(X)$ is the blow-up of a point on a smooth surface F ; the table gives the invariants of F , see [Log].

3.2 Intersecting conics with a K3 surface

Let $S \subset X$ be a K3 surface, defined as a general anticanonical section. Then the image of the natural map $j : F(X) \rightarrow \text{Hilb}_2 S$ is a possibly singular Lagrangian surface. In fact j can be an embedding only if $g \geq 6$. Our precise result is the following:

Proposition 4 *Let X be a general prime Fano threefold of genus g , and S a general hyperplane section. Then:*

1. $j : F(X) \rightarrow \text{Hilb}_2 S$ is an immersion.
2. j is injective for $g \geq 7$ (but not for $g \leq 6$).

Proof. The injectivity of j is equivalent to the non-existence of bisecant conics meeting along S . For $g \geq 7$ this was observed in [Voi], Section 2. For $g \leq 6$ this is no longer true, see the Remark below.

Nevertheless we can check that j is always immersive by applying the criterion of Corollary 2. Let $q \in F(X)$ be a smooth conic. Since $F(X)$ is smooth at q and the normal bundle N of q in X has trivial determinant, we must have $N = \mathcal{O}_q \oplus \mathcal{O}_q$ or $N = \mathcal{O}_q(1) \oplus \mathcal{O}_q(-1)$. In both cases the evaluation map

$$ev_{q \cap S} : H^0(N) \rightarrow N \otimes_{\mathcal{O}_X} \mathcal{O}_{q \cap S}$$

is injective, where $q \cap S$ is the union of two distinct points or the tangent direction to q at some point.

Now we consider the case of a singular conic q . A simple dimension count shows that X being general, q cannot be a double line. So we let $q = \ell \cup m$, with vertex $p = \ell \cap m$. Since q is

a locally complete intersection, its normal bundle $N = N_{q/X} = \text{Hom}(I_q/I_q^2, \mathcal{O}_q)$ is a locally free \mathcal{O}_q -module of rank two. Twisting the exact sequence $0 \rightarrow \mathcal{O}_q \rightarrow \mathcal{O}_\ell \oplus \mathcal{O}_m \rightarrow \mathcal{O}_p \rightarrow 0$ by N , we get

$$0 \rightarrow N \rightarrow N(\ell) \oplus N(m) \rightarrow N_p \rightarrow 0,$$

where $N(\ell) = N \otimes_{\mathcal{O}_q} \mathcal{O}_\ell$ is a locally-free \mathcal{O}_ℓ -module. Note that since $I_q \supset I_\ell I_m$, $I_m N$ is also a \mathcal{O}_ℓ -module, again locally free of rank two. In fact $I_m N = N(\ell)(-p)$, and we have a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N_p & \longrightarrow & N(\ell)_p \oplus N(m)_p & \longrightarrow & N_p \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & N & \longrightarrow & N(\ell) \oplus N(m) & \longrightarrow & N_p \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \\ & & I_m N \oplus I_\ell N & = & N(\ell)(-p) \oplus N(m)(-p) & & \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

Beware that $I_m N$ is not isomorphic to N_ℓ , the normal bundle to ℓ , although these two sheaves are obviously isomorphic outside p . In fact, there is an exact sequence

$$0 \longrightarrow I_m N \longrightarrow N_\ell \longrightarrow N_{\ell,p}/\text{Im}T_p m = \mathcal{O}_p \rightarrow 0.$$

For X general, the curve $\Gamma(X)$ of lines on X is smooth, hence $N_\ell \simeq \mathcal{O}_\ell \oplus \mathcal{O}_\ell(-1)$. So there are two possibilities: either $I_m N \simeq \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(-1)$ or $I_m N \simeq \mathcal{O}_\ell \oplus \mathcal{O}_\ell(-2)$, and the latter case occurs if and only if the map

$$H^0(N_\ell) \longrightarrow N_{\ell,p}/\text{Im}T_p m$$

is not injective. Anyway, we get three possibilities:

1. $N(\ell)$ and $N(m)$ are both trivial, hence N itself is trivial;
2. N_ℓ is trivial but $N(m) = \mathcal{O}_m(1) \oplus \mathcal{O}_m(-1)$ (or the opposite): then $H^0(N) \simeq H^0(N(m))$, and each section is defined on restriction to ℓ by its value at p ;
3. $N(\ell) = \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(-1)$ and $N(m) = \mathcal{O}_m(1) \oplus \mathcal{O}_m(-1)$. Note this does not imply that $N = \mathcal{O}_q(1) \oplus \mathcal{O}_q(-1)$: this is true only if the two distinguished directions in N_p fit together, which is in fact excluded by the condition that $h^0(N) = 2$.

