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# PFAFFIAN LINES AND VECTOR BUNDLES ON FANO THREEFOLDS OF GENUS 8

A. ILIEV AND L. MANIVEL

ABSTRACT. Let  $X$  be a general complex Fano threefold of genus 8. We prove that the moduli space of rank two semistable sheaves on  $X$  with Chern numbers  $c_1 = 1$ ,  $c_2 = 6$  and  $c_3 = 0$  is isomorphic to the Fano surface  $F(X)$  of conics on  $X$ . This surface is smooth and isomorphic to the Fano surface of lines in the orthogonal to  $X$  cubic threefold. Inside  $F(X)$ , the non-locally free sheaves are parameterized by a smooth curve of genus 26 isomorphic to the base of the family of lines on  $X$ .

## 1. INTRODUCTION

A vector bundle  $E$  on a smooth complex projective  $n$ -fold  $X$  is *without intermediate cohomology* if  $h^i(X, E(k)) = 0$  for any  $i \neq 0, n$  and  $k \in \mathbb{Z}$ . By a well known criterion of Horrocks, such a bundle must split if  $X = \mathbb{P}^n$ . If  $X = \mathbb{Q}^n$  is a smooth quadric and  $E$  is indecomposable, it must be a twist of a spinor bundle, see [Kn, Ot]. In addition, as seen by Buchweitz, Greuel and Schreyer, these are the only smooth  $n$ -folds with a finite number, up to twist, of indecomposable vector bundles without intermediate cohomology.

Until now more or less complete descriptions of these bundles have been obtained for only some restricted classes of varieties, see [IM4] for a more comprehensive account of the known results. In particular, arithmetically Cohen-Macaulay (aCM) vector bundles (that is, indecomposable rank two bundles with no intermediate cohomology) on prime Fano threefolds have attracted considerable attention; we refer to [IP] for general facts about Fano varieties.

Prime Fano threefolds  $Y_d$  of index two and degree  $d$  exist for  $1 \leq d \leq 5$ . The classification of aCM bundles on  $Y_d$  is known for  $d \geq 3$  [AC]. Although the classification of aCM bundles on  $Y_1$  (the double Veronese cone) and  $Y_2$  (the quartic double solid) can easily be derived by similar methods, it still remains unwritten. The degree  $d = -K_X^3 = 2g - 2$  of a prime Fano threefold  $X = X_d$  of index one is always even – the integer  $g$  is called the genus of  $X$ . Such Fano's exist for  $2 \leq g \leq 12$ ,  $g \neq 11$ , see e.g. [Mu4]. The classification of all possible aCM bundles on these threefolds (but not the existence of all of them) is given by C. Madonna in [Ma] (see also [AC]). There are finitely many possible Chern numbers for a normalized aCM bundle on a prime Fano threefold  $X$ , and then a finite number of families for each possible choice, whose general member is a stable vector bundle obtained by Serre's construction from a subcanonical curve on  $X$ . Madonna deduces a list of 91 possible pairs  $(c_1, c_2)$ , with  $-1 \leq c_1 \leq 3$ , corresponding to lines, conics and certain elliptic, canonical or half-canonical curves (provided such curves on  $X_d$  exist).

For a pair  $(c_1, c_2)$  from the lists of Arrondo-Costa and Madonna, denote by  $M_X(2; c_1, c_2)$  the Maruyama moduli space of semistable rank two coherent sheaves on the Fano threefold  $X$  with these Chern classes, and  $c_3 = 0$ . In some cases this moduli space happens to consist of a single point, corresponding to a unique bundle  $E$ . Such is the moduli space  $M_X(2; 1, 5)$

for the Fano threefold  $X = X_{14}$ , and the bundle  $E$  defines a unique embedding of  $X_{14}$  in the Grassmannian  $G(2, 6)$  parameterizing planes in a six dimensional complex vector space. The study of rigid bundles on the Fano threefolds, K3 surfaces and canonical curves of genus 6, 7, 8, 9, 10 is the base of Mukai's classifications by embeddings into homogeneous varieties.

The next step is the study of moduli spaces  $M_X(2; c_1, c_2)$  of non-rigid aCM bundles on prime Fano threefolds. The first attempts in this direction have been made in [MT1] and [IM1], with the study of the moduli space  $M_Y(2; 0, 2)$  on a cubic threefold  $Y = Y_3$  based on a parallel study of the family of subcanonical curves – elliptic quintics – that appear as zero-loci of sections of the general  $E \in M_Y(2; 0, 2)$ . The Abel-Jacobi map sends the family of elliptic quintics on  $Y$  onto the 5-dimensional intermediate Jacobian  $J(Y)$ , and one can deduce that  $M_Y(2; 0, 2)$  is birational to  $J(Y)$ . This statement was made more precise by S. Druel [D], who proved that  $M_Y(2; 0, 2)$  is isomorphic to the blow-up of  $J(Y)$  along a copy of the Fano surface of lines in  $Y$ ; see [B1] for a joint presentation on both approaches.

From some other point of view the same moduli space  $M_Y(2; 0, 2)$  compactifies the set of Pfaffian representations of the 3-fold  $Y$ , an idea due originally to Adler, see [AR]. As shown in [IM2], a similar question can be stated and answered similarly also for the quartic threefold  $X = X_4 \subset \mathbb{P}^4$ , which by itself is a prime Fano 3-fold of index 1: the set of Pfaffian representations of the 3-fold quartic  $X$  is compactified by the 7-dimensional moduli space of aCM bundles  $M_X(2; 3, 14)$  on  $X$ . We refer to the paper [B2] of Beauville for a modern and more general view on determinantal and Pfaffian representations of homogeneous forms.

The ideas from [MT1, IM2] have been used in [IM3] and [IM4] in the study of the moduli spaces  $M_X(2; 1, 5)$  and  $M_X(2; 1, 6)$  coming correspondingly from elliptic quintics and elliptic sextics on the prime Fano 3-fold  $X = X_{12}$  of index 1 and genus seven. To this circle of works one can include the new parameterization given by Tikhomirov (see [Tih]) of the theta divisor  $\Theta$  for the quartic double solid  $Y = Y_2$  by elliptic quintics, that in particular yields a birationality between  $\Theta$  and a component of the moduli space  $M_Y(2; 0, 3)$ , see also [MT2]. More recent, and from a different point of view, is the study by Arrondo and Faenzi of the aCM bundles on the prime Fano 3-folds  $Y_5$  and  $X_{12}$  in terms of monads, see [AF] and [F].

Note the common weak point in all these descriptions: they only consider the open subset of stable vector bundles in the moduli space.

In this paper we give a full description of the moduli space  $M_X(2; 1, 6)$  on the *general* Fano threefold  $X = X_{14}$  of index one and genus eight. We combine the geometric approach used in [IM1]-[IM4] for other Fano threefolds with the ideas of S. Druel [D] to get not only all the vector bundles but also all the non locally free sheaves parameterized by this moduli space. Our main result (Theorem 7.2) is that  $M_X(2; 1, 6)$  is isomorphic to the smooth surface  $F(X)$  parameterizing the conics contained in  $X$ . Inside  $F(X)$ , the non locally free sheaves are parameterized by a smooth curve of genus 26 isomorphic to the family  $\Gamma(X)$  of lines in  $X$ .

Our proof is rather indirect, and we use the orthogonal cubic threefold  $Y$  of  $X$  as an essential tool. Remember that  $X$  is a generic linear section of the Grassmannian  $G(2, 6) \subset \mathbb{P}^{14}$ . The orthogonal cubic threefold  $Y$  is then obtained as the section of the cubic Pfaffian hypersurface by the orthogonal linear subspace of the dual projective space. We prove

that the surface  $F(X)$  of conics in  $X$  is isomorphic to the Fano surface  $F(Y)$  of lines in  $Y$ . Given a vector bundle in  $M_X(2; 1, 6)$ , we prove that it is generated by global sections, and next we show how to construct a line  $\ell \subset Y$  from an elliptic sextic obtained as the zero locus of a general section. Conversely, a given line  $\ell \subset Y$  defines uniquely a codimension two singular linear section of  $G(2, 6)$ , a special rational projection of which turns out to be isomorphic, somewhat unexpectedly, to the Grassmannian  $G(2, 5)$ . Pulling-back the tautological rank two bundle on  $G(2, 5)$ , we get a vector bundle  $E_\ell \in M_X(2; 1, 6)$ . Moreover, these two processes are inverse to each other.

The Abel-Jacobi map allows to conclude the proof:  $M_X(2; 1, 6)$  is mapped bijectively onto  $F(X)$ , which by itself is embedded in the intermediate Jacobian  $J(X)$  of  $X$ . Since both  $M_X(2; 1, 6)$  and  $F(X)$  are smooth surfaces, they are isomorphic.

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## 2. PRELIMINARIES I : SOME GRASSMANNIAN GEOMETRY

**2.1. The Grassmannian  $G(2, 6)$ .** Lots of important geometric properties of the Fano threefold  $X_{14}$  come from the geometry of the Grassmannian  $G(2, 6)$ . Here we state some of them that will be used later.

We denote by  $V$  a six-dimensional complex vector space, and by  $G(2, 6) = G(2, V) \hookrightarrow \mathbb{P}(\wedge^2 V) = \mathbb{P}^{14}$  the Grassmannian of planes in  $V$ , in its Plücker embedding. This is a smooth Fano manifold of dimension 8, degree 14 and index 6.

The secant variety of  $G(2, 6)$  in  $\mathbb{P}^{14}$  is the cubic hypersurface of skew-symmetric tensors of rank at most four. Its equation is given by the Pfaffian. The action of  $PGL_6$  in  $\mathbb{P}^{14}$  has only three orbits, defined by the rank:  $G(2, 6)$ , its complement in the Pfaffian hypersurface, and the complement of this hypersurface.

**2.2. Lines and conics in  $G(2, 6)$ .** A line in  $G(2, 6)$  is of the form  $\mathbb{P}(\ell \wedge L)$  for some line  $\ell$  in  $V$  and some three dimensional subspace  $L$  of  $V$  containing  $\ell$ . The Fano variety of lines in  $G(2, 6)$  can thus be identified with the flag variety  $F(1, 3; 6)$ , with obvious notations.

Similarly, the family of planes in  $G(2, 6)$  has two connected components, both homogeneous under the action of  $PGL_6$ . Of course these planes contain lots of conics of  $G(2, 6)$ . Apart from these, smooth conics are defined by two points of  $G(2, 6)$  plus two concurring tangents. We deduce that we can find independent vectors  $e_0, e_1, e_2, e_3 \in V$  such that the span of our conic is generated by  $e_0 \wedge e_1, e_2 \wedge e_3, e_0 \wedge e_2 + e_1 \wedge e_3$ . Similarly, a singular but reduced conic whose linear span is not contained in the Grassmannian will be a union of two lines  $\langle e_0 \wedge e_1, e_0 \wedge e_2 \rangle$  and  $\langle e_0 \wedge e_1, e_1 \wedge e_3 \rangle$ . The case of double lines is settled by the following lemma:

**Lemma 2.1.** *Let  $L$  be a projective plane, meeting  $G(2, 6)$  along a double line. Then we can find four independent vectors  $e_0, e_1, e_2, e_3$  in  $V$  such that  $L$  is generated by  $e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3 + e_1 \wedge e_2$ .*

*Proof.* Our hypothesis is that we have a line  $\ell = \langle e_0 \wedge e_1, e_0 \wedge e_2 \rangle$ , contained in the plane  $L \subset \mathbb{P}^{14}$ , such that the scheme intersection of  $L$  with  $G(2, 6)$  is a double structure on  $\ell$ .

Complete  $e_0, e_1, e_2$  with three other vectors  $e_3, e_4, e_5$  to get a basis of  $V$ . Then  $L$  contains a unique tensor of the form  $\omega = e_0 \wedge u + \phi$  where  $u \in \langle e_3, e_4, e_5 \rangle$  and  $\phi$  does not involve  $e_0$ . For  $\Omega = x_1 e_0 \wedge e_1 + x_2 e_0 \wedge e_2 + y\omega$ , the vanishing of

$$\Omega \wedge \Omega = y(e_0 \wedge (x_1 e_1 + x_2 e_2 + yu) \wedge \phi) + y^2 \phi \wedge \phi$$

defines our scheme-theoretic intersection. In order to get a double line, this equation must reduce to  $y^2 = 0$ , and we thus need that  $e_1 \wedge \phi = e_2 \wedge \phi = 0$ . Hence  $\phi$  must be a multiple of  $e_1 \wedge e_2$ . Finally, for  $L$  not to be contained in  $G(2, 6)$  we need  $u$  not to be contained in  $\langle e_0, e_1, e_2 \rangle$ .  $\square$

**2.3. Lines in the Pfaffian hypersurface.** We will be interested in pencils of skew-symmetric tensors of constant rank four, i.e. lines in the Pfaffian hypersurface that do not meet the Grassmannian  $G(2, 6)$ .

We define an *A-line* to be a line generated by tensors of the form

$$\begin{aligned} e_0 \wedge e_2 + e_1 \wedge e_3, \\ e_0 \wedge e_4 + e_1 \wedge e_5, \end{aligned}$$

for some basis  $e_0, \dots, e_5$  of  $V$ .

Similarly, a *B-line* will be generated by tensors of the form

$$\begin{aligned} e_0 \wedge e_2 + e_1 \wedge e_3, \\ e_0 \wedge e_3 + e_1 \wedge e_4, \end{aligned}$$

for some independent vectors  $e_0, \dots, e_4$  of  $V$ .

