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ABEL-JACOBI MAPS FOR HYPERSURFACES AND NON COMMUTATIVE CALABI-YAU'S

A. KUZNETSOV, L. MANIVEL, AND D. MARKUSHEVICH

Abstract. It is well known that the Fano scheme of lines on a cubic 4-fold is a symplectic variety. We generalize this fact by constructing a closed $(2n-4)$-form on the Fano scheme of lines on a $(2n-2)$-dimensional hypersurface Y_n of degree n. We provide several definitions of this form — via the Abel–Jacobi map, via Hochschild homology, and via the linkage class — and compute it explicitly for $n=4$. In the special case of a Pfaffian hypersurface Y_n we show that the Fano scheme is birational to a certain moduli space of sheaves of a $(2n-4)$-dimensional Calabi–Yau variety X arising naturally in the context of homological projective duality, and that the constructed form is induced by the holomorphic volume form on X. This remains true for a general non Pfaffian hypersurface but the dual Calabi-Yau becomes non commutative.

Introduction

Let Y be a projective or compact Kähler manifold and $p : Z \to B$ a family of k-cycles on Y, parameterized by a smooth base B. The Abel–Jacobi map of the family $p : Z \to B$ is the homomorphism

$$\mathrm{AJ} : H^\bullet(Y,\mathbb{Z}) \longrightarrow H^{\bullet-2k}(B,\mathbb{Z}) \ , \quad \mathrm{AJ}(c) = p_* q^*(c),$$

where $q : Z \to Y$ is the natural projection. The cohomology classes on Y can be represented by closed forms, and AJ can be interpreted on the level of de Rham or Dolbeault cohomology via the map of integration over the fibers $Z_b = p^{-1}(b)$ of p : for a closed (i,j)-form ω on Y, representing a class $[\omega] \in H^{i,j}(Y,\mathbb{C})$, the formula $b \mapsto \int_{Z_b} \omega$ defines the closed $(i-k, j-k)$-form $\mathrm{AJ}(\omega)$ on B.

If Y is a nonsingular hypersurface in $\mathbb{P}^n$, then the only interesting piece of the cohomology of Y is the primitive part of $H^{n-1}(Y)$. Clemens and Griffiths [CG] studied the Abel–Jacobi map when Y is a cubic threefold in $\mathbb{P}^4$ and $B = F(Y)$ is the Fano surface of Y, that is the base of the family of lines on it. They showed that AJ is an isomorphism between the Hodge structures on $H^3(Y)$ and $H^1(F(Y))$, and deduced the nonrationality of Y. Beauville and Donagi [BD] considered the case of a smooth cubic fourfold in $\mathbb{P}^5$. They proved that AJ provides an isomorphism of polarized Hodge structures between the primitive cohomologies $H^4(Y)_{prim}$ and $H^2(F(Y))_{prim}$. In particular, $H^{3,1}(Y) \simeq H^{2,0}(F(Y))$ is 1-dimensional, that is $F(Y)$ carries a holomorphic 2-form $\alpha \in H^{2,0}(F(Y))$, unique up to proportionality. Looking at a special cubic Y, whose equation is the Pfaffian of a 6-by-6 matrix of linear forms, they identified $F(Y)$ with the Hilbert square $X^{[2]}$ of the "orthogonal" K3 surface $X = Y^\perp$, which is a transversal linear section of $G(2,6)$. They deduced from this that α is nondegenerate, hence symplectic, and moreover, that for any smooth cubic Y, the Fano scheme $F(Y)$ is an irreducible symplectic fourfold, obtained by deformation of the complex structure on $X^{[2]}$.

We partly generalize the work of Beauville–Donagi to higher dimensions, in replacing the cubic Y in $\mathbb{P}^5$ by a hypersurface Y_n of degree n in $\mathbb{P}^{2n-1}$. Its index, that is the maximal integer dividing the canonical class K_{Y_n} is also n. We consider the scheme of lines on

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Y_n (denoted by $F(Y_n)$ and called the Fano scheme of Y_n) and investigate the Abel–Jacobi map $\mathrm{AJ} : H^\bullet(Y_n,\mathbb{Z}) \longrightarrow H^{\bullet-2}(F(Y_n),\mathbb{Z})$, the only nontrivial part of which is the restriction to $H^{2n-2}(Y_n,\mathbb{Z})_{prim}$. We show that

$$\mathrm{AJ} : H^{2n-2}(Y_n,\mathbb{Z})_{prim} \longrightarrow H^{2n-4}(F(Y_n),\mathbb{Z})$$

is compatible with natural polarizations on both sides. One would expect then that AJ induces an isomorphism of polarized lattices $H^{2n-2}(Y_n,\mathbb{Z})_{prim}$ and $H^{2n-4}(F(Y_n),\mathbb{Z})_{prim}$, similarly to the case $n = 3$. However, it turns out that this is not true in general. Firstly, we do not know whether the cohomology of $F(Y_n)$ is torsion-free. Secondly, it follows from the results of Shimada that the image of the Abel–Jacobi map coincides, modulo torsion, with the saturated sublattice $H^{2n-4}(F(Y_n),\mathbb{Z})_{van} \subset H^{2n-4}(F(Y_n),\mathbb{Z})_{prim}$ of vanishing cohomology, defined as the kernel of the Gysin map associated to the embedding $F(Y_n) \hookrightarrow G(2,2n)$. We verify that in the case $n = 4$ the rank of the Gysin map is 2 and hence $H^{2n-4}(F(Y_n),\mathbb{Z})_{van}$ is of corank 1 in $H^{2n-4}(F(Y_n),\mathbb{Z})_{prim}$.

The above results have the following interesting consequence. Using the Griffiths residue isomorphism one can easily see that $h^{2n-3,1}(Y_n) = 1$. So, Shimada's result implies that $h^{2n-4,0}(F(Y_n)) = 1$. The image α_ω of a generator $\omega \in H^{2n-3,1}(Y_n)$ under the Abel–Jacobi map is a nonzero holomorphic form $\alpha_\omega \in H^{2n-4,0}(F(Y_n)) = H^0(F(Y_n),\Omega^{2n-4}_{F(Y_n)})$. As we mentioned above, for $n = 3$, it is symplectic. Moreover, when Y_3 is specialized into a Pfaffian cubic, $F(Y_3)$ becomes the Hilbert scheme $X^{[2]}$ of a $K3$ surface X, and α_ω is induced by the holomorphic volume element of X.

We show that for $n \geq 4$ the situation is pretty similar. First of all, to each Pfaffian hypersurface Y_n we associate an "orthogonal" Calabi–Yau manifold X_n of dimension $2n-4$ and show that $F(Y_n)$ is birational to a certain subvariety $H(X_n)$ of the Hilbert scheme $X_n^{[C_{n-1}]}$, where C_{n-1} is a Catalan number. Further, we explain that for arbitrary (non-Pfaffian) degree n hypersurface in $\mathbb{P}^{2n-1}$ the orthogonal Calabi–Yau variety exists on a categorical (or noncommutative) level. This means that for any Y_n there exists a triangulated category $\mathcal{C}_n$, which is a $(2n-4)$-Calabi–Yau category, and for Y_n Pfaffian this category is expected to be equivalent to $\mathcal{D}^b(X_n)$. Actually, a generic Pfaffian hypersurface Y_n is singular in codimension 5 (a phenomenon which does not occur in the case of a cubic fourfold), so we have to replace Y_n by a noncommutative crepant resolution $\tilde{Y}_n$, and $\mathcal{C}_n$ is defined as the orthogonal complement to the exceptional collection $\mathcal{O}$, $\mathcal{O}(1)$, …, $\mathcal{O}(n-1)$ in the bounded derived category of coherent sheaves on $\tilde{Y}_n$, so that there is a semiorthogonal decomposition

$$\mathcal{D}^b(\tilde{Y}_n) = \langle \mathcal{C}_n, \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1)\rangle.$$

We show that the Fano scheme $F(Y_n)$ parameterizes certain objects in the triangulated category $\mathcal{C}_n$, so it can be considered as a moduli space of coherent sheaves (or of complexes) on the corresponding non-commutative Calabi–Yau variety. We show that the form α_ω constructed on $F(Y_n)$ via the Abel–Jacobi map coincides with the form naturally arising on $F(Y_n)$ considered as a moduli space of complexes on a Calabi–Yau variety.

For this we use a homological interpretation of the Abel–Jacobi map. Since we want this interpretation to have a sense in a noncommutative setup, we have to use a version of homology defined for noncommutative varieties as well as for commutative ones. So, we use the Hochschild (or cyclic) homology.

The Hochschild homology $\mathsf{HH}(Y)$ of a smooth projective variety Y was introduced by Markarian in [Ma]. The exponential of the universal Atiyah class of Y provides an isomorphism

$$\exp_Y : \mathsf{HH}(Y) \to H^\bullet(Y,\Omega^\bullet_Y) = H^\bullet(Y,\mathbb{C}).$$

On the other hand, Hochschild homology is functorial with respect to functors between derived categories. In particular, any object $\mathcal{E}$ on the product $Y \times \mathcal{M}$ of two smooth projective varieties

Y and $\mathcal{M}$ induces a kernel functor $\Phi_{\mathcal{E}} : \mathcal{D}^b(Y) \to \mathcal{D}^b(\mathcal{M})$ and a map of Hochschild homology $\phi_{\mathcal{E}} : \mathsf{HH}(Y) \to \mathsf{HH}(\mathcal{M})$. This map is compatible with the isomorphisms $\exp_Y$ and $\exp_{\mathcal{M}}$ and the modified Abel–Jacobi map:

$$\widetilde{\mathrm{AJ}}_{\mathrm{ch}(\mathcal{E})} \circ \exp_Y = \exp_{\mathcal{M}} \circ \phi_{\mathcal{E}},$$

where $\widetilde{\mathrm{AJ}}_\xi : H^\bullet(Y, \mathbb{C}) \to H^\bullet(\mathcal{M}, \mathbb{C})$ is defined for any $\xi \in H^\bullet(Y \times \mathcal{M}, \mathbb{C})$ by $\widetilde{\mathrm{AJ}}_\xi(c) = p_*(q^*(c) \wedge \xi \wedge \mathrm{td}(Y))$, $p : Y \times \mathcal{M} \to \mathcal{M}$ and $q : Y \times \mathcal{M} \to Y$ being the projections. So, the Hochschild homology of smooth projective varieties can replace their Dolbeault cohomology.

On the other hand, Hochschild homology can be defined for any triangulated category which is equivalent to a semiorthogonal component of the derived category of coherent sheaves on a smooth projective variety [Ku6]. Thus it can be defined for smooth noncommutative varieties as well. In particular, $\mathsf{HH}(\mathcal{C}_n)$ is well defined for any Y_n and in the Pfaffian case $\mathsf{HH}(\mathcal{C}_n)$ is expected to be isomorphic to $\mathsf{HH}(X_n) = H^\bullet(X_n, \mathbb{C})$. Moreover, it is shown in *loc. cit.* that Hochschild homology is *additive* with respect to semiorthogonal decompositions. So, $\mathsf{HH}(\mathcal{C}_n)$ is a direct summand in $\mathsf{HH}(Y_n)$. We check that the corresponding projection $\mathsf{HH}(Y_n) \to \mathsf{HH}(\mathcal{C}_n)$ takes the form $\omega \in H^{2n-3,1}(Y_n)$ to the holomorphic volume form of $\mathcal{C}_n$. We also check that for an appropriate object $\mathcal{E} \in \mathcal{D}^b(Y_n \times F(Y_n))$ the map $\phi_{\mathcal{E}} : \mathsf{HH}(Y_n) \to \mathsf{HH}(F(Y_n))$ takes ω to α_ω, and at the same time factors through the projection $\mathsf{HH}(Y_n) \to \mathsf{HH}(\mathcal{C}_n)$, so the form α_ω indeed comes from the holomorphic volume form on $\mathcal{C}_n$.

For Pfaffian hypersurfaces we describe the relation of the form α_ω on $F(Y_n)$ with the holomorphic volume form of the corresponding Calabi–Yau variety X_n more explicitly. We show that the Fano scheme $F(Y_n)$ is birational to a certain moduli space $H(X_n)$ of sheaves on X_n, a subvariety of a Hilbert scheme of points of X_n. We expect that under this birational correspondence the form α_ω goes to the form on $H(X_n)$ induced by the holomorphic volume form of X_n.

Although we deal mostly with the Fano scheme of lines on Y_n, our approach can be used for any moduli space $\mathcal{M}$ of sheaves on Y_n. Using the corresponding Abel–Jacobi map (or its Hochschild homology incarnation) one can define a closed $(2n-4)$-form on $\mathcal{M}$, for which we use the same notation α_ω. However, the most convenient way for an explicit computation of the form α_ω is via the divisorial linkage class.

For any sheaf $\mathcal{F}$ on Y_n the *divisorial linkage class* $\epsilon_{\mathcal{F}}$ of $\mathcal{F}$ is the extension class of the complex $Li^*i_*\mathcal{F}$ in $\mathrm{Ext}^1(L_0i^*i_*\mathcal{F}[0], L_1i^*i_*\mathcal{F}[1]) = \mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}^\vee_{Y_n/\mathbb{P}^{2n-1}})$, where i is the embedding $Y_n \to \mathbb{P}^{2n-1}$. Since the degree of Y_n equals to the index, we have $\omega_{Y_n} \cong \mathcal{N}^\vee_{Y_n/\mathbb{P}^{2n-1}}$, so we can consider $\epsilon_{\mathcal{F}}$ as an element of $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \omega_{Y_n})$. We show that the value $\alpha_\omega|_m$ at a point $m = [\mathcal{F}]$ of a moduli space $\mathcal{M}$ can be computed as the following composition

$$\underbrace{\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \times \ldots \times \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})}_{2n-4 \text{ times}} \xrightarrow{\text{Yoneda}} \mathrm{Ext}^{2n-4}(\mathcal{F}, \mathcal{F}) \xrightarrow{\epsilon_{\mathcal{F}}} \mathrm{Ext}^{2n-2}(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}^\vee_{Y_n/\mathbb{P}^{2n-1}})$$

$$\simeq \mathrm{Ext}^{2n-2}(\mathcal{F}, \mathcal{F} \otimes \omega_{Y_n}) \xrightarrow{\mathrm{Tr}} H^{2n-2}(Y_n, \omega_{Y_n}) = \mathbb{C}. \quad (1)$$

We illustrate this approach by an explicit calculation of the 4-form α_ω on the 7-dimensional Fano scheme $F(Y_4)$ of a generic 6-dimensional quartic $Y_4 \subset \mathbb{P}^7$.

We also address the question of nondegeneracy of the form α_ω constructed by the above procedure. For $n = 3$, one can prove that the form is nondegenerate by using its noncommutative Calabi–Yau interpretation (see [KuM]). For bigger n, however, the question is not so easy. First of all, it is not clear at all what the nondegeneracy means for a p-form when $p > 2$. Several approaches are possible here. We use the following one. Given a p-form α on a n-dimensional vector space V $(n \geq p)$ with even $p = 2k$, one can define the 2-rank of α as the rank of the

induced bilinear form on $\wedge^k V$ (which has the same parity as k), and α is nondegenerate if it has maximal 2-rank with k and n fixed.

For particular k, n, one can use other notions of rank. For example, if $n = 7$ and $k = 4$, there is a symmetric bilinear form q_α on V associated to α and defined up to proportionality (see Section 7). Then we can define the rank of α as the rank of q_α. These notions of rank and 2-rank classify, with a single exception, the orbits of 4-forms on V under the action of $GL(V)$. In particular there is an open orbit made of 4-forms of maximal rank and 2-rank, whose stabilizers are (up to a finite group) copies of the complex exceptional group $\mathbf{G_2}$. Moreover, there is a unique codimension one orbit, made of 4-forms that we call "minimally degenerate", characterized by the fact that their rank is equal to four. We show that the 4-form α_ω on the 7-dimensional Fano scheme $F(Y_4)$ of a generic 6-dimensional quartic $Y_4 \subset \mathbb{P}^7$ has rank 4, that is, it is minimally degenerate.

As we have already remarked, $F(Y_4)$ is (birational to) a moduli space of sheaves on a Calabi–Yau 4-fold X. It is quite an intriguing question, given a p-dimensional Calabi–Yau manifold and a moduli space M of sheaves on V, what does the p-form induced on M by the holomorphic volume element of V look like? Thomas constructed in [Th] an example of a Calabi–Yau threefold V and a 3-dimensional moduli space M, on which the induced 3-form is nondegenerate, so that M is a new Calabi–Yau threefold associated to V, a quasi-mirror of it. It would be interesting to find other moduli spaces on X, either related to Y_4 or not, and find out, to which degree their 4-forms are (non)degenerate.

In Section 1, we define the Abel–Jacobi map in the transcendental setting and describe its properties for the case of the Fano scheme of lines $F(Y_n)$ on a hypersurface Y_n. In particular, we use it to define the $(2n-4)$-form α_ω on $F(Y_n)$. For $n = 4$, that is when Y is a 6-dimensional quartic, we also determine the relevant Hodge numbers of Y and F. In Section 2 we introduce the Hochschild homology interpretation of the Abel–Jacobi map and show that the form α_ω on $F(Y_n)$ is induced by the holomorphic volume form of a noncommutative Calabi–Yau variety associated to Y_n.

In Sections 3–4 we investigate a special case of Pfaffian hypersurfaces. In Section 3, we discuss the homological projective duality between the Grassmannian of lines and the Pfaffian variety. We define the Calabi–Yau linear section $X = X_{2n-4}$ of the Grassmannian $\mathrm{Gr}(2, 2n)$ associated to the Pfaffian hypersurface $Y = Y_n \subset \mathbb{P}^{2n-1}$, and formulate a conjectural relation between their derived categories of coherent sheaves. In Section 4, it is proved that the Fano scheme $F(Y)$ is irreducible and has a natural crepant resolution for a generic Pfaffian Y. A moduli space $H(X)$ of sheaves on X is constructed, which is birational to $F(Y)$. These results base upon some general facts from geometry of lines on Pfaffian varieties defined by rank-$2k$ Pfaffians in $\mathbb{P}(\wedge^2 W^*)$ ($2k \le 2n = \dim W$), and we gather such facts in the Appendix. In particular, we construct there a natural crepant resolution of singularities of the variety of lines on the Pfaffian hypersurface ($k = n - 1$).

In Section 5, we define the linkage class $\epsilon_{\mathcal{F}}$, show that it factors through the Atiyah class and prove that formula (1) provides the value of α_{2n-4}. We also prove a nonvanishing result for $\epsilon_{\mathcal{F}}$.

In the last two Sections 6 and 7 we consider another special case, $n = 4$. In Section 6, an explicit calculation of α_4 in coordinates is done for the quartic 6-fold Y_4. In the last Section 7, we describe the classification of the orbits of $GL(7)$ in $\wedge^4 \mathbb{C}^7$ and show that α_4 is minimally degenerate at the generic point of the 7-fold $F(Y_4)$.

In what follows, the base field k is an algebraically closed field of characteristic 0; sometimes we assume that $k = \mathbb{C}$. A *variety* is a reduced irreducible (separated) scheme of finite type over k, and an *algebraic space* is assumed to be locally of finite type over k, but not necessarily separated.

