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Cubic hypersurfaces and integrable systems

Atanas Iliev, Laurent Manivel

October 14, 2008

Abstract

Together with the cubic and quartic threefolds, the cubic fivefolds are the only hypersurfaces of odd dimension bigger than one for which the intermediate Jacobian is a nonzero principally polarized abelian variety (p.p.a.v.). In this paper we show that the family of 21-dimensional intermediate Jacobians of cubic fivefolds containing a given cubic fourfold X is generically an algebraic integrable system. In the proof we apply an integrability criterion, introduced and used by Donagi and Markman to find a similar integrable system over the family of cubic threefolds in X . To enter in the conditions of this criterion, we write down explicitly the symplectic structure, known by Beauville and Donagi, on the family $F(X)$ of lines on the general cubic fourfold X , and prove that the family of planes on a cubic fivefold containing X is embedded as a Lagrangian surface in $F(X)$. By a symplectic reduction we deduce that our integrable system induces on the nodal boundary another integrable system, interpreted generically as the family of 20-dimensional intermediate Jacobians of Fano threefolds of genus four contained in X . Along the way we prove an Abel-Jacobi type isomorphism for the Fano surface of conics in the general Fano threefold of genus four, and compute the numerical invariants of this surface.

1 Introduction

1.1 Background

For a smooth compact Kähler manifold Y of dimension n and any positive integer $q \leq n$, its q -th intermediate Jacobian

$$J_q(Y) = (H^{q-1,q}(Y) \oplus \dots \oplus H^{0,2q-1}(Y))/H^{2q-1}(Y, \mathbb{Z})$$

is a complex torus. Because of the skew-symmetry of the Riemann-Hodge bilinear relations the tori $J_q(Y)$ aren't in general abelian varieties, except for the Picard variety $J_1(Y) = \text{Pic}^0(Y)$ and the Albanese variety $J_n(Y) = \text{Alb}(Y)$, see Ch.2 §6 in [CG]. If $n = \dim Y = 2q - 1$ is odd then $J(Y) = J_q(Y)$ is simply called the *intermediate Jacobian* of Y . In the particular case when $Y \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree d , the intermediate Jacobian $J(Y)$ happens to be a non trivial abelian variety only when $n = 1$ and $d \geq 3$, $n = 3$ and $d = 3, 4$ (i.e. when Y is a cubic or quartic 3-fold), or $n = 5$ and $d = 3$ (i.e. when Y is a cubic 5-fold), see p.43 in [Col].

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In [DM2] Donagi and Markman studied certain complex algebraic analogs of integrable systems in mechanics. These analogs take as models the integrable Hamiltonian systems $f : M \rightarrow B$, where the phase space M is a real symplectic $2n$ -fold, and the general fibers of f are compact Lagrangian n -folds in M , which by the Arnold-Liouville theorem are n -fold real tori. In §7 of [DM2] are given general criteria (the cubic conditions) which ensure when an algebraic family $f : M \rightarrow B$ of complex tori represents an algebraic integrable system; for more details see §2 in [DM2] or §2 in [Fr].

The main application of these criteria is the observation that the relative intermediate Jacobians over the gauged moduli spaces of Calabi-Yau threefolds are in fact such algebraic integrable systems, see [DM1]. This gives rise to the question whether besides the Calabi-Yau integrable systems there exist other nontrivial examples of families of intermediate Jacobians that are also algebraic integrable systems, see e.g. Question 4.6 in [AS]. It seems that until now, the only known such nontrivial example (besides the integrable systems coming from families of Jacobians of curves) is given in Example 8.22 of [DM2]:

If $X \subset \mathbb{P}^5$ is a smooth cubic fourfold, and $B = \mathcal{H}_s$ is the open subset in the dual projective space parameterizing the smooth hyperplane sections Y_b of X , then the relative intermediate Jacobian $f : \mathcal{J}(\mathcal{Y}) \rightarrow \mathcal{H}_s$ is an algebraic integrable system.

This example is based on an abstract procedure, described in §8 of [DM2], that generates algebraic integrable systems in the context of Lagrangian deformations. More precisely, for a given complex symplectic variety W , its Lagrangian Hilbert scheme parameterizes the Lagrangian subvarieties of W . By a result of Voisin and Ran the deformations of a smooth Lagrangian subvariety $F \subset W$ are unobstructed and the component \mathcal{H} of the Lagrangian Hilbert scheme of W containing F is generically smooth, see [Voi, Ran]. Consider the universal family $\mathcal{F} \rightarrow \mathcal{H}$ and its Picard bundle $\text{Pic}^o \mathcal{F} \rightarrow \mathcal{H}$. In this context, Theorem 8.1 in [DM2] states:

(DM) *If W is a smooth complex symplectic variety and F is a smooth Lagrangian subvariety of W , then the symplectic structure on W generates a natural symplectic structure on the relative Picard $\text{Pic}^o \mathcal{F} \rightarrow \mathcal{H}_s$ over the base \mathcal{H}_s of smooth Lagrangian deformations of F in W , making the relative Picard $\text{Pic}^o \mathcal{F} \rightarrow \mathcal{H}_s$ an algebraic integrable Hamiltonian system.*

In the cited Example 8.22, the above criterion **(DM)** is applied in the case where $W = F(X)$ is the family of lines on a smooth cubic hypersurface $X \subset \mathbb{P}^5$ and $F = F(Y) \subset F(X)$ is the Fano surface of lines on a smooth hyperplane section Y of X . To enter in the conditions of **(DM)**, the following results are used:

1. *The family $F(X)$ is a symplectic 4-fold, see [BD],*
2. *$F(Y) \subset F(X)$ is a Lagrangian surface in $F(X)$, see Ex.7 in §3 of [Voi],*
3. *For a smooth cubic 3-fold Y , its intermediate Jacobian $J(Y)$