Both types of lines are contained in the tangent space to the Grassmannian  $G(2, 6)$  at the unique plane  $e_0 \wedge e_1$ . The main difference between the two types is that a B-line is contained, contrary to an A-line, in the projective span of a copy of  $G(2, 5)$  inside  $G(2, 6)$ . As proved in [MMe], every line of skew-symmetric tensors of constant rank four is an A-line or a B-line; moreover, the B-lines describe a hypersurface in the closure of the 22-dimensional family of A-lines.

For future use we note the following easy result, which can be obtained by an explicit computation. If  $\ell$  is an A-line or a B-line, a point on  $\ell$  represents a skew-symmetric form of rank four on  $V^*$ , whose kernel defines a projective line. Denote by  $Q^\ell \subset \mathbb{P}V^*$  the union of these lines.

**Lemma 2.2.** *If  $\ell$  is an A-line, then  $Q^\ell$  is a smooth quadric surface in  $\mathbb{P}V^*$ . If  $\ell$  is a B-line, then  $Q^\ell$  is a quadratic cone.*

### 3. PRELIMINARIES II : PRIME FANO THREEFOLDS OF GENUS 8

**3.1. Prime Fano threefolds from  $G(2, 6)$ .** Since the Grassmannian  $G(2, 6)$  has dimension 8 and index 6, any smooth transverse linear section

$$X = X_{14} = G(2, 6) \cap \mathbb{P}_X^9 \subset \mathbb{P}^{14}$$

is a Fano threefold of index one and degree  $d = 14$ , hence of genus  $g = d/2 + 1 = 8$ , i.e. the smooth codimension 2 linear sections of  $X$  are canonical curves of genus 8. As shown independently by Gushel' and Mukai, see [G], [Mu4]:

*Any smooth prime Fano threefold  $X = X_{14}$  of index one and genus 8 is a transverse section of  $G(2, 6) \subset \mathbb{P}^{14}$  by a linear space  $\mathbb{P}_X^9 \subset \mathbb{P}^{14}$ .*

In Mukai's notation, the embedding  $X \hookrightarrow G(2, 6)$  is given by the unique rank 2 stable vector bundle  $E_o \in M_X(2; 1, 5)$ . This bundle  $E_o$  is defined by the Serre construction from any elliptic quintic on  $X$ , see [Mu4].

The even Betti numbers of  $X$  are  $b_2 = b_4 = 1$ , canonical generators being given by the class of a hyperplane section and the class of a line in  $X$ , respectively. The odd Betti

numbers are  $b_1 = 0$  and  $b_3 = 10$ . In particular the intermediate Jacobian of  $X$  is a five dimensional abelian variety  $J(X)$ .

This can be seen as follows. Consider the normal exact sequence

$$0 \rightarrow TX \rightarrow TG_X \rightarrow \mathcal{O}_X(1)^{\oplus 5} \rightarrow 0,$$

where  $TG$  is the tangent bundle of the Grassmannian and  $TG_X$  is its restriction to  $X$ . Note that  $TX(-1) = \Omega_X^2$  since  $K_X = \mathcal{O}_X(-1)$ . Twisting the previous sequence by  $\mathcal{O}_X(-1)$  and taking cohomology, we deduce an exact sequence

$$H^0(TG_X(-1)) \rightarrow H^0(\mathcal{O}_X^{\oplus 5}) \rightarrow H^1(\Omega_X^2) \rightarrow H^1(TG_X(-1)).$$

Since  $X$  is a linear section of the Grassmannian there is an associated Koszul complex that we can use to deduce that  $H^i(TG_X(-1)) = 0$  whenever  $H^{i+j}(TG(-j-1)) = 0$  for  $0 \leq j \leq 5$ . For  $i = 0$  and  $i = 1$  this easily follows from Bott's theorem (see e.g. [Man], Proposition 2). We conclude that  $h^{1,2}(X) = 5$ , as claimed.

**3.2. The orthogonal cubic threefold.** For the Fano threefold  $X = G(2, 6) \cap \mathbb{P}_X^9$ , consider the orthogonal four dimensional space in the dual projective space  $\mathbb{P}(\wedge^2 V^*)$ . This four dimensional projective space meets the Pfaffian hypersurface in  $\mathbb{P}(\wedge^2 V^*)$  along a cubic threefold  $Y \subset \mathbb{P}_Y^4$ , the *orthogonal cubic threefold* of the Fano threefold  $X$ .

For  $X$  general,  $\mathbb{P}_Y^4$  is a general subspace in  $\mathbb{P}^{14}$ , hence does not meet the Grassmannian  $G(2, V^*)$ , whose codimension is 6. Therefore any point  $y \in Y$  represents a skew-symmetric form of rank four on  $V$ , whose kernel is a well-defined projective line  $n_y \subset \mathbb{P}^5$ .

Since the representation of  $X$  as a linear section of  $G(2, 6)$  is unique up to the action of  $PGL_6$  on the codimension five spaces in  $\mathbb{P}^{14}$ , the dual cubic threefold  $Y$  is uniquely defined up to the action of  $PGL_6$ . Moreover,  $Y$  is smooth whenever  $X$  is smooth, see [Pu].

**3.3. The Palatini quartic.** For  $x \in X$ , denote by  $\ell_x \subset \mathbb{P}^5$  the corresponding line. Let

$$\begin{aligned} W &= \cup\{\ell_x : x \in X\} \subset \mathbb{P}^5, \\ V &= \cup\{n_y : y \in Y\} \subset \mathbb{P}^5. \end{aligned}$$

Both  $W$  and  $V$  are subvarieties of  $\mathbb{P}^5$  swept out by lines, and in fact they coincide. More precisely, the following takes place (see §50 in [AR] or [Pu]):

(i)  $W = V$  is an irreducible quartic hypersurface in  $\mathbb{P}^5$ , whose singular locus is a curve  $\Gamma(W)$  of degree 25;

(ii) through any point  $p \in W - \Gamma(W)$  passes a unique line  $\ell_x, x \in X$  and a unique kernel line  $n_y, y \in Y$ ; and any point  $v \in \Gamma(W)$  is the vertex of exactly one line  $\ell \subset X$ .

The quartic hypersurface  $W \subset \mathbb{P}^5$  is known as the *Palatini quartic* of the prime Fano threefold  $X$  of genus 8. The assertion (ii) above implies that a general hyperplane section of  $W$  is a singular quartic threefold with 25 singular points which is birational to both  $X$  and  $Y$ . In particular  $X$  and  $Y$  are birational (and both unirational but not rational).

This has an interesting consequence, observed by Puts, that will be of crucial use later.

**Proposition 3.1.** *The intermediate Jacobians of  $X$  and  $Y$  are isomorphic.*

*Proof.* We briefly recall the proof given in [Pu].

Let  $H$  be a general hyperplane in  $\mathbb{P}^5$ .

First, define the rational map  $u_X : X \dashrightarrow W \cap H$  by mapping  $x \in X$  to the intersection point of the line  $\ell_x$  with  $H$ . The map  $u_X$  is well defined at  $x \in X \subset G(2, 6)$  if the line

$\ell_x$  of  $x$  is not contained in  $H$ , that is outside the intersection  $C = X \cap G(2, H)$ . Since  $H$  is general this is a smooth elliptic quintic. By (i) - (ii), the hyperplane  $H$  intersects the curve  $\Gamma(W) = \text{Sing}(W)$  at 25 points which are the vertices of 25 lines  $\ell_1, \dots, \ell_{25}$  on  $X$ ; these  $\ell_i$  are the same as the 25 lines on  $X$  that intersect the elliptic quintic  $C$ .

Similarly, the rational map  $u_Y : Y \dashrightarrow W \cap H$  sending  $y \in Y$  to the intersection point of the kernel line  $n_y$  with  $H$  is well defined outside the subset  $B$  of these  $y \in Y$  for which the kernel line  $n_y$  lies in  $H$ . This  $B$  is a smooth elliptic quintic on  $Y$ , which has exactly 25 bisecant lines  $m_1, \dots, m_{25}$  that lie on  $Y$ .

By (ii), both  $u_X$  and  $u_Y$  are birational. Moreover, there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{u_X} & W \cap H & \xleftarrow{u_Y} & Y \\ \uparrow & & & & \uparrow \\ X' & \leftarrow & Z & \rightarrow & Y' \end{array}$$

where the vertical maps  $X' \rightarrow X$  and  $Y' \rightarrow Y$  are the blow-ups of the elliptic quintics  $C \subset X$  and  $B \subset Y$ . The map  $Z \rightarrow X'$  blows up the proper preimages  $\ell'_i \subset X'$  of the 25 lines  $\ell_i$  to 25 exceptional divisors  $E_i \subset Z$  all isomorphic to the smooth quadric surface  $Q$ , with fibers of  $E_i \rightarrow \ell'_i$  corresponding to one of the two families of lines  $\Phi$  and  $\Psi$  (say  $\Phi$ ) on  $Q$ . The map  $Z \rightarrow Y'$  blows down the 25 divisors  $E_i \cong Q$  to the proper preimages  $m'_i \subset Y'$  of the 25 lines  $m_i$ , with fibers of  $E_i \rightarrow m'_i$  corresponding to the lines on  $Q$  from the family  $\Psi$ , see [Pu].

Since  $X' \rightarrow X$  and  $Y' \rightarrow Y$  are blowups of the nonrational curves  $C$  and  $B$ , and  $Z \rightarrow X'$  and  $Z \rightarrow Y'$  are compositions of blowups of rational curves, then

$$J(X) + J(C) \simeq J(X') \simeq J(Z) \simeq J(Y') \simeq J(Y) + J(B)$$

as principally polarized abelian varieties, see [CG]. Since  $J(Y)$  is not isomorphic to the jacobian of a curve, neither to the sum of jacobians of curves (ibid.), then  $J(X) \simeq J(Y)$  and  $J(C) \simeq J(B)$ . In particular, the elliptic curves  $C$  and  $B$  are isomorphic (see also §1, Ch.III in [Is]).  $\square$

#### 4. LINES ON $X_{14}$ AND ITS ORTHOGONAL CUBIC

**4.1. Lines on the orthogonal cubic threefold.** We recall some known facts about the family of lines on the 3-dimensional cubic hypersurface. The basic reference about cubic threefolds and intermediate Jacobians of threefolds is the paper [CG] of Clemens and Griffiths.

*Let  $Y \subset \mathbb{P}^4$  be a general cubic 3-fold. Then*

(i) *The family  $F(Y)$  of lines on  $Y$  is a smooth irreducible surface of general type, of geometric genus  $p_g = 10$ , of irregularity  $q = 5$ , and such that  $K^2 = 45$ . The canonical system  $|K|$  is very ample and defines the Plücker embedding  $F(Y) \subset G(2, 5) \rightarrow \mathbb{P}^9$ .*

(ii) *The intermediate Jacobian  $J(Y) = H^{2,1}(Y)^*/H_3(Y, \mathbb{Z})$  is an abelian variety of dimension 5, and the Abel-Jacobi map  $\Phi : F(Y) \rightarrow J(Y)$  is an embedding. In particular,  $F(Y)$  contains no rational curve.*

When  $Y$  is the orthogonal cubic threefold to a Fano threefold  $X$  of genus 8, a line in  $Y$  is a pencil of skew-symmetric forms on  $V^*$ , all of rank four. Since the B-lines form a codimension one family of lines of skew-symmetric forms of constant rank four, we get:

(iii) *The family of B-lines in the orthogonal cubic threefold  $Y$  to a general Fano threefold  $X$  of genus 8 is a smooth curve  $\Gamma(Y) \subset F(Y)$ .*

For the smoothness of  $\Gamma(Y)$ , see [AR], Corollary (49.8) and Lemmas (51.3) and (51.15). Notice that the curve  $\Gamma(Y)$  is not invariantly defined by the cubic threefold  $Y$ ; as a subset of  $F(Y)$  it depends on the choice of  $X$  in the 5-dimensional family of Fano threefolds whose orthogonal cubic is  $Y$ .

Other characterizations of B-lines are given in [AR]. They are the lines  $\ell \subset Y$  such that the kernel lines  $n_y, y \in \ell$  have a common point (see Definition (49.1) in [AR]). Equivalently, they are the *jumping lines* of the restriction to  $Y$  of the tautological rank two bundle (see Lemma (49.2) in [AR]).

**4.2. Lines in the general Fano threefold  $X_{14}$ .** It is well known that the scheme  $\Gamma(X)$  of lines on the general Fano threefold  $X$  of genus 8 is a smooth irreducible curve of genus 26, see e.g. Prop. 4.2.2 and Th. 4.2.7 in [IP]; in particular the normal bundle to any line  $\ell \subset X$  is  $N_{\ell/X} = \mathcal{O}_\ell \oplus \mathcal{O}_\ell(-1)$ , *ibid.* This implies that  $h^0(N_{\ell/X}) = \text{ext}^1(\mathcal{O}_\ell, \mathcal{O}_\ell) = 1$  and  $h^1(N_{\ell/X}) = \text{ext}^2(\mathcal{O}_\ell, \mathcal{O}_\ell) = 0$ .

Let  $\ell$  be a line in  $X$ , and let  $H(\ell) \subset V^*$  be the hyperplane orthogonal to the vertex  $v(\ell)$ . Then  $\mathbb{P}(\wedge^2 H(\ell)) \simeq \mathbb{P}^9$  has codimension three in  $\ell^\perp$ , which also contains  $\mathbb{P}_Y^4$ . Therefore these two subspaces of  $\ell^\perp$  have to meet along some projective space of dimension at least one. Moreover this space is contained in  $Y$  since  $H(\ell)$  has dimension five, and in dimension five a skew-symmetric form has rank at most four. But  $Y$  does not contain any plane, so  $\mathbb{P}(\wedge^2 H(\ell)) \cap \mathbb{P}_Y^4$  is a projective line  $\ell'$  in  $Y$ , which is obviously a B-line.