Now we need to check that the evaluation map $ev_x \oplus ev_y : H^0(N) \longrightarrow N_x \oplus N_y$ is injective, if $x = \ell \cap S$ and $y = m \cap S$ are distinct; if they coincide, we want $ev_x^1 : H^0(N) \longrightarrow N \otimes \mathcal{O}_q/\mathcal{M}_x^2$ to be injective. For the three possibilities above this is a straightforward verification. \square

Corollary 5 *For X a general prime Fano threefold of genus $g \geq 7$, and S a general hyperplane section, $j(F(X))$ is a smooth Lagrangian surface in $\text{Hilb}_2 S$.*

Remark. For $g = 6$ recall that X is a complete intersection of multidegree $\delta = (2, 1, 1)$ in $G(2, 5)$. For any conic q in X , there is a unique hyperplane H in \mathbb{C}^5 such that q is contained in $G(2, H)$. Since the intersection of X with $G(2, H)$ has degree four, it must be the union of q with an involutive conic q' , which is bisecant to q . By an obvious dimension count, there must exist a finite number of conics q on X , whose two common points with the involutive conic q' lie on S – and in this case $j(q) = j(q')$. This shows that $j(F(X))$ is not smooth but has finitely many double points.

3.3 Getting back from $j(F(X))$ to X

It is not true in general that X is uniquely determined by the surface $F(X)$. For example, the Fano surface of the general X_{22} is isomorphic to \mathbb{P}^2 , but the Fano 3-folds X_{22} have 6 moduli, see e.g. Corollary 1.2 in [Sch]. Similarly, one can see that the Fano surface of the general X_{18} is the Jacobian of the general genus 2 curve and has 3 moduli, while X_{18} has 10 moduli.

Here we consider a slightly different question: does $j(F(X)) \subset \text{Hilb}_2 S$ define X uniquely among those Fano threefolds that contain S ?

Theorem 6 *For $7 \leq g \leq 10$, let S be a general K3 surface of genus g . Then a general Fano threefold X containing a fixed S as a hyperplane section, is uniquely determined by the Lagrangian surface $j(F(X)) \subset \text{Hilb}_2 S$.*

We do not know whether this statement is still true for $g \leq 6$.

Proof. We want to recover $X \supset S$ from the embedded Lagrangian surface $j(F(X)) \subset \text{Hilb}_2 S$. By [Muk1], S and X have linear embeddings in a rational homogeneous variety $\Sigma = G/P$ of some simple adjoint Lie group G , and this embedding is unique up to the action of G . Therefore we may assume that $S \subset X \subset G/P$.

Now we have to consider separately the different values of g .

$g = 10$. Here $G = G_2$ and G/P is the adjoint G_2 -variety, the variety G_2^{ad} parametrizing highest root spaces in the projectivized Lie algebra $\mathbb{P}\mathfrak{g}_2$. By this we mean the following: choose a maximal torus T of G . Then the adjoint action of T on the Lie algebra $\mathfrak{g} = \mathfrak{g}_2$ of G can be diagonalized as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Here \mathfrak{t} denotes the Lie algebra of T . Moreover the lines \mathfrak{g}_α are the root spaces of \mathfrak{g} , they are parametrized by the set Φ of roots, which can be seen as linear forms on \mathfrak{t} (see [Bou]). Once we split Φ into positive and negative roots, we get a highest root $\tilde{\alpha}$. Then the adjoint G_2 -variety can be defined as the G_2 -orbit of the line $\mathfrak{g}_{\tilde{\alpha}}$ in $\mathbb{P}\mathfrak{g}_2 \simeq \mathbb{P}^{13}$.

In order to describe the geometry of G_2^{ad} , we need to remember that the group G_2 is the automorphism group of the (complexified) algebra \mathbb{O} of Cayley octonions. This action preserves the unit element 1 and also the scalar product on \mathbb{O} . So it also preserves the orthogonal space to 1, which is the space $Im\mathbb{O}$ of imaginary octonions. This yields the minimal representation of G_2 , of dimension 7. Moreover the projectivization $\mathbb{P}(Im\mathbb{O})$ contains the smooth quadric Q defined by the scalar product. The basic geometry of Q and G_2^{ad} , as G_2 -varieties, can be described as follows (see [LM], 6.1):

1. G_2^{ad} can be described as a subvariety of $G_Q(2, 7)$, the variety of projective lines on Q . The special lines parametrized by G_2^{ad} are those on which the octonionic multiplication is identically zero. We have a diagram

$$\begin{array}{ccc} G_2^{ad} & \hookrightarrow & \mathbb{P}\mathfrak{g}_2 \\ \cap & & \cap \\ G_Q(2, 7) & \hookrightarrow & \mathbb{P}^{507}, \end{array}$$

and $G_2^{ad} = G_Q(2, 7) \cap \mathbb{P}\mathfrak{g}_2$.

2. Let x be a point in G_2^{ad} . Then the lines in G_2^{ad} passing through x generate a cone K_x over a rational normal cubic curve C_x . The linear span of this set of lines is a hyperplane H_x in $\mathbb{P}T_x G_2^{ad}$ (the contact hyperplane).
3. For any point $y = [Y] \in Q$, the set of $z = [Z] \in Q$ such that $YZ = 0$ in $Im\mathbb{O}$ is a projective plane containing y . This plane defines a line $L_y \subset G_2^{ad}$. The variety of lines in G_2^{ad} is isomorphic to Q .