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1. Abel–Jacobi map of the family of lines

Let $n \geq 3$, and let $Y = Y_n$ be a nonsingular hypersurface of degree n in $\mathbb{P}^{2n-1}$. Let $F(Y)$ be the Fano scheme of Y. One can think of it either as the Hilbert scheme of lines in Y, or as the locus of the Grassmannian $G = G(2, 2n)$ parameterizing the lines in $\mathbb{P}^{2n-1}$ that are contained in Y. Unlike the case of a cubic hypersurface treated by Clemens–Griffiths [CG], where the Fano scheme of lines is smooth whenever the hypersurface is smooth, the Fano scheme of a smooth higher degree hypersurface may be singular and even non-reduced, see [T] for the case of a quartic threefold. Thus we will assume in the sequel that Y is generic, and this assumption is essential.

Lemma 1.1. *Let Y be a generic hypersurface of degree n in $\mathbb{P}^{2n-1}$. Then $F(Y)$ is a nonsingular projective variety of dimension $3n - 5$.*

Proof. Denote by V the vector space $\mathbb{C}^{2n}$ whose projectivization contains Y. So $\mathbb{P}^{2n-1} = \mathbb{P}(V)$, and $Y = Y_f$ is defined by an homogeneous form $f \in S^d V^*$. The Grassmannian $G = G(2, 2n)$ of lines in $\mathbb{P}^{2n-1}$ is naturally embedded into $\mathbb{P}(\wedge^2 V)$ via the Plücker embedding and carries the tautological quotient bundle T such that $H^0(G, S^n T^*)$ is canonically isomorphic to $S^n V^*$ for any $n \geq 1$. According to [AK1], $F(Y)$ is the scheme of zeros of the section s_f of $S^d T^*$ corresponding to f under the above canonical isomorphism (with $n = d$). As T^* is generated by global sections, so is $S^d T^*$, and by Kleiman's generalization of Bertini's Theorem [K], for generic f, the section s_f is transversal to the zero section of $S^d T^*$. Thus the zero locus of s_f is smooth and is of expected codimension $n + 1 = \operatorname{rk} S^d T^*$ whenever it is nonempty. The fact that s_f has nonempty zero locus follows from the results of Barth–Van de Ven [BvdV] and Debarre–Manivel [DM]. □

Following [BD], we define the Abel–Jacobi map on the middle-dimensional integer cohomology of Y by the formula

$$\mathrm{AJ} : H^{2n-2}(Y, \mathbb{Z}) \longrightarrow H^{2n-4}(F, \mathbb{Z}), \quad \mathrm{AJ}(c) = p_* q^*(c), \tag{2}$$

where $F = F(Y)$, $p : Z \to F$, $q : Z \to Y$ are the natural projections from the universal family of lines Z in Y,

$$Z = \{(\ell, x) \in F \times Y \mid x \in \ell\}.$$

We have defined the Abel–Jacobi map on the integral cohomology. Going over to the complex coefficients, we can represent the cohomology classes on Y by closed forms of type (i, j), and AJ lifts to the level of closed forms as the map of integration over the fibers p. This allows us to consider it as a morphism of Hodge structures, shifting the weight by -2.

We will denote by h the class of a hyperplane section of Y, and by σ_i, σ_{ij} the Schubert classes on $G = \mathrm{Gr}(2, 2n)$, as well as their restrictions to $F \subset G$. The primitive parts of the above cohomology groups are defined by

$$H^{2n-2}(Y, \mathbb{Z})_{prim} = \{x \in H^{2n-2}(Y, \mathbb{Z}) \mid xh = 0\},$$
$$H^{2n-4}(F, \mathbb{Z})_{prim} = \{u \in H^{2n-4}(F, \mathbb{Z}) \mid u\sigma_1^n = 0\}.$$

We introduce also the vanishing part of the cohomology of F:

$$H^{2n-4}(F, \mathbb{Z})_{van} = \ker \big(H^{2n-4}(F, \mathbb{Z}) \xrightarrow{\text{Gysin}} H^{6n-4}(G, \mathbb{Z})\big),$$

where the Gysin map is associated to the natural embedding $F \hookrightarrow G$. Obviously, $H^{2n-4}(F,\mathbb{Z})_{van} \subset H^{2n-4}(F,\mathbb{Z})_{prim}$. We will see that though in the Beauville–Donagi case ($n=3$) this inclusion is an isomorphism, this does not hold for $n=4$, in which case $H^4(F,\mathbb{Z})_{van}$ is of codimension 1 in $H^4(F,\mathbb{Z})_{prim}$. One can also define the vanishing cohomology of Y as the kernel of the Gysin map associated to the embedding into $\mathbb{P}^{2n-1}$, but it will coincide with the primitive cohomology. The following result was proved by Shimada:

Theorem 1.2 (Shimada [Sh])**.** *The Abel–Jacobi map induces an isomorphism*

$$H^{2n-2}(Y,\mathbb{Z})_{prim} \simeq H^{2n-4}(F,\mathbb{Z})_{van}/(\text{torsion}).$$

Now, following the approach of [BD], we will show that AJ is a morphism of *polarized* Hodge structures, that is, it respects natural bilinear forms on the primitive cohomology.

Remark that $Z = \mathbb{P}(T_F)$, where T_F is the restriction to F of the universal rank-2 bundle T over G, and the tautological sheaf $\mathcal{O}_{Z/F}(1)$ coincides with $q^*\mathcal{O}_Y(1)$. By the theory of Chern classes, $H^\bullet(Z,\mathbb{Z}) = p^*H^\bullet(F,\mathbb{Z})[q^*h]$ with a single relation

$$(q^*h)^2 = p^*\sigma_1 q^*h - p^*\sigma_{11}.$$

In particular, we can write

$$q^*(x) = p^*x_{2n-4}q^*h - p^*x_{2n-2}$$

for any $x \in H^{2n-2}(Y,\mathbb{Z})$, where $x_i \in H^{2i}(F,\mathbb{Z})$ and $x_{2n-4} = \mathrm{AJ}(x)$.

Let us define on $H^{2n-4}(F,\mathbb{Z})$ the symmetric bilinear form ϕ by

$$\phi(u,v) = \sigma_1^{n-1}uv \quad \forall \quad u,\ v \in H^{2n-4}(F,\mathbb{Z}).$$

Using the above description of $H^\bullet(Z,\mathbb{Z})$, one can easily prove the following lemma (by the same arguments as in [BD]):

Lemma 1.3. *(i) An element $x \in H^{2n-2}(Y,\mathbb{Z})$ is primitive if and only if*

$$x_{2n-4}\sigma_{11} = 0, \quad x_{2n-2} = x_{2n-4}\sigma_1.$$

(ii) For any $x, y \in H^{2n-2}(Y,\mathbb{Z})_{prim}$,

$$xy = -\frac{1}{n!}\phi(\mathrm{AJ}(x),\mathrm{AJ}(y)).$$

The intersection form $(x,y) \mapsto xy$ on $H^{2n-2}(Y,\mathbb{Z})$ is nondegenerate, so part (ii) gives an easy proof of the injectivity of AJ on $H^{2n-2}(Y,\mathbb{Z})_{prim}$. But the fact that the image is contained in $H^{2n-4}(F,\mathbb{Z})_{van}$ (and hence in $H^{2n-4}(F,\mathbb{Z})_{prim}$) and the surjectivity modulo torsion are the nontrivial counterparts of Shimada's result.

Corollary 1.4. $\mathrm{AJ}|_{H^{2n-2}(Y,\mathbb{Z})_{prim}}$ *is a Hodge isometry onto a saturated sublattice $H^{2n-4}(F,\mathbb{Z})_{van}$ of $H^{2n-4}(F,\mathbb{Z})_{prim}$, polarized by the bilinear form $-\frac{1}{n!}\phi$.*

As the image of $H^{2n-3,1}(Y) \subset H^{2n-2}(Y,\mathbb{C})_{prim}$ under AJ is in $H^{2n-4,0}(F)$, we obtain:

Corollary 1.5. *F carries a nonzero $(2n-4)$-form $\alpha \in H^0(F,\Omega^{2n-4})$. It is defined uniquely up to proportionality as a generator of the one-dimensional vector space* $\mathrm{AJ}(H^{2n-3,1}(Y)) = H^{2n-4,0}(F)$.

Proof. The Hodge numbers of Y are given by the Griffiths residue theorem (see e. g. [G]): for a smooth degree-d hypersurface $Y_f \subset \mathbb{P}^N$, the primitive Hodge cohomology $H^{N-p,p-1}(Y_d)_{\mathrm{prim}}$ is identified with the homogeneous component of degree $pd-N-1$ of the Jacobian ring $R_f = k[x_0,\dots,x_N]/(\partial f/\partial x_0,\dots,\partial f/\partial x_N)$. Here one gets $h^{2n-3,1}(Y) = 1$, and Shimada's Theorem implies the result. □

No we will turn to the example $n = 4$. We will calculate some of the Hodge numbers of $F = F(Y_4)$. In particular, we will show that $\operatorname{rk} H^4(F,\mathbb{Z})_{prim} = \operatorname{rk} H^6(Y_4,\mathbb{Z})_{prim} + 1$. The discrepancy is due to the fact that the ambient Grassmannian $G = \mathrm{Gr}(2,8)$ has a rank two cohomology $H^4(G,\mathbb{Z})$, and its restriction to F is injective, which implies that the Gysin map $H^4(F,\mathbb{Q}) \to H^8(G,\mathbb{Q})$ is surjective.

Proposition 1.6. *Let $Y = Y_4$ be a generic quartic 6-fold in $\mathbb{P}^7$, and $F = F(Y)$ its Fano scheme. Then we have the following results for their Hodge numbers:*

(i) $h^{6,0}(Y) = 0,\ h^{5,1}(Y) = 1,\ h^{4,2}(Y) = 266,\ h^{3,3}(Y) = 1108.$

(ii) $h^{0,0}(F) = 1$ *(that is, F is connected),* $h^{4,0}(F) = 1$, $h^{7,0}(F) = 336$, *and* $h^{i,0}(F) = 0$ *for i different from* $0, 4, 7$.

(iii) $h^{1,1}(F) = 1$, $h^{1,2}(F) = 0$, $h^{1,3}(F) = 266$, $h^{2,2}(F) = 1109$.

Proof. The Hodge numbers of Y are computed via the Griffiths residue isomorphism.

To compute the Hodge numbers of F, we will use the fact that F is the zero-locus in the Grassmannian $G = \mathrm{Gr}(2,8)$ of a general section of the vector bundle S^4T^*, where T is the tautological rank two bundle on G. This bundle has rank five and is generated by global sections. We can therefore use the conormal sequence

$$0 \to S^4T_{|F} \to \Omega^1_{G|F} \to \Omega^1_F \to 0,$$

as well as the Koszul complex

$$0 \to \wedge^5(S^4T) \to \wedge^4(S^4T) \to \wedge^3(S^4T) \to \wedge^2(S^4T) \to S^4T \to \mathcal{O}_G \to \mathcal{O}_F \to 0$$

in order to compute the cohomology of the restriction to F of vector bundles on G. The wedge powers of S^4T are readily computed in terms of Schur powers (note that since T has rank two, we simply have $\Sigma^{a,b}T = S^{a-b}T \otimes \mathcal{O}(-b)$):

$$\begin{aligned}
\wedge^2(S^4T) &= \Sigma^{7,1}T \oplus \Sigma^{5,3}T \\
\wedge^3(S^4T) &= \Sigma^{9,3}T \oplus \Sigma^{7,5}T \\
\wedge^4(S^4T) &= \Sigma^{10,6}T \\
\wedge^5(S^4T) &= \Sigma^{10,10}T
\end{aligned}$$

In order to compute the cohomology of S^4T and Ω^1_G restricted to F, we will twist the Koszul resolution of $\mathcal{O}_F$ by these bundles and use Bott's theorem on G. Recall that if Q denotes the rank 6 quotient vector bundle on G, we have $\Omega^1_G \simeq Q^* \otimes T$. Bott's theorem computes the cohomology of any tensor product of Schur powers of Q and T as follows. Let α and β be two non-increasing sequences of relative integers, of respective lengths 6 and 2. Let $\rho = (8,7,6,5,4,3,2,1)$ and consider the sequence $(\alpha,\beta) + \rho$. Call it *regular* if its entries are pairwise distinct. In that case there is a unique permutation w such that $w((\alpha,\beta)+\rho)$ is strictly decreasing. Then $\lambda = w((\alpha,\beta)+\rho) - \rho$ is non-increasing. Bott's theorem asserts, if V is the ambient eight-dimensional space, that

$$H^q(G, \Sigma^\alpha Q \otimes \Sigma^\beta T) = \Sigma^\lambda V$$

if $(\alpha,\beta)+\rho$ is regular and $q = \ell(w)$, and zero otherwise. For example, if $\beta = (a,b)$, we get that $H^q(G,\Sigma^\beta T) \neq 0$ if and only if either

$$\begin{aligned}
q = 0, &\qquad 0 \geq a \geq b; \\
q = 6, &\qquad a \geq 7, 1 \geq b; \\
q = 12, &\qquad a \geq b \geq 8.
\end{aligned}$$

We can easily deduce the cohomology groups of $\mathcal{O}_F$. Indeed, $H^q(G,\wedge^i(S^4T))$ is non zero only for $(i,q) = (0,0),(2,6),(5,12)$, which implies that $H^q(F,\mathcal{O}_F)$ is non zero exactly for $q = 0,4,7$.

In particular

$$H^4(F,\mathcal{O}_F)\simeq H^6(G,\wedge^2(S^4T))\simeq\mathbb{C}.$$

Let us turn to the cohomology groups of the cotangent bundle of F. Bott's theorem implies that the only non zero groups among the $H^q(G,S^4T\otimes\wedge^i(S^4T))$ appear in bidegree $(i,q)=(1,6),(2,6),(4,12),(5,12)$. In particular $q-i$ is always bigger than three and we can deduce that $H^q(F,S^4T_{|F})=0$ for $q\leq 3$. Moreover, Bott's theorem gives

$$\begin{array}{rcl} H^6(G,S^4T\otimes S^4T) & = & End(V),\\ H^6(G,S^4T\otimes\wedge^2(S^4T)) & = & S^4V, \end{array}$$

and we deduce an exact sequence

$$0\to H^4(F,S^4T_{|F})\to S^4V\to End(V)\to H^5(F,S^4T_{|F})\to 0.$$

The middle map $S^4V\stackrel{\psi_f}{\to}End(V)$ must be dual to the map $End(V)\stackrel{\phi_f}{\to}S^4V^*$ mapping u to $u(f)$, where f denotes an equation of Y. Indeed, we can do the same computation in family, with a variable f, and use the $GL(V)$-equivariance to ensure that the map ϕ_f depends linearly on f. And there is, up to scalar, a unique equivariant map from S^4V^* to $Hom(End(V),S^4V^*)$. We can conclude that for f general, ψ_f is surjective, and that its kernel $H^4(F,S^4T_{|F})\simeq H^{4,2}(Y)^*$. Indeed, this is just a reformulation of the Griffiths isomorphism.

There remains to compute the cohomology groups of $\Omega^1_{G|F}$. Applying Bott's theorem as above, one checks that $H^q(G,\Omega^1_G\otimes\wedge^i(S^4T))$ is non zero only for $(i,q)=(0,1),(2,7),(5,12)$, which implies that $H^q(F,\Omega^1_{G|F})$ is non zero only for $q=1,5,7$. But then we can deduce from the conormal exact sequence that

$$H^{1,3}(F)\simeq H^4(F,S^4T_{|F})\simeq H^{2,4}(Y).$$

Finally we need to compute $H^{2,2}(F)$. Let us denote by K the kernel of the natural map $\Omega^2_{G|F}\to\Omega^2_F$. Bott's theorem implies that $H^q(G,\Omega^2_G\otimes\wedge^i(S^4T))$ is non zero only for $(i,q)=(0,2),(2,7),(2,8),(3,8),(4,12),(5,12)$. In particular the only non zero group $H^q(F,\Omega^2_{G|F})$ for $q\leq 4$ is

$$H^2(F,\Omega^2_{G|F})\simeq H^2(G,\Omega^2_G)=\mathbb{C}^2,$$

and we get an exact sequence

$$H^2(F,K)\to H^{2,2}(G)\to H^{2,2}(F)\to H^3(F,K)\to 0.$$

In order to compute the cohomology groups of K, consider the exact sequence

$$0\to S^2(S^4T)_{|F}\to S^4T\otimes\Omega^1_{G|F}\to K\to 0.$$

Applying Bott's theorem once again, we check that $H^q(G,S^4T\otimes\Omega^1_G\otimes\wedge^i(S^4T))$ is non zero only for $(i,q)=(1,6),(1,7),(2,7),(3,12),(4,12)$, which implies that $H^q(F,S^4T_{|F}\otimes\Omega^1_{G|F})=0$ for $q\leq 4$. Therefore $H^q(F,K)\simeq H^{q+1}(F,S^2(S^4T)_{|F})$ for $q\leq 3$. To compute the latter, we consider the groups $H^q(G,S^2(S^4T)\otimes\wedge^i(S^4T))$ and check by Bott's theorem that they are non zero exactly for $(i,q)=(0,6),(1,6),(2,6)$ or $q=12$. We have

$$\begin{array}{rcl} H^6(G,S^2(S^4T)) & = & \Sigma^{21111110}V\simeq sl(V),\\ H^6(G,S^2(S^4T)\otimes S^4T) & = & \Sigma^{61111110}V\oplus\Sigma^{51111111}V\simeq S^5V\otimes V^*,\\ H^6(G,S^2(S^4T)\otimes\wedge^2(S^4T)) & = & \Sigma^{91111111}V\simeq S^8V. \end{array}$$

Hence the exact sequence

$$0\to H^3(F,K)\to S^8V\stackrel{\alpha_f}{\to}S^5V\otimes V^*.$$

As above we can argue that the map α_f must be dual to the natural map β_f obtained as the composition

$$S^5V\otimes V^*\to S^5V\otimes S^3V\to S^8V$$

deduced from the map $V^* \to S^3V$ given by the differential of f. Otherwise said, we are just multiplying quintic polynomials with the derivatives of f, so that Griffiths' residue theorem tells us precisely that the cokernel of β_f is isomorphic to $H^{3,3}(Y)_{prim}$. Since this cokernel is dual to the kernel of α_f, we finally get

$$H^{2,2}(F) \simeq H^{2,2}(G) \oplus H^3(F,K) \simeq H^{2,2}(G) \oplus H^{3,3}(Y)_{prim}.$$

□

Corollary 1.7. *For generic $Y = Y_4$, F is a smooth connected 7-dimensional projective variety, and $H^0(F, \Omega^4_F)$ is 1-dimensional, generated by the 4-form α_ω, the Abel–Jacobi image of a generator ω of $H^{5,1}(Y) \simeq \mathbb{C}$.*

In the sequel we will investigate this 4-form in more detail. In particular, we believe that F is a moduli space of sheaves on some (categorical deformation of a) Calabi–Yau 4-fold X, and α_ω has a different intepretation as a 4-form induced by the holomorphic volume element of X. We can produce X only for special quartics Y, namely, the Pfaffian ones.

2. Abel–Jacobi map and Hochschild homology

The natural context for the Abel–Jacobi map is the Hochschild (or cyclic) homology. We will first recall its definition.

Definition 2.1 ([Ma])**.** Let X be a smooth projective variety. The Hochschild homology of X is defined as

$$\mathsf{HH}(X) = \mathbb{H}^\bullet(X \times X, \Delta_*\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \Delta_*\mathcal{O}_X),$$

where $\Delta : X \to X \times X$ is the diagonal embedding, and $\mathbb{H}^\bullet$ is the hypercohomology.