Conversely, let  $\ell' \subset Y$  be a B-line, hence contained in the span  $\mathbb{P}(\wedge^2 H)$  of  $G(2, H)$ , a copy of  $G(2, 5)$ , for some hyperplane  $H$  of  $V^*$ . The orthogonal to  $H$  is point  $v \in \mathbb{P}^5$ , and the Schubert cycle of lines passing through  $v$  is a  $\mathbb{P}_v^4$  contained in  $G(2, 6)$ . Both  $\mathbb{P}_v^4$  and  $\mathbb{P}_X^9$  are contained in the 12-space  $(\ell')^\perp$ , so they must meet each other along a linear space  $\ell$  of dimension at least one. Since  $\ell$  is contained in  $X = G(2, 6) \cap \mathbb{P}_X^9$ , and since  $X$  contains no planes, then  $\ell$  must be a line. Together with the preceding, this yields the following:

**Proposition 4.1.** *The curve  $\Gamma(X)$  of lines in  $X$  is isomorphic to the curve  $\Gamma(Y) \subset F(Y)$  of B-lines in the orthogonal cubic  $Y$  of  $X$ .*

Notice that if  $\ell$  is a line in  $X$ , its vertex  $v(\ell)$  is a singular point of the Palatini quartic  $W$ , see Lemma (51.5) in App.V of [AR]. Moreover the map  $\Gamma(X) \rightarrow \Gamma(W)$  sending a line to its vertex is injective, since two lines with the same vertex would generate a plane contained in  $X$ . This implies:

**Proposition 4.2.** *The curve  $\Gamma(X)$  of lines in  $X$  is isomorphic to the singular locus  $\Gamma(W)$  of the Palatini quartic of  $X$ .*

The fact that for  $X$  general,  $\Gamma(W)$  is a smooth curve of genus 26 was already observed by Puts [Pu].

**4.3. Conics on  $X_{14}$ .** We denote by  $F(X)$  the Hilbert scheme of conics on  $X$ , that is, the Hilbert scheme of closed subschemes of  $X$  with Hilbert polynomial  $P(n) = 2n + 1$ . It is known that  $F(X)$  is reduced of pure dimension two: it is called traditionally the *Fano surface* of  $X$ . Any general point of  $F(X)$  parametrizes a smooth conic  $q \subset X$  with normal bundle  $N_{q/X} = \mathcal{O}_q \oplus \mathcal{O}_q$ . Moreover the closed subset of singular conics (pairs of intersecting lines, or double lines) is of pure dimension 1. See §4.2 in [IP].

Of course any conic on  $X$  is a conic on  $G(2, 6)$ , and the different types of conics in  $G(2, 6)$  can easily be described. We can distinguish three types:



- (1) Conics whose linear span is a plane contained in  $G(2, 6)$ ;
- (2) Reduced plane sections of a copy of  $G(2, 4)$  in  $G(2, 6)$ ;
- (3) Double lines.

Clearly conics of the first type cannot be contained in a general Fano threefold  $X$ , which contains no planes. Double lines are handled as follows:

**Lemma 4.3.** *The general Fano threefold  $X_{14}$  contains no double lines.*

*Proof.* Double lines on  $G(2, 6)$  were described in Lemma 2.1. An immediate consequence is that there is a  $\mathbb{P}^3$  of projective planes whose scheme intersection with  $G(2, 6)$  is a double line supported on a given line in  $G(2, 6)$ . Since the Fano variety of lines on the Grassmannian is 11-dimensional, we get a family of planes of dimension 14, and the family of  $\mathbb{P}^9$ 's containing a plane in this family has dimension  $14 + 7 \times 5 = 49$ . This is one less than the dimension of the Grassmann variety of  $\mathbb{P}^9$ 's in  $\mathbb{P}^{14}$ , so our claim follows.  $\square$

**Proposition 4.4.** *The Fano surface  $F(X)$  of conics on the general  $X$  is smooth.*

*Proof.* Let  $q \subset X$  be a reduced conic. To prove that the  $F(X)$  is smooth and two-dimensional at  $q$ , we will check that  $h^1(N_{q/X}) = 0$  and  $h^0(N_{q/X}) = 2$ , where  $N_{q/X}$  denotes the rank two normal bundle of  $q$  in  $X$  (note that  $q$  can be singular but is always a locally complete intersection).

*First case :  $q$  is smooth.* We know that  $q = G(q) \cap X$ , where  $G(q)$ , the Grassmannian of lines in  $\mathbb{P}_q^3$ , is a four dimensional quadric in  $G = G(2, 6)$ . We have an exact sequence

$$0 \rightarrow N_{q/G(q)} \rightarrow N_{q/G} \rightarrow N_{G(q)/G|_q} \rightarrow 0.$$

Since  $q$  is a transverse linear section of  $G(q)$ , we have  $N_{q/G(q)} = \mathcal{O}(2)^3$ . The restriction of the tautological bundle  $T^*$  to  $q$  is globally generated and has degree two, thus it must decompose into  $\mathcal{O} \oplus \mathcal{O}(2)$  or  $\mathcal{O}(1) \oplus \mathcal{O}(1)$ . In the first case, the lines parametrized by  $q$  would have a common point, and the linear span of  $q$  would be contained in  $G$ , hence in  $X$ , which is impossible. So we must be in the second case, and since  $G(q)$  is the zero locus of a section of  $T^* \oplus T^*$  on  $G$ , we deduce that  $N_{G(q)/G|_q} = \mathcal{O}(1)^4$ .

Now, consider the commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & Tq & \rightarrow & TG(q)|_q & \rightarrow & N_{q/G(q)} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & TX|_q & \rightarrow & TG|_q & \rightarrow & N_{X/G|_q} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & N_{q/X} & \rightarrow & N_{G(q)/G|_q} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Suppose that the image  $\bar{z} \in N_{q/X}$  of some  $z \in TX|_q$ , is mapped to zero in  $N_{G(q)/G|_q}$ . From the diagram, we deduce that  $z \in TG(q)|_q$ , hence  $z \in Tq = TX|_q \cap TG(q)|_q$ . But this means that  $\bar{z} = 0$ . We conclude that the map  $N_{q/X} \rightarrow N_{G(q)/G|_q} = \mathcal{O}(1)^4$  is injective. Since  $N_{q/X}$  has degree zero, this implies that  $N_{q/X} = \mathcal{O} \oplus \mathcal{O}$  or  $N_{q/X} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$ , hence our claim.

*Second case:  $q$  is singular.* So  $q$  is the union of two coplanar but distinct lines  $\ell$  and  $m$ , meeting at a point  $p$ . We prove that  $Ext^2(I_q, I_q) = 0$  and  $Ext^1(I_q, I_q)$  is two

dimensional. We begin with the short exact sequence  $0 \rightarrow I_q \rightarrow I_\ell \rightarrow \mathcal{O}_m(-p) \rightarrow 0$ . Applying  $\text{Hom}(I_q, \cdot)$ , we get the long exact sequence

$$(1) \quad \begin{aligned} 0 \rightarrow \text{Hom}(I_q, I_q) &\rightarrow \text{Hom}(I_q, I_\ell) \rightarrow \text{Hom}(I_q, \mathcal{O}_m(-p)) \rightarrow \\ &\rightarrow \text{Ext}^1(I_q, I_q) \rightarrow \text{Ext}^1(I_q, I_\ell) \rightarrow \text{Ext}^1(I_q, \mathcal{O}_m(-p)) \rightarrow \\ &\rightarrow \text{Ext}^2(I_q, I_q) \rightarrow \text{Ext}^2(I_q, I_\ell) \rightarrow \text{Ext}^2(I_q, \mathcal{O}_m(-p)). \end{aligned}$$

Applying  $\text{Hom}(\cdot, \mathcal{O}_m(-p))$ , we obtain

$$(2) \quad \begin{aligned} 0 \rightarrow \text{Hom}(\mathcal{O}_m, \mathcal{O}_m) &\rightarrow \text{Hom}(I_\ell, \mathcal{O}_m(-p)) \rightarrow \text{Hom}(I_q, \mathcal{O}_m(-p)) \rightarrow \\ &\rightarrow \text{Ext}^1(\mathcal{O}_m, \mathcal{O}_m) \rightarrow \text{Ext}^1(I_\ell, \mathcal{O}_m(-p)) \rightarrow \text{Ext}^1(I_q, \mathcal{O}_m(-p)) \rightarrow \\ &\rightarrow \text{Ext}^2(\mathcal{O}_m, \mathcal{O}_m) \rightarrow \text{Ext}^2(I_\ell, \mathcal{O}_m(-p)) \rightarrow \text{Ext}^2(I_q, \mathcal{O}_m(-p)). \end{aligned}$$

Note that  $\text{Hom}(\mathcal{O}_m, \mathcal{O}_m) = \text{Hom}(I_\ell, \mathcal{O}_m(-p)) = \mathbb{C}$ , while  $\text{Hom}(I_q, \mathcal{O}_m(-p)) = 0$ .

To compute  $\text{Ext}^k(\mathcal{O}_m(-p), \mathcal{O}_m(-p)) = \text{Ext}^k(\mathcal{O}_m, \mathcal{O}_m)$ , we use the short exact sequence  $0 \rightarrow I_m \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_m \rightarrow 0$ . Since  $\text{Ext}^k(\mathcal{O}_X, I_m) = H^k(I_m) = 0$  for  $k \geq 0$  we deduce that  $\text{Ext}^k(\mathcal{O}_m, I_m) = \text{Ext}^{k-1}(I_m, I_m)$ . Using 4.2, we get that this group is equal to  $\mathbb{C}$  for  $k = 1, 2$  and to zero for  $k = 0, 3$ .

Then we apply  $\text{Hom}(\mathcal{O}_m, \cdot)$  to the previous short sequence:

$$(3) \quad \begin{aligned} 0 \rightarrow \text{Hom}(\mathcal{O}_m, I_m) &\rightarrow \text{Hom}(\mathcal{O}_m, \mathcal{O}_X) \rightarrow \text{Hom}(\mathcal{O}_m, \mathcal{O}_m) \rightarrow \\ &\rightarrow \text{Ext}^1(\mathcal{O}_m, I_m) \rightarrow \text{Ext}^1(\mathcal{O}_m, \mathcal{O}_X) \rightarrow \text{Ext}^1(\mathcal{O}_m, \mathcal{O}_m) \rightarrow \\ &\rightarrow \text{Ext}^2(\mathcal{O}_m, I_m) \rightarrow \text{Ext}^2(\mathcal{O}_m, \mathcal{O}_X) \rightarrow \text{Ext}^2(\mathcal{O}_m, \mathcal{O}_m). \end{aligned}$$

We deduce that  $\text{Ext}^k(\mathcal{O}_m, \mathcal{O}_m) = \mathbb{C}$  for  $k = 0, 1$ , and zero for  $k = 2$ .

Now we apply the functor  $\text{Hom}(\cdot, \mathcal{O}_m(-p))$  to the sequence  $0 \rightarrow I_\ell \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_\ell \rightarrow 0$ . Since  $\text{Ext}^k(\mathcal{O}_X, \mathcal{O}_m(-p)) = H^k(\mathbb{P}^1, \mathcal{O}(-1)) = 0$  for all  $k$ , we deduce that  $\text{Ext}^k(I_\ell, \mathcal{O}_m(-p)) = \text{Ext}^{k+1}(\mathcal{O}_\ell, \mathcal{O}_m(-p))$ . To compute the latter, we use the existence of a spectral sequence

$$H^i(X, \mathcal{E}xt^j(\mathcal{O}_\ell, \mathcal{O}_m(-p))) \Rightarrow \text{Ext}^{i+j}(\mathcal{O}_\ell, \mathcal{O}_m(-p)).$$

Since the sheaf  $\mathcal{E}xt^j(\mathcal{O}_\ell, \mathcal{O}_m(-p))$  is supported at  $p$ , we deduce that

$$\text{Ext}^{k+1}(\mathcal{O}_\ell, \mathcal{O}_m(-p)) = H^0(X, \mathcal{E}xt^{k+1}(\mathcal{O}_\ell, \mathcal{O}_m(-p))).$$

So we just need to compute the rank of the fiber of this local  $\mathcal{E}xt$ -sheaf at the point  $p = \ell \cap m$ . Locally around that point,  $\ell$  and  $m$  are given by some equations  $x = y = 0$  and  $x = z = 0$  respectively, and  $\mathcal{O}_\ell$  has a free resolution given by the Koszul complex defined by  $x = z = 0$ . Applying  $\mathcal{H}om(\cdot, \mathcal{O}_m(-p))$  we get a complex whose cohomology is readily computed. We deduce that the rank of  $\mathcal{E}xt^{k+1}(\mathcal{O}_\ell, \mathcal{O}_m(-p))$  at  $p$  is one for  $k = 1, 2$ , and zero for  $k = 3$ . Thus  $\text{Ext}^k(I_\ell, \mathcal{O}_m(-p)) = \mathbb{C}$  for  $k = 0, 1$  and zero for  $k = 2$ .

