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Laurent Manivel

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AN EXTENSION OF THE CAYLEY-SYLVESTER FORMULA

L. MANIVEL

ABSTRACT. We extend the Cayley-Sylvester formula for symmetric powers of $SL_2(\mathbb{C})$ -modules, to plethysms defined by rectangle partitions. Ordinary partitions are replaced by plane partitions, and an extension of the Hermite reciprocity law follows.

1. Let S_d denote the irreducible $SL_2(\mathbb{C})$ -module of dimension $d + 1$, which can be identified with the space of homogeneous polynomials of degree d in two variables. It is a classical result in invariant theory that the decomposition of symmetric powers of S_d into irreducible $SL_2(\mathbb{C})$ -modules can be computed in terms of partitions. Let $P(d, n; m)$ denote the number of partitions of size m inside the rectangle $d \times n$. Such a partition μ is a non-increasing sequence $\mu_1 \geq \dots \geq \mu_d$ of non-negative integers, bounded by n , and the size of μ is just the sum of these integers. The *Cayley-Sylvester formula* (see [6]) states that the multiplicity of S_e inside $S^n S_d$ is

$$[S^n S_d, S_e] = P(n, d; \frac{nd - e}{2}) - P(n, d; \frac{nd - e}{2} - 1).$$

An easy consequence is:

The Hermite reciprocity law.

For all integers n and d , $S^n S_d = S^d S_n$ as $SL_2(\mathbb{C})$ -modules.

Indeed, represent a partition μ by its *Ferrer diagram* [2]. A symmetry with respect to the diagonal gives the Ferrer diagram of a partition μ^* of the same size in the rectangle $d \times n$.

2. We will extend the Hermite reciprocity to more general plethysms involving rectangle partitions. Recall that for any partition λ , we can define the plethysm $S^\lambda S_d$ and try to decompose it into irreducible $SL_2(\mathbb{C})$ -modules. Michel Brion noticed that the proof of the Cayley-Sylvester formula can be adapted mutatis mutandis to that situation. To state the resulting formula, let $P(\lambda, d; m)$ denote the number of semistandard tableaux of shape λ and weight (j_0, \dots, j_d) , with $j_1 + 2j_2 + \dots + dj_d = m$. Then ([1], Proposition 5.2):

$$[S^\lambda S_d, S_e] = P(\lambda, d; \frac{nd - e}{2}) - P(\lambda, d; \frac{nd - e}{2} - 1).$$

Such numbers of tableaux are not very easy to compute in general, but for rectangle partitions we notice they encode more pleasant objects. Let $\lambda = r(k, n)$, the partition with k non zero parts, all equal to n , and let T be some semi-standard tableau of shape $r(n, k)$ and weight (j_0, \dots, j_d) .

Consider the first line of T : this is a non-decreasing sequence of length n , of non-negative integers not greater than $d - k + 1$ (since the last column is strictly increasing from top to bottom, and bounded by d). We may see the numbers in that sequence as the parts of some partition $\mu(1)$ contained in the rectangle $n \times (d - k + 1)$. Now consider the second line of T , which is again a non-decreasing sequence of length n , of integers now between 1 and $d - k + 1$. Subtract 1 to each of these, and consider the resulting sequence as encoding a partition $\mu(2)$, which as $\mu(1)$ is contained in the rectangle $n \times (d - k + 1)$.

We observe that the first two lines of T form a semi-standard tableau if and only if $\mu(1) \subset \mu(2)$ (which means that the i -th part of $\mu(1)$ is bounded by the i -th part of $\mu(2)$, or that the diagram of $\mu(2)$ contains that of $\mu(1)$). Indeed, to check the required property we just need to verify that for each number k in the second line of T , there is no more integers greater than or equal to k in the first line, than integers strictly greater than k in the second line. But this means that $\mu(1)$ does not have more parts greater than k than $\mu(2)$, and asking that for all k is the same as requiring that $\mu(1) \subset \mu(2)$.

We can reiterate this observation for each line of T : the i -th line is made of integers comprised between $i - 1$ and $d - k + i$. Subtracting $i - 1$ we get a partition $\mu(i)$, and the tableau T is semi-standard if and only if

$$\mu(1) \subset \mu(2) \subset \cdots \subset \mu(k) \subset r(n, d - k + 1).$$

But such a sequence of partitions can be considered as a *plane partition* Δ_T contained in a box $n \times (d - k + 1) \times k$, such that $\mu(i)$ is just the slice of Δ_T of height $k - i + 1$. We will denote by $PP(n, d - k + 1, k; m)$ the number of plane partitions of size m inside the box $n \times (d - k + 1) \times k$.