Lemma 7 *Consider two general points x, y in the adjoint variety G_2^{ad} . Then there is a unique conic in G_2^{ad} passing through x and y .*

Proof of the lemma. We first prove that there exists at least one conic in G_2^{ad} through x and y . To see this, observe that the adjoint variety being homogeneous, we may let x be defined by the line $\mathfrak{g}_{\bar{\alpha}}$. Then, by the Bruhat decomposition, the stabilizer $P \subset G_2$ of that line acts on the adjoint variety with an open orbit, containing the point defined by the line $\mathfrak{g}_{-\bar{\alpha}}$. So we may suppose that this line defines our point y .

Now the Lie subalgebra of \mathfrak{g}_2 generated by x and y is $\mathfrak{g}_{\bar{\alpha}} \oplus \mathfrak{g}_{-\bar{\alpha}} \oplus [\mathfrak{g}_{\bar{\alpha}}, \mathfrak{g}_{-\bar{\alpha}}]$, a copy of \mathfrak{sl}_2 , generating in G_2 a copy of SL_2 . Then the SL_2 -orbits of x and y have the same closure, which is a conic passing through x and y in G_2^{ad} .

We must check that this conic is unique. But it is already true that through two general points of $G_Q(2,7)$ there exists a unique conic, which can be constructed as follows. The two points x and y represent two skew lines $\ell_x, \ell_y \subset Q$. These two lines generate $P_{x,y}^3 = \text{Span}(\ell_x \cup \ell_y)$ intersecting Q along a smooth quadric surface $S_{x,y}$. One of the two rulings of this surface contains ℓ_x and ℓ_y ; it defines the unique conic q in $G_Q(2,7)$ passing through x and y . \square

From $j(F(X)) \subset \text{Hilb}_2 S$ we can therefore recover the Fano surface $F(X)$, and then the threefold X itself since it is swept out by the conics it contains. This concludes the proof of the Theorem. \square

For the proof of Proposition 3 we need a more precise result.

Lemma 8 *Let ξ be a punctual subscheme of length two of G_2^{ad} . Suppose that the line generated by ξ is not contained in G_2^{ad} . Then there is a unique conic in G_2^{ad} containing ξ .*

Proof. The existence follows from the previous lemma so we just need to prove the uniqueness. Let $d(\xi) \in \{3, 4\}$ be the dimension of the minimal subspace $D(\xi)$ of $Im\mathbb{O}$ such that the line \mathbb{P}_ξ^1 generated by ξ is contained in $\mathbb{P}(\wedge^2 D(\xi))$. Let $Q_\xi = Q \cap \mathbb{P}D(\xi)$.

If $d(\xi) = 4$ and Q_ξ is smooth, then the argument we used in the proof of the previous lemma applies, and there is a unique conic in G_2^{ad} passing through ξ .

If $d(\xi) = 4$ but Q_ξ is singular, a conic q through ξ must be singular, say $q = L_u + L_v$ for some $u, v \in Q$. The support of ξ must $x \in L_u \cap L_v$ (beware that the two lines may coincide). Then L_u and L_v can be seen as two points c_u and c_v of the rational cubic C_x in $\mathbb{P}H_x$. If $u \neq v$, the line \mathbb{P}_ξ^1 defines a point in $\mathbb{P}H_x$ which stands on the bisecant to C_x joining c_u and c_v . Moreover this point is neither c_u nor c_v , since otherwise the line \mathbb{P}_ξ^1 would be contained in K_x , hence in G_2^{ad} . If $u = v$, then q is a double line, and its supporting plane is the cone over the tangent line to C_x at $c_u = c_v$. Moreover \mathbb{P}_ξ^1 defines a point on this tangent line, and this point cannot be c_u . In both cases, the uniqueness of q follows from the fact that any point of $\mathbb{P}H_x - C_x$ belongs to a unique bisecant (including tangent lines) to C_x .

If $d(\xi) = 3$, the line \mathbb{P}_ξ^1 is contained in $G_Q(2, 7)$. But since G_2^{ad} is a linear section of $G_Q(2, 7)$, this implies that \mathbb{P}_ξ^1 is also contained in G_2^{ad} , a contradiction. \square

$g = 9$. Here $G = Sp_6$ and $G/P = LG(3, 6)$, the Lagrangian Grassmannian.

Recall from [Ili, IR] that a conic $q \subset LG(3, 6) = LG(\mathbb{P}^2, \mathbb{P}^5)$ has a vertex-point $v(q) \in \mathbb{P}^5$, defined as the unique intersection point of the Lagrangian planes \mathbb{P}_t^2 , $t \in q$.

A general conic $q \subset X$ is smooth and intersects S at two different points x, y , from which the vertex can be recovered as $v(q) = \mathbb{P}_x^2 \cap \mathbb{P}_y^2$. In particular, $j(F(X))$ determines the vertex surface $v(F(X)) \subset \mathbb{P}^5$. By [IR], Proposition 2.8.5, X can be recovered as the set of projective planes in \mathbb{P}^5 that cut the vertex surface along a finite scheme of length 12.