From the long exact sequence (2), we deduce that  $\text{Ext}^k(I_q, \mathcal{O}_m(-p)) = 0$  for  $k = 1, 2$ , and consequently from (1) that  $\text{Ext}^k(I_q, I_q) = \text{Ext}^k(I_q, I_\ell)$  again for  $k = 1, 2$ . To compute this group, we apply the functor  $\text{Hom}(\cdot, I_\ell)$  to the short sequence  $0 \rightarrow I_q \rightarrow I_\ell \rightarrow \mathcal{O}_m(-p) \rightarrow 0$ . This gives

$$(4) \quad \begin{aligned} 0 \rightarrow \text{Hom}(\mathcal{O}_m(-p), I_\ell) &\rightarrow \text{Hom}(I_\ell, I_\ell) \rightarrow \text{Hom}(I_q, I_\ell) \rightarrow \\ &\rightarrow \text{Ext}^1(\mathcal{O}_m(-p), I_\ell) \rightarrow \text{Ext}^1(I_\ell, I_\ell) \rightarrow \text{Ext}^1(I_q, I_\ell) \rightarrow \\ &\rightarrow \text{Ext}^2(\mathcal{O}_m(-p), I_\ell) \rightarrow \text{Ext}^2(I_\ell, I_\ell) \rightarrow \text{Ext}^2(I_q, I_\ell) \rightarrow 0. \end{aligned}$$

The last arrow is surjective because  $\text{Ext}^3(\mathcal{O}_m(-p), I_\ell)$ , since  $m$  is a locally complete intersection of codimension two. Since  $\ell$  is supposed to define a smooth point of the

family of lines on  $X$ , we know that  $Ext^2(I_\ell, I_\ell) = 0$  and  $Ext^1(I_\ell, I_\ell) = \mathbb{C}$ . Finally, we apply  $Hom(\mathcal{O}_m(-p), \cdot)$  to  $0 \rightarrow I_\ell \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_\ell \rightarrow 0$  to obtain

$$(5) \quad \begin{aligned} 0 &\rightarrow Hom(\mathcal{O}_m(-p), I_\ell) \rightarrow Hom(\mathcal{O}_m(-p), \mathcal{O}_X) \rightarrow Hom(\mathcal{O}_m(-p), \mathcal{O}_\ell) \rightarrow \\ &\rightarrow Ext^1(\mathcal{O}_m(-p), I_\ell) \rightarrow Ext^1(\mathcal{O}_m(-p), \mathcal{O}_X) \rightarrow Ext^1(\mathcal{O}_m(-p), \mathcal{O}_\ell) \rightarrow \\ &\rightarrow Ext^2(\mathcal{O}_m(-p), I_\ell) \rightarrow Ext^2(\mathcal{O}_m(-p), \mathcal{O}_X) \rightarrow Ext^2(\mathcal{O}_m(-p), \mathcal{O}_\ell) \rightarrow 0. \end{aligned}$$

By Serre duality,  $Ext^k(\mathcal{O}_m(-p), \mathcal{O}_X) = Ext^k(\mathcal{O}_m(-2p), \omega_X)$  is dual to the cohomology group  $H^{3-k}(X, \mathcal{O}_m(-2p)) = H^{3-k}(\mathbb{P}^1, \mathcal{O}(-2))$ , hence equal to  $\mathbb{C}$  for  $k = 2$  and zero otherwise. We deduce that  $Ext^1(\mathcal{O}_m(-p), I_\ell) = 0$  and  $Ext^2(\mathcal{O}_m(-p), I_\ell) = \mathbb{C}$ . Using (4) we get that  $Ext^2(I_q, I_\ell) = Ext^2(I_q, I_q) = 0$ , while  $Ext^1(I_q, I_\ell) = Ext^1(I_q, I_q)$  is two-dimensional. This concludes the proof that  $F(X)$  is smooth at  $q$ .  $\square$

Let  $q \in F(X)$  be a conic in  $X$ , possibly singular. There is a unique codimension two subspace  $L$  of  $V$  such that  $q$  is a linear section of  $G(2, L)$ , and the linear space  $\mathbb{P}(\wedge^2 L) \simeq \mathbb{P}^5$  meets  $\mathbb{P}_X^9$  along the span  $\mathbb{P}_q^9$  of the conic. Dually, this implies that the orthogonal to  $\mathbb{P}(\wedge^2 L)$ , which is the linear space  $\mathbb{P}(L^\perp \wedge V^*) \simeq \mathbb{P}^8$ , meets  $\mathbb{P}_Y^4$  along a line  $\ell$ . But a skew-symmetric form in  $L^\perp \wedge V^*$  has rank at most four, so the line  $\ell$  is in fact contained in  $Y$ .

Conversely, a line  $\ell$  in  $Y$ , of type A or B, is contained in the tangent space  $\mathbb{P}(M \wedge V^*)$  of the Grassmannian at a unique line  $m = \mathbb{P}M$ . Dually, the orthogonal  $\mathbb{P}(\wedge^2 M^\perp)$  to this tangent space is contained in  $\ell^\perp$ . The linear span  $\mathbb{P}_X^9$  of  $X$ , which has codimension three in  $\ell^\perp$ , has to cut the four dimensional quadric  $G(2, M^\perp) \subset \mathbb{P}(\wedge^2 M^\perp)$  at least along a conic. But since  $X$  contains no surfaces of degree two, the intersection  $G(2, M^\perp) \cap \mathbb{P}_X^9 = q$  is nothing more than a conic. We have proved:

**Proposition 4.5.** *The Fano surface  $F(X)$  of conics in  $X$  is isomorphic with the Fano surface  $F(Y)$  of lines in the orthogonal cubic  $Y$ .*

Recall that the Albanese variety of  $F(Y)$  is isomorphic to the intermediate Jacobian  $J(Y)$  of  $Y$  [CG]. Moreover, by Proposition 3.1,  $J(Y)$  is isomorphic to the intermediate Jacobian  $J(X)$  of  $X$ . Since  $F(X)$  is smooth, the Abel-Jacobi map  $F(X) \rightarrow J(X)$  is algebraic.

How is the Abel-Jacobi image of  $F(X)$  related with  $F(Y)$  in  $J(X) \simeq J(Y)$ ?

Let  $\ell$  be a line on  $Y$ . We have seen in Lemma 2.2 that the union of the kernels  $n_y$  of the points  $y \in \ell$  is a two-dimensional quadric  $Q^\ell$  of rank at least three, contained in the Palatini quartic  $W$ . The linear span of  $Q^\ell$  is a codimension two linear space  $\mathbb{P}_\ell^3 \subset \mathbb{P}^5$ .

On the side of  $X$ , we have a conic  $q_\ell$  parametrizing lines in  $\mathbb{P}^5$  whose union is again a quadric  $Q_\ell$ , also contained in  $W \cap \mathbb{P}_\ell^3$ . Note that when the conic  $q_\ell$  is a union of two lines,  $Q_\ell$  is a union of two planes. In particular  $Q_\ell \neq Q^\ell$ , and this must remain true for any general  $\ell$ .

Note moreover that  $W$  cannot contain  $\mathbb{P}_\ell^3$ , since otherwise  $Sing(W) \cap \mathbb{P}_\ell^3$  would contain a surface or a curve of degree nine, in contradiction with the fact that  $\Gamma(W) = Sing(W)$  is a smooth curve of degree 25.

Since  $W$  has degree four, we deduce that

$$W \cap \mathbb{P}_\ell^3 = Q_\ell + Q^\ell$$

for all  $\ell \in F(Y)$ . Let  $H \subset \mathbb{P}^5$  be the general hyperplane that we used in Proposition 3.1 to define a birationality between  $X$  and  $Y$ , and then an isomorphism between  $J(X)$  and

$J(Y)$ . The image of  $\ell$  by  $u_Y$  is the plane conic  $Q^\ell \cap H$ . The image of  $q_\ell$  by  $u_X$  is the plane conic  $Q_\ell \cap H$ . Since the sum of these two cycles is a linear section of  $W \cap H$ , the Abel-Jacobi image of  $\ell$  in  $J(Y)$  is equal to minus the Abel-Jacobi image of  $q_\ell$  in  $J(X)$ , once we have identified the two intermediate Jacobians via Proposition 3.1.

Since the Abel-Jacobi map embeds  $F(Y)$  in  $J(Y)$ , and since  $F(X)$  and  $F(Y)$  are isomorphic by Proposition 4.5, we deduce:

**Proposition 4.6.** *The Abel-Jacobi map from  $F(X)$  to  $J(X)$  is an embedding. Moreover, the images of  $F(X)$  in  $J(X)$  and  $F(Y)$  in  $J(Y)$  are the same up to multiplication by  $-1$ .*

## 5. THE MODULI SPACE $M_X(2; 1, 6)$

In this section  $X = X_{14} = G(2, 6) \cap \mathbb{P}_X^9$  is again a general smooth prime Fano threefold of genus 8, and we study the moduli space  $M_X(2; 1, 6)$  of semistable rank two torsion free sheaves  $E$  on  $X$  with Chern numbers  $c_1 = 1$ ,  $c_2 = 6$  and  $c_3 = 0$ . In particular  $c_1(E) = h$  is the hyperplane class of  $X$ , and since  $h$  is not divisible in  $\text{Pic}(X) = \mathbb{Z}h$ , any such sheaf  $E$  is in fact stable, and the stability of  $E$  is equivalent to its slope stability.

**5.1. Vector bundles in  $M_X(2; 1, 6)$ .** First we study the locally free sheaves  $E$  from the moduli space  $M_X(2; 1, 6)$ . Note that by the stability assumption the bundle  $E^* = E(-1) = \text{Hom}(\mathcal{O}_X(1), E)$  has no sections. Moreover, the Riemann-Roch formula for vector bundles on threefolds gives  $\chi(E) = 5$ , see Ch.15.2: Example 15.2.5 in [Fu].

**Lemma 5.1.**  $h^2(E) = 0$  and  $h^0(E) > 0$ .

*Proof.* Since  $E(-2) = E^*(-1)$  has no sections,  $h^3(E) = 0$  by Serre duality. Suppose that  $h^2(E) = h^1(E^*(-1)) \neq 0$ . Then we would have a non trivial extension

$$0 \rightarrow \mathcal{O}_X(-1) \rightarrow F \rightarrow E \rightarrow 0,$$

where the rank three vector bundle  $F$  has  $c_1(F) = 0$ ,  $c_2(F) = -8$  and  $c_3(F) = 6$ . Thus  $\Delta(F) = \text{deg}(2c_1(F)^2 - 6c_2(F)) = 48 > 0$ , so by Bogomolov's inequality [Bo],  $F$  cannot be semistable. So either  $\mathcal{O}_X(1)$  maps non trivially to  $F$ , hence to  $E$  – a contradiction!, or  $F$  maps non trivially to  $\mathcal{O}_X(-1)$  and the extension splits, again a contradiction. Thus  $h^2(E) = 0$  and  $h^0(E) = 5 + h^1(E) > 0$ .  $\square$

Now we choose a general hyperplane section  $S$  of  $X$ , and denote by  $E_S$  the restriction of  $E$  to  $S$ . The Picard group of this K3 surface is generated by  $\mathcal{O}_S(1)$ , and by Maruyama's restriction theorem  $E_S$  remains semistable, hence stable, with Chern classes  $c_1(E_S) = h_S$ , the hyperplane class of  $S$ , and  $c_2(E_S) = 6$ , see [Mar].

**Proposition 5.2.**

- (1)  $h^0(E_S) = 5$ , and  $h^i(E_S) = 0$  for  $i > 0$ .
- (2)  $h^0(E) = 5$  and  $h^1(E) = h^1(E(-1)) = 0$ .

*Proof.* (1) By Riemann-Roch for vector bundles on surfaces  $\chi(E_S) = 5$  (see Example 15.2.2 in [Fu]); and since  $H^2(E_S)$  and  $H^0(E_S(-1))$  vanish by the stability of  $E_S$  then  $h^0(E_S) = 5 + h^1(E_S) \geq 5$ .

In particular  $E_S$  has non-trivial sections; let  $s$  be one of them, and let  $Z = Z(s)$  be the zero-scheme of  $s$ . The stability of  $E_S$  implies that  $Z$  is either of codimension 2 in  $S$  or empty. But in the latter case  $E_S$  would be an extension of  $\mathcal{O}_S$  by  $\mathcal{O}_S(1)$ , so this extension would split, thus contradicting the stability.

Therefore the Koszul complex of  $s$  determines  $E_S$  as an extension

$$0 \rightarrow \mathcal{O}_S \rightarrow E_S \rightarrow \mathcal{I}_Z(1) \rightarrow 0,$$

where  $\mathcal{I}_Z$  is the ideal sheaf of  $Z$ , a zero dimensional scheme of length six. Since  $E_S$  is locally free, Theorem 3.13 in [L] implies that  $Z$  has the following property: every hyperplane containing a colength one subscheme of  $Z$  contains  $Z$ . This means that  $Z$  is of type  $Z_6^k$ , i.e.  $\deg(Z) = 6$ ,  $h^0(\mathcal{I}_Z(1)) = 3 + k$  and  $h^1(\mathcal{I}_Z(1)) = k$  for some  $k \geq 1$ .

**Lemma 5.3.** *Let  $S$  be a smooth K3 surface in  $G(2, 6)$ , with  $\text{Pic}(S) = \mathbb{Z}\mathcal{O}_S(1)$ . Then*

- (i)  $S$  contains no subscheme of type  $Z_3^1$  or  $Z_4^1$ ;
- (ii)  $S$  contains no subscheme of type  $Z_6^k$  for  $k > 1$ .

*Proof.* (i) Since  $S$  is an intersection of quadrics, if it contains a subscheme of type  $Z_3^1$  then it contains the line spanned by this finite scheme. Next, suppose that  $S$  contains a subscheme  $Z$  of type  $Z_4^1$ . Then  $Z$  is the complete intersection of two quadrics in  $\mathbb{P}_Z^2 = \text{Span}(Z)$ , and  $\mathbb{P}_Z^2 \cap S = Z$  (otherwise  $S$  would contain a conic or a line). But since  $\mathbb{P}_Z^2 \cap G(2, 6) = \mathbb{P}_Z^2 \cap S = Z$ , then  $\mathbb{P}_Z^2$  will be a purely 4-secant plane to  $G(2, 6)$  which is impossible – any 4-secant plane to  $G(2, 6)$  must either lie in  $G(2, 6)$  or intersect  $G(2, 6)$  along a 1-cycle of degree 2.