The Hochschild homology of X is closely related to its Hodge diamond. To state this relation, we need the notion of Atiyah class [Ill].

Definition 2.2. Let X be an algebraic variety, $\Delta : X \to X \times X$ the diagonal embedding, $\mathcal{I}_\Delta$ the ideal sheaf of the diagonal $\Delta(X) \subset X \times X$, and $\Delta(X)^{(2)} \subset X \times X$ the second infinitesimal neighborhood of the diagonal, that is the closed subscheme of $X \times X$ defined by the ideal sheaf $\mathcal{I}^2_\Delta$. Then $\mathcal{I}_\Delta/\mathcal{I}^2_\Delta \cong \mathcal{N}^\vee_{\Delta(X)/Y\times X} \cong \Omega^1_X$, and there is a natural exact sequence

$$0 \to \Delta_*\Omega^1_X \to \mathcal{O}_{\Delta(X)^{(2)}} \to \Delta_*\mathcal{O}_X \to 0. \tag{3}$$

The extension class

$$\widetilde{\mathrm{At}_X} \in \mathrm{Ext}^1(\Delta_*\mathcal{O}_X, \Delta_*\Omega^1_X).$$

of this exact triple is called the universal Atiyah class on X. Further, let $\mathcal{F}$ be a sheaf on X or an object of $\mathcal{D}^b(X)$, and

$$0 \to \mathcal{F} \otimes \Omega^1_X \to \mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{O}_{\Delta(X)^{(2)}}) \to \mathcal{F} \to 0 \tag{4}$$

the exact triple obtained by applying $\mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathcal{F} \otimes \bullet)$ to (3). The extension class $\mathrm{At}_\mathcal{F} \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega^1_X)$ of (4) is called the Atiyah class of $\mathcal{F}$, and the object in the middle of (4) is the sheaf (or the complex) of first jets of $\mathcal{F}$.

Theorem 2.3 ([Ma])**.** *There is an isomorphism $I_X : \mathsf{HH}_i(X) \cong \oplus_{k-l=i} H^k(X, \Omega^l_X)$.*

The isomorphism I_X is given by the exponent of the universal Atiyah class. Explicitly, the projection $\mathsf{HH}(X) \to H^\bullet(X, \Omega^l_X)$ is given by the composition

$$\Delta_*\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \Delta_*\mathcal{O}_X \xrightarrow{\frac{1}{l!}\widetilde{\mathrm{At}_X}^{\wedge l}} \Delta_*\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \Delta_*\Omega^l_X[l] \longrightarrow \Delta_*\Omega^l_X[l].$$

Another important point about Hochschild homology is its functoriality. We denote by $\mathcal{D}^b(X)$ the derived category of bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology [Ve]. For any kernel functor $\Phi_K : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$, $K \in \mathcal{D}^b(X \times Y)$, one can define an operator $\phi_K : \mathsf{HH}(X) \to \mathsf{HH}(Y)$ such that the composition of functors gives the composition of operators. In particular, for a morphism $f : X \to Y$ there are pullback and pushforward maps $f^* : \mathsf{HH}(Y) \to \mathsf{HH}(X)$ and $f_* : \mathsf{HH}(X) \to \mathsf{HH}(Y)$ respectively, and for any object $\mathcal{E} \in \mathcal{D}^b(X)$ there is a map $\phi_{\Delta_*\mathcal{E}} : \mathsf{HH}(X) \to \mathsf{HH}(X)$ induced by the tensor product with $\mathcal{E}$ functor.

The following result is contained essentially in the paper of Markarian [Ma]. See also [Ra] and [MS] for a more detailed exposition.

Theorem 2.4. (*i*) *Pullbacks commute with the isomorphisms I:*

$$f^* \circ I_Y = I_X \circ f^*.$$

(*ii*) *Pushforwards commute with the isomorphisms I up to the multiplication by the Todd genus:*

$$f_*(I_X(-) \wedge \mathrm{td}_X / f^*\mathrm{td}_Y) = I_Y(f_*(-)),$$

where td_X *is the Todd genus of* X *and* td_Y *is the Todd genus of* Y.
(*iii*) *The map induced by the tensor product with* $\mathcal{E}$ *turns into multiplication with* $\mathrm{ch}(\mathcal{E})$ *after application of the isomorphism I:*

$$I_X(\phi_{\Delta_*\mathcal{E}}(-)) = I_X(-) \wedge \mathrm{ch}(\mathcal{E}).$$

Now assume that $\mathcal{M}$ is a fine moduli space of stable sheaves on X. Assume that $\mathcal{M}$ is smooth and projective. Let $\mathcal{E}$ be the universal sheaf on $\mathcal{M} \times X$. Considering $\mathcal{E}$ as a kernel, we obtain the kernel functor $\Phi_{\mathcal{E}} : \mathcal{D}^b(X) \to \mathcal{D}^b(\mathcal{M})$ and the corresponding map on the Hochschild homology $\phi_{\mathcal{E}} : \mathsf{HH}(X) \to \mathsf{HH}(\mathcal{M})$.

Proposition 2.5. *Take any* $\omega \in H^{n-k-l}(X, \Omega_X^{n-l}) \subset \mathsf{HH}_{-k}(X)$, *where* $n = \dim X$, *and consider the corresponding* k*-form* $\alpha_\omega \in H^0(\mathcal{M}, \Omega_X^k) \subset \mathsf{HH}_{-k}(\mathcal{M})$. *Then* $\alpha_\omega = \phi_{\mathcal{E}}(\omega)$, *up to constant.*

Proof. Compare the definition of α_ω with the definition of $\phi_{\mathcal{E}}$. By construction, α_ω is determined via a Künneth component of the l-th Newton class of $\mathcal{E}$:

$$\alpha_\omega \otimes [X] = \gamma^{(k,0),(n,n)} \left(\mathrm{Tr} \left(\frac{1}{(k+l)!} \mathrm{At}_{\mathcal{E}}^{k+l} \right) \wedge \mathrm{pr}_2^* \omega \right) \in H^0(\mathcal{M}, \Omega_X^k) \otimes H^n(X, \Omega_X^n)$$

where $[X] \in H^n(X, \Omega_X^n)$ is the fundamental class of X,

$$\mathrm{Tr} : \mathrm{Ext}^{k+l}(\mathcal{E}, \mathcal{E} \otimes \Omega^{k+l}) \to H^{k+l}(\mathcal{M} \times X, \Omega^{k+l})$$

is the trace map, and

$$\gamma^{(k,l),(s,t)} : H^{k+l}(\mathcal{M} \times X, \Omega^{k+l}) \to H^l(\mathcal{M}, \Omega_X^k) \otimes H^t(X, \Omega_X^s)$$

is the projection to a Künneth component. □

Now assume that the moduli space $\mathcal{M}$ parameterizes subschemes of X, that is $\mathcal{M}$ is a component of the Hilbert scheme of X. The example of the Fano scheme we started with is of this kind, for $F(Y)$ can be interpreted as the moduli space of sheaves parameterizing the structure (or ideal) sheaves of lines contained in Y. In this case the universal sheaf $\mathcal{E}$ is just the structure sheaf $\mathcal{O}_Z$ of the universal subscheme $Z \subset \mathcal{M} \times X$, and the corresponding kernel functor $\Phi_{\mathcal{O}_Z}$ boils down to the composition p_*q^*, where $p : Z \to \mathcal{M}$ and $q : Z \to X$ are the projections. So, it is natural to expect a relation to the Abel–Jacobi map. Theorem 2.4 implies that, if we want the map $\phi_{\mathcal{O}_Z}$ to be compatible with the Atiyah–Jacobi map under the isomorphism I of Hochschild and Dolbeault homologies, we need either to modify the isomorphism I by multiplying it with the Todd genus, or to modify the Abel–Jacobi map. We choose the second way.

For each $\xi \in H^\bullet(X \times \mathcal{M}, \mathbb{C})$ we define the modified Abel–Jacobi map by the formula

$$\widetilde{\mathrm{AJ}}_\xi : H^\bullet(X, \mathbb{C}) \longrightarrow H^\bullet(\mathcal{M}, \mathbb{C}), \quad \widetilde{\mathrm{AJ}}(c) = p_*(q^*(c) \cdot \xi \cdot \mathrm{td}_X). \tag{5}$$

Proposition 2.6 (cf. [MS]). *For any $K \in \mathcal{D}^b(X \times \mathcal{M})$ there is a commutative diagram*

$$\begin{array}{ccc} \mathsf{HH}(X) & \xrightarrow{\phi_K} & \mathsf{HH}(\mathcal{M}) \\ \downarrow \cong & & \downarrow \cong \\ \oplus H^p(X, \Omega_X^q) & & \oplus H^p(\mathcal{M}, \Omega_{\mathcal{M}}^q) \\ \downarrow \cong & & \downarrow \cong \\ \oplus H^k(X, \mathbb{C}) & \xrightarrow{\widetilde{\mathrm{AJ}}_{\mathrm{ch}(K)}} & \oplus H^k(\mathcal{M}, \mathbb{C}) \end{array}$$

where the bottom arrow is the modified Abel–Jacobi map.

Proof. Follows from Theorem 2.4 and the multiplicativity of the Todd genus: $\mathrm{td}(X \times \mathcal{M}) = \mathrm{td}(X) \cdot \mathrm{td}(\mathcal{M})$. □

It is well known (see e.g. [Or], Theorem 2.1.8) that $\mathsf{HH}(X) \cong \mathsf{HH}(Y)$ if the derived categories of X and Y are equivalent. Thus Hochschild homology is an invariant of the derived category and it is natural to generalize the definition to any triangulated category. Actually, we will need this in a particular case.

Let $\mathcal{A} \subset \mathcal{D}^b(X)$ be an admissible subcategory (i.e. a component of a semiorthogonal decomposition of $\mathcal{D}^b(X)$). It is proved in [Ku5] that there exists an object $K(\mathcal{A}) \in \mathcal{D}^b(X \times X)$ such that the corresponding kernel functor $\Phi_{K(\mathcal{A})} : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ is the projection onto the subcategory $\mathcal{A}$.

Definition 2.7 ([Ku6]). Let $\mathcal{A} \subset \mathcal{D}^b(X)$ be an admissible subcategory. The Hochschild homology of $\mathcal{A}$ is defined as

$$\mathsf{HH}(\mathcal{A}) = \mathbb{H}^\bullet(X \times X, K(\mathcal{A}) \overset{\mathbb{L}}{\otimes} K(\mathcal{A})^*),$$

where $K(\mathcal{A})^*$ is the kernel of the left adjoint functor to the projection onto the subcategory $\mathcal{A}$.

A very important property of Hochschild homology is the following additivity Theorem.

Theorem 2.8 ([Ku6]). *If $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition, then*

$$\mathsf{HH}(X) = \mathsf{HH}(\mathcal{A}_1) \oplus \mathsf{HH}(\mathcal{A}_2) \oplus \cdots \oplus \mathsf{HH}(\mathcal{A}_n).$$

Recall that a triangulated category $\mathcal{T}$ is called an m-Calabi–Yau category, if the shift by m functor $[m]$ is a Serre functor in $\mathcal{T}$, i.e. if we are given bifunctorial isomorphisms

$$\mathrm{Hom}(F, G) \cong \mathrm{Hom}(G, F[m])^*$$

for all $F, G \in \mathcal{T}$.

Remark 2.9. One can also define the Hochschild cohomology $\mathsf{HH}^\bullet(\mathcal{T})$ of an admissible subcategory $\mathcal{T} \subset \mathcal{D}^b(X)$. The Hochschild cohomology is naturally a ring, while the Hochschild homology is a module over it.

A triangulated category $\mathcal{T}$ is called *connected* if $\mathsf{HH}^{<0}(\mathcal{T}) = 0$ and $\mathsf{HH}^0(\mathcal{T}) = \mathbb{C}$. One has $\mathsf{HH}^0(\mathcal{D}^b(X)) = H^0(X, \mathcal{O}_X)$, so the derived category of a smooth projective variety is connected if and only if the variety itself is connected.

Lemma 2.10 ([Ku6]). *Let $\mathcal{T}$ be a connected m-Calabi–Yau category, equivalent to an admissible subcategory in the derived category of a smooth projective variety, then $\mathsf{HH}_i(\mathcal{T}) = 0$ unless $-m \le i \le m$. Moreover $\dim \mathsf{HH}_{-m}(\mathcal{T}) = 1$.*

Let $\mathcal{T}$ be a connected m-Calabi–Yau category and let $\alpha = \alpha_{\mathcal{T}}$ be the generator of the space $\mathsf{HH}_{-m}(\mathcal{T})$. By the above Lemma $\alpha_{\mathcal{T}}$ is defined uniquely up to a constant. We will call $\alpha_{\mathcal{T}}$ the holomorphic volume form of $\mathcal{T}$.

Remark 2.11. If $\mathcal{T}$ is an m-Calabi–Yau category then the Hochschild homology is a free module over the Hochschild cohomology ring, and $\alpha_{\mathcal{T}}$ is a generator.

Now we are coming back to our original setting: $Y = Y_n$ is a general nonsingular hypersurface of degree n in $\mathbb{P}^{2n-1}$, $F = F(Y)$ is its Fano scheme, $q : Z \to Y$ is the universal family of lines in Y, $p : Z \to F$ the natural projection.

Theorem 2.12 ([Ku1]). *The line bundles $\mathcal{O}_Y, \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(n-1)$ form an exceptional collection in $\mathcal{D}^b(Y)$, so that we have a semiorthogonal decomposition*

$$\mathcal{D}^b(Y_n) = \langle \mathcal{C}_n, \mathcal{O}_Y, \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(n-1)\rangle,$$

where $\mathcal{C}_n = \{F \in \mathcal{D}^b(Y) \mid \mathbb{H}^\bullet(Y, F) = \mathbb{H}^\bullet(Y, F(-1)) = \dots = \mathbb{H}^\bullet(Y, F(1-n)) = 0\}$. Moreover, $\mathcal{C}_n$ is a connected $(2n-4)$-Calabi–Yau category.

Corollary 2.13. *We have*

$$\mathsf{HH}_i(\mathcal{C}_n) = \begin{cases} H^{n-1+t,n-1-t}(Y, \mathbb{C}), & \text{if } i = 2t \neq 0, \\ H^{n-1,n-1}(Y, \mathbb{C}) \oplus \mathbb{C}^{n-2}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The category generated by an exceptional bundle is equivalent to the derived category of vector spaces, that is, to the derived category of coherent sheaves on a point. Thus, $\mathsf{HH}_\bullet(\langle \mathcal{O}_Y(k)\rangle) = \mathbb{C}$ (sitting in degree 0) for any k. Combining with Theorems 2.8 and 2.12 we conclude that

$$\mathsf{HH}_i(\mathcal{D}^b(Y)) = \begin{cases} \mathsf{HH}_i(\mathcal{C}_n) \oplus \mathbb{C}^n, & \text{if } i = 0 \\ \mathsf{HH}_i(\mathcal{C}_n), & \text{otherwise.} \end{cases}$$

On the other hand, by Theorem 2.3 we have

$$\mathsf{HH}_i(\mathcal{D}^b(Y)) = \begin{cases} H^{n-1+t,n-1-t}(Y, \mathbb{C}), & \text{if } i = 2t \neq 0, \\ H^{n-1,n-1}(Y, \mathbb{C}) \oplus \mathbb{C}^{2n-2}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

So, the Corollary follows. □

Our final goal in this section is to show that the form α_ω on $F(Y)$ constructed above is induced by the form $\alpha_{\mathcal{C}_n} \in \mathsf{HH}_{4-2n}(\mathcal{C}_n)$.

First of all choose a line $\ell = \mathbb{P}(T) \subset \mathbb{P}(V)$ on $Y \subset \mathbb{P}(V)$. Then ℓ is the zero locus of a regular section of a vector bundle $V/T \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$ on $\mathbb{P}(V)$, hence on $\mathbb{P}(V)$ we have the following Koszul resolution

$$\left\{\Lambda^{2n-2}T^\perp \otimes \mathcal{O}_{\mathbb{P}(V)}(2-2n) \to \dots \to \Lambda^2 T^\perp \otimes \mathcal{O}_{\mathbb{P}(V)}(-2) \to T^\perp \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)}\right\} \cong \mathcal{O}_\ell.$$

Restricting the resolution to Y, twisting by $\mathcal{O}_Y(k-2)$ and truncating in term $\Lambda^{k-2}T^\perp \otimes \mathcal{O}_Y$ we obtain a complex

$$\left\{\Lambda^{k-2}T^\perp \otimes \mathcal{O}_Y \to \dots \to T^\perp \otimes \mathcal{O}_Y(k-3) \to \mathcal{O}_Y(k-2) \to \mathcal{O}_\ell(k-2)\right\} \tag{6}$$

which we consider as an object of $\mathcal{D}^b(Y)$ and denote by $\mathcal{G}^k_\ell$.

Remark 2.14. Note that the restriction of the Koszul complex to Y has two cohomology sheaves, $\mathcal{O}_\ell$ at the rightmost term and $L_1 i^* i_* \mathcal{O}_\ell \cong \mathcal{O}_\ell(-n)$ at the term next to it. Therefore, $\mathcal{G}_\ell^k$ is indeed a complex with two cohomology sheaves, a torsion free sheaf at the leftmost term and $\mathcal{O}_\ell(k-n-2)$ at the third from the right term.

Lemma 2.15. *We have $\mathcal{G}_\ell^k \in \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(k-1)\rangle^\perp$ for $k \le n$. In particular, $\mathcal{G}_\ell^n \in \mathcal{C}_n$.*

Proof. First of all, note that $\mathcal{G}_\ell^1 = \mathcal{O}_\ell(-1)$ is right orthogonal to $\mathcal{O}_Y$, since we have $\operatorname{Ext}^\bullet(\mathcal{O}_Y(1), \mathcal{O}_\ell) = H^\bullet(Y, \mathcal{O}_\ell(-1)) = 0$. On the other hand, we have an exact triangle

$$\mathcal{G}_\ell^k \to \Lambda^{k-2} T^\perp \otimes \mathcal{O}_Y \to \mathcal{G}_\ell^{k-1}(1)$$

It follows by induction that $\mathcal{G}_\ell^k \in \langle \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(k-1)\rangle^\perp$, so it remains to check that $\operatorname{Ext}^\bullet(\mathcal{O}_Y, \mathcal{G}_\ell^k) = H^\bullet(Y, \mathcal{G}_\ell^k) = 0$. For this we apply the functor $H^\bullet(Y, -)$ to the complex (6). We get a complex

$$\Lambda^{k-2} T^\perp \to \Lambda^{k-3} T^\perp \otimes V^* \to \dots \to \Lambda^2 T^\perp \otimes S^{k-4} V^* \to T^\perp \otimes S^{k-3} V^* \to S^{k-2} V^* \to S^{k-2} T^*$$

which is well known to be exact. So, $H^\bullet(Y, \mathcal{G}_\ell^k) = 0$ and we are done. □

Denote by $\mathbb{L}$ the endofunctor of the category $\mathcal{D}^b(Y)$ defined as the composition of a twist by $\mathcal{O}_Y(1)$ and a left mutation through $\mathcal{O}_Y$. In other words, for any $F \in \mathcal{D}^b(Y)$ there is an exact triangle

$$H^\bullet(Y, F) \otimes \mathcal{O}_Y \to F(1) \to \mathbb{L}(F).$$

Then $\mathcal{G}_\ell^k = \mathbb{L}^{k-1}(\mathcal{O}_\ell(-1))$. Moreover, one can check (see [Ku1]) that $\mathbb{L}$ induces an autoequivalence of the category $\mathcal{C}_n$ such that $\mathbb{L}^n \cong [2-n]$ on $\mathcal{C}_n$. On the other hand, repeating the arguments of Proposition 5.4 of [KuM] one can check that $\mathbb{L}$ induces a chain of isomorphisms

$$\operatorname{Ext}^1(\mathcal{O}_\ell, \mathcal{O}_\ell) = \operatorname{Ext}^1(\mathcal{O}_\ell(-1), \mathcal{O}_\ell(-1)) \to \operatorname{Ext}^1(\mathcal{G}_\ell^2, \mathcal{G}_\ell^2) \to \dots \to \operatorname{Ext}^1(\mathcal{G}_\ell^n, \mathcal{G}_\ell^n).$$

It follows that the form defined on $F(Y)$ by the structure sheaf $\mathcal{O}_Z$ of the universal line coincides (up to a sign) with the form defined by the family $\{\mathcal{G}_\ell^n\}_{\ell \in F(Y)}$, that is given by the kernel functor $\Phi_{\mathcal{G}^n}$, where $\mathcal{G}^n = (\mathrm{id}_{\mathcal{D}^b(Y)} \times \mathbb{L})^{n-1}(\mathcal{O}_Z(-1,0)) \in D^b(Y \times F(Y))$ (alternatively, one can define $\mathcal{G}^n$ by a universal version of the complex (6)). On the other hand, since $\mathcal{G}_\ell^n \in \mathcal{C}_n$ for any ℓ, it follows that the map $\phi_{\mathcal{G}_n} : \mathsf{HH}(Y) \to \mathsf{HH}(F(Y))$ factors through the projection $\mathsf{HH}(Y) \to \mathsf{HH}(\mathcal{C}_n)$.