Note that for a general Fano threefold X of genus 8, the vertex surface $v(F(X)) \subset \mathbb{P}^5$ is a smooth surface isomorphic to $F(X)$. Indeed $v(F(X))$ is smooth ([IR], Corollary 2.7.9), and the map $v : F(X) \rightarrow v(F(X))$ is bijective ([Ili], Lemma 4.5.c).

$g = 8$. Here $G = PSL_6$ and $G/P = G(2, 6)$ is the Grassmannian of (projective) lines.

If $S \subset X \subset G(2, 6) \subset \mathbb{P}(\wedge^2 \mathbb{C}^6)$, let $L_X \subset L_S$ be the orthogonal in $\mathbb{P}(\wedge^2 \mathbb{C}^6)^*$ of the linear spans of S and X . They are projective spaces of dimension four and five respectively. We want to show that, the embedding $j(F(X)) \subset Hilb_2 S$ being given, the hyperplane L_X of L_S is uniquely determined.

A conic q in $G(2, 6)$ whose linear span is not contained in $G(2, 6)$, defines a unique 4-space $V \subset \mathbb{C}^6$ such that $q \subset G(2, V)$. Moreover, q can be obtained as the intersection of the four-dimensional quadric $G(2, V) \subset \mathbb{P} \wedge^2 V$ with some projective plane. In particular, $G(2, V)$ contains some conic of X if and only if $\mathbb{P} \wedge^2 V$ meets L_X^\perp along a projective space of dimension two at least. We thus consider on $G = G(4, 6)$ the tautological rank four vector bundle T . By restriction of skew-symmetric forms, we have morphisms of vector bundles

$$\begin{array}{ccc} L_S \otimes \mathcal{O}_G & \xrightarrow{\phi_S} & \wedge^2 T^* \\ \uparrow & & \parallel \\ L_X \otimes \mathcal{O}_G & \xrightarrow{\phi_X} & \wedge^2 T^* \end{array}$$

Since ϕ_X is a general morphism from $\mathcal{O}_G^{\oplus 5}$ to the rank six vector bundle $\wedge^2 T^*$, its degeneracy locus $\Sigma(X) = D_3(\phi_X)$ has codimension $(5-3)(6-3) = 6$ and is singular exactly along $D_2(\phi_X)$, which has codimension $(5-2)(6-2) = 12 > 8$. So $\Sigma(X)$ is a smooth surface, and since it is in bijective correspondence with $F(X)$ these two surfaces must be isomorphic.

On the other hand, the degeneracy locus $\Sigma(S) = D_4(\phi_S)$ has codimension $(6-4)(6-4) = 4$ and is singular exactly along $D_3(\phi_S)$, which has codimension $(6-3)(6-3) = 9 > 8$. So $\Sigma(S)$ is a smooth four-dimensional subvariety of $G(4, 6)$. On $\Sigma(S)$ the morphism ϕ_S has constant rank four, so in the exact sequence

$$0 \longrightarrow Ker\phi_S \longrightarrow L_S \otimes \mathcal{O}_G \xrightarrow{\phi_S} \wedge^2 T^* \longrightarrow Coker\phi_S \longrightarrow 0,$$

$Ker\phi_S$ and $Coker\phi_S$ are rank two vector bundles. Note that if we take determinants in this sequence, we get

$$\det(Coker\phi_S) \det(Ker\phi_S)^{-1} = \det(\wedge^2 T^*) = \mathcal{O}_G(3).$$

Now the normal bundle of $\Sigma(S)$ in G is naturally identified with $Hom(Ker\phi_S, Coker\phi_S)$, whose determinant is $\det(Coker\phi_S)^2 \det(Ker\phi_S)^{-2} = \mathcal{O}_G(6) = K_G^{-1}$. Therefore $\Sigma(S)$ has trivial

canonical bundle. Now the natural map $\Sigma(S) \rightarrow \text{Hilb}_2 S$ is finite, hence étale, hence an isomorphism since $\text{Hilb}_2 S$ is simply connected. We conclude that the embedding $j(F(X)) \subset \text{Hilb}_2 S$ is the same as the embedding $\Sigma(X) \subset \Sigma(S)$.

But from L_S , its degeneracy locus $\Sigma(S)$ and the embedding $\Sigma(X) \subset \Sigma(S)$, the hyperplane L_X of L_S can be recovered as follows. At any point $x \in \Sigma(X)$, the morphisms ϕ_S and ϕ_X have the same kernel, which in particular is contained in L_X . So we must have

$$L_X = \text{Span}(\text{Ker} \phi_{S,x}, x \in \Sigma(X)) \subset L_S.$$

And then of course X can be recovered as $X = G(2, 6) \cap L_X^\perp$.

$g = 7$. Here $G = \text{Spin}_{10}$ and $G/P = \Sigma \subset \mathbb{P}\Delta_+$ is the spinor variety, naturally embedded in the projectivization of a half-spin representation Δ_+ . The spinor variety parametrizes one of the two connected families of maximal linear spaces in a smooth quadric $\mathbb{Q}^8 \subset \mathbb{P}^9$. The dimension of Σ is 10. The dimension of Δ_+ is sixteen, and its dual is the other non-equivalent half-spin representation, Δ_- (see [Bou] for more details).