(ii) Since on  $S$  there are no subschemes of type  $Z_4^1$ , then  $S$  contains no subschemes of type  $Z_5^2$  or  $Z_6^3$ . Next, suppose that on  $S$  there exists a 0-scheme  $Z$  of type  $Z_6^2$ . Then  $\mathbb{P}_Z^3 = \text{Span}(Z)$  will be a 3-space in  $\mathbb{P}_S^8 = \text{Span}(S)$  such that  $\mathbb{P}_Z^3 \cap S \supset Z$ .

Consider a subscheme  $Y \subset Z$  of length 5. It must be of type  $Z_5^1$ . Consider the sheaf  $E_Y$  obtained from  $Y$  by the Serre construction, that is by the non trivial extension

$$0 \rightarrow \mathcal{O}_S \rightarrow E_Y \rightarrow I_Y(1) \rightarrow 0.$$

This extension is unique since  $\text{Ext}^1(I_Y(1), \mathcal{O}_S) = H^1(I_Y(1)) = \mathbb{C}$ . Since  $Y$  contains no subschemes of type  $Z_4^1$ ,  $E_Y$  is a vector bundle [L]. We check that it must be stable. If  $L$  is a line bundle on  $S$  with a non trivial sheaf homomorphism  $L \rightarrow E_Y$ , either the composition  $L \rightarrow I_Y(1)$  is non trivial or  $L$  maps to  $\mathcal{O}_S \subset E_Y$ ; Since  $\text{Pic}(S) = \mathbb{Z}\mathcal{O}_S(1)$ , in both cases we conclude that  $L = \mathcal{O}_S(\ell)$  for some  $\ell \leq 0$ , which implies the stability. Using the argument of [Mu4], Theorem 4.5 (ii), we conclude that  $E_Y$  is isomorphic to the restriction  $T_S^*$  of the tautological vector bundle on the ambient Grassmannian. In particular  $E_Y$  is generated by global sections. But then  $I_Y(1)$  is also generated by global sections, which means that  $S \cap \mathbb{P}_Z^3 = Y$ , a contradiction.  $\square$

Applying this result to our punctual scheme  $Z$ , which is contained in a general K3 surface in  $G(2, 6)$ , we conclude that it must be of type  $Z_6^1$ , that is  $h^0(\mathcal{I}_Z(1)) = 4$ ,  $h^1(\mathcal{I}_Z(1)) = 1$  and  $Z$  spans a  $\mathbb{P}^4$  in  $\mathbb{P}_S^8 = \text{Span}(S)$ . From the exact sequence

$$0 \rightarrow H^1(E_S) \rightarrow H^1(\mathcal{I}_Z(1)) \rightarrow H^2(\mathcal{O}_S) \rightarrow 0$$

we deduce that  $h^1(E_S) = 0$ , hence  $h^0(E_S) = 5$ .

Now (2) follows from the exact sequence

$$0 \rightarrow E(-1) \rightarrow E \rightarrow E_S \rightarrow 0,$$

since by 5.1 we get  $5 \leq h^0(E) \leq h^0(E_S) = 5$ . This implies that  $h^1(E) = 0$ , hence also  $h^1(E(-1)) = 0$ .  $\square$

**Proposition 5.4.** *Any vector bundle in  $M_X(2; 1, 6)$  is globally generated.*

*Proof.* Let  $x \in X$  be any point. A crucial fact will be the following:

**Lemma 5.5.** *In the linear system  $|I_x(1)|$  of hyperplane sections of  $X$  passing through  $x$ , the set of smooth sections  $S$  such that  $\text{Pic}(S) = \mathbb{Z}\mathcal{O}_S(1)$  is a non empty open subset for the countable Zariski topology.*

*Proof.* A Lefschetz pencil of hyperplane sections of  $X$  is defined by a line in the dual projective space, cutting the dual variety  $X^*$  transversely at smooth points (see [V], Proposition 14.9). Fix a point  $x \in X$ , and denote by  $H_x$  the family of hyperplane sections of  $X$  containing  $x$ . The singular locus of  $X^*$  cannot coincide with its intersection with the hyperplane  $H_x$ , so there exists a Lefschetz pencil of hyperplane sections of  $X$  all passing through  $x$ . Then the proof of the Noether-Lefschetz theorem by monodromy applies verbatim (see [V], Corollaire 15.28 and Théorème 15.33).  $\square$

Let  $S \subset X$  be such a hyperplane section through  $x$ . Since  $h^1(E_S(-1)) = 0$  then  $h^0(E) \simeq h^0(E_S)$ . Thus to prove that  $E$  is generated by global sections at  $x$ , we just need to show that  $E_S$  is generated by global sections. Consider once again the Koszul complex

$$0 \rightarrow \mathcal{O}_S \rightarrow E_S \rightarrow \mathcal{I}_Z(1) \rightarrow 0,$$

defined by some non trivial section of  $E_S$ . The bundle  $E_S$  will be generated if and only if  $\mathcal{I}_Z(1)$  is generated. So we need to prove that  $Z$  is cut out on  $X$ , scheme-theoretically, by its linear span – in other words, that  $Z$  cannot be contained in a  $Z'$  of type  $Z_7^2$ .

Suppose the contrary. Then  $h^1(I_{Z'}(1)) = 2$ , and by [Mo](page 22),  $Z'$  defines a rank three sheaf  $F$  over  $S$  as the universal extension

$$0 \rightarrow \mathcal{O}_S \otimes H^1(I_{Z'}(1)) \rightarrow F \rightarrow I_{Z'}(1) \rightarrow 0.$$

Since by Lemma 5.3  $S$  does not contain any subscheme of type  $Z_6^2$ , then any proper subscheme  $Z'' \subset Z'$  has  $h^1(I_{Z''}(1)) \leq 1$ , and by [Mo](the Lemma on page 23) the sheaf  $F$  will be locally free, i.e. a vector bundle.

Next, consider the induced cohomology sequence

$$0 \rightarrow H^1(F) \rightarrow H^1(I_{Z'}(1)) \rightarrow H^1(I_{Z'}(1)) \rightarrow H^2(F) \rightarrow 0.$$

Since  $F$  has been defined as a universal extension, the middle map must be an isomorphism, hence  $h^1(F) = h^2(F) = 0$ , and  $h^0(F) = 6$ ; in particular  $\chi(F) = 6$ .

We shall see that  $F$  is stable. Suppose it is not. Since  $c_1(F)$  is the hyperplane class of  $S$  and  $\text{Pic}(S) = \mathbb{Z}\mathcal{O}_S(1)$ , then either there exists a non trivial map  $\mathcal{O}_S(1) \rightarrow F$ , or there exists a non trivial map  $F \rightarrow \mathcal{O}_S$ .

But in the first case the induced map to  $I_{Z'}(1)$  vanishes and the map factors through  $\mathcal{O}_S \otimes H^1(I_{Z'}(1))$ , a contradiction. Therefore there exists a non trivial map  $F \rightarrow \mathcal{O}_S$ , and denote this map by  $f$ . But then the restriction of  $f$  to  $\mathcal{O}_S \otimes H^1(I_{Z'}(1))$  cannot be zero, since otherwise  $f$  would descend to  $I_{Z'}(1)$ , which is impossible. Therefore  $f$  is surjective, and since  $h^2(\mathcal{O}_S) = 1$  then  $h^2(F) > 0$ , again a contradiction.

Thus  $F$  is a stable vector bundle of rank three with Chern numbers  $c_1 = 1$  and  $c_2 = 7$ , on the smooth K3 surface  $S = S_{14}$  with  $\text{Pic}(S) = \mathcal{O}_S(1)$ . In other words, the stable bundle  $F$  belongs to the Mukai's moduli space  $M_S(r, L, s) = M_S(3, \mathcal{O}_S(1), 3)$  of simple sheaves  $E$  on  $S$ , with  $\text{rank}(E) = r = 3$  and  $s = \chi(E) - r = 3$  (see above); in particular this moduli space must be non-empty. But by Theorem 0.1 in Mukai [Mu1],  $M_S(3, \mathcal{O}(1), 3)$  should have dimension  $= \text{deg } S - 2.r.s = 14 - 2.3.3 + 2 = -2$ , contradiction.  $\square$

**Corollary 5.6.** *Any vector bundle in  $M_X(2; 1, 6)$  is arithmetically Cohen-Macaulay.*

*Proof.* We first check that  $E(2)$  is Castelnuovo-Mumford regular, that is,  $h^1(E(1)) = h^2(E) = h^3(E(-1)) = 0$ . By Serre duality,  $h^3(E(-1)) = h^0(E^*) = h^0(E(-1)) = 0$ . The vanishing of  $h^2(E)$  has already been checked in 5.1, and

$$H^1(E(1)) = H^1(K_X \otimes E \otimes \det E \otimes \mathcal{O}_X(1)),$$

so the vanishing of  $H^1(E(1))$  follows from Griffiths' vanishing theorem, see e.g. [Dem].

The Castelnuovo-Mumford regularity of  $E(2)$  ensures that

$$H^1(E(k+1)) = H^2(E(k)) = H^1(E(-k-2)) = 0 \quad \forall k \geq 0.$$

Since  $H^1(E) = H^1(E(-1)) = 0$  by Proposition 5.2, then  $H^1(E(k)) = 0$  for all  $k \in \mathbb{Z}$ . Together with the Serre duality, the last yields  $H^2(E(k)) = 0$  for all  $k \in \mathbb{Z}$ .  $\square$

Note that  $E$  itself is not Castelnuovo-Mumford regular, even  $E(1)$  is not since  $h^3(E(-2)) = h^0(E^*(1)) = h^0(E) \neq 0$ . That's what makes the proof of Proposition 5.4 rather intricate.

**5.2. Elliptic sextics.** Since a vector bundle  $E$  in  $M_X(2; 1, 6)$  is globally generated, a general section of  $E$  will vanish along a smooth curve  $C$ , an elliptic sextic in  $G(2, 6)$ . The associated Koszul complex is

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow I_C(1) \rightarrow 0,$$

and since  $h^2(\mathcal{O}_X) = 0$  and  $h^1(E) = 0$  by Proposition 5.2, we deduce that  $h^1(I_C(1)) = 0$ . This means that  $C$  is *projectively normal*.

Consider the tautological rank two vector bundle  $T^*$  on  $G(2, 6)$ , and restrict it to  $C$ . By the Atiyah classification of rank two bundles on elliptic curves (see [A]), and the fact that  $T^*$  is generated by global sections, one of the following two possibilities must take place:

- (1)  $T_C^* = L \oplus M$  is a direct sum of two line bundles of degrees  $(3, 3)$ ,  $(2, 4)$  or  $(0, 6)$ ;
- (2)  $T_C^* = F \otimes N$  is a direct product of a degree three line bundle  $N$  with the unique vector bundle  $F$  on  $C$  obtained as a non trivial self-extension of  $\mathcal{O}_C$ .

We say correspondingly that  $C$  is split of type  $(a, b)$ , or unsplit.

If  $C$  is split, in fact it cannot be of type  $(0, 6)$ . Indeed, this would imply that the restriction of  $T$  to  $C$  has a constant factor. This would mean that the lines in  $\mathbb{P}^5$  parameterized by  $C$  contain some fixed vector. But such lines are parameterized by a  $\mathbb{P}^4$  in  $G(2, 6)$ , and we would conclude that  $C$  is not projectively normal, a contradiction.

So the space of global sections

$$H^0(C, T_C^*) = H^0(C, L) \oplus H^0(C, M)$$

has dimension 6, which is the same as the dimension of  $H^0(G(2, 6), T^*) = V^*$ . Note that the non-injectivity of the restriction map

$$H^0(G(2, 6), T^*) \rightarrow H^0(C, T_C^*)$$

implies that  $C$  is contained in a copy of  $G(2, 5)$ . If this is not the case,  $H^0(C, L)$  and  $H^0(C, M)$  can be identified with the orthogonal spaces  $A^\perp$  and  $B^\perp$  to two supplementary spaces  $A$  and  $B$  in  $V$ , and then the curve  $C$  must be contained in the intersection of  $G(2, V)$  with the Segre variety

$$\Sigma_C = \mathbb{P}A \times \mathbb{P}B \subset \mathbb{P}(A \otimes B) \subset \mathbb{P}(\wedge^2 V).$$

**Lemma 5.7.** *Let  $C$  be a smooth projectively normal elliptic sextic in  $X$ . Then  $C$  is split of type  $(3, 3)$ , or unsplit.*

*Proof.* First we notice that  $C$  cannot be contained in a copy  $G(2, H)$  of  $G(2, 5)$ . This is simply because  $G(2, 5)$  has degree 5, while  $C$  has degree 6. So the intersection of  $G(2, H)$  with  $\mathbb{P}_X^9$  cannot be proper. But then  $X$  contains a surface of degree 5, contradicting the fact that its Picard group is generated by the hyperplane class – or  $X$  is contained in  $G(2, H)$ , which is clearly impossible for  $X$  general.