Corollary 2.16. *The form $\alpha_\omega \in H^0(F(Y), \Omega^{2n-4}_{F(Y)})$ is induced by the holomorphic volume form $\alpha_{\mathcal{C}_n}$ of the Calabi–Yau category $\mathcal{C}_n$.*

3. Pfaffian hypersurfaces and their duals

Now we consider a very special type of degree n hypersurfaces $Y_n \subset \mathbb{P}^{2n-1}$. A hypersurface Y_n is called *Pfaffian* if its equation can be written as the Pfaffian of a $2n$-by-$2n$ matrix of linear forms. Such hypersurfaces are not generic, in particular they are singular for $n \ge 4$. However, they deserve special consideration because an analogue of the Calabi–Yau category $\mathcal{C}_n$ for these hypersurfaces has a geometric interpretation. Actually, it is equivalent to the derived category of a certain Calabi–Yau linear section of the Grassmannian $\mathrm{Gr}(2, 2n)$. The relation between Pfaffian hypersurfaces and linear sections of the Grassmannian comes naturally in the context of homological projective duality [Ku2].

Homological projective duality (HP-duality for short) is a certain duality on the set of smooth (non-commutative) varieties equipped with a map into a projective space and a compatible semiorthogonal decomposition of its derived category (called Lefschetz decomposition). It associates to a smooth variety X with a map $f : X \to \mathbb{P}^N$ and a Lefschetz decomposition $\mathcal{A}_\bullet$, a smooth variety Y with a map into the dual projective space $g : Y \to \check{\mathbb{P}}^N$ and the dual Lefschetz decomposition $\mathcal{B}_\bullet$. Classical projective duality can be considered as a quasiclassical

limit of HP-duality since in the case where the map $f : X \to \mathbb{P}^N$ is a closed embedding, the classically projectively dual variety $X^\vee \subset \check{\mathbb{P}}^N$ coincides with the set of critical values of the map $g : Y \to \check{\mathbb{P}}^N$ from the HP-dual variety Y. But the most important property of HP-duality is a strong relation between the derived categories of linear sections of the HP-dual varieties X and Y. Choose a projective subspace $\mathbb{P}(V) \subset \check{\mathbb{P}}^N$ and let $\mathbb{P}(V^\perp) \subset \mathbb{P}^N$ be its orthogonal complement. Let $X_V = X \times_{\mathbb{P}^N} \mathbb{P}(V^\perp)$ and $Y_V = Y \times_{\check{\mathbb{P}}^N} \mathbb{P}(V)$ be the corresponding linear sections. Then the derived categories $\mathcal{D}^b(X_V)$ and $\mathcal{D}^b(Y_V)$ have semiorthogonal decompositions, one part of each coming from the Lefschetz decomposition $\mathcal{A}_\bullet$ (or $\mathcal{B}_\bullet$) of the ambient variety (this part is called *trivial*), and the other parts (*nontrivial*) being equivalent. In some cases one of the trivial parts is zero, so the nontrivial part coincides with the whole derived category and one obtains a fully faithful embedding of one of categories $\mathcal{D}^b(X_V)$ or $\mathcal{D}^b(Y_V)$ into the other, with the orthogonal complement being trivial (in the above sense).

Though on the categorical level one can always describe the HP-dual variety Y for any X, it is a very difficult problem to find a geometrical description for it. There are not so many examples for which the answer is known. However, Pfaffian varieties are among them.

Let W be a vector space of dimension $2n$. Consider the space $\mathbb{P}(\Lambda^2 W^*)$ of skew-forms on W. Let $\mathrm{Pf}(W^*) \subset \mathbb{P}(\Lambda^2 W^*)$ denote the hypersurface of degenerate skew-forms. It is called *the Pfaffian hypersurface*. More generally, for each $1 \le k \le n-1$ let $\mathrm{Pf}_k(W^*)$ denote the subvariety of $\mathbb{P}(\Lambda^2 W^*)$ consisting of skew-forms of rank less or equal to $2n-2k$. These varieties are called the *generalized Pfaffian varieties*. They form a chain

$$\mathrm{Gr}(2, W^*) = \mathrm{Pf}_{n-1}(W^*) \subset \mathrm{Pf}_{n-2}(W^*) \subset \cdots \subset \mathrm{Pf}_2(W^*) \subset \mathrm{Pf}_1(W^*) = \mathrm{Pf}(W^*).$$

It is easy to see that for $k < n-1$ the Pfaffian variety $\mathrm{Pf}_k(W^*)$ is singular, its singular locus being the next Pfaffian variety $\mathrm{Pf}_{k+1}(W^*)$. Moreover, it is easy to see that $\mathrm{Pf}_k(W^*)$ is the closure of a $\mathrm{GL}(W)$-orbit on $\mathbb{P}(\Lambda^2 W^*)$, and that $\mathrm{Pf}_k(W^*)$ is the $(n-2-k)$-th secant variety of $\mathrm{Gr}(2, W^*)$.

The sheaf of ideals of $\mathrm{Pf}_k(W^*)$ is the image of the map

$$\Lambda^{2(k-1)} W \otimes \mathcal{O}_{\mathbb{P}(\Lambda^2 W^*)}(-(n-k+1)) \xrightarrow{\sigma_k} \mathcal{O}_{\mathbb{P}(\Lambda^2 W^*)} \tag{7}$$

given by Pfaffians of principal $2(n-k+1) \times 2(n-k+1)$-minors of a skew-form. Alternatively, σ_k can be described as the unique $\mathrm{GL}(W)$-semiinvariant element in the space

$$H^0(\mathbb{P}(\Lambda^2 W^*), \Lambda^{2(k-1)} W^* \otimes \mathcal{O}_{\mathbb{P}(\Lambda^2 W^*)}(n-k+1)) = \Lambda^{2(k-1)} W^* \otimes S^{n-k+1}(\Lambda^2 W^*)$$

Another interesting fact is that the class of Pfaffian varieties is self dual with respect to projective duality. More precisely, it is easy to see that $(\mathrm{Pf}_k(W^*))^\vee = \mathrm{Pf}_{n-1-k}(W)$. This classical statement has the following extension.

Conjecture 3.1 ([Ku4]). *The Pfaffian varieties* $\mathrm{Pf}_k(W^*)$ *admit categorical resolutions of singularities* $\widetilde{\mathrm{Pf}}_k(W^*)$ *such that* $\widetilde{\mathrm{Pf}}_k(W^*)$ *is Homologically Projectively Dual to* $\widetilde{\mathrm{Pf}}_{n-1-k}(W)$. *In particular, the Grassmannian of lines* $\mathrm{Gr}(2, W)$ *is Homologically Projectively Dual to a certain categorical resolution of singularities* $\widetilde{\mathrm{Pf}}(W^*)$ *of the Pfaffian hypersurface* $\mathrm{Pf}(W^*)$.

This conjecture was proved in [Ku4] for $n = 3$.

Now consider the derived categories of linear sections of $X = \mathrm{Gr}(2, W)$ and of $\tilde{Y} = \widetilde{\mathrm{Pf}}(W^*)$ corresponding to a generic linear subspace $V \subset \Lambda^2 W^*$ of dimension $\dim V = 2n$. Then X_V is a linear section of the Grassmannian X and it is easy to see that its canonical class is zero. On the other hand, $\tilde{Y}_V$ is a categorical resolution of a degree n hypersurface $Y_V \subset \mathbb{P}(V) = \mathbb{P}^{2n-1}$. If $\mathbb{P}(V)$ does not intersect the singular locus $\mathrm{Sing}\,\mathrm{Pf}(W^*)$ (which is possible, by dimension reasons, only for $n \le 3$), then $\tilde{Y}_V = Y_V$, and in other cases the resolution $\tilde{Y}_V$ of Y_V is nontrivial. Both

X_V and $\tilde{Y}_V$ being linear sections of HP-dual varieties come with semiorthogonal decompositions of their derived categories. The one for $\mathcal{D}^b(X_V)$ turns out to be very simple: in this case there is no trivial part, so the nontrivial part coincides with the whole category $\mathcal{D}^b(X_V)$. As for $\mathcal{D}^b(Y_V)$, the trivial part is present here and is given by the exceptional collection $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1)$. So, in using the properties of HP-duality one can deduce from Conjecture 3.1 the following

Conjecture 3.2. *Let $V \subset \Lambda^2 W^*$ be a vector subspace of dimension $\dim V = 2n = \dim W$. Let $V^\perp \subset \Lambda^2 W$ be the orthogonal. Then the derived category of coherent sheaves on a categorical resolution of singularities $\tilde{Y}_V$ of the Pfaffian hypersurface $Y_V = \mathbb{P}(V) \cap \mathrm{Pf}(W^*) \subset \mathbb{P}(V)$ has a semiorthogonal decomposition with nontrivial component equivalent to the derived category of the dual linear section of the Grassmanian $X_V = \mathbb{P}(V^\perp) \cap \mathrm{Gr}(2, W)$. More precisely,*

$$\mathcal{D}^b(\widetilde{Y_V}) = \langle \mathcal{D}^b(X_V), \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1) \rangle.$$

See [Ku4] in case $n = 3$.

If the above Conjectures are true then for a Pfaffian hypersurface Y_n the Calabi–Yau category $\mathcal{C}_n = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1) \rangle^\perp \subset \mathcal{D}^b(\tilde{Y}_n)$ is equivalent to $\mathcal{D}^b(X_n)$. Thus, for any smooth Y_n the category $\mathcal{C}_n$ can be considered as a *noncommutative deformation* of the derived category of the Calabi–Yau variety X_n. In case $n = 3$ there are examples of other special cubics Y_3 for which $\mathcal{C}_3$ is equivalent to the derived category of a (commutative) K3-surface. It is a fascinating problem to find other special Y_n's (preferably smooth ones), for which $\mathcal{C}_n$ becomes the derived category of a (commutative) Calabi–Yau manifold.

The semiorthogonal decomposition of Conjecture 3.2 suggests that any moduli space of sheaves on Y_V can be represented as a moduli space on X_V. In the next two sections, we will show that this is the case for the Fano scheme of lines on Y_V.

4. Fano scheme of a Pfaffian variety

Let us fix the notation for this section. We let $W = \mathbb{C}^{2n}$, $V = \mathbb{C}^{2n}$, consider a linear embedding $V \to \Lambda^2 W^*$, and denote by $X = X_V = \mathrm{Gr}(2, W) \cap \mathbb{P}(V^\perp)$ and $Y = Y_V = \mathbb{P}(V) \cap \mathrm{Pf}(W^*)$ the corresponding linear sections.

The goal of this section is to show that the Fano scheme of lines on $Y = Y_V$ can be interpreted as a certain moduli space on $X = X_V$. We start by considering the Fano scheme $F(\mathrm{Pf}(W^*))$ of lines on $\mathrm{Pf}(W^*)$. As it is proved in Appendix A the variety

$$\tilde{F}_1 = \{(U, L) \in \mathrm{Gr}(n+1, W) \times \mathrm{Gr}(2, \Lambda^2 W^*) \mid L \subset \mathrm{Ker}(\Lambda^2 W^* \to \Lambda^2 U^*)\}.$$

is a resolution of $F(\mathrm{Pf}(W^*))$.

Let $F(Y)$ be the Fano scheme of lines on Y and put $\tilde{F}(Y) = F(Y) \times_{F(\mathrm{Pf}(W^*))} \tilde{F}_1$, which we call the resolved Fano scheme of lines on Y. Let $F_0(Y) \subset F(Y)$ be the open subscheme consisting of lines which do not intersect $Y \cap \mathrm{Pf}_2(W^*) \subset Y$. Then the projection $\pi : \tilde{F}_1 \to F(\mathrm{Pf}(W^*))$ restricts to a projection $\pi : \tilde{F}(Y) \to F(Y)$ which is an isomorphism over $F_0(Y)$ by the proof of Proposition A.6.

Lemma 4.1. *Let $n \geq 3$. For generic $V \subset \Lambda^2 W^*$ the resolved Fano scheme $\tilde{F}(Y)$ of lines on $Y = \mathbb{P}(V) \cap \mathrm{Pf}(W^*)$ is smooth and connected and $F_0(Y)$ is nonempty. In particular, $F(Y)$ is irreducible, $F_0(Y)$ is dense in $F(Y)$ and $\pi : \tilde{F}(Y) \to F(Y)$ is birational.*

Proof. To check the smoothness of $F(Y)$ we consider the universal Pfaffian variety and its resolved Fano scheme of lines. In other words, we consider $\mathrm{Gr}(2n, \Lambda^2 W^*)$ and

$$\mathcal{Y} = \mathbb{P}_{\mathrm{Gr}(2n, \Lambda^2 W^*)}(\mathcal{V}) \times_{\mathbb{P}(\Lambda^2 W^*)} \mathrm{Pf}(W), \qquad \tilde{\mathcal{F}}(\mathcal{Y}) = \mathrm{Gr}_{\mathrm{Gr}(2n, \Lambda^2 W^*)}(2, \mathcal{V}) \times_{\mathrm{Gr}(2, \Lambda^2 W^*)} \tilde{F}(\mathrm{Pf}(W)),$$

where we denote the tautological bundle on $\mathrm{Gr}(2n, \Lambda^2 W^*)$ by $\mathcal{V}$. We have canonical projections $\mathcal{Y} \to \mathrm{Gr}(2n, \Lambda^2 W^*)$ and $\tilde{\mathcal{F}}(\mathcal{Y}) \to \mathrm{Gr}(2n, \Lambda^2 W^*)$ and it is clear that their fibers over $V \in \mathrm{Gr}(2n, \Lambda^2 W^*)$ are the corresponding Pfaffian variety and its resolved Fano scheme of lines.

Note that $\tilde{\mathcal{F}}(\mathcal{Y})$ is smooth. Indeed, considering the projection $\tilde{\mathcal{F}}(\mathcal{Y}) \to \tilde{F}(\mathrm{Pf}(W^*))$ we see that its fiber over a point $(U, L) \in \tilde{F}(\mathrm{Pf}(W^*))$ is just the set of all $V \in \mathrm{Gr}(2n, \Lambda^2 W^*)$ such that $L \subset V \subset \Lambda^2 W^*$. In other words, $\tilde{\mathcal{F}}(\mathcal{Y}) = \mathrm{Gr}_{\tilde{F}(\mathrm{Pf}(W^*))}(2n-2, \Lambda^2 W^*/L)$.

The smoothness of $\tilde{F}(Y)$ and nonemptyness of $F_0(Y)$ for general Y follow because the general fiber of a morphism of smooth varieties is smooth and has a nontrivial intersection with a given open subset.

Now let us verify the connectedness of $\tilde{F}(Y)$. Since the fibers of the projection $\tilde{F}(Y) \to F(Y)$ are connected, it suffices to check that $F(Y)$ is connected. And for this it suffices to check that $H^0(F(Y), \mathcal{O}_{\tilde{F}(Y)}) = 0$. But $F(Y) \subset \mathrm{Gr}(2, V)$ is the zero locus of a regular section of the vector bundle $S^n L^*$, so we have the following Koszul resolution

$$\cdots \to \Lambda^2(S^n L) \to S^n L \to \mathcal{O} \to \mathcal{O}_{F(Y)} \to 0.$$

Looking at the hypercohomology spectral sequence it is easy to note that it suffices to check that $H^q(\mathrm{Gr}(2, V), \Lambda^t(S^n L)) = 0$ for $q \le t$ and $t > 0$. But the Bott–Borel–Weil theorem implies that $H^q(\mathrm{Gr}(2, V), \Lambda^t(S^n L))$ can be nontrivial only for $q = \dim V - 2 = 2n-2$ and $q = 2(\dim V - 2) = 4n-4$. On the other hand $r(S^n L) = n+1$, so $t \le n+1$. Since for $n \ge 3$ we have $2n-2 \ge n+1$ we see that $H^q(\mathrm{Gr}(2, V), \Lambda^t(S^n L)) = 0$ for $q \le t$ and $t > 0$ unless $n = 3$ and $q = t = 4$. In the latter case $H^4(\mathrm{Gr}(2,6), \Lambda^4(S^3 L)) = H^4(\mathrm{Gr}(2,6), \Sigma^{6,6} L) = H^4(\mathrm{Gr}(2,6), \mathcal{O}(-6)) = 0$ (in this case the nontrivial cohomology is $H^8 = \mathbb{C} \neq 0$), so we conclude that $H^0(F(Y), \mathcal{O}_{\tilde{F}(Y)}) = 0$ and that $\tilde{F}(Y)$ is connected.

Since $\tilde{F}(Y)$ is smooth and connected we conclude that $\tilde{F}(Y)$ is irreducible. Therefore the Fano scheme $F(Y) = \pi(\tilde{F}(Y))$ is irreducible as well. And since $F_0(Y) \neq \varnothing$ the map $\pi : \tilde{F}(Y) \to F(Y)$ is birational. □

Denote by ρ the projection $\tilde{F}(Y) \to \mathrm{Gr}(n+1, W)$. Our further goal is to identify the image of $\tilde{F}(Y)$ in $\mathrm{Gr}(n+1, W)$. Consider the map $\varphi : V \otimes \mathcal{O}_{\mathrm{Gr}(n+1,W)} \to \Lambda^2 W^* \otimes \mathcal{O}_{\mathrm{Gr}(n+1,W)} \to \Lambda^2 U^*$ on the Grassmannian $\mathrm{Gr}(n+1, W)$. Denote

$$\begin{aligned} G(Y) &= \{U \in \mathrm{Gr}(n+1, W) \mid \mathrm{rank}(\varphi_U) \le 2n-2\}, \\ G_0(Y) &= \{U \in \mathrm{Gr}(n+1, W) \mid \mathrm{rank}(\varphi_U) = 2n-2\}. \end{aligned}$$

Lemma 4.2. *We have $\rho(F_0(Y)) = G(Y)$. Moreover, the projection $\rho : \tilde{F}(Y) \to G(Y)$ is an isomorphism over $G_0(Y)$.*

Proof. For any point $(U, L) \in \tilde{F}(Y)$ we have $\varphi_{g(L)}(L) = 0$ by definition of $\tilde{F}$. Hence at any such point the rank of φ is less than or equal to $\dim V/L = 2n-2$. On the other hand, assume that $\mathrm{rank}\, \varphi \le 2n-2$ at a point $U \in \mathrm{Gr}(n+1, W)$. Then there is a two-dimensional subspace $L \subset V$ such that $\varphi_U(L) = 0$, which means that $(U, L) \in \tilde{F}(Y)$. □

We have the following diagram

$$\begin{array}{ccc} & \tilde{F}(Y) & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \rho} \\ F_0(Y) \subset F(Y) & & G(Y) \supset G_0(Y) \end{array}$$

where both maps π and ρ are isomorphisms over open subsets $F_0(Y)$ and $G_0(Y)$ respectively.