If $S \subset X \subset \Sigma \subset \mathbb{P}\Delta_+$, let $L_X \subset L_S$ be the orthogonal spaces in $\mathbb{P}\Delta_-$ to the linear spans of S and X . They are projective spaces of dimension seven and eight respectively. Again we want to show that, the embedding $j(F(X)) \subset \text{Hilb}_2 S$ being given, the hyperplane L_X of L_S is uniquely determined.

A conic q in Σ whose linear span is not contained in Σ , has a vertex $v = v(q) \in \mathbb{P}^9$, which is the unique common point to the maximal linear spaces in \mathbb{Q}^8 parametrized by q (see [IMr]). The set Σ_v of such maximal linear spaces in \mathbb{Q}^8 , passing through v , is a copy of the spinor variety of Spin_8 . But then we have the famous Cartan triality: there exists an outer automorphism of Spin_8 permuting its two half-spin representations with the usual vector representation of SO_8 (seen as a Spin_8 -module through the double cover $\text{Spin}_8 \rightarrow SO_8$). In particular, this implies that Σ_v is just a six-dimensional quadric $Q_v \subset \Sigma$. Note that the conic q can be obtained as the intersection of Q_v with some projective plane. In particular, Q_v contains some conic of X if and only if its linear span P_v meets L_X^\perp along a projective space of dimension two at least.

These linear spaces P_v fit into a vector bundle on the eight-dimensional quadric \mathbb{Q}^8 in \mathbb{P}^9 , a spinor bundle S of rank eight [Ott]. We get on \mathbb{Q}^8 the morphisms of vector bundles

$$\begin{array}{ccc} L_S \otimes \mathcal{O}_G & \xrightarrow{\phi_S} & S^* \\ \uparrow & & \parallel \\ L_X \otimes \mathcal{O}_G & \xrightarrow{\phi_X} & S^* \end{array}$$

Exactly as in genus eight, the degeneracy locus $\Sigma(X) = D_3(\phi_X)$ is a smooth surface isomorphic to $F(X)$. Also the degeneracy locus $\Sigma(S) = D_2(\phi_S)$ is a smooth four-dimensional variety with trivial canonical bundle (because $\det S = \mathcal{O}_Q(4)$), in fact isomorphic to $\text{Hilb}_2 S$. And from the embedding $\Sigma(X) \subset \Sigma(S)$, which is the same as $j(F(X)) \subset \text{Hilb}_2 S$, we can recover L_X in L_S as before.

This concludes the proof. □

4 Fano-Integrable Systems

Thanks to the main results of the previous section, we are now in position to apply the Donagi-Markman construction [DM2]. The upshot will be new Lagrangian fibrations over families of prime Fano threefolds with a fixed “boundary”, whose fibers are intermediate Jacobians.

4.1 The Abel-Jacobi isomorphism for conics on prime Fano's

The intermediate Jacobian $J(X)$ a prime Fano threefold of genus g is an Abelian variety of dimension $h^{1,2}(X) = b_3(X)/2$ (see [CG] for the definition and general properties of intermediate Jacobians).

This dimension can be computed case by case. Note that for $7 \leq g \leq 10$, we can use the fact that X is a linear section of a d -dimensional homogeneous space $\Sigma \subset \mathbb{P}^{d+g-2}$. After twisting the normal exact sequence by $K_X = \mathcal{O}_X(-1)$, we get

$$0 \longrightarrow \Omega_X^2 \longrightarrow T\Sigma \otimes \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X^{\oplus d-3} \longrightarrow 0.$$

With the help of the Bott-Borel-Weil theorem one can check that $H^i(\Sigma, T\Sigma \otimes \mathcal{O}_\Sigma(-i-1)) = H^{i+1}(\Sigma, T\Sigma \otimes \mathcal{O}_\Sigma(-i-1)) = 0$ for $0 \leq i \leq d-3$. Using the Koszul resolution of \mathcal{O}_X this implies that $H^0(X, T\Sigma \otimes \mathcal{O}_X(-1)) = H^1(X, T\Sigma \otimes \mathcal{O}_X(-1)) = 0$. Hence the isomorphism

$$H^1(\Omega_X^2) = \mathbb{C}^{d-3} = L_X,$$

if we denote by L_X the space of linear forms defining X in Σ .

Note also that if X has no automorphism, which is the general case, we get an exact sequence

$$0 \longrightarrow H^0(X, T\Sigma|_X) \simeq \mathfrak{g} \longrightarrow L_X \otimes L_X^\perp \longrightarrow H^1(X, TX) \longrightarrow 0,$$

where \mathfrak{g} denotes the Lie algebra of the Lie group G ; hence $h^1(TX) = (g+2)(d-3) - \dim G$. The analogue sequence for S is

$$0 \longrightarrow H^0(S, T\Sigma|_S) \simeq \mathfrak{g} \longrightarrow L_S \otimes L_S^\perp \longrightarrow H^1(S, TS) \longrightarrow \mathbb{C} \longrightarrow 0.$$

This gives $h^1(TS) - 1 = (g+1)(d-2) - \dim G = 19$, and thus

$$h^1(TX) = 15 + d - g = h^1(\Omega_X^2) + 18 - g.$$

A basic property of the intermediate Jacobian is that for any family $\mathcal{C} \rightarrow F$ of curves in X , parameterized by a smooth base F , there is an induced *Abel-Jacobi mapping* $F \rightarrow J(X)$.