We must exclude the possibility that  $C$  be of type  $(2, 4)$ . Suppose it is. Then all the lines  $\ell_x$ , for  $x \in C$ , intersect a fixed line  $D$  in  $\mathbb{P}^5$ , identified with the image of  $C$  by a complete linear system of type  $g_2^1$ . In particular, through a general point of  $D$  pass two lines  $\ell_x$  and  $\ell_{x'}$ . But then the line  $D$  must be contained in the singular locus  $\Gamma(W)$  of the Palatani quartic (see 3.3), in contradiction with the fact that  $\Gamma(W)$  is a smooth curve of genus 26.  $\square$

A split elliptic sextic  $C$  of type  $(3, 3)$  has two unisecant planes, that is, two planes meeting all the lines parameterized by  $C$ : with the notations above, they are the planes  $\mathbb{P}A$  and  $\mathbb{P}B$ . In the unsplit case there is a unique such plane.

We denote by  $\mathcal{S}(X)$  the family of smooth projectively normal elliptic sextic curves  $C \subset X$ , an open subset of the Hilbert scheme of  $X$ . Let  $\mathcal{S}(X)_{un}$  be the subfamily of these  $C \in \mathcal{S}(X)$  that have only one unisecant plane.

**Lemma 5.8.** *For  $X$  general, the subfamily  $\mathcal{S}(X)_{un} \subset \mathcal{S}(X)$  has dimension at most 5.*

*Proof.* Let  $C \subset G(2, 6)$  be a smooth projectively normal elliptic sextic curve with only one unisecant plane. Then the restriction  $T_C^*$  of the tautological rank two bundle fits into an unsplit extension  $0 \rightarrow L \rightarrow T_C^* \rightarrow L \rightarrow 0$  for some degree three line bundle  $L$  on  $C$ . This line bundle maps  $C$  isomorphically to a plane cubic  $E$  inside  $\mathbb{P}H^0(L)^*$ .

Taking global sections, we get  $0 \rightarrow H^0(L) \rightarrow H^0(T_C^*) \rightarrow H^0(L) \rightarrow 0$ . If the restriction map  $H^0(T^*) = V^* \rightarrow H^0(T_C^*)$  is not injective, then  $C$  is contained in a  $G(2, 5)$ . Otherwise  $H^0(T_C^*) = V^*$ , and we deduce a projection map  $\mu : V \rightarrow H^0(L)^* \hookrightarrow V$ . Let  $\Lambda \subset V$  be transverse to  $M := H^0(L)^*$ , so that the restriction of  $\mu$  to  $\Lambda$ , that we denote by  $\nu$ , is an isomorphism. From the commutative diagram

$$\begin{array}{ccccc} L^* & \rightarrow & T_C & \rightarrow & L^* \\ \downarrow & & \downarrow & & \downarrow \\ M \otimes \mathcal{O}_C & \rightarrow & V \otimes \mathcal{O}_C & \rightarrow & M \otimes \mathcal{O}_C, \end{array}$$

whose vertical maps are injective, we deduce that over a point  $[e] \in E \simeq C$ , where  $e \in M$ , the two-dimensional subspace  $T_C$  of  $V$  is generated by  $e$  and a vector  $f$  mapping to  $e$  by  $\mu$ . So  $f$  must be of the form  $\nu^{-1}(e) + \theta(e)$  for some  $\theta(e) \in M$ , defined up to translation by some multiple of  $e$ . Thus  $\theta$  has to be interpreted as a global section of the vector bundle  $\mathcal{H}om(L^*, M \otimes \mathcal{O}_C/L^*)$ . From the exact sequence  $0 \rightarrow \mathcal{O}_C \rightarrow M \otimes L \rightarrow \mathcal{H}om(L^*, M \otimes \mathcal{O}_C/L^*) \rightarrow 0$  we deduce the sequence

$$0 \rightarrow \mathbb{C} \rightarrow \text{End}(M) \rightarrow \mathcal{H}om(L^*, M \otimes \mathcal{O}_C/L^*) \rightarrow \mathbb{C} \rightarrow 0.$$

In particular  $\mathcal{H}om(L^*, M \otimes \mathcal{O}_C/L^*)$  has dimension 9.

We can now count the number of parameters for  $C$ : we have 9 parameters for the three-space  $H^0(L)^* \subset V$ , 9 parameters for the cubic curve  $E \subset \mathbb{P}H^0(L)^*$ , then 9 again for  $\mu$  and 9 for  $\theta$ , minus one since multiplying  $\mu^{-1}$  and  $\theta$  by a same scalar does not change the resulting curve: this makes 35 parameters. Each of our curves  $C$  spans a  $\mathbb{P}^5$ , and there is a 20-dimensional family of  $\mathbb{P}^9$ 's containing this  $\mathbb{P}^5$ . Since the Grassmannian of  $\mathbb{P}^9$ 's in  $\mathbb{P}^{14}$  has dimension 50, our claim follows.  $\square$



**Corollary 5.9.** *Any smooth projectively normal elliptic sextic in  $X$  with only one unisecant plane can be deformed into an elliptic sextic with two unisecant planes.*

*Proof.* Let  $C \in \mathcal{S}(X)_{un}$ . The dimension at  $C$  of the family  $\mathcal{S}(X)$  is at least  $\chi(N_{C/X})$  (see e.g. [S], Corollary 8.5), which by Riemann-Roch is equal to  $\deg(C) = 6$ . So by Lemma 5.8, the curve  $C$  can be deformed outside  $\mathcal{S}(X)_{un}$ .  $\square$

**Definition.** *Let  $M_X(2; 1, 6)^0$  denote the open subset of vector bundles in  $M_X(2; 1, 6)$  having a section whose zero locus is a split elliptic sextic.*

By the previous statement any vector bundle in  $M_X(2; 1, 6)$  is in the closure of the open set  $M_X(2; 1, 6)^0$ . In fact we will finally conclude that any vector bundle in  $M_X(2; 1, 6)$  belongs to  $M_X(2; 1, 6)^0$ , see Theorem 7.2.

Let  $C \in \mathcal{S}(X)$  be a split elliptic sextic, with two unisecant planes  $\mathbb{P}A$  and  $\mathbb{P}B$ . The linear span of the Segre variety  $\Sigma_C = \mathbb{P}A \times \mathbb{P}B$  is  $\mathbb{P}(A \otimes B) \subset \mathbb{P}(\wedge^2 V)$ , whose orthogonal in the dual space is

$$\mathbb{P}(\wedge^2 A^\perp \oplus \wedge^2 B^\perp) \subset \mathbb{P}(\wedge^2 V^*).$$

Since the span of  $X$  meets  $\mathbb{P}(A \otimes B)$  along the five dimensional linear span of  $C$ , their orthogonal spaces meet each other along a line  $\ell \subset Y = X^\perp$ . Note that  $\ell$  only depends on the vector bundle  $E$  defined by  $C$ , since the zero-loci of the sections of  $E$  are parameterized by the projective space of its global sections, while  $F(Y)$ , being embedded in the abelian variety  $F(Y)$ , contains no rational curve. We thus get a map  $M_X(2; 1, 6)^0 \rightarrow F(Y)$ . An inverse of this map will be constructed in the next section.

We now use what we learned on elliptic sextics to prove that  $M_X(2; 1, 6)^0$  is a smooth open subset of the moduli space.

**Proposition 5.10.** *Let  $E \in M_X(2; 1, 6)^0$  be a vector bundle. Then*

$$h^0(\text{End}(E)) = 1, \quad h^1(\text{End}(E)) = 2, \quad h^2(\text{End}(E)) = h^3(\text{End}(E)) = 0.$$

*In particular  $M_X(2; 1, 6)$  is smooth at  $E$ .*

*Proof.* By Serre duality  $h^2(\text{End}(E)) = h^1(\text{End}(E)(-1)) = h^1(E \otimes E(-2))$ . Consider the Koszul complex

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow I_C(1) \rightarrow 0$$

of a section of  $E$  vanishing along a smooth elliptic sextic  $C$  with two unisecant planes. Twisting this sequence by  $E(-2)$  and using 5.6, we are reduced to prove that  $h^1(E \otimes I_C(-1)) = 0$ , thus that  $h^0(E_C(-1)) = h^0(E_C^*) = 0$ .

Suppose the contrary. Since  $E$ , hence  $E_C$ , is globally generated, we deduce that  $E_C$  splits. Since it is isomorphic to the normal bundle of  $C$  in  $X$ , we get that  $N_{C/X} = \mathcal{O}_C \oplus \mathcal{O}_C(1)$ . Now  $C$  is a linear section of a Segre variety  $\Sigma_C = \Sigma \simeq \mathbb{P}A \times \mathbb{P}B$  by the linear span of  $X$ , meeting the linear span of  $\Sigma$  along the span of  $C$ . Let  $G$  denote the intersection of the Grassmannian with the linear span of  $X$  and  $\Sigma$ , a codimension two

linear space. We have the following diagram of normal bundles on  $C$ :

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow & & & \\
 & & & N_{X/G}|_C = \mathcal{O}_C(1)^{\oplus 3} & & & \\
 & & & \uparrow & & & \\
 0 \rightarrow N_{C/\Sigma} = \mathcal{O}_C(1)^{\oplus 3} & \rightarrow & & N_{C/G} & \rightarrow & N_{\Sigma/G}|_C & \rightarrow 0 \\
 & & & \uparrow & & & \\
 & & & N_{C/X} & & & \\
 & & & \uparrow & & & \\
 & & & 0 & & & 
 \end{array}$$

The induced map from  $N_{C/\Sigma}$  to  $N_{X/G}|_C$  is an isomorphism, hence also the induced map from  $N_{C/X}$  to  $N_{\Sigma/G}|_C$ , and we get an injective map from  $N_{C/X}$  to  $N_{\Sigma/G(2,6)}$  restricted to  $C$ . Remember that  $T_C^* = L \oplus M$  for some line bundles  $L$  and  $M$  of degree three, whose spaces of global sections are  $H^0(L) = A^\perp$  and  $H^0(M) = B^\perp$ . In particular  $\mathcal{O}_C(1) = L \otimes M$ . We compute that the normal bundle of  $\Sigma$  inside  $G(2, 6)$  is

$$N_{\Sigma/G(2,6)} = (L \otimes B/L^*) \oplus (M \otimes A/M^*).$$

Suppose we have a non trivial morphism from  $\mathcal{O}_C(1)$ , which we supposed to be a factor of  $N_{C/X}$ , to one of these direct factors, say the first one. Then we deduce that  $\text{Hom}(M, B/L^*) \neq 0$ . But the rank two vector bundle  $B/L^*$  on  $C$  is globally generated and has degree three, and from the Atiyah classification we deduce that  $B/L^* = M \oplus \mathcal{O}_C$ . Then the restriction to  $C$  of the tautological quotient bundle  $Q_C$ , which is isomorphic to  $B/L^* \oplus A/M^*$ , has a trivial factor. This implies that the linear span of  $C$  is contained in a copy of  $G(2, 5)$ , a contradiction.

That the vanishing of  $h^2(\text{End}(E))$  implies the smoothness of  $M_X(2; 1, 6)$  at  $E$  is well-known, see e.g. [HL], Theorem 4.5.4.  $\square$

**5.3. Non locally free sheaves in  $M_X(2; 1, 6)$ .** In this section we give a complete description of the non locally free sheaves in our moduli space  $M_X(2; 1, 6)$ . A similar study has been made by Druel on the cubic threefold. Our discussion follows closely that of [D], but Mukai's theorems on Fano threefolds and K3 surfaces of genus 8 will play a crucial role.

**Proposition 5.11.** *If  $E \in M_X(2; 1, 6)$  is not locally free, there exists a unique line  $\ell \subset X$  such that  $E$  fits into an exact sequence*

$$0 \rightarrow E \rightarrow T_X^* \rightarrow \mathcal{O}_\ell \rightarrow 0.$$

Note that  $T^*$  restricted to  $\ell$  is isomorphic to  $\mathcal{O}_\ell \oplus \mathcal{O}_\ell(1)$ , so that the rightmost arrow is uniquely defined up to scalar. Thus the line  $\ell$  defines  $E$  uniquely.

*Proof.* Let  $F = E^{**}$  denote the bidual of  $E$ , and let  $R = F/E$ . Since  $F$  is reflexive, the singular locus  $S(F)$  has codimension at least three, see Lemma 1.1.10 in Ch.2 of [OSS]. Therefore the restriction  $F_S$  of  $F$  to the general hyperplane section  $S$  of  $X$  is locally free, and the restriction  $R_S$  of  $R$  has finite support. We have  $c_1(F_S) = 1$  and  $c_2(F_S) = 6 - \text{length}(R_S)$ , so by Riemann-Roch  $\chi(F_S) = 2\chi(\mathcal{O}_S) + (c_1(F_S)^2 - 2c_2(F_S))/2 = \text{length}(R_S) + 5$ .

This rank two vector bundle  $F_S$  is semistable by Maruyama's restriction theorem. By Mukai ([Mu1], Theorem 0.1), the moduli space of simple sheaves on  $S$  with these invariants is smooth at  $F_S$  and has dimension  $c_1(F_S)^2 - 4\chi(F_S) + 10 = 4 - 4\text{length}(R_S)$ . In particular

$$\text{length}(R_S) \leq 1.$$

First, we shall see that  $\text{length}(R_S)$  cannot be zero. Suppose the contrary. Then  $R$  has finite support and  $c_2(F) = 6$ . Moreover, by Riemann-Roch  $\text{length}(R) = \chi(F(-1)) = c_3(F)/2$ . Since  $\chi(F_S) = 5$  and  $h^2(F_S) = h^0(F_S^*) = 0$  by stability, the vector bundle  $F_S$  has a non trivial section. The corresponding Koszul complex gives

$$0 \rightarrow \mathcal{O}_S \rightarrow F_S \rightarrow I_Z(1) \rightarrow 0,$$

where the finite scheme  $Z$  has type  $Z_6^1$ . The associated long exact sequence

$$0 \rightarrow H^1(F_S) \rightarrow H^1(I_Z(1)) \rightarrow H^2(\mathcal{O}_S) \rightarrow 0$$

gives  $H^1(F_S) = 0$ . Moreover,  $H^1(F_S(k)) = H^1(I_Z(k+1))$  for  $k > 0$ . But this is zero, since the fact that  $Z$  has type  $Z_6^1$  easily implies that the restriction map  $H^0(\mathcal{O}_S(k+1)) \rightarrow H^0(\mathcal{O}_Z(k+1))$  is surjective. Now the exact sequence

$$0 \rightarrow F(-1) \rightarrow F \rightarrow F_S \rightarrow 0$$

twisted by  $\mathcal{O}_X(k)$ , together with the vanishing of  $H^1(F_S(k))$  for all  $k \geq 0$ , imply that  $H^2(F(k-1))$  embeds inside  $H^2(F(k))$ , hence it is always zero since it certainly vanishes for  $k$  large enough. In particular  $h^2(F(-1)) = 0$ , and since  $h^0(F(-1)) = 0$  by stability, we conclude that  $\chi(F(-1)) \leq 0$ . Hence  $c_3(F) = 0$ , so that  $F$  is locally free by [H2], and  $\text{length}(R) = 0$ ; so that  $E$  is isomorphic to  $F$ . But then  $E$  is locally free – contradiction.