Lemma 4.3. *For generic Y the set $G_0(Y)$ is nonempty. In particular, for generic Y the projection $\rho : \tilde{F}(Y) \to G(Y)$ is birational.*

Proof. Consider the universal versions of $G(Y)$ and $G_0(Y)$:

$$\begin{aligned} \mathcal{G}(\mathcal{Y}) &= \{(V,U) \in \mathrm{Gr}(2n, \Lambda^2 W^*) \times \mathrm{Gr}(n+1, W) \mid \mathrm{rank}(\varphi : V \to \Lambda^2 U^*) \le 2n-2\}, \\ \mathcal{G}_0(\mathcal{Y}) &= \{(V,U) \in \mathrm{Gr}(2n, \Lambda^2 W^*) \times \mathrm{Gr}(n+1, W) \mid \mathrm{rank}(\varphi : V \to \Lambda^2 U^*) = 2n-2\}. \end{aligned}$$

It is easy to see that $\mathcal{G}(Y)$ is irreducible and $\mathcal{G}_0(Y)$ is open in $\mathcal{G}(Y)$. Hence the general fiber of $\mathcal{G}_0(Y)$ over $\mathrm{Gr}(2n, \Lambda^2 W^*)$, which is nothing but $G_0(Y)$, is nonempty. □

Our further goal is to identify $G(Y)$ for $Y = Y_V$ in terms of the corresponding Calabi–Yau linear section $X = X_V$ of the Grassmannian $\mathrm{Gr}(2, W)$. Let

$$d_n = \deg \mathrm{Gr}(2, n+1), \qquad c_n = (n^2 - 3n + 4)/2 = (n-1)(n-2)/2 + 1$$

Actually, d_n is the Catalan number C_{n-1}, but we do not need it.

Consider the Hilbert scheme $\mathrm{Hilb}_{d_n}(X)$ of Artinian subschemes of X of length d_n and its subvariety

$$H(X) = \left\{ Z \in \mathrm{Hilb}_{d_n}(X) \;\middle|\; \begin{array}{l} \mathrm{rank}(W^* \to H^0(Z, \mathcal{S}^*_{|Z})) \le n+1, \\ \mathrm{rank}(\Lambda^2 W^* \to H^0(Z, \Lambda^2 \mathcal{S}^*_{|Z})) \le c_n \end{array} \right\}.$$

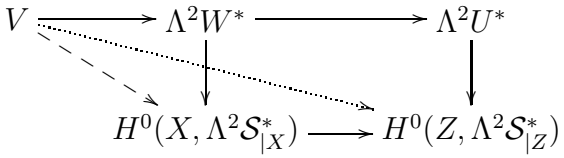
where $\mathcal{S}$ is the tautological bundle on $G(2, W)$. If for a point $Z \in H(X)$ the rank of the map $W^* \to H^0(Z, \mathcal{S}^*_{|Z})$ equals $n+1$, we can associate to Z the kernel of the map $W^* \to H^0(Z, \mathcal{S}^*_{|Z})$. Thus we obtain a rational map $H(X) \to \mathrm{Gr}(n-1, W^*) = \mathrm{Gr}(n+1, W)$. More precisely, let

$$\tilde{H}(X) = \left\{ (U, Z) \in \mathrm{Gr}(n+1, W) \times \mathrm{Hilb}_{d_n}(X) \;\middle|\; \begin{array}{l} Z \in \mathrm{Hilb}_{d_n}(X \cap \mathrm{Gr}(2, U)), \\ \mathrm{rank}(\Lambda^2 U^* \to H^0(Z, \Lambda^2 \mathcal{S}^*_{|Z})) \le c_n \end{array} \right\}.$$

We have projections $\tilde{H}(X) \to \mathrm{Gr}(n+1, W)$ and $\tilde{H}(X) \to H(X)$.

Lemma 4.4. *The image of $\tilde{H}(X)$ in $\mathrm{Gr}(n+1, W)$ coincides with $G(Y)$.*

Proof. Let $(U, Z) \in \tilde{H}(X)$. Since $Z \subset X \cap \mathrm{Gr}(2, U)$ we have commutative diagram



Note that the dashed map vanishes by definition of X. Therefore, the dotted arrow is also zero. In other words, the composition

$$V \to \Lambda^2 U^* \to H^0(Z, \Lambda^2 \mathcal{S}^*_{|Z})$$

is zero. On the other hand, the rank of the second map here is not greater than c_n by definition of $\tilde{H}(X)$. Therefore the rank of the map $V \to \Lambda^2 U^*$ is not greater than $\dim \Lambda^2 U^* - c_n = n(n+1)/2 - (n^2 - 3n + 4)/2 = 2n - 2$, so we see that $U \in G(Y)$ by definition of $G(Y)$.

Vice versa, assume that $U \in G(Y)$. Then $X \cap \mathrm{Gr}(2, U) \subset \mathrm{Gr}(2, W)$ is the linear section of $\mathrm{Gr}(2, U)$ by the image of $V \subset \Lambda^2 W^*$ in $\Lambda^2 U^*$. By definition of $G(Y)$ the dimension of this image is not greater than $2n - 2$, so $X \cap \mathrm{Gr}(2, U)$ is a linear section of the Grassmannian $\mathrm{Gr}(2, U) = \mathrm{Gr}(2, n+1)$ of codimension $\le 2n - 2$. But since $\dim \mathrm{Gr}(2, n+1) = 2n - 2$ such an intersection contains not less than $\deg \mathrm{Gr}(2, n+1) = d_n$ points, so we conclude that there exists a subscheme $Z \subset X \cap \mathrm{Gr}(2, U)$ of length d_n. Then $(U, Z) \in \tilde{H}(X)$. □

Denote the maps from $\tilde{H}(X)$ to $H(X)$ and $G(Y)$ by π' and ρ' respectively. Consider the open subsets

$$G'_0(Y) = \{U \in \mathrm{Gr}(n+1, W) \mid X \cap \mathrm{Gr}(2, U) \text{ is zero-dimensional}\},$$
$$H_0(X) = \{Z \in H(X) \mid \mathrm{rank}(W^* \to H^0(Z, \mathcal{S}^*_{|Z})) = n+1\}.$$

We have a diagram

$$\begin{array}{ccc} & \tilde{H}(X) & \\ {\scriptstyle \rho'}\swarrow & & \searrow{\scriptstyle \pi'} \\ G'_0(Y) \subset G(Y) & & H(X) \supset H_0(X) \end{array}$$

Also denote

$$\tilde{H}'_0(X) = \rho'^{-1}(G'_0(Y)) = \left\{ (U, Z) \;\middle|\; \begin{array}{l} Z = X \cap \mathrm{Gr}(2, U) \in \mathrm{Hilb}_{d_n}, \\ \mathrm{rank}(\Lambda^2 U^* \to H^0(Z, \Lambda^2 \mathcal{S}^*_{|Z})) \le c_n \end{array} \right\},$$
$$\tilde{H}_0(X) = \pi'^{-1}(H_0(X)) = \left\{ Z \in \mathrm{Hilb}_{d_n}(X) \;\middle|\; \begin{array}{l} \mathrm{rank}(W^* \to H^0(Z, \mathcal{S}^*_{|Z})) = n+1, \\ \mathrm{rank}(\Lambda^2 W^* \to H^0(Z, \Lambda^2 \mathcal{S}^*_{|Z})) \le c_n \end{array} \right\}.$$

Lemma 4.5. *If $n \ge 3$ then for a generic X the set $\tilde{H}'_0(X) \cap \tilde{H}_0(X)$ is nonempty. In particular, the sets $\tilde{H}(X)$ and $H(X)$ have irreducible components $\tilde{H}_1(X)$ and $H_1(X)$ such that restriction of the maps ρ' and π' to $\tilde{H}_1(X)$ are birational transformations $\tilde{H}_1(X) \to G(Y)$ and $\tilde{H}_1(X) \to H_1(X)$.*

Proof. Once again, consider the universal versions of $\tilde{H}(X)$, $\tilde{H}_0(X)$ and $\tilde{H}'_0(X)$:

$$\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}'_0 \subset \tilde{\mathcal{H}} = \{(V, U, Z) \in \mathrm{Gr}(2n, \Lambda^2 W^*) \times \mathrm{Gr}(n+1, W) \times \mathrm{Hilb}_{d_n}(\mathrm{Gr}(2, W)) \mid$$
$$\mid Z \subset \mathbb{P}(V^\perp) \cap \mathrm{Gr}(2, U) \subset \mathrm{Gr}(2, W) \text{ and } \mathrm{rank}(\Lambda^2 U^* \to H^0(Z, \Lambda^2 \mathcal{S}^*_{|Z})) \le c_n\}.$$

It is easy to see that $\tilde{\mathcal{H}}_0 \cap \tilde{\mathcal{H}}'_0 \subset \tilde{\mathcal{H}}$ is a nonempty open subset and its projection to $\mathrm{Gr}(2n, \Lambda^2 W^*)$ is dominant. It follows that the generic fiber of $\tilde{\mathcal{H}}_0 \cap \tilde{\mathcal{H}}'_0$ over $\mathrm{Gr}(2n, \Lambda^2 W^*)$ is nonempty, so $\tilde{H}_0(X) \cap \tilde{H}'_0(X)$ is nonempty for generic X. Define $\tilde{H}_1(X)$ to be the closure of $\tilde{H}_0(X) \cap \tilde{H}'_0(X)$ in $\tilde{H}(X)$ and $H_1(X) = \pi'(\tilde{H}_1(X))$. Then the claim becomes obvious. □

Corollary 4.6. *For a generic $V \subset \Lambda^2 W^*$, the varieties $F(Y)$, $G(Y)$ and $H(X)$ are birational.*

Since $H(X)$ is a moduli space on a Calabi–Yau manifold X of dimension $2n-4$, it has a natural $(2n-4)$-form.

Conjecture 4.7. *The birational isomorphism of $F(Y)$ and $H(X)$ is compatible with $(2n-4)$-forms.*

We conclude this section by describing the above construction in cases $n = 2$ and $n = 3$ more explicitly.

If $n = 2$ then $\mathrm{Pf}(\mathbb{C}^4) = \mathrm{Gr}(2, 4)$, so $Y = \mathbb{P}^3 \cap \mathrm{Gr}(2, 4)$ is a quadric in $\mathbb{P}^3$, and $X = \mathrm{Gr}(2, 4) \cap \mathbb{P}^1$ is a pair of points. In this case $d_n = 1$, $c_n = 1$, so $H(X) = X$ is a pair of points. However, the map π' in this case is not birational, it is a $\mathbb{P}^1$-fibration in fact, so $\tilde{H}(X)$ is a union of two $\mathbb{P}^1$, as well as $G(Y)$ and $F(Y)$. So, in case $n = 2$ our construction shows that the variety of lines on a 2-dimensional quadric is $\mathbb{P}^1 \sqcup \mathbb{P}^1$.

If $n = 3$ then $Y = \mathbb{P}^5 \cap \mathrm{Pf}(\mathbb{C}^6)$ is a Pfaffian cubic fourfold, and $X = \mathrm{Gr}(2, 6) \cap \mathbb{P}^8$ is a K3-surface. In this case $d = 2$, $c_n = 2$ and it is easy to see that all the maps π, ρ, ρ', π' described above are biregular isomorphisms. Moreover, the conditions $\mathrm{rank}(W^* \to H^0(Z, \mathcal{S}^*_{|Z})) \le 4$, $\mathrm{rank}(\Lambda^2 W^* \to H^0(Z, \Lambda^2 \mathcal{S}^*_{|Z})) \le 2$ defining $H(X) \subset \mathrm{Hilb}_2(X)$ are void since Z is a length 2 subscheme in X, so $H(X) = \mathrm{Hilb}_2(X)$ and our construction gives the classical isomorphism between the Fano scheme $F(Y)$ of lines of Y and $\mathrm{Hilb}_2(X)$.

5. Exterior forms via the linkage class

We will start by defining the divisorial linkage class (see [KuM], Section 3 for a more general notion of k-th linkage classes associated to any closed embedding $i : Y \hookrightarrow W$). In what follows, Y is a hypersurface (that is, an effective Cartier divisor) in a nonsingular algebraic variety W. We reserve the calligraphic letters $\mathcal{F}$, $\mathcal{G}$,... for coherent $\mathcal{O}_Y$-modules or objects of $\mathcal{D}^b(Y)$ and the block letters F, G, ... for coherent $\mathcal{O}_W$-modules or objects of $\mathcal{D}^b(W)$.

The restriction (or pullback) functor $i^* : \mathrm{Coh}(W) \to \mathrm{Coh}(Y)$ has a left derived functor $Li^* : \mathcal{D}^b(W) \to \mathcal{D}^b(Y)$. It can be described as follows: let $\mathcal{R}(F)$ denote a locally free resolution of any $F \in \mathcal{D}^b(W)$. Then $Li^*(F)$ is quasi-isomorphic to $i^*\mathcal{R}(F)$. The cohomology $h^k Li^*(F)$ of the complex $Li^*(F)$ is denoted by $L_{-k}i^*(F)$ or $L^k i^*(F)$. Assume that F is a sheaf, that is a complex concentrated at grade 0. Then the cohomology of $Li^*(F)$ can only appear at $k \leq 0$. More exactly, we can write $Li^*(F) = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_W} f^{-1}\mathcal{R}(F)$ and compute $L_k i^*(F)$ using the symmetry of the tensor product on its arguments together with the fact that $\mathcal{O}_Y$, as an $\mathcal{O}_W$-module, has a locally free resolution of the form $\mathcal{R}(i_*\mathcal{O}_Y) = [\mathcal{O}_W(-Y) \to \mathcal{O}_W]$. We obtain that $L_0 i^*(F) = i^*(F)$, $L_1 i^*(F) = i^*(F) \otimes \mathcal{O}_W(-Y)|_Y = i^*(F) \otimes \mathcal{N}^\vee_{Y/W}$, where $\mathcal{N}^\vee_{Y/W}$ denotes the conormal sheaf of Y in W, and all the other cohomologies are zero.

If we consider $\mathcal{R}(i_*\mathcal{O}_Y) \otimes_{f^{-1}\mathcal{O}_W} f^{-1}F$ and $\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_W} f^{-1}\mathcal{R}(F)$ as complexes of $\mathcal{O}_W$-modules, then they are quasi-isomorphic to each other and to the direct sum of their cohomologies. But they are different as complexes of $\mathcal{O}_Y$-modules: the first one is quasi-isomorphic to the direct sum of its cohomologies, the second may be a nontrivial extension. More exactly, let $\tau^\bullet Li^*(F)$ be the canonical filtration on $Li^*(F)$. Then $\tau^{-1}Li^*(F)$ and $\mathrm{gr}^0_\tau Li^*(F)$ are complexes concentrated at only one grade, so they are quasi-isomorphic to their cohomologies:

$$\tau^{-1}Li^*(F) \underset{\mathrm{qis}}{\simeq} L_1 i^*(F)[1], \qquad \mathrm{gr}^0_\tau Li^*(F) \underset{\mathrm{qis}}{\simeq} L_0 i^*(F).$$

Thus the exact triple

$$0 \to \tau^{-1}Li^*(F) \to \tau^0 Li^*(F) \to \mathrm{gr}^0_\tau Li^*(F) \to 0 \tag{8}$$

defines the extension class

$$\epsilon_F \in \mathrm{Ext}^1(L_0 i^*(F), L_1 i^*(F)[1]) = \mathrm{Ext}^2(L_0 i^*(F), L_1 i^*(F)) = \mathrm{Ext}^2(i^*F, i^*F \otimes \mathcal{N}^\vee_{Y/W}). \tag{9}$$

Definition 5.1. The extension class (9) of the exact triple (8) is called the divisorial linkage class of F with respect to the embedding $i : Y \hookrightarrow W$. To make explicit its dependence on the embedding, we will also denote it by $\epsilon_F^{Y/W}$. By abuse of notation, the divisorial linkage class of $i_*\mathcal{F}$ for an $\mathcal{O}_Y$-module $\mathcal{F}$ will be denoted $\epsilon_\mathcal{F}$ or $\epsilon_\mathcal{F}^{Y/W}$.

Remark 5.2. In practice, to compute ϵ_F, one can use a partial resolution $0 \to G \to E \to F \to 0$ with E locally free and G torsion free instead of the full resolution $\mathcal{R}(F)$. Tensoring by $\mathcal{O}_Y$, we get the 4-term exact sequence

$$0 \to i^*\,\mathcal{T}or_1(i_*\mathcal{O}_Y, F) \to i^*G \to i^*E \to i^*F \to 0,$$

whose extension class in $\mathrm{Ext}^2(i^*F, i^*\,\mathcal{T}or_1(i_*\mathcal{O}_Y, F)) = \mathrm{Ext}^2(i^*F, i^*F \otimes \mathcal{N}^\vee_{Y/W})$ is ϵ_F.

Remark 5.3. In [KuM], a more abstract definition of $\epsilon_\mathcal{F}$ is given, valid for an object $\mathcal{F}$ of $\mathcal{D}^b(Y)$. It is a morphism in the derived category included into the following distinguished triangle: $Li^*i_*\mathcal{F} \longrightarrow \mathcal{F} \xrightarrow{\epsilon_\mathcal{F}} \mathcal{F} \otimes \mathcal{N}^\vee_{Y/W}[2]$.

The following example seems to be the easiest one for which the linkage class is nonzero.

Example 5.4. Take for i the Segre embedding $Y = \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow W = \mathbb{P}^3$ with image given by the equation $xw - yz = 0$, and set $\mathcal{F} = \mathcal{O}_Y(1, 0)$. Then $F = i_*\mathcal{F}$ has a resolution $[\mathcal{O}_W(-1)^{\oplus 2} \to$

$\mathcal{O}_W^{\oplus 2}]$ with the map given by the matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$. It follows that $Li^*F \underset{\text{qis}}{\simeq} [\mathcal{O}_Y(-1,-1)^{\oplus 2} \to \mathcal{O}_Y^{\oplus 2}]$ with the map given by the same matrix, $L_1 i^*(F) = \mathcal{O}_Y(-1,-2)$, $L_0 i^*(F) = \mathcal{F} = \mathcal{O}_Y(1,0)$, and, by Remark 5.2, $\epsilon_F \in H^2(Y, \mathcal{O}_Y(-2,-2))$ is the extension class of the exact quadruple

$$0 \to \mathcal{O}_Y(-1,-2) \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} \mathcal{O}_Y(-1,-1)^{\oplus 2} \xrightarrow{\begin{pmatrix} x & y \\ z & w \end{pmatrix}} \mathcal{O}_Y^{\oplus 2} \xrightarrow{\begin{pmatrix} w & -y \end{pmatrix}} \mathcal{O}_Y(1,0) \to 0$$

We have $\epsilon_F = 0 \iff Li^*F \underset{\text{qis}}{\simeq} \mathcal{O}_Y(1,0) \oplus \mathcal{O}_Y(-1,-2)[1] \implies H^0(Y, \mathcal{O}_Y(-1,0) \otimes Li^*F) = \mathbb{C}$. But from the resolution,

$$H^0(Y, \mathcal{O}_Y(-1,0) \otimes Li^*F) = H^0(Y, [\mathcal{O}_Y(-2,-1)^{\oplus 2} \to \mathcal{O}_Y(-1,0)^{\oplus 2}]) = 0,$$

hence $\epsilon_F \neq 0$.