Note that the tableau T has weight (j_0, \dots, j_d) , which means that it contains each integer s exactly j_s times. In particular, the sum $j_1 + 2j_2 + \cdots + dj_d$ is just the sum of all the integers in T . But, by definition of $\mu(i)$, the sum of the integers in the i -th line of T is just $|\mu(i)| + (i - 1)n$, so that

$$j_1 + 2j_2 + \cdots + dj_d = |\mu(1)| + \cdots + |\mu(k)| + \frac{k(k-1)}{2}n.$$

Thus the tableau T is such that $j_1 + 2j_2 + \cdots + dj_d = (nkd - e)/2$ if and only if the plane partition Δ_T has size $|\mu(1)| + \cdots + |\mu(k)| = (nk(d - k + 1) - e)/2$. Changing d into $d + k - 1$ we immediately deduce from these observations the main result of this note:

Theorem. *The multiplicity of S_e inside the plethysm $S^{r(k,n)}S_{d+k-1}$ is*

$$[S^{r(k,n)}S_{d+k-1}, S_e] = PP(n, d, k; \frac{ndk - e}{2}) - PP(n, d, k; \frac{ndk - e}{2} - 1).$$

This is the exact analogue of the Cayley-Sylvester formula, when we replace ordinary partitions by plane partitions.

We deduce our extension of Hermite's reciprocity law, but note that since we now deal with three-dimensional objects the symmetry is richer:

Corollary. *The plethysm $S^{r(k,n)}S_{d+k-1}$ is completely symmetric in n, d, k , that is,*

$$\begin{array}{ccccc} S^{r(k,n)}S_{d+k-1} & = & S^{r(n,d)}S_{k+n-1} & = & S^{r(d,k)}S_{n+d-1} \\ \parallel & & \parallel & & \parallel \\ S^{r(d,n)}S_{d+k-1} & = & S^{r(k,d)}S_{k+n-1} & = & S^{r(n,k)}S_{n+d-1} \end{array}$$

We thank the referee for suggesting a more direct approach to this corollary, as follows. The character of the $GL_2(\mathbb{C})$ -module $S^{r(k,n)}S_{d+k-1}$ is given by the plethysm $s_{r(k,n)} \circ h_{d+k-1}$ of Schur functions, evaluated on two variables x_1, x_2 that we may suppose to be $x_1 = 1$ and $x_2 = q$. We get the specialization $s_{r(k,n)}(1, q, q^2, \dots, q^{d+k-1})$, which can be evaluated by applying the formula given in Proposition 1.4.10 of [3]:

$$s_{r(k,n)}(1, q, q^2, \dots, q^{d+k-1}) = q^{n \binom{k}{2}} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} \frac{1 - q^{d+k+i-j}}{1 - q^{i+j-1}}.$$

Denote the product on the right hand side by $P_{k,n}^d(q)$. Since the factor $q^{n\binom{k}{2}}$ only accounts for the determinantal representation of $GL_2(\mathbb{C})$, we are reduced to prove that $P_{k,n}^d(q)$ is completely symmetric in k, n, d . Because of the obvious symmetry with respect to k and n , we just need to check that $P_{k,n}^d(q) = P_{d,n}^k(q)$, which amounts to the identity

$$\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} (1 - q^{d+k+i-j}) \prod_{\substack{1 \leq \ell \leq n \\ 1 \leq m \leq d}} (1 - q^{\ell+m-1}) = \prod_{\substack{1 \leq \ell \leq n \\ 1 \leq m \leq d}} (1 - q^{d+k+\ell-m}) \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} (1 - q^{i+j-1}).$$

Supposing that $k \leq d$, this simplifies to

$$\prod_{\substack{1 \leq \ell \leq n \\ k+1 \leq m \leq d}} (1 - q^{\ell+m-1}) = \prod_{\substack{1 \leq \ell \leq n \\ k+1 \leq m \leq d}} (1 - q^{d+k+\ell-m})$$

which immediately follows from letting $m = d + k + 1 - m'$ in either one of these products.

Remarks.

1. Even for $k = 1$ we get more than Hermite's reciprocity, namely $S^n S_d = \Lambda^n S_{d+n-1}$. This was first observed by Murnaghan [5].

2. For $k = 2$, the identity $S^{n,n} S_{d+1} = S^{d,d} S_{n+1}$ was obtained by Matthias Meulien as a consequence of the very interesting fact that a certain algebra related to the geometry of pencils of binary forms of degree d , is a Gorenstein algebra ([4], Proposition 1.1.5). It would be nice to have a similar interpretation of our more general reciprocity.

3. The identity $S^{r(k,n)} S_{d+k-1} = S^{r(d,n)} S_{d+k-1}$ is actually obvious, since S_{d+k-1} has dimension $d + k$, and $r(d, n)$ is the complement of $r(k, n)$ inside $r(d + k, n)$. In the statement of the corollary the three obvious identities are the vertical ones.

4. Our correspondance between plane partitions and semistandard tableaux of rectangular shape gives an immediate proof of the following fact: the number of plane partitions contained in the box $n \times d \times k$ is equal to

$$PP(n, d, k) = \frac{(n + d + k - 1)!!(n - 1)!!(d - 1)!!(k - 1)!!}{(n + d - 1)!!(d + k - 1)!!(k + n - 1)!!}.$$

Indeed, we just need to apply the classical formula ([3], Corollary 1.4.11) that computes the number of semistandard tableaux of shape $r(n, k)$ numbered by integers between 1 and $d + k$.

Here $k!! := k!(k - 1)! \cdots 2!1!$ is the usual double factorial. This result, originally due to MacMahon, is a generalization of the well-known fact that the number of ordinary partitions inscribed in a rectangle $n \times d$ is given by a binomial coefficient.

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