Therefore the only possibility left is  $\text{length}(R_S) = 1$ , so  $R_S = \mathcal{O}_p$  for some point  $p \in S$ .

In this case,  $R$  must be supported on the union of a line  $\ell$  with a finite set, and must have multiplicity one on that line. By Riemann-Roch  $\chi(R(-1)) = \chi(F(-1)) = c_3(F)/2$ . The restriction  $F_S$  has Chern classes  $c_1(F_S) = h$  and  $c_2(F_S) = 5$ . Therefore by [Mu4],  $F_S$  is the same as the restriction  $T_S^*$  to  $S \subset G(2, 6)$  of the tautological bundle  $T^*$  on  $G(2, 6)$ . Since  $S$  is a linear section of  $G(2, 6)$ , we have a Koszul complex

$$0 \rightarrow T^*(-6) \rightarrow \cdots \rightarrow T^*(-1)^{\oplus 6} \rightarrow T^* \rightarrow F_S \rightarrow 0.$$

Thus  $h^1(F_S(k)) = 0$  as soon as  $h^{q+1}(T^*(k-q)) = 0$  for any  $q = 0, \dots, 6$ . By Bott's theorem this holds true for any  $k \in \mathbb{Z}$ . As in the previous case we conclude that  $h^2(F(-1)) = 0$ , hence  $\chi(F(-1)) \leq 0$ , hence  $c_3(F) = 0$  and  $F$  is locally free.

Since  $c_1(F) = h$  and  $c_2(F) = 5$ , then again by [Mu4] (see e.g. Theorem 1.10), the bundle  $F$  must be the same as  $T_X^*$ , the restriction to  $X$  of the tautological bundle  $T^*$  on  $G(2, 6)$ .

On the other hand, from the exact sequence

$$0 \rightarrow E_S \rightarrow T_S^* \rightarrow \mathcal{O}_p \rightarrow 0,$$

we deduce that  $h^1(E_S(k)) = 0$  for  $k \geq 0$ , hence  $h^2(E(-1)) = 0$ , and then  $h^1(E(-1)) = 0$  since  $\chi(E(-1)) = 0$ ,  $h^0(E(-1)) = 0$  by stability and  $h^3(E(-1)) = h^0(E^*) = 0$  by Serre duality for sheaves. Thus  $h^0(R(-1)) = 0$ . Since Riemann-Roch gives  $\chi(R(n)) = n + 1$ , we conclude by [D], Lemme 3.2, that  $R = \mathcal{O}_\ell$ .  $\square$

**Proposition 5.12.** *Let  $E \in M_X(2; 1, 6)$  be a non locally free sheaf. Then*

$$\operatorname{hom}(E, E) = 1, \quad \operatorname{ext}^1(E, E) = 2, \quad \operatorname{ext}^2(E, E) = \operatorname{ext}^3(E, E) = 0.$$

*In particular the moduli space  $M_X(2; 1, 6)$  is smooth at  $E$ .*

*Proof.* Since  $E$  is stable,  $\operatorname{hom}(E, E) = 1$ . By Serre duality we deduce that  $\operatorname{ext}^3(E, E) = \operatorname{hom}(E, E(-1))$  must vanish, since otherwise we would clearly get  $\operatorname{hom}(E, E) > 1$ . By Riemann-Roch  $\operatorname{hom}(E, E) - \operatorname{ext}^1(E, E) + \operatorname{ext}^2(E, E) - \operatorname{ext}^3(E, E) = -1$ , so we just need to check that  $\operatorname{ext}^1(E, E) = 2$ .

By 5.11 there is a line  $\ell$  on  $X$  such that  $E$  fits into an exact sequence

$$0 \rightarrow E \rightarrow T_X^* \rightarrow \mathcal{O}_\ell \rightarrow 0.$$

We have  $\operatorname{ext}^i(T_X^*, T_X^*) = 0$  for  $i > 0$ : use the Koszul complex of  $X$  and Bott's theorem as in 3.1 (this means that the moduli space  $M_X(2; 1, 5)$ , which by Mukai's theorem reduces to one point, is smooth). Tensoring the previous short exact sequence by  $T_X$  and taking cohomology, we deduce that  $h^i(T_X \otimes E) = 0$  for  $i > 0$ . Applying the functor  $\operatorname{Hom}(\cdot, E)$  to the same short sequence, we deduce that

$$\operatorname{Ext}^1(E, E) \simeq \operatorname{Ext}^2(\mathcal{O}_\ell, E).$$

Now we apply  $\operatorname{Hom}(\mathcal{O}_\ell, \cdot)$  and obtain

$$\begin{aligned} \operatorname{Ext}^1(\mathcal{O}_\ell, T_X^*) &\rightarrow \operatorname{Ext}^1(\mathcal{O}_\ell, \mathcal{O}_\ell) \rightarrow \\ &\rightarrow \operatorname{Ext}^2(\mathcal{O}_\ell, E) \rightarrow \operatorname{Ext}^2(\mathcal{O}_\ell, T_X^*) \rightarrow \operatorname{Ext}^2(\mathcal{O}_\ell, \mathcal{O}_\ell) \end{aligned}$$

To compute  $\operatorname{Ext}^i(\mathcal{O}_\ell, T_X^*)$ , recall that it can be obtained as the abutment of the spectral sequence with order two terms

$$E_2^{j, i-j} = H^j(\mathcal{E}xt^{i-j}(\mathcal{O}_\ell, T_X^*)) = H^j(\mathcal{E}xt^{i-j}(\mathcal{O}_\ell, \omega_X) \otimes T_X^*(1)).$$

Since  $\ell$  is smooth of codimension two, the sheaf  $\mathcal{E}xt^k(\mathcal{O}_\ell, \omega_X) = 0$  for  $k < 2$ , and  $\mathcal{E}xt^2(\mathcal{O}_\ell, \omega_X) = \omega_\ell$ . We deduce that  $\operatorname{Ext}^1(\mathcal{O}_\ell, T_X^*) = 0$  and

$$\operatorname{Ext}^2(\mathcal{O}_\ell, T_X^*) = H^0(\mathcal{E}xt^2(\mathcal{O}_\ell, \omega_X) \otimes T_X^*(1)) = H^0(T_\ell^*(-1)) = \mathbb{C}.$$

Now recall that  $\Gamma(X)$ , the family of lines on  $X$  is a smooth irreducible curve, see 4.2, and that  $\operatorname{ext}^1(\mathcal{O}_\ell, \mathcal{O}_\ell) = 1$  and  $\operatorname{ext}^2(\mathcal{O}_\ell, \mathcal{O}_\ell) = 0$  for any line  $\ell$ . Putting all this in the long exact sequence above, we finally get that

$$\operatorname{ext}^1(E, E) = \operatorname{ext}^2(\mathcal{O}_\ell, E) = \operatorname{ext}^1(\mathcal{O}_\ell, \mathcal{O}_\ell) + \operatorname{ext}^2(\mathcal{O}_\ell, T_X^*) = 1 + 1 = 2,$$

which concludes the proof.  $\square$

## 6. VECTOR BUNDLES AND PROJECTIONS

Every vector bundle in  $M_X(2; 1, 6)$  can be obtained by the Serre construction from a smooth elliptic sextic in  $X$ . In this section we give an alternative construction.

**6.1. Projections associated to lines.** Let  $e_0, \dots, e_5$  be a basis of  $V = \mathbb{C}^6$ , and let  $f_0, \dots, f_5$  be the dual basis. Consider the  $\Lambda$ -line  $\ell$  in  $\mathbb{P}(\Lambda^2 V^*)$  generated by

$$\begin{aligned} f_0 \wedge f_2 + f_1 \wedge f_3, \\ f_0 \wedge f_4 + f_1 \wedge f_5. \end{aligned}$$

Its orthogonal  $\ell^\perp$  is a special codimension two subspace in  $\mathbb{P}^{14} = \mathbb{P}(\Lambda^2 V)$ .

**Lemma 6.1.** *The singular locus of the linear section  $G(2, 6) \cap \ell^\perp$  is a smooth plane conic  $q^\ell$ , parameterizing the singular points of the sections of  $G(2, 6)$  by a hyperplane in the pencil  $\ell$ . The projective span of  $q^\ell$  is a plane  $\pi^\ell$  which is not contained in  $G(2, 6)$ .*

*Proof.* This singular locus is the set of points  $x \wedge y$  on  $G(2, 6) \cap \ell^\perp$  at which the tangent space to  $G(2, 6)$  is not transverse to  $\ell^\perp$ . This means that one of the linear forms  $f_0 \wedge f + f_1 \wedge f'$  of  $\ell$  vanishes on the affine tangent space, which is  $x \wedge V + y \wedge V$ . This is possible only if the plane  $\langle x, y \rangle$  is orthogonal to the linear forms  $f_0, f_1, f, f'$ . Note that this defines uniquely the corresponding point of  $G(2, 6)$ , as the singular point of the intersection of  $G(2, 6)$  with the hyperplane orthogonal to  $f_0 \wedge f + f_1 \wedge f'$ . Explicitly, letting  $f = sf_2 + tf_4$  and  $f' = sf_3 + tf_5$ , this singular point is given by  $x = te_2 - se_4$  and  $y = te_3 - se_5$ , so that  $x \wedge y = t^2 e_2 \wedge e_3 - st(e_2 \wedge e_5 - e_3 \wedge e_4) + s^2 e_4 \wedge e_5$  describes a smooth conic, as claimed.  $\square$

Now we project  $G(2, 6) \cap \ell^\perp \subset \ell^\perp$  linearly from the projective plane  $\pi^\ell$ . The image of this projection is a six-dimensional variety  $G_\ell \subset \mathbb{P}^9$ .

**Proposition 6.2.** *The variety  $G_\ell \subset \mathbb{P}^9$  is projectively equivalent to the Grassmannian  $G(2, 5)$  in the Plücker embedding.*

*Proof.* Keeping the previous notations, the projective plane  $\pi^\ell$  is generated by  $e_2 \wedge e_3, e_2 \wedge e_5 - e_3 \wedge e_4$  and  $e_4 \wedge e_5$ . To describe the variety  $G_\ell$ , we first parameterize an open subset of  $G(2, 6)$  as follows: any plane transverse to  $\langle e_0, e_1, e_3, e_5 \rangle$  has a unique basis of the form

$$\begin{aligned} e_2 + \alpha_0 e_0 + \alpha_1 e_1 + \alpha_3 e_3 + \alpha_5 e_5, \\ e_4 + \beta_0 e_0 + \beta_1 e_1 + \beta_3 e_3 + \beta_5 e_5. \end{aligned}$$

This gives affine coordinates on  $G(2, 6)$ . Let  $\delta_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$ . The linear section  $G(2, 6) \cap \ell^\perp$  is defined by the conditions

$$\beta_0 - \delta_{13} = \alpha_0 + \delta_{15} = 0.$$

The image  $G_\ell$  of the projection of this section along  $\pi^\ell$  is, in suitable coordinates, the closure of the set of points in  $\mathbb{P}^9$  of the form

$$[1, \alpha_1, \beta_1, \alpha_3 + \beta_5, \delta_{01}, \delta_{03}, \delta_{05}, \delta_{13}, \delta_{15}, \delta_{35}].$$

We can already say that this is a unirational variety, locally parameterized by  $\alpha_1, \alpha_3, \alpha_5, \beta_1, \beta_3, \beta_5$ . Indeed,  $\alpha_0$  and  $\beta_0$  are functions of these parameters, hence also  $\delta_{01}, \delta_{03}, \delta_{05}$ .

Let us denote by  $X, Y, Z, T, U_{01}, U_{03}, U_{05}, U_{13}, U_{15}, U_{35}$  our homogeneous coordinates on  $\mathbb{P}^9$ .