Now we go over to the situation when $Y = Y_n$ is a hypersurface of degree n in $W = \mathbb{P}^{2n-1}$ and $\mathcal{F}$ is a sheaf representing a point $m = [\mathcal{F}]$ of some connected component M of the smooth locus of the moduli space of stable (or simple) sheaves on Y. There is a canonical isomorphism $T_m M = \operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$, and formula (1) defines a p-linear form $\alpha_p(m)$ on $T_m M$ with $p = 2n-4$. It depends on the choice of the isomorphism $\mathcal{N}_{Y/W}^\vee \simeq \omega_Y$, where ω_Y is the dualizing sheaf of Y. Fixing once and forever such an isomorphism, the forms $\alpha_p(m)$ fit to a well-defined cross-section α_p of the vector bundle $\wedge^p \mathcal{T}^\vee M = \Omega_M^{2n-4}$. One way to see that it is a regular section is to relativize the definition of $\epsilon_{\mathcal{F}}$ and formula (1) in flat families of sheaves. We will use another approach, consisting in relating $\epsilon_{\mathcal{F}}$ to the Atiyah class of $\mathcal{F}$.

We will use the Atiyah class of torsion sheaves on Y with support Z which is a locally complete intersection subscheme in Y. Let $i : Z \hookrightarrow Y$ be the natural embedding. For any $\mathcal{F}, \mathcal{G} \in \mathcal{D}^b(Z)$, we have a spectral sequence

$$E_2^{pq} = \operatorname{Ext}^p(L_q i^* i_* \mathcal{F}, \mathcal{G}) \implies \operatorname{Ext}^{p+q}(i_*\mathcal{F}, i_*\mathcal{G}),$$

and a canonical isomorphism $L_q i^* i_* \mathcal{F} \xrightarrow{\text{can}}{\sim} \mathcal{F} \otimes \wedge^q \mathcal{N}_{Z/Y}^\vee$. This provides a natural map $\operatorname{Ext}^1(i_*\mathcal{F}, i_*\mathcal{G}) \to \operatorname{Hom}(\mathcal{F} \otimes \mathcal{N}_{Y/W}^\vee, \mathcal{G})$.

Assume now that, moreover, $\mathcal{F}$ is a locally free sheaf on Z. Then there is a canonical isomorphism $i^* \mathcal{E}xt^q(i_*\mathcal{F}, i_*\mathcal{G}) = \mathcal{H}om(\mathcal{F} \otimes \wedge^q \mathcal{N}_{Z/Y}^\vee, \mathcal{G})$, and we obtain another natural map between the same objects:

$$\operatorname{Ext}^1(i_*\mathcal{F}, i_*\mathcal{G}) \to H^0(Y, \mathcal{E}xt^1(i_*\mathcal{F}, i_*\mathcal{G})) = H^0(Z, i^*\mathcal{E}xt^1(i_*\mathcal{F}, i_*\mathcal{G})) = \\ = H^0(Z, \mathcal{H}om(\mathcal{F} \otimes \mathcal{N}_{Y/W}^\vee, \mathcal{G})) = \operatorname{Hom}(\mathcal{F} \otimes \mathcal{N}_{Y/W}^\vee, \mathcal{G}),$$

where the first arrow comes from the local-to-global spectral sequence for Exts. One can prove that these two maps coincide.

Theorem 5.5. *Let Z be a locally complete intersection subscheme in Y and $i : Z \hookrightarrow Y$ the natural embedding. Let $\mathcal{F} \in \mathcal{D}^b(Z)$. Then the image of the Atiyah class $\operatorname{At}_{i_*\mathcal{F}} \in \operatorname{Ext}^1(i_*\mathcal{F}, i_*\mathcal{F} \otimes \Omega_Y^1)$ in $\operatorname{Hom}(\mathcal{F} \otimes \mathcal{N}_{Z/Y}^\vee, \mathcal{F} \otimes \Omega_Y^1|_Z)$ coincides with $1_{\mathcal{F}} \otimes \kappa_{Z/Y}$, where $\kappa_{Z/Y}$ denotes the natural map of sheaves $\mathcal{N}_{Z/Y}^\vee \to \Omega_Y^1|_Z$.*

Proof. See [KuM], Theorem 3.2 (iii). □

When Y is a hypersurface in a smooth variety W, consider the conormal bundle sequence

$$0 \longrightarrow \mathcal{N}_{Y/W}^\vee \xrightarrow{\kappa_{Y/W}} \Omega_W^1|_Y \xrightarrow{\rho_{Y/W}} \Omega_Y^1 \longrightarrow 0. \tag{10}$$

Denote by $\nu_{Y/W}$ the extension class of (10) in $\operatorname{Ext}^1(\Omega_Y^1, \mathcal{N}_{Y/W}^\vee)$.

Theorem 5.6. *Let Y be a hypersurface in a smooth variety W, $i : Y \to W$ the natural embedding, and $\mathcal{F} \in \mathcal{D}^b(Y)$. Then the linkage class $\epsilon_{\mathcal{F}}^{Y/W} \in \mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}_{Y/W}^\vee)$ factors through the Atiyah class $\mathrm{At}_{\mathcal{F}} \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega_Y^1)$:*

$$\epsilon_{\mathcal{F}}^{Y/W} = (1_{\mathcal{F}} \otimes \nu_{Y/W}) \circ \mathrm{At}_{\mathcal{F}}\,.$$

Proof. See [KuM], Theorem 3.2 (i). □

In the next theorem, we deal with exterior forms on a moduli space M of sheaves on a projective variety Y. Speaking about exterior forms on M, we will always mean that M is smooth. On the other hand, it is not important to us whether M is separated or not. Thus we will take for M an open subset of the smooth locus of either one of the following two moduli spaces: either the moduli space of H-stable sheaves on Y in the sense of the definition of Simpson [Si] for some ample divisor class H on Y, or the moduli space of simple sheaves on Y as defined in [AK2]. The former is a quasi-projective scheme over k, and the latter is a possibly non-separated algebraic space.

Corollary 5.7. *Let $Y = Y_n$ be a smooth hypersurface of degree n in $W = \mathbb{P}^{2n-1}$ and M a connected component of the smooth locus of the moduli space of stable or simple sheaves on Y. Let us fix an isomorphism $\mathcal{N}_{Y/W}^\vee \simeq \omega_Y$. Then there exists a closed regular p-form $\alpha_p \in H^0(M, \Omega_M^p)$, where $p = 2n-4$, such that its value $\alpha_p(m)$ at any point $m = [\mathcal{F}] \in M$ represented by a sheaf $\mathcal{F}$ is given by formula (1). For $n = 4$, and $M = F(Y)$, α_p is proportional to the 4-form on $F(Y)$ defined in Corollary 1.7.*

Proof. As in the proof of Theorem 2.2 of [KuM], we can shrink M to the biholomorphic image of a polydisc in it, choose a universal sheaf $\boldsymbol{\mathcal{F}}$ over $M \times Y$ and represent $(p+1)\alpha_p$ as the Künneth component $\gamma^{2n-4,2n-2} \in H^0(M, \Omega_M^{2n-4}) \otimes H^{2n-2}(Y, \Omega_Y^{2n-2})$ of

$$\gamma = \mathrm{Tr}(\mathrm{At}_{\boldsymbol{\mathcal{F}}}^{\wedge(2n-3)}) \wedge \nu_{Y/W} \in H^{2n-2}(M \times Y, \Omega_{M\times Y}^{4n-6}),$$

where $\nu_{Y/W}$ is viewed here as an element of $H^1(Y, \Omega_Y^{2n-3}) \simeq \mathrm{Ext}^1(\Omega_Y^1, \mathcal{N}_{Y/W}^\vee)$. Then γ is de Rham closed by [HL], Sect. 10.1.6. In the same way as in loc. cit., this fact together with the projectivity of Y implies that all the Künneth components of γ are d_M-closed, where d_M denotes the de Rham differential on M. □

To conclude this section, we will prove a nonvanishing theorem for $\epsilon_{\mathcal{F}}$ which is a direct generalization of Proposition 4.1 of [KuM] from $n = 3$ to all $n \geq 3$. We use here a more general definition of the linkage class, referred to in Remark 5.3.

Proposition 5.8. *Assume that $\mathcal{F}, \mathcal{G} \in \mathcal{C}_n$, that is $H^p(Y, \mathcal{F}(-k)) = H^p(Y, \mathcal{G}(-k)) = 0$ for all $p \in \mathbb{Z}$ and $k = 0, 1, \ldots, n-1$. Then the multiplication by the linkage class $\epsilon_{\mathcal{G}} \in \mathrm{Ext}^2(\mathcal{G}, \mathcal{G}(-n))$ induces an isomorphism $\mathrm{Ext}^p(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{p+2}(\mathcal{F}, \mathcal{G}(-n))$ for all $p \in \mathbb{Z}$.*

Proof. Consider the Beilinson spectral sequence for $i_*\mathcal{G}$ (see [Bei, OSS])

$$E_1^{-p,q} = H^q(\mathbb{P}^{2n-1}, i_*\mathcal{G}(-p)) \otimes \Omega_{\mathbb{P}^{2n-1}}^p(p) \Longrightarrow i_*\mathcal{G},$$

where $i : Y \hookrightarrow \mathbb{P}^{2n-1}$ is the natural embedding. By the assumption on $\mathcal{G}$, we have $E_1^{0,q} = E_1^{-1,q} = \ldots = E_1^{-n+1,q} = 0$ for all q. Hence the derived pullback $Li^*i_*\mathcal{G}$ is contained in the triangulated subcategory of $\mathcal{D}^b(\mathrm{Coh}(Y))$ generated by $i^*\Omega_{\mathbb{P}^{2n-1}}^n(n)$, $i^*\Omega_{\mathbb{P}^{2n-1}}^{n+1}(n+1)$, … , $i^*\Omega_{\mathbb{P}^{2n-1}}^{2n-1}(2n-1)$. Let $V = \mathbb{C}^{2n}$, so that $\mathbb{P}^{2n-1} = \mathbb{P}(V)$. The Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2n-1}} \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}(1) \longrightarrow \mathcal{T}_{\mathbb{P}^{2n-1}} \longrightarrow 0$$

and the isomorphism $\Omega^{2n-1-k}_{\mathbb{P}^{2n-1}} \simeq \wedge^k \mathcal{T}_{\mathbb{P}^{2n-1}}(-2n)$ lead to the following resolutions for the sheaves $\Omega^{2n-1-k}(2n-1-k)$ $(k = 0, 1, \ldots, n-1)$:

$$0 \to \mathcal{O}(-k-1) \to \quad \ldots \quad \to \wedge^{k-1} V \otimes \mathcal{O}(-2) \to \wedge^k V \otimes \mathcal{O}(-1) \to \Omega^{2n-1-k}(2n-1-k) \to 0 \ .$$

Hence this subcategory coincides with the subcategory of $\mathcal{D}^b(\mathrm{Coh}(Y))$ generated by $\mathcal{O}_Y(-1)$, $\ldots$, $\mathcal{O}_Y(-n)$. By Serre duality on Y and by the hypothesis on $\mathcal{F}$, we obtain:

$$\mathrm{Ext}^p(\mathcal{F}, \mathcal{O}_Y(-k-1)) \cong H^{2n-1-p}(\mathcal{F}(-k))^\vee = 0$$

for all $k = 0, 1, \ldots, n-1$. Hence $\mathrm{Ext}^\bullet(\mathcal{F}, Li^* i_* \mathcal{G}) = 0$. It remains to note that we have a distinguished triangle $Li^* i_* \mathcal{G} \longrightarrow \mathcal{G} \xrightarrow{\epsilon_{\mathcal{G}}} \mathcal{G}(-n)[2]$. Applying the functor $\mathrm{Hom}(\mathcal{F}, -)$, we deduce the proposition. □

For $n = 3$, this nonvanishing property implies the nondegeneracy of α_{2n-4} on moduli spaces M that parameterize sheaves from $\mathcal{C}_n$. The same argument implies only partial results for bigger n. For example:

Corollary 5.9. *Let $n = 4$, so that Y is a quartic in $\mathbb{P}^7$, and assume that a moduli space M parameterizes sheaves from $\mathcal{C}_n$. Then the 2-rank of α_4 at a point $m \in M$ representing a sheaf $\mathcal{F}$ coincides with the dimension of the image of the Yoneda coupling*

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \times \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \longrightarrow \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}).$$

Proof. Similar to the proof of Theorem 4.3 in [KuM]. □

6. An explicit calculation

Throughout this section, Y will denote a generic quartic hypersurface in $\mathbb{P}^7$ (a quartic 6-fold). Let $F = F(Y)$ be the Fano scheme of Y, that is the Hilbert scheme of lines in Y. We have seen that F is a smooth connected projective variety of dimension 7. We will denote by $\{\ell\}$ the point of $F(Y)$ representing a line $\ell \subset Y$.

The constructions of Corollaries 1.7 and 5.7 provide a closed 4-form $\alpha = \alpha_4 \in H^0(F(Y), \Omega^4_{F(Y)})$. We will produce an explicit formula for its value $\alpha(m)$ at a point $m = \{\ell\}$.

Proposition 6.1. *Let Y be a generic quartic hypersurface in $\mathbb{P}^7$. Then there is an algebraic subset $R(Y)$ in $F(Y)$ of codimension ≥ 3 such that $\mathcal{N}_{\ell/Y} \simeq \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus 3}$ for all $\{\ell\} \in F(Y) \setminus R(Y)$, and $\mathcal{N}_{\ell/Y} \simeq \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O} \oplus \mathcal{O}(-1)$ if $\{\ell\} \in R(Y)$.*

Proof. Denote by V the vector space $\mathbb{C}^8$ whose projectivization contains Y. So $\mathbb{P}^7 = \mathbb{P}(V)$, and $Y = Y_f$ is defined by an octal quartic form $f \in S^4 V^*$.

The possible types of the normal bundle follow easily from the two exact triples

$$0 \to \mathcal{T}_\ell \to \mathcal{T}_Y|_\ell \to \mathcal{N}_{\ell/Y} \to 0 \ , \quad 0 \to \mathcal{T}_Y|_\ell \to \mathcal{T}_{\mathbb{P}^7}|_\ell \to \mathcal{N}_{Y/\mathbb{P}^7}|_\ell \to 0.$$

Indeed, the tangent space $T_{\{\ell\}} F(Y)$ is of dimension $h^0(\mathcal{N}_{\ell/Y})$, so the smoothness of $F(Y)$ implies that $h^1(\mathcal{N}_{\ell/Y}) = 0$ for all ℓ. Hence the Grothendieck splitting of $\mathcal{N}_{\ell/Y}$ has no summands $\mathcal{O}(a)$ with $a < -1$. As $\mathcal{T}_{\mathbb{P}^7}|_\ell \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 6}$ and $\mathcal{T}_Y|_\ell$ has an injective map to it, $\mathcal{T}_Y|_\ell$ is the sum of sheaves $\mathcal{O}(a_i)$ with $2 \geq a_1 \geq 1 \geq a_2 \geq \cdots \geq a_6 \geq -1$, and $\sum a_i = \deg \mathcal{T}_Y|_\ell = 4$. Moreover, $a_1 = 2$ because $\mathcal{T}_\ell \simeq \mathcal{O}(2)$ has a nonzero map to $\mathcal{T}_Y|_\ell$. This leaves only two possible choices $(a_i) = (2, 1^2, 0^3)$ or $(2, 1^3, 0, -1)$. Splitting off $\mathcal{O}(a_1) \simeq \mathcal{T}_\ell$, we obtain the two possible normal bundles $\mathcal{N}_{\ell/Y}$.

Thus the lines in Y are of two types. The summands $\mathcal{O}(1)$ of $\mathcal{N}_{\ell/Y}$ span the subbundle corresponding to infinitesimal deformations of ℓ inside the projective subspace $\bigcap_{P \in \ell} T_P Y$, the intersection of the tangent hyperplanes to Y at all the points of ℓ. This is a cubic pencil of

hyperplanes, and generically there are 4 linearly independent ones. Hence $\bigcap_{P\in\ell} T_P Y \simeq \mathbb{P}^3$ for ℓ in an open subset of $F(Y)$. These are lines of the first type. Their complement $R(Y)$ in $F(Y)$ is the set of lines of the second type; for them $\bigcap_{P\in\ell} T_P Y \simeq \mathbb{P}^4$. It remains to estimate the codimension of $R(Y)$.

Let $g = g_f : Y \to Y^* \subset \mathbb{P}^{7*}$, $P \mapsto T_P Y$ be the Gauss map, given in homogeneous coordinates by $(x_0 : \cdots : x_7) \mapsto (\partial f/\partial x_0 : \cdots : \partial f/\partial x_7)$, where f is the quartic form defining Y. As the partial derivatives $\partial f/\partial x_i$ have no common zero on Y, the Gauss map is finite. Its restriction to a line ℓ in Y is given by a cubic pencil without fixed points, so the image $g(\ell)$ is either a cubic rational curve, or a line. In the latter case, $\bigcap_{P\in\ell} T_P Y$ would be 5-dimensional, which is impossible by the above. Hence $g(\ell)$ is always a rational cubic, and $\{\ell\} \in R(Y)$ if and only if $g(\ell)$ has exactly one singular point. Thus if we define the algebraic set

$$\tilde{R}(Y) = \{(\ell, P) \in F(Y) \times Y \ \mid \ P \in \ell,\ g(P) \in \operatorname{Sing} g(\ell)\},$$

then the projection $\mathrm{pr}_1 : \tilde{R}(Y) \to F(Y)$ is finite of degree 2 and has $R(Y)$ as its image, so $\dim R(Y) = \dim \tilde{R}(Y)$. We will deduce the dimension of $\tilde{R}(Y)$ by a standard dimension count.

Consider two incidence varieties:

$$I_1 = \{(P, Q, f) \in \mathbb{P}(V) \times \mathbb{P}(V) \times \mathbb{P}(S^4 V^*) \mid P \neq Q,\ \nabla_P f \sim \nabla_Q f,\ \overline{PQ} \subset Y_f\},$$

where $\nabla_P f = (\partial f/\partial x_0, \cdots, \partial f/\partial x_7)|_P$ is the gradient of f at P, the sign $\sim$ stands for proportionality, and $\overline{PQ}$ denotes the line passing through P, Q, and

$$I_2 = \{([P, v], f) \in \mathbb{P}(\mathcal{T}_{\mathbb{P}^7}) \times \mathbb{P}(S^4 V^*) \mid P \in Y_f,\ v \in T_P Y_f,\ v \neq 0,\ (\nabla_P f, v) = 0,\ \overline{Pv} \subset Y_f\},$$

where $[P, v]$ is the proportionality class of v considered as a point of $\mathbb{P}(T_P \mathbb{P}^7)$, and $\overline{Pv}$ is the line through P in the direction of v.