Theorem 9 *Let X be a general prime Fano threefold of genus g , with a smooth surface of conics $F(X)$. Then the Abel-Jacobi mapping $F(X) \rightarrow J(X)$ induces an isomorphism*

$$\text{Alb}F(X) \simeq J(X).$$

This is due to several people.

The case $g = 12$ is trivial since in that case $h^{1,2}(X) = 0$ while $F(X)$ is a projective plane. For $g = 10$, $F(X)$ is already isomorphic to $J(X)$ so there is nothing to prove.

For $g = 9$, $F(X)$ is a ruled surface over the orthogonal plane quartic curve C_X , therefore $\text{Alb}F(X) = J(C_X)$. The fact that $J(C_X) \simeq J(X)$ is proved in [IR], Corollary 2.8.10.

For $g = 8$, $F(X)$ is isomorphic to the Fano surface $F(Y)$ of lines in the orthogonal cubic threefold Y . The intermediate Jacobians of X and Y are naturally isomorphic. The Fano surface $F(X)$ embeds in $J(X)$, $F(Y)$ embeds in $J(Y)$, and their images coincide up to multiplication by -1 ([IM1], Proposition 4.6). So the Abel-Jacobi isomorphism for $F(X)$ follows from the corresponding statement for $F(Y)$, which is due to Clemens and Griffiths [CG].

For $g = 7$, $F(X)$ is isomorphic to $Sym^2 C_X$, where C_X is the $Spin_{10}$ -orthogonal curve. This implies that $AlbF(X) \simeq J(C_X)$. As known by §8 of [Muk2] the jacobian $J(C_X)$ is isomorphic to $J(X)$, see also [IMr].

For $g = 5$ Welters proved in [Wel] that the Abel-Jacobi map is an isogeny. It follows from the work of Picco Botta [PB], that it is in fact an isomorphism. The method of proof is the degeneration technique of Clemens and Letizia, which was used in [Let] for $g = 3$ (quartic threefolds), in [CV] for $g = 2$ (sextic double solids), and also in [IM2] for $g = 4$.

For $g = 6$, it is proved in [Log], Theorem 6.12, that the Abel-Jacobi map $AlbF(X) \rightarrow J(X)$ is an isogeny. In fact the Clemens-Letizia method also applies in that case. All the necessary conditions (a precise description of the Fano surface for a general nodal Fano threefold) are provided by section 5 of [Log].

4.2 The intermediate Jacobian integrable systems for prime Fano's

Consider a projective symplectic variety Y of dimension $2n$, a smooth variety L of dimension n and a closed Lagrangian immersion $f : L \rightarrow Y$. Then:

1. If $\mathcal{L} \rightarrow \mathcal{H}$ is a local semi-universal deformation of f , with fixed target Y , then $\mathcal{H} = Def(f/Y)$ is smooth ([Ran], Corollary 3.4).
2. The tangent space to \mathcal{H} at L is

$$T_f \mathcal{H} = H^0(L, N_f) \simeq H^0(L, \Omega_L^1),$$

where the bundle N_f is defined by the exact sequence $0 \rightarrow TL \rightarrow f^*TY \rightarrow N_f \rightarrow 0$; indeed the fact that f is Lagrangian yields an isomorphism $N_f \simeq \Omega_L^1$.

3. Any small deformation of f remains a Lagrangian immersion in Y , see [Voi].
4. The relative Picard bundle $Pic^0 \mathcal{L} \rightarrow \mathcal{H}$ is a Lagrangian fibration (or an algebraic completely integrable Hamiltonian system), see [DM2]. In fact Donagi and Markman only treat the case where j is an embedding, that is, they consider the deformations of a smooth Lagrangian subvariety in a symplectic manifold. Nevertheless, their proof extends verbatim to the setting of Lagrangian immersions.

Let X be a general prime Fano threefold of genus g and S a K3 surface in X . We can apply the results above to $f = j : L = F(X) \rightarrow Y = Hilb_2 S$, since we have proved in Proposition 4 that j is indeed a Lagrangian immersion. We get that the family $Pic^0 \mathcal{L}$ over $Def(j/Hilb_2 S)$ is a Lagrangian fibration.

Now we want to deform X with S fixed. The tangent space to the corresponding deformation functor is $H^1(X, TX \otimes \mathcal{I}_S)$ (see [Be4]). Since $\mathcal{I}_S = \mathcal{O}_X(-S) = K_X$, this is also $H^1(\Omega_X^2)$, the universal covering of $J(X)$.

We begin with a simple observation.