**Lemma 6.3.**  *$G_\ell$  has five quadratic equations, explicitly given by*

$$(6) \quad XU_{01} + YU_{13} + ZU_{15} = 0,$$

$$(7) \quad XU_{03} + ZU_{35} + TU_{13} = 0,$$

$$(8) \quad XU_{05} + YU_{35} - TU_{15} = 0,$$

$$(9) \quad YU_{03} + ZU_{05} - TU_{01} = 0,$$

$$(10) \quad U_{01}U_{35} - U_{01}U_{15} + U_{05}U_{13} = 0.$$

*Proof of the lemma.* We first note that  $\delta_{01} = \alpha_0 \beta_1 - \beta_0 \alpha_1 = -\delta_{15} \beta_1 - \delta_{13} \alpha_1$ , which gives (1). We also have

$$\begin{aligned} \delta_{03} + \delta_{15} \beta_3 + \delta_{13} \alpha_3 &= 0, \\ \delta_{05} + \delta_{15} \beta_5 + \delta_{13} \alpha_5 &= 0. \end{aligned}$$

Combining these identities to the Plücker relations

$$\begin{aligned}\delta_{13}\beta_5 - \delta_{15}\beta_3 + \delta_{35}\beta_1 &= 0, \\ \delta_{13}\alpha_5 - \delta_{15}\alpha_3 + \delta_{35}\alpha_1 &= 0,\end{aligned}$$

we get the two equations

$$\begin{aligned}\delta_{03} + (\alpha_3 + \beta_5)\delta_{13} + \beta_1\delta_{35} &= 0, \\ \delta_{05} + (\alpha_3 + \beta_5)\delta_{15} + \alpha_1\delta_{35} &= 0,\end{aligned}$$

which are (2) and (3). Two other Plücker relations are

$$\begin{aligned}\delta_{15}\beta_0 + \delta_{01}\beta_5 - \delta_{05}\beta_1 &= 0, \\ \delta_{01}\alpha_3 - \delta_{03}\alpha_1 + \delta_{13}\alpha_0 &= 0.\end{aligned}$$

Adding them, we get (4). Finally (5) is itself a Plücker relation.  $\square$

To conclude the proof of the Proposition, we need to check that these quadrics are the Plücker equations of  $G(2, 5)$  in a slightly disguised form. This will imply that the 6-dimensional variety  $G_\ell$  is contained in, hence equal to, a copy of  $G(2, 5)$ .

We make the following substitution:

$$\begin{aligned}\delta_{01} &= \Delta_{12} & \delta_{13} &= \Delta_{14} & X &= \Delta_{45} \\ \delta_{03} &= \Delta_{13} & \delta_{15} &= \Delta_{24} & Y &= -\Delta_{25} \\ \delta_{05} &= \Delta_{23} & \delta_{35} &= \Delta_{34} & Z &= \Delta_{15} \\ & & & & T &= -\Delta_{35}.\end{aligned}$$

The five quadrics of the Lemma become

$$\begin{aligned}\Delta_{12}\Delta_{45} - \Delta_{14}\Delta_{25} + \Delta_{15}\Delta_{24} &= 0, \\ \Delta_{13}\Delta_{45} + \Delta_{15}\Delta_{34} - \Delta_{14}\Delta_{35} &= 0, \\ \Delta_{23}\Delta_{45} + \Delta_{25}\Delta_{34} - \Delta_{24}\Delta_{35} &= 0, \\ \Delta_{13}\Delta_{25} + \Delta_{15}\Delta_{23} - \Delta_{12}\Delta_{35} &= 0, \\ \Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} &= 0.\end{aligned}$$

The proof is complete.  $\square$

**6.2. Induced vector bundles.** Let as above  $X$  be a general prime Fano threefold of index one and of genus 8, and let  $Y$  be its orthogonal cubic threefold. For an  $A$ -line  $\ell \subset Y$ , consider the linear projection

$$f_\ell : G(2, 6) \cap \ell^\perp \dashrightarrow G_\ell \simeq G(2, 5).$$

It can be easily seen that  $f_\ell$  is a birational map, which is regular outside the conic  $q^\ell$  swept out by the singular points of the hyperplane sections  $H_y \subset G(2, 6)$  defined by the points  $y \in \ell$ .

The Fano threefold  $X$  in  $G(2, 6)$  can't meet the conic  $q^\ell$ , otherwise  $X$  would be singular at its intersection points with  $q^\ell$ . Therefore the restriction

$$f_{\ell, X} : X \longrightarrow G_\ell$$

of  $f_\ell$  to  $X$  is regular (and a birational map to its image). In particular, the pull-back of the tautological bundle  $T^*$  on  $G(2, 5) = G_\ell$  restricts to a rank two vector bundle  $E_\ell$  on  $X$ .

**Lemma 6.4.** *We have  $c_1(E_\ell) = h$  and  $c_2(E_\ell) = 6$ .*

*Proof.* Clearly  $c_1(E_\ell) = h$  since  $c_1(T^*)$  is the hyperplane class of  $G(2, 5)$  and  $E_\ell$  is its pull-back by a linear projection.

Let  $A, B$  denote complementary three-spaces in  $V$  such that the line  $\ell \subset \mathbb{P}(\wedge^2 A^\perp \oplus \wedge^2 B^\perp)$ . Then  $\ell^\perp$  contains the Segre variety  $\Sigma = \mathbb{P}A \times \mathbb{P}B$ , which meets the plane  $\pi^\ell$  precisely along the conic  $q^\ell$ . The image of  $\Sigma$  by the projection  $f_\ell$  is thus a four dimensional quadric  $Q \subset G_\ell$ . Such a quadric is a copy of  $G(2, 4)$  inside  $G_\ell \simeq G(2, 5)$ . In particular, it can be described as the zero locus of a general section of the tautological bundle  $T^*$  on  $G_\ell$ . The induced section of  $E_\ell$  vanishes along  $\Sigma \cap X$ , and this intersection is generically transverse; hence  $\Sigma \cap X$  represents the second Chern class of  $E_\ell$ . Since  $\Sigma$  has degree 6, we conclude that  $c_2(E_\ell) = 6$ .  $\square$

**Lemma 6.5.** *The vector bundle  $E_\ell$  is stable.*

*Proof.* We just need to check that there is no embedding  $\mathcal{O}_X(1) \hookrightarrow E_\ell$ , or equivalently that  $h^0(E_\ell(-1)) = h^0(E_\ell^*) = 0$ . But since  $E_\ell$  is globally generated, a non zero section of  $E_\ell^*$  cannot vanish, and we would deduce that  $E_\ell = \mathcal{O}_X \oplus \mathcal{O}_X(1)$ . Then the tautological bundle restricted to  $f_\ell(X)$  would have a trivial factor, which would mean that  $f_\ell(X)$  is a  $\mathbb{P}^3$  inside  $G_\ell$  – a contradiction.  $\square$

Now let  $E$  in  $M_X(2; 1, 6)^0$  be any vector bundle. A general section of  $E$  vanishes along a smooth split elliptic sextic  $C$  with two unisecant planes. These two planes define complementary three-spaces  $A, B$  in  $V$ , and  $\mathbb{P}(\wedge^2 A^\perp \oplus \wedge^2 B^\perp)$  cuts the cubic  $Y$  orthogonal to  $X$  along an A-line  $\ell$ .

But by the proof of Lemma 6.4, we know that the vector bundle  $E_\ell$  has a section whose zero locus is precisely  $C$ . Since there is a unique vector bundle in  $M_X(2; 1, 6)$  with a section vanishing along  $C$ , we get  $E_\ell \simeq E$  and we deduce:

**Proposition 6.6.** *The correspondence  $\ell \mapsto E_\ell$  defines a bijection between A-lines in the orthogonal cubic  $Y$ , and vector bundles in  $M_X(2; 1, 6)^0$ .*

**Corollary 6.7.**  *$M_X(2; 1, 6)$  is an irreducible surface.*

*Proof.* Non locally free sheaves in  $M_X(2; 1, 6)$  describe a curve contained, by 5.12, in the smooth locus of a two-dimensional component of  $M_X(2; 1, 6)$ . In particular they belong to the closure of the open subset of vector bundles, hence by 5.9 to the closure of  $M_X(2; 1, 6)^0$ . Finally, Proposition 6.6 implies that  $M_X(2; 1, 6)^0$  is irreducible, and by 5.10 it is a smooth surface.  $\square$

## 7. THE MAIN RESULT

Let again  $J(X)$  denote the intermediate Jacobian of  $X$ . As in [D], Theorem 4.8, once we fix a base point  $E_0 \in M_X(2; 1, 6)$ , the second Chern class defines a morphism

$$c_2 : M_X(2; 1, 6) \longrightarrow J(X),$$

which is uniquely defined up to a translation.

Consider a vector bundle  $E_\ell \in M_X(2; 1, 6)^0$ . A general section of  $E_\ell$  vanishes along a smooth split elliptic sextic  $C$ , and  $c_2(E) = [C]$ . Let  $\mathbb{P}A$  be one of the two unisecant planes to  $C$  – we can suppose that the line  $\ell$  can be defined as at the beginning of 6.1, where the dual basis  $e_0, \dots, e_5$  of  $f_0, \dots, f_5$  is such that  $A = \langle e_0, e_2, e_4 \rangle$ . Consider the intersection with  $X$  of the Schubert cycle  $\sigma_{20}(\mathbb{P}A)$ , the degree 9 cycle of lines meeting  $\mathbb{P}A$ .

This intersection is the union of  $C$  with a cubic curve  $D$ . A point in  $D$  but not in  $C$  is defined by a tensor  $\omega = \alpha + \beta$ , where  $\alpha \in \wedge^2 A$  and  $\beta \in A \otimes B$ . Since  $X \subset \ell^\perp$ ,  $\alpha$  must be a non zero multiple of  $e_2 \wedge e_4$ . Then for  $\omega$  to have rank two,  $\beta$  must not involve  $e_0$ . In particular,  $D$  is contained in  $G(2, H) \simeq G(2, 5)$ , if  $H$  is the hyperplane generated by  $e_1, \dots, e_5$ . More precisely, we can write  $\omega = e_2 \wedge e_4 + e_2 \wedge u_2 + e_4 \wedge u_4$ , with  $u_2, u_4 \in B$ . But then  $u_2$  and  $u_4$  must be collinear, which implies that the line  $\langle e_2, e_4 \rangle$  in  $\mathbb{P}^5$  meets the line defined by  $\omega$ . We deduce:

**Lemma 7.1.** *Let  $C$  be a projectively normal elliptic sextic in  $X$ , with a unisecant plane  $\mathbb{P}A$ . Let  $D$  be the residual cubic curve of  $C$  in the Schubert cycle  $\sigma_{20}(\mathbb{P}A) \cap X$ .*

*Then there is a line  $ax(D)$  and a hyperplane  $H$  in  $\mathbb{P}^5$ , with  $ax(D) \subset H$ , such that  $D$  is contained in  $G(2, H)$  and  $ax(D)$  is a unisecant line of  $D$ .*

Of course we have checked this only for a split elliptic sextic, but by continuity the statement holds also in the unsplit case. We call the line  $ax(D)$  the *axis* of the cubic curve  $D$ .

Now we notice that  $G(2, 5)$  has degree 5, and contains both the cubic  $D$  and the conic  $q_\ell$ . Thus the intersection  $G(2, H) \cap X = D \cup q_\ell$  and, up to some fixed translations, we have

$$c_2(E_\ell) = [C] = -[D] = [q_\ell] \in J(X).$$

**Theorem 7.2.** *The map  $c_2$  defines an isomorphism between the moduli space  $M_X(2; 1, 6)$  and a translate in  $J(X)$  of the Fano surface  $F(X)$  of conics in  $X$ .*

*Proof.* We have just seen that the map  $c_2$  sends  $M_X(2; 1, 6)^0$  to (a translate of) the open set of conics  $q_\ell$  in  $F(X)$ , which by 4.6 corresponds to the set of A-lines in  $F(Y)$ ; and by 6.6 this is a bijection. Since  $M_X(2; 1, 6)$  is irreducible by 6.7, we deduce that  $c_2$  maps  $M_X(2; 1, 6)$  to  $F(X)$  in  $J(X)$ . Moreover, it follows from 5.11 that non locally free sheaves in  $M_X(2; 1, 6)$  are mapped bijectively to the image of the curve  $\Gamma(X)$  of lines in  $X$ , which is in bijection with the curve of B-lines in  $F(Y)$  by 4.6.

Let  $M_X(2; 1, 6)^1$  denote the locally closed subset of vector bundles that do not belong to  $M_X(2; 1, 6)^0$ . The complement of  $M_X(2; 1, 6)^1$  in  $M_X(2; 1, 6)$  is mapped bijectively by  $c_2$  to the Fano surface  $F(X) \subset J(X)$ .

Suppose that  $M_X(2; 1, 6)^1$  is not empty. A fiber of  $c_2$  passing through a point of  $M_X(2; 1, 6)^1$  must contain another point, not in  $M_X(2; 1, 6)^1$ . Since the fibers are connected we deduce that  $c_2$  contracts a curve in the moduli space to a same conic  $q_\ell$ . We thus have a five-dimensional family of elliptic sextics  $C$  mapping to  $q_\ell$ , that is, with a unisecant plane  $\mathbb{P}A$  such that  $\sigma_{20}(\mathbb{P}A) = C \cup D$ ,  $D \subset G(2, H)$  for a hyperplane  $H$ , and  $G(2, H) \cap X = D \cup q_\ell$ .

But remember from the proof of Proposition 4.6 that the Palatini quartic  $W$  meets the linear space  $\mathbb{P}_\ell^3$  along the union of two quadrics  $Q_\ell$  and  $Q^\ell$ . The axis of  $D$  is contained in  $W \cap \mathbb{P}_\ell^3$ , so we have a one parameter family of such lines. The unisecant plane of  $C$  contains the axis of  $D$ , so we have three more parameters for this plane and then the curve  $C$  is determined. Hence a total of four parameters for the elliptic sextics mapping to a given conic in  $X$ , and we deduce a contradiction.

Since  $M_X(2; 1, 6)^1 = \emptyset$ , by 5.10 and 5.12 the moduli space  $M_X(2; 1, 6)$  is smooth. Moreover  $c_2$  defines a bijection between the smooth surfaces  $M_X(2; 1, 6)$  and  $F(X)$ , so it must be an isomorphism.  $\square$



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