The proportionality $\nabla_P f \sim \nabla_Q f$ can be interpreted as the coincidence of the tangent spaces $T_P Y_f = T_Q Y_f$ as soon as both gradients are nonzero. The part of I_1 for which Y_f is nonsingular parametrizes all the triples (P, Q, f) for which $g_f(\overline{PQ})$ has a node at $g_f(P) = g_f(Q)$. Similarly, the part of I_2 for which Y_f is nonsingular parametrizes the pairs $([P, v], f)$ for which $g_f(\overline{Pv})$ has a cusp at $g_f(P)$. Thus, assuming $Y = Y_f$ nonsingular, we can represent $\tilde{R}(Y)$ as the union of two algebraic sets $\pi_i^{-1}(f)$, where $\pi_i : I_i \to \mathbb{P}(S^4 V^*)$ $(i = 1, 2)$ is the natural projection.

Looking at the other projection $\mathrm{pr}_{12} : I_1 \to \mathbb{P}(V) \times \mathbb{P}(V) \setminus (\text{diagonal})$, we find that all of its fibers are isomorphic to each other and irreducible of dimension 319, hence I_1 is irreducible of dimension 333. As $\dim \mathbb{P}(S^4 V^*) = 329$, we conclude that $\pi_1^{-1}(f)$ is either empty, or is of dimension $333 - 329 = 4$ for generic f. Similarly, I_2 is irreducible and $\dim I_2 = 332$, so $\pi_2^{-1}(f)$ is either empty or is of dimension 3 for generic f. Hence $\dim \tilde{R}(Y) \leq 4$ for generic f, as was to be proved. $\square$

Fix now a generic quartic $Y \subset \mathbb{P}^7 = \mathbb{P}(V)$ and a line ℓ of first type in Y. Choose homogeneous coordinates in $\mathbb{P}^7$ in such a way that $\ell = \{x_0 = \cdots = x_5 = 0\}$. Then the equation f of Y can be written in the form

$$f = x_0 f_0(x_0, \ldots, x_7) + \cdots + x_5 f_5(x_0, \ldots, x_7),$$

where the f_i are cubic forms. Denote by $\overline{f}_i = \overline{f}_i(x_6, x_7)$ the restriction of f_i to ℓ. The fact that ℓ is of the first type implies that the 6 forms $\overline{f}_i = \frac{\partial f}{\partial x_i}|_\ell$ generate the whole 4-dimensional vector space of binary cubics, and by a linear change of coordinates $x_0, \ldots, x_5$, we can arrange the things so that

$$\overline{f}_0 = x_6^3,\ \overline{f}_1 = x_6^2 x_7,\ \overline{f}_2 = x_6 x_7^2,\ \overline{f}_3 = x_7^3,\ \overline{f}_4 = \overline{f}_5 = 0.$$

Now we will construct an explicit Grothendieck splitting of $\mathcal{T}_Y|_\ell$ and $\mathcal{N}_{\ell/Y}$. We have $\mathbb{P}(V) = V^0/\mathbb{C}^*$, where $V^0 = V \setminus \{0\}$. The symbols $\partial/\partial x_i$, dx_i have a natural meaning as vector fields,

respectively 1-forms on V or V^0. A rational section of $\mathcal{T}_{\mathbb{P}^7}(k)$ can be represented in the form $\sum \phi_i \partial/\partial x_i \mod E$, where ϕ_i are rational homogeneous functions in x_j of degree $k+1$, and $E = \sum x_i \partial/\partial x_i$ is the Euler vector field. Let us denote $\partial/\partial x_i \mod E$ by ∂_i. Then $\partial_0, \dots, \partial_7$ form a basis of $H^0(\mathcal{T}_{\mathbb{P}^7}(-1))$, and $H^0(\mathcal{T}_{\mathbb{P}^7})$ is generated by $x_i\partial_j$, $0 \le i, j \le 7$, with a single linear relation $\sum x_i \partial_i = 0$.

A rational vector field $\sum \phi_i \partial_i$ is tangent to Y if and only if $\sum \phi_i \partial f/\partial x_i|_Y = 0$. Similarly, an expression $\sum \psi_i dx_i$, where ψ_i are rational homogeneous functions in x_j of degree $k-1$, represents a rational section of $\Omega^1_{\mathbb{P}^7}(k)$ whenever $\sum x_i\psi_i = 0$, and then $\sum \psi_i dx_i|_Y \mod df$ represents a rational section of $\Omega^1_Y(k)$. We will use the overbar to denote the restriction of sections of $\Omega^1_{\mathbb{P}^7}(k)$ or $\mathcal{T}_{\mathbb{P}^7}(k)$ to sections of $\Omega^1_Y(k)|_\ell$ or $\mathcal{T}_Y(k)|_\ell$. With this notation, we have the following Grothendieck splitting:

$$\mathcal{T}_Y|_\ell = \mathcal{O}_\ell(2)e_0 \oplus \mathcal{O}_\ell(1)e_1 \oplus \mathcal{O}_\ell(1)e_2 \oplus \mathcal{O}_\ell e_3 \oplus \mathcal{O}_\ell e_4 \oplus \mathcal{O}_\ell e_5, \tag{11}$$

where the e_i are the following sections of appropriate twists of $\mathcal{T}_Y|_\ell$:

$$e_0 = x_6^{-1}\overline{\partial}_7 = -x_7^{-1}\overline{\partial}_6,\ e_1 = \overline{\partial}_4,\ e_2 = \overline{\partial}_5,$$
$$e_3 = x_6\overline{\partial}_1 - x_7\overline{\partial}_0,\ e_4 = x_6\overline{\partial}_2 - x_7\overline{\partial}_1,\ e_5 = x_6\overline{\partial}_3 - x_7\overline{\partial}_2. \tag{12}$$

Thus $e_0 \in H^0(\mathcal{T}_Y(-2)|_\ell)$, e_1 and e_2 are elements of $H^0(\mathcal{T}_Y(-1)|_\ell)$, and the remaining e_i are sections of $\mathcal{T}_Y|_\ell$. We can identify $\mathcal{N}_{\ell/Y}$ as the sum of the last five summands. Denote by $\check{e}_i$ the dual basis of $H^0(\ell, (\Omega^1_Y|_\ell) \otimes k(\ell))$, so that $(\check{e}_i, e_j) = \delta_{ij}$. Then

$$\check{e}_0 = \overline{x_6dx_7 - x_7dx_6} \in H^0(\Omega^1_Y(2)|_\ell),$$
$$\check{e}_1 = d\overline{x}_4,\ \check{e}_2 = d\overline{x}_5 \in H^0(\Omega^1_Y(1)|_\ell),\quad H^0(\Omega^1_Y|_\ell) = \langle \check{e}_3, \check{e}_4, \check{e}_5\rangle,\ \check{e}_3 = -\frac{d\overline{x}_0}{x_7},$$
$$\check{e}_4 = \frac{1}{2}\left(-\frac{x_6}{x_7^2}d\overline{x}_0 - \frac{1}{x_7}d\overline{x}_1 + \frac{1}{x_6}d\overline{x}_2 + \frac{x_7}{x_6^2}d\overline{x}_3\right),\quad \check{e}_5 = \frac{d\overline{x}_3}{x_6}. \tag{13}$$

Here $d\overline{x}_i$ are well-defined for $i = 0, \dots, 5$ by the formula

$$d\overline{x}_i = \overline{dx_i - \frac{x_i}{x_j}dx_j} = \left(dx_i - \frac{x_i}{x_j}dx_j\right)\Big|_\ell \mod df|_\ell \quad (j = 6 \text{ or } 7). \tag{14}$$

The Grothendieck splitting of $\Omega^1_Y|_\ell$ has the form

$$\Omega^1_Y|_\ell = \mathcal{O}_\ell(-2)\check{e}_0 \oplus \mathcal{O}_\ell(-1)\check{e}_1 \oplus \mathcal{O}_\ell(-1)\check{e}_2 \oplus \mathcal{O}_\ell\check{e}_3 \oplus \mathcal{O}_\ell\check{e}_4 \oplus \mathcal{O}_\ell\check{e}_5 \tag{15}$$

and the last four summands give a Grothendieck splitting of $\mathcal{N}^\vee_{\ell/Y}$.

Let us now compute the composition (1) with $p = 4$ on four sections $\xi_i \in \mathrm{Ext}^1(\mathcal{O}_\ell, \mathcal{O}_\ell) = H^0(\mathcal{N}_{\ell/Y})$, $i = 1, \dots, 4$,

$$\xi_i = (a_{i0}x_6 + a_{i1}x_7)e_1 + (b_{i0}x_6 + b_{i1}x_7)e_2 + c_{i1}e_3 + c_{i2}e_4 + c_{i3}e_5, \tag{16}$$

where a_{ij}, b_{ij}, c_{ij} are elements of k.

Theorem 6.2. *In the above notation, the 4-form $\alpha_4(m)$ defined by (1) at a point $m = [\mathcal{O}_\ell]$ of the Fano scheme $F(Y)$ is given up to a constant factor by the formula*

$$\alpha_4(m)(\xi_1, \dots, \xi_4) = \begin{vmatrix} a_{10} & b_{10} & c_{12} & c_{13} \\ \vdots & \vdots & \vdots & \vdots \\ a_{40} & b_{40} & c_{42} & c_{43} \end{vmatrix} - \begin{vmatrix} a_{10} & b_{11} & c_{11} & c_{13} \\ \vdots & \vdots & \vdots & \vdots \\ a_{40} & b_{41} & c_{41} & c_{43} \end{vmatrix}$$
$$- \begin{vmatrix} a_{11} & b_{10} & c_{11} & c_{13} \\ \vdots & \vdots & \vdots & \vdots \\ a_{41} & b_{40} & c_{41} & c_{43} \end{vmatrix} + \begin{vmatrix} a_{11} & b_{11} & c_{11} & c_{12} \\ \vdots & \vdots & \vdots & \vdots \\ a_{41} & b_{41} & c_{41} & c_{42} \end{vmatrix}. \tag{17}$$

Proof. By Theorem 5.6, $\alpha_4(m)$ is the composition of the maps in the upper line of the diagram

$$\begin{array}{ccccccccc}
\mathrm{Ext}^1(\mathcal{O}_\ell,\mathcal{O}_\ell)^4 & \xrightarrow{\text{Yoneda}} & \mathrm{Ext}^4(\mathcal{O}_\ell,\mathcal{O}_\ell) & \xrightarrow{\mathrm{At}_{\mathcal{O}_\ell}} & \mathrm{Ext}^5(\mathcal{O}_\ell,\mathcal{O}_\ell\otimes\Omega^1_Y) & \xrightarrow{\nu} & \mathrm{Ext}^6(\mathcal{O}_\ell,\mathcal{O}_\ell\otimes\omega_Y) & \xrightarrow{\mathrm{Tr}} & k \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \| \\
H^0(\mathcal{N}_{\ell/Y})^4 & \xrightarrow{\wedge} & H^0(\wedge^4\mathcal{N}_{\ell/Y}) & \xrightarrow{\kappa} & H^0(\wedge^5\mathcal{N}_{\ell/Y}\otimes\Omega^1_Y) & \xrightarrow{\nu} & H^1(\omega_\ell) & \xrightarrow{\mathrm{Tr}} & k
\end{array}$$

Here we are using the canonical isomorphisms $\mathcal{E}xt^{\,q}(\mathcal{O}_\ell,\mathcal{O}_\ell)=\wedge^q\mathcal{N}_{\ell/Y}$, and the vanishing $h^i(\wedge^q\mathcal{N}_{\ell/Y})=h^i(\wedge^4\mathcal{N}_{\ell/Y}\otimes\Omega^1_Y)=0$ for $i>0$ implies that the vertical maps, coming from the local-to-global spectral sequence, are also isomorphisms. Further, by [KuM], Lemma 1.3.2, the Yoneda coupling on the Ext-sheaves corresponds to the wedge product under these isomorphisms. The symbols κ, ν were defined in Theorems 5.5, 5.6. After applying $\nu\in\mathrm{Ext}^1(\Omega^1_Y,\mathcal{N}^\vee_{Y/\mathbb{P}^7})$ in both lines, we also use the isomorphisms $\mathcal{N}^\vee_{Y/\mathbb{P}^7}\simeq\omega_Y$, $\wedge^5\mathcal{N}_{\ell/Y}\otimes\mathcal{N}^\vee_{Y/\mathbb{P}^7}\simeq\omega_\ell$; the constant factor mentioned in the statement of the theorem depends only on the choice of these isomorphisms.

Thus we can calculate $\alpha_4(m)$ in following the bottom line of the digram, in which the Yoneda product turns into the wedge one. We have $\kappa=e_1\otimes\check{e}_1+\cdots+e_5\otimes\check{e}_5$, and the image of $(\xi_1,\dots,\xi_4)$ in $H^0(\wedge^5\mathcal{N}_{\ell/Y}\otimes\Omega^1_Y)$ is

$$\psi=\xi_1\wedge\xi_2\wedge\xi_3\wedge\xi_4\wedge\kappa=e_1\wedge\ldots\wedge e_5\otimes\begin{vmatrix} a_{10}x_6+a_{11}x_7 & b_{10}x_6+b_{11}x_7 & c_{11} & c_{12} & c_{13}\\ \vdots & \vdots & \vdots & \vdots & \\ a_{40}x_6+a_{41}x_7 & b_{40}x_6+b_{41}x_7 & c_{41} & c_{42} & c_{43}\\ \check{e}_1 & \check{e}_2 & \check{e}_3 & \check{e}_4 & \check{e}_5 \end{vmatrix},$$

where the determinant is understood as its row expansion with respect to the last row. We can equally omit the factor $e_1\wedge\ldots\wedge e_5$, for it is a nowhere vanishing section of $\det\mathcal{N}_{\ell/Y}(-2)\simeq\mathcal{O}_\ell$. Then ψ is just the section of $\Omega^1|_\ell(2)$ given by the determinant in the last formula.

Next we have to couple ψ with ν. The Ext-group to which ν belongs can be represented as $H^1(\mathcal{T}_Y(-4))$ or else $H^1(\Omega^5_Y)$. As we know from Proposition 1.6, $h^{51}(Y)=1$. Thus we have to determine the image of the 1-dimensional $H^1(\Omega^5_Y)$ inside the 14-dimensional $H^1(\Omega^5_Y|_\ell)$. Keeping in mind that $\Omega^5_Y\simeq\mathcal{T}_Y(-4)$, we use the natural exact triple

$$0\to\mathcal{T}_Y(-4)\to\mathcal{T}_{\mathbb{P}^7}(-4)|_Y\to\mathcal{O}_Y\to 0.$$

Then we see that $H^1(\mathcal{T}_Y(-4))=\beta_Y(H^0(\mathcal{O}_Y))$, where β_Y is the Bockstein homomorphism, and the commutativity of the maps of the exact triple with the restriction to ℓ implies that

$$\mathrm{im}\left(H^1(\mathcal{T}_Y(-4))\xrightarrow{\text{restriction}}H^1(\mathcal{T}_Y(-4)|_\ell)\right)\;=\;\mathrm{im}\left(H^0(\mathcal{O}_\ell)\xrightarrow{\beta_\ell}H^1(\mathcal{T}_Y(-4)|_\ell)\right),$$

where β_ℓ is the Bockstein homomorphism of the restricted exact triple

$$0\longrightarrow\mathcal{T}_Y(-4)|_\ell\longrightarrow\mathcal{T}_{\mathbb{P}^7}(-4)|_\ell\longrightarrow\mathcal{O}_\ell\longrightarrow 0.$$

The surjection in this exact triple is induced by $\overline{\partial}_i\mapsto(\partial f/\partial x_i)|_\ell$, so that $x_6^{-3}\overline{\partial}_0\mapsto 1$ and $x_7^{-3}\overline{\partial}_3\mapsto 1$. Taking the standard covering of $\ell=\mathbb{P}^1$ by the open affine sets $U_6=\{x_6\neq 0\}$, $U_7=\{x_7\neq 0\}$, we get a Čech representative for $\beta_\ell(1)\in H^1(\mathcal{T}_Y(-4)|_\ell)$ of the form

$$x_7^{-3}\overline{\partial}_3-x_6^{-3}\overline{\partial}_0=\frac{x_7^2e_3+x_6x_7e_4+x_6^2e_5}{x_6^3x_7^3}\in\Gamma(U_6\cap U_7,\mathcal{T}_Y(-4)|_\ell).\qquad(18)$$

The wanted quantity $\alpha_4(m)(\xi_1,\dots,\xi_4)$ is by construction $\mathrm{Tr}(\nu\wedge\psi)$ when ν is considered as an element of $H^1(\Omega^5_Y|_\ell)$, but (18) represents it as an element of $H^1(\mathcal{T}_Y(-4)|_\ell)$, and the wedge product becomes the contraction of 1-forms with vector fields. Thus we are computing $\mathrm{Tr}(\nu,\psi)$, which is nothing but the image of $(\nu,\,\psi)$ under the Serre coupling

$$H^1(\mathcal{T}_Y(-4)|_\ell)\times H^0(\Omega^1_Y(2)|_\ell)\longrightarrow H^1(\mathcal{O}_\ell(-2)).$$

On the level of Čech cocycles, the value of the coupling on a pair $(\sum_i \phi_i e_i, \sum_j \psi_j \check{e}_j)$ is the coefficient of $\frac{1}{x_6x_7}$ in the expression $\sum_i \phi_i\psi_i$. This is due to the fact that the cohomology class of the cocycle $\frac{1}{x_6x_7}$ generates $H^1(\mathcal{O}_\ell(-2))$, and the classes of all the other cocycles of the form $\frac{1}{x_6^i x_7^j}$ $(i+j=2,\ (i,j)\neq(1,1))$ are zero. We obtain that $\alpha_4(m)(\xi_1,\dots,\xi_4)$ is the coefficient of $\frac{1}{x_6x_7}$ in the product

$$\left(\frac{x_7^2e_3 + x_6x_7e_4 + x_6^2e_5}{x_6^3x_7^3}\ , \begin{vmatrix} a_{10}x_6+a_{11}x_7 & b_{10}x_6+b_{11}x_7 & c_{11} & c_{12} & c_{13} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{40}x_6+a_{41}x_7 & b_{40}x_6+b_{41}x_7 & c_{41} & c_{42} & c_{43} \\ \check{e}_1 & \check{e}_2 & \check{e}_3 & \check{e}_4 & \check{e}_5 \end{vmatrix} \right),$$

which coincides with (17). □

Corollary 6.3. *Let Y be a smooth quartic hypersurface in $\mathbb{P}^7$, and ℓ a line of first type in Y. Then $F(Y)$ is smooth at $m=[\mathcal{O}_\ell]$, and there exists a basis $(a_0,\ a_1,\ b_0,\ b_1,\ c_1,\ c_2,\ c_3)$ of the $\mathcal{O}_{F(Y)}$-module $\Omega^1_{F(Y)}$ over an open neighborhood U of m such that the 4-form α_4 defined on U by Theorem 5.7 is given by*

$$\alpha_4 = a_0\wedge b_0\wedge c_2\wedge c_3 - a_0\wedge b_1\wedge c_1\wedge c_3 - a_1\wedge b_0\wedge c_1\wedge c_3 + a_1\wedge b_1\wedge c_1\wedge c_2.$$

7. Nondegeneracy of exterior forms

The classification of trilinear alternating forms in seven complex variables goes back to Schouten (1931). There are exactly ten orbits of GL_7 (including zero); in particular, as was already known to E. Cartan, there is an open orbit of forms whose stabilizer is (up to a finite group) a form of the exceptional group G_2 (see [CH]). Moreover the normal form of a generic alternating 3-form encodes the multiplication table of the Cayley octonion algebra.