Proposition 10 *There exists a semi-universal algebraic family \mathcal{H}_S parameterizing deformations of X containing S .*

Proof. Suppose for example that $7 \leq g \leq 10$ (in which case j is an embedding). Recall that by Mukai's theorem, S can be described as a linear section $S = \Sigma \cap L_S^\perp$ of a d -dimensional

homogeneous space Σ by a projective space of codimension $d - 2$. The set of prime Fano threefolds X containing S as an anticanonical section, is then identified with an open subset of the orthogonal projective space $L_S \simeq \mathbb{P}^{d-3}$.

In the complete intersection case, consider for example the general quartic threefold $X \subset \mathbb{P}^4$ admitting S as a hyperplane section. The set of such X 's is given by the set of quartic polynomials of the form $k = k_S + x_0c$, where x_0 is an equation of the hyperplane H spanned by S in \mathbb{P}^4 , k_S is an equation of S in H , and c is any cubic form. So the smooth quartic hypersurfaces of \mathbb{P}^4 containing S are parameterized by an open subset of a 35-dimensional affine space. Moreover the 5-dimensional subgroup Γ_H of GL_5 , acting trivially on H , acts on this affine space, and any linear subspace transverse to the orbit of X defines a semi-universal, 30-dimensional, deformation of X with S fixed. The other cases are similar. \square

We have a natural map from the germ (\mathcal{H}_S, X) to $Def(j/Hilb_2S)$, whose differential

$$H^1(X, TX(-S)) = H^1(X, \Omega_X^2) \longrightarrow H^0(F(X), \Omega_{F(X)}^1)$$

is an isomorphism when the Abel-Jacobi isomorphism does hold. In particular we can pull-back our Lagrangian fibration to \mathcal{H}_S , first locally, and then everything globalizes since the fibration is defined canonically – more precisely, we can pull it back to the open subset \mathcal{H}_S^0 on which X is smooth, $F(X)$ is also smooth and the Abel-Jacobi isomorphism theorem does hold. But then over \mathcal{H}_S^0 , $Pic^0 F(X) \simeq Alb F(X)^* \simeq J(X)^* \simeq J(X)$, the intermediate Jacobian being self-dual. Since this isomorphism certainly holds in families, we finally deduce our main result:

Theorem 11 *Let S be a general K3 surface of genus g , with $2 \leq g \leq 10$. Let $\mathcal{X} \rightarrow \mathcal{H}_S$ be an algebraic semi-universal family of smooth Fano threefolds containing S .*

Then the family of intermediate Jacobians $J(\mathcal{X}) \rightarrow \mathcal{H}_S$ is a Lagrangian fibration over a dense open subset \mathcal{H}_S^0 of \mathcal{H}_S .

For $g \geq 7$ the situation is even cleaner: one can take for \mathcal{H}_S an open subset of the projective space L_S , and by Theorem 6 there is a birational isomorphism between L_S and the component \mathcal{H} of the Hilbert scheme of $Hilb_2S$ parameterizing the Lagrangian surfaces $jF(X)$, for X a general Fano threefold containing S .

References

- [Be1] A. Beauville, *Variétés Kählériennes dont la première classe de Chern est nulle*, J. Differential Geom. **18**, 755-782 (1983).
- [Be2] A. Beauville, *Systèmes hamiltoniens complètement intégrables associés aux surfaces K3*, in Problems in the theory of surfaces and their classification (Cortona, 1988), 25-31, Sympos. Math. XXXII, Academic Press, London 1991.
- [Be3] A. Beauville, *Vector bundles on the cubic threefold*, in Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), 71–86, Contemp. Math. **312**, A.M.S. 2002.
- [Be4] A. Beauville, *Fano threefolds and K3 surfaces*, The Fano Conference, 175-184, Univ. Torino, Turin 2004.
- [Bot] F. Bottacin, *Poisson structures on Hilbert schemes of points on a surface and integrable systems*, Manuscripta Math. **97**, 517-527 (1998).

- [Bou] N. Bourbaki, *Groupes et algèbres de Lie*, Hermann 1968.
- [CV] G. Ceresa, A. Verra, *The Abel-Jacobi isomorphism for the sextic double solid*, Pacific J. Math. **124** No.1, 85-105 (1986).
- [CG] H. Clemens, Ph.A. Griffiths, *The intermediate jacobian of the cubic threefold*, Annals of Math. **95**, 281-356 (1972).
- [Col] A. Collino, *The Abel-Jacobi isomorphism for the cubic fivefold*, Pacific J. Math. **122** No.1, 43-55 (1986).
- [CMW] A. Collino, J.P. Murre, G.E. Welters, *On the family of conics lying on a quartic threefold*, Rend. Sem. Mat. Univ. Politec. Torino **38** No.1, 151-181 (1980).
- [DM1] R. Donagi, E. Markman, *Cubics, integrable systems, and Calabi-Yau threefolds*, Proceedings of the Hirzebruch 65 conference on algebraic geometry, Bar-Ilan University, Ramat Gan, Israel, May 2-7, 1993; Israel Math. Conf. Proc. **9**, 199-221 (1996)
- [DM2] R. Donagi, E. Markman, *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, in Francaviglia, M. (ed.) et al., *Integrable systems and quantum groups*, Lecture Notes in Math. **1620**, 1-119 (1996).
- [HT] B. Hassett, Yu. Tschinkel, *Abelian fibrations and rational points on symmetric products*, International J. Math. **11** No.9, 1163-1176 (2000).
- [Ili] A. Iliev, *The $\mathbb{S}p_3$ -Grassmannian and duality for prime Fano threefolds of genus 9*, Manuscripta Math. **112** No.1, 29-53 (2003).
- [IM1] A. Iliev, L. Manivel, *Pfaffian lines and vector bundles on Fano threefolds of genus 8*, e-print: math.AG/0504595, to appear in: J. Alg. Geom.
- [IM2] A. Iliev, L. Manivel, *Cubic hypersurfaces and integrable systems*, e-print: math.AG/0605260.
- [IMr] A. Iliev, D. Markushevich, *Elliptic curves and rank-2 vector bundles on the prime Fano threefold of genus 7*, Advances in Geom. **4** No.3, 287-318 (2004).
- [IR] A. Iliev, K. Ranestad, *Geometry of the Lagrangian Grassmannian $LG(3, 6)$ with applications to Brill-Noether loci*, Michigan Math. J. **53** No.2, 383-417 (2005).
- [IP] V. Iskovskikh, Yu.G. Prokhorov, *Fano varieties*, in Algebraic Geometry V (A. N. Parsin, I. R. Shafarevich, Eds.), Encyclopaedia of Math. Sci. **47**, Springer 1998.
- [KLS] D. Kaledin, M. Lehn, Ch. Sorger, *Singular symplectic moduli spaces*, e-print: math.AG/0504202.
- [Kuz] A. Kuznetsov, *Derived category of V_{12} Fano threefolds*, e-print: math.AG/0310008.
- [LM] J.M. Landsberg, L. Manivel, *On the projective geometry of rational homogeneous varieties*, Comment. Math. Helv. **78**, No. 1, 65-100 (2003).
- [Let] M. Letizia, *The Abel-Jacobi mapping for the quartic threefold*, Inventiones Math. **75** No.3, 477-492 (1984).
- [Log] D. Logachev, *Fano threefolds of genus 6*, e-print: math.AG/0407147.
- [Mar] D. Markushevich, *An integrable system of $K3$ -Fano flags*, arXiv:math/0703166.
- [Msh] D. Matsushita, *On fibre space structures of a projective irreducible symplectic manifold*, Topology **38** No.1, 79-83 (1999); *Addendum: Topology* **40** No.2, 431-432 (2001).

- [Msk] T. Matsusaka, *On a characterization of a Jacobian variety*, Mem. Coll. Sci., Univ. Kyoto, Ser.A **32**, 1-19 (1959).
- [Muk1] S. Mukai, *Curves, K3 surfaces and Fano 3-folds of genus ≤ 10* , in Algebraic geometry and commutative algebra, Vol. I, 357-377, Kinokuniya, Tokyo 1988.
- [Muk2] S. Mukai, *Non-Abelian Brill-Noether theory and Fano 3-folds*, Sugaku Expositions **14** No.2, 125-153 (2001).
- [Ott] G. Ottaviani, *Spinor bundles on quadrics*, Trans. Amer. Math. Soc. **307** No.1, 301-316 (1988).
- [PB] L. Picco Botta, *On the intersection of three quadrics*, J. Reine Angew. Math. **399**, 188-207 (1989).
- [Ran] Z. Ran, *Hodge theory and deformations of maps*, Compositio Math. **97** No.3, 309-328 (1995).
- [Saw] J. Sawon, *Lagrangian fibrations on Hilbert schemes of points on K3 surfaces*, e-print: math.AG/0509224.
- [Sch] F.-O. Schreyer, *Geometry and algebra of prime Fano 3-folds of genus 12*, Compositio Math. **127** No.3, 297-319 (2001).
- [Tak] K. Takeuchi, *Some birational maps of Fano 3-folds*, Compositio Math. **71**, 265-283 (1989).
- [Tyu] A.N. Tyurin, *Non-abelian analogues of Abel's theorem*, Izv. Math. **65** No.1, 123-180 (2001).
- [Voi] C. Voisin, *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, London Math. Soc. Lect. Note Ser. **179**, 294-303 (1992).
- [Wel] G.E. Welters, *Abel-Jacobi isogenies for certain types of Fano threefolds*, Mathematical Centre Tracts **141**, Mathematisch Centrum, Amsterdam 1981.

Atanas Iliev

Institute of Mathematics
 Bulgarian Academy of Sciences
 Acad. G. Bonchev Str., bl.8
 1113 Sofia, Bulgaria
e-mail: ailiev@math.bas.bg

Laurent Manivel

Institut Fourier
 UMR 5582 (UJF-CNRS), BP 74
 38402 St Martin d'Hères Cedex, France
e-mail: Laurent.Manivel@ujf-grenoble.fr