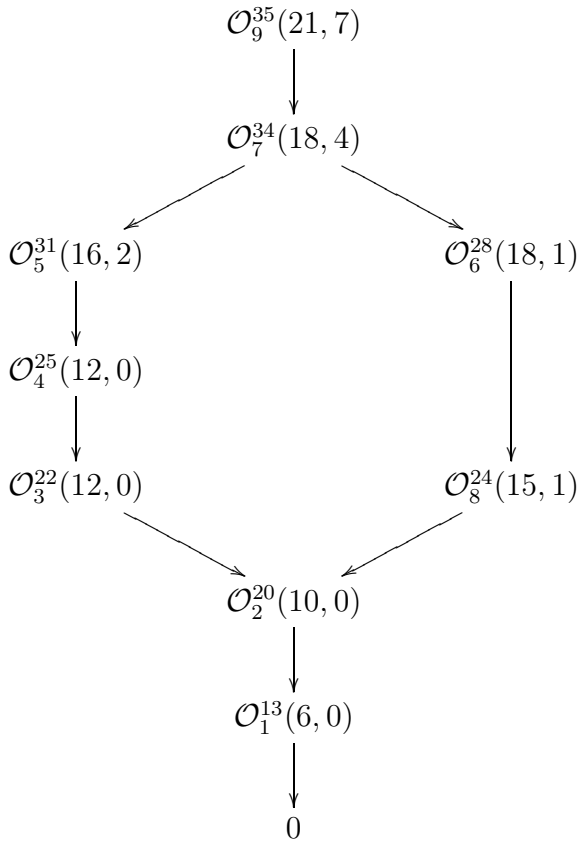
The classification of 4-forms in seven complex variables is almost equivalent to that of 3-forms, because of the isomorphism $\wedge^4U \simeq \wedge^3U^\vee\otimes\det(U)$ for a seven dimensional vector space U. More precisely, the projective classifications are completely equivalent, and it follows that $GL(U)$ has the same orbits in $\wedge^4U$ and in $\wedge^3U^\vee$, with isomorphic stabilizers up to finite groups.

One way to distinguish the orbits is to observe that there exists a GL_7-equivariant map $S^3(\wedge^3U^\vee)\to S^2U^\vee$ defined up to constant. Indeed, let us choose a generator Ω of $\det(U)^\vee$. To each $\omega\in\wedge^3U^\vee$ we associate the quadratic form q_ω on U defined by the formula

$$\omega\wedge(u\lrcorner\,\omega)\wedge(u\lrcorner\,\omega) = q_\omega(u)\Omega \qquad \forall u\in U,$$

where $u\lrcorner\,\omega$ denotes the 2-form obtained by contracting ω with u. Then q_ω is non-degenerate if and only if ω belongs to the open orbit, and this yields the classical embedding $G_2\subset SO_7$. On the complement of this open orbit the rank of q_ω drops to four. More precisely, it is equal to four exactly on the codimension one orbit.

One can check that the orbit structure of $\wedge^4U$ is as follows, where we denote by $\mathcal{O}^d_k(r,\rho)$ the orbit corresponding to the 3-form denoted f_k in [CH], Table 1; d is the dimension of the orbit, r is the 2-rank and ρ the rank.



Comparing the expression of our 4-form in Corollary 6.3, we see that it does not belong to the open orbit $\mathcal{O}_9^{35}$, so it does not define a G_2-structure. Nevertheless, it belongs to the codimension one orbit $\mathcal{O}_7^{34}$, and in this sense it can be said "minimally degenerate".

Appendix A. Lines on Pfaffian varieties

Let $F(\mathrm{Pf}_k(W^*))$ denote the Fano scheme of lines on $\mathrm{Pf}_k(W^*)$. By definition it is a subscheme of $\mathrm{Gr}(2,\Lambda^2 W^*)$ with equations given by σ_k. In other words, the sheaf of ideals of $F(\mathrm{Pf}_k(W^*))$ is the image of the map

$$\Lambda^{2(k-1)}W\otimes S^{-(n-k+1)}L\xrightarrow{\sigma_k}\mathcal{O}_{\mathrm{Gr}(2,\Lambda^2W^*)},$$

where σ_k is defined in (7) and L is the tautological bundle on $\mathrm{Gr}(2,\Lambda^2W^*)$.

Consider the product $\mathrm{Gr}(n+k,W)\times\mathrm{Gr}(2,\Lambda^2W^*)$ and the subvariety

$$\tilde{F}_k=\{(U,L)\in\mathrm{Gr}(n+k,W)\times\mathrm{Gr}(2,\Lambda^2W^*)\mid L\subset\mathrm{Ker}(\Lambda^2W^*\to\Lambda^2U^*)\}. \tag{19}$$

The projection π of $\mathrm{Gr}(n+k,W)\times\mathrm{Gr}(2,\Lambda^2W^*)$ onto the second factor maps $\tilde{F}_k$ into $F(\mathrm{Pf}_k(W^*))$. The goal of this section is to prove the following

Theorem A.1. *The map $\pi:\tilde{F}_k\to F(\mathrm{Pf}_k(W^*))$ is a resolution of singularities. For $k=1$ this resolution is crepant.*

We start with the following simple observation.

Lemma A.2. *$\tilde{F}_k$ is a smooth connected algebraic subvariety of codimension $(n+k-1)(n+k)$ in $\mathrm{Gr}(n+k,W)\times\mathrm{Gr}(2,\Lambda^2W^*)$.*

Proof. Consider the projection $\tilde{F}_k \to \mathrm{Gr}(n+k, W)$. It is clear that its fiber over a point $U \in \mathrm{Gr}(n+k, W)$ is the Grassmannian $\mathrm{Gr}(2, \mathrm{Ker}(\Lambda^2 W^* \to \Lambda^2 U^*))$. Therefore $\tilde{F}_k$ is smooth and connected. This also allows to compute the codimension. □

The most complicated part is the surjectivity of π.

Proposition A.3. *The map $\pi : \tilde{F}_k \to F(\mathrm{Pf}_k(W^*))$ is surjective.*

Proof. By definition $\tilde{F}_k$ is the zero locus of the canonical global section ϕ of the vector bundle $\Lambda^2 U^* \boxtimes L^*$ where U and L denote the tautological bundles on the Grassmannians $\mathrm{Gr}(n+k, W)$ and $\mathrm{Gr}(2, \Lambda^2 W^*)$ respectively. By Lemma A.2 this section is regular, hence we have the Koszul resolution

$$\cdots \to \Lambda^2(\Lambda^2 U \boxtimes L) \to \Lambda^2 U \boxtimes L \to \mathcal{O} \to \mathcal{O}_{\tilde{F}_k} \to 0. \tag{20}$$

Its t-th term equals

$$\Lambda^t(\Lambda^2 U \boxtimes L) = \bigoplus_{a+b=t,\ a \geq b} \Sigma^{(a,b)'}(\Lambda^2 U) \boxtimes \Sigma^{(a,b)} L,$$

where Σ^α is the Schur functor associated with the partition α and α' denotes the transposed partition of α. Let π be the projection of the product $\mathrm{Gr}(n+k, W) \times \mathrm{Gr}(2, \Lambda^2 W^*)$ onto the second factor. We are going to apply the pushforward functor π_* to the above Koszul resolution. For this we will need the following vanishing result.

Lemma A.4. *We have $H^q(\mathrm{Gr}(n+k, W), \Sigma^{(a,b)'}(\Lambda^2 U)) = 0$ for all $q \geq a+b-1$ with only two exceptions:*

$$H^{n-k}(\mathrm{Gr}(n+k, W), \Sigma^{(n-k+1,0)'}(\Lambda^2 U)) = \Lambda^{2k-2} W, \qquad H^0(\mathrm{Gr}(n+k, W), \Sigma^{(0,0)'}(\Lambda^2 U)) = \mathbb{C}.$$

We postpone the proof of the Lemma, and now finish with the Proposition. Note that

$$\pi_*(\Sigma^{(a,b)'}(\Lambda^2 U) \boxtimes \Sigma^{(a,b)} L) \cong H^\bullet(\mathrm{Gr}(n+k, W), \Sigma^{(a,b)'}(\Lambda^2 U)) \otimes \Sigma^{(a,b)} L.$$

Therefore, the spectral sequence of hyperdirect images together with the above Lemma shows that we have the following exact sequence

$$\ldots \longrightarrow \Lambda^{2(k-1)} W \otimes S^{-(n-k+1)} L \xrightarrow{\xi_k} \mathcal{O}_{\mathrm{Gr}(2, \Lambda^2 W^*)} \longrightarrow \pi_* \mathcal{O}_{\tilde{F}_k} \longrightarrow 0,$$

so it follows that $\pi(\tilde{F}_k)$ is the zero locus of a section ξ_k of the vector bundle $\Lambda^{2(k-1)} W^* \otimes S^{n-k+1} L^*$ on $\mathrm{Gr}(2, \Lambda^2 W^*)$. Moreover, since $\tilde{F}_k$ is $\mathrm{GL}(W)$-invariant it follows that ξ_k should be $\mathrm{GL}(W)$-semi-invariant. But as it was mentioned in Section 3, the only $\mathrm{GL}(W)$-semi-invariant section there is σ_k, so $\xi_k = \sigma_k$ and we conclude that $\pi(\tilde{F}_k) = F(\mathrm{Pf}_k(W^*))$. In particular, the projection π restricts to a surjective map $\pi : \tilde{F}_k \to F(\mathrm{Pf}_k(W^*))$, which is precisely what is claimed in the Proposition. □

Corollary A.5. *The Fano scheme $F(\mathrm{Pf}_k(W^*))$ is irreducible.*

Proof of the Lemma A.4. Note that $\Sigma^{(a,b)'}(\Lambda^2 U) \subset \Lambda^a(\Lambda^2 U) \otimes \Lambda^b(\Lambda^2 U)$. The decomposition of $\Lambda^a(\Lambda^2 U)$ into irreducible components is

$$\Lambda^a(\Lambda^2 U) = \bigoplus_{\lambda \in W(a)} \Sigma^{d(\lambda)} U,$$

the sum being taken over the set $W(a)$ of non decreasing sequences $\lambda = (\lambda_1 \geq \cdots \lambda_c \geq c)$ such that $a = |\lambda| + \frac{c(c-1)}{2}$, and we use the notation

$$d(\lambda) = (\lambda_1, \ldots, \lambda_c, c^{\lambda_c - c + 1}, (c-1)^{\lambda_{c-1} - \lambda_c}, \ldots, 1^{\lambda_1 - \lambda_2}).$$

Now assume that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+k})$ is a Young diagram with $n+k$ rows and even number $2m$ of boxes such that $\Sigma^\alpha U$ is a component of $\Sigma^{(a,b)'}(\Lambda^2 U)$. Then $H^q(\mathrm{Gr}(n+k, W), \Sigma^\alpha U) \neq 0$ only if

$$q = p(n-k), \quad \alpha_1 \geq \cdots \geq \alpha_p \geq n-k+p, \quad \text{and} \quad p \geq \alpha_{p+1} \geq \cdots \geq \alpha_{n+k} \quad \text{for some } p.$$

Moreover, $\Sigma^\alpha U$ must be a component of the tensor product of $\Sigma^{d(\lambda)}U$ and $\Sigma^{d(\mu)}U$ for some $\lambda = (\lambda_1 \geq \cdots \geq \lambda_c \geq c) \in W(a)$ and some $\mu = (\mu_1 \geq \cdots \geq \mu_d \geq d) \in W(b)$.

If $p = 0$ then $\alpha = 0$, so we get the case $H^0(\mathrm{Gr}(n+k, W), \Sigma^{(0,0)'}(\Lambda^2 U)) = \mathbb{C}$.

Assume that $p \geq 1$. Recall that by the Littlewood-Richardson rule, the diagram of α is obtained by adding $2b$ boxes to the diagram of $d(\lambda)$ (with certain restrictions imposed by $d(\mu)$). In particular $\alpha_c \geq d(\lambda)_c \geq c$, and therefore $p \geq c$ (and symmetrically $p \geq d$). Because of the special form of the partition $d(\lambda)$, this implies that

$$\begin{aligned} d(\lambda)_{p+1} + \cdots + d(\lambda)_{n+k} &= \tfrac{1}{2}|d(\lambda)| - \tfrac{c(c+1)}{2} - (d(\lambda)_{c+2} + \cdots + d(\lambda)_p) \\ &\geq a - \tfrac{c(c+1)}{2} - c(p-c-1) = a + \tfrac{c(c+1)}{2} - cp. \end{aligned}$$

And of course we have a similar estimate for $d(\lambda)$.

Now, we observe that following the Littlewood-Richardson rule, the diagram of α is obtained from that of $d(\lambda)$ by adding some numbered boxes, with for each i, exactly $d(\mu)_i$ boxes numbered i. Moreover the j-th line can only contain boxes numbered by some $i \leq j$, so all the boxes numbered k must appear on line k or below. This implies that

$$\begin{aligned} \alpha_{p+1} + \cdots + \alpha_{n+k} &\geq d(\lambda)_{p+1} + \cdots + d(\lambda)_{n+k} + d(\mu)_{p+1} + \cdots + d(\mu)_{n+k} \\ &\geq a + \tfrac{c(c+1)}{2} - cp + b + \tfrac{d(d+1)}{2} - dp. \end{aligned}$$

But since $\alpha_1 \geq \cdots \geq \alpha_p \geq n-k+p$, we deduce that

$$\begin{aligned} a+b &\geq p(n-k+p) + \tfrac{c(c+1)}{2} - cp + \tfrac{d(d+1)}{2} - dp \\ &= q + \tfrac{(p-c)^2}{2} + \tfrac{(p-d)^2}{2} + \tfrac{c+d}{2} \geq q+p. \end{aligned}$$

Under the hypothesis that $q \geq a+b-1$, we deduce that $p = c = d = 1$, and the only possibility is $\alpha = (n-k+1, 1, \ldots, 1, 0, \ldots, 0)$ (the number of ones being $n-k+1$). Then $H^{n-k}(\mathrm{Gr}(n+k, W), \Sigma^\alpha U) = \Lambda^{2k-2} W$, which gives the second case, since $\Sigma^\alpha U$ is a component of $\Sigma^{(n-k+1,0)'}(\Lambda^2 U)$ (with multiplicity one), but of no other $\Sigma^{(a,b)'}(\Lambda^2 U)$. □

Proposition A.6. *The map $\pi : \tilde{F}_k \to F(\mathrm{Pf}_k(W^*))$ is birational.*

Proof. Since π is surjective and proper, and $F(\mathrm{Pf}_k(W^*))$ is irreducible, it suffices to check that it is an isomorphism over an open subset of $F(\mathrm{Pf}_k(W^*))$. Let $F_k^0 \subset F(\mathrm{Pf}_k(W^*))$ be the open subset consisting of lines which do not intersect the locus of skew-forms of rank $\leq 2n-2k-2$. Let L be such a line. Note that if $U \subset W$ is a subspace of dimension $n+k$ such that $\phi_{L,U} = 0$, then for each point in L the space U contains a $2k$-dimenisonal subspace lying in the kernel of the corresponding skew-form. But if all skew-forms in L have rank $2n-2k$ then U contains all their kernels. Thus it suffices to check that for such L the linear hull of the kernels of all forms in L has dimension $n+k$. For this we consider the following exact sequence on $\mathbb{P}(L)$:

$$0 \to K(-1) \to W \otimes \mathcal{O}_{\mathbb{P}(L)}(-1) \to W^* \otimes \mathcal{O}_{\mathbb{P}(L)} \to K^* \to 0,$$

where the middle map is the evaluation of a skew-form on a vector and K is the bundle of kernels of skew-forms. Then the codimension of the linear hull of kernels coincides with H^0 of the image I of the middle map. Computing the determinant of this exact sequence we see that $\deg K^* = n-k$. Since K^* is a vector bundle of rank $2k$ on $\mathbb{P}(L) = \mathbb{P}^1$ generated by global sections we conclude by Riemann–Roch that $\dim H^0(\mathbb{P}(L), K^*) = n+k$. On the other hand, from the above exact sequence it follows that the map $H^0(\mathbb{P}(L), W^* \otimes \mathcal{O}_{\mathbb{P}(L)}) \to H^0(\mathbb{P}(L), K^*)$

is surjective, hence $\dim H^0(\mathbb{P}(L), I) = 2n - (n+k) = n-k$. So the codimension of the linear hull of kernels is $n-k$, hence the dimension is $n+k$ as required. □

The next goal is to investigate where the map π is not an isomorphism. For this we need the following.

Lemma A.7. *Let $L \in F(\mathrm{Pf}_k(W^*))$ be a line on $\mathrm{Pf}_k(W^*)$. If $L \cap \mathrm{Pf}_{k+1}(W^*) \neq \varnothing$ then $\pi^{-1}(L)$ contains a line.*

Proof. Let $U \subset W$ be a subspace of dimension $n+k$ isotropic for L and assume that the form λ corresponding to a point of L has rank strictly less than $2n-2k$. Then there exists a vector subspace $U' \subset W$ of dimension $n+k+1$ containing U and isotropic for λ. Let $U' = U \oplus \mathbb{C}w$. Let λ' be another skew-form in L. Then $\lambda'(-, w)$ is a linear form on U. Let U'' be its kernel. Then it is clear that each vector subspace of dimension $n+k$ in U' containing U'' is isotropic both for λ and λ'. These subspaces form a line in $\mathrm{Gr}(n+k, W)$ which is contained in $\pi^{-1}(L)$. □

By Lemma A.2 variety $\tilde{F}_k$ is smooth and by Proposition A.6 the map $\pi : \tilde{F}_k \to F(\mathrm{Pf}_k(W^*))$ is birational, hence $\tilde{F}_k$ is a resolution of $F(\mathrm{Pf}_k(W^*))$. The final statement of the Theorem is given by the following

Lemma A.8. *The resolution $\pi : \tilde{F}_1 \to F(\mathrm{Pf}(W^*))$ is crepant.*

Proof. Compute the canonical class of $\tilde{F}_1$. Recall that $\tilde{F}_1$ is the zero locus of a regular section of $\Lambda^2 U^* \boxtimes L^*$ on $\mathrm{Gr}(n+1, W) \times \mathrm{Gr}(2, \Lambda^2 W^*)$. We have

$$\omega_{\mathrm{Gr}(n+1,W)\times\mathrm{Gr}(2,\Lambda^2 W^*)} = \mathcal{O}(-2n, -n(2n-1)),$$
$$\det(\Lambda^2 U^* \boxtimes L^*) = \det(\Lambda^2 U^*)^{\otimes 2} \otimes (\det L^*)^{\otimes n(n+1)/2} = \mathcal{O}(2n, n(n+1)/2),$$

so by adjunction we get $\omega_{\tilde{F}_1} \cong \mathcal{O}(0, -3n(n-1)/2)$. Thus the canonical class of $\tilde{F}_1$ is a pullback from $F(\mathrm{Pf}(W^*)) \subset \mathrm{Gr}(2, \Lambda^2 W^*)$, so $\pi : \tilde{F}_1 \to F(\mathrm{Pf}(W^*))$ is crepant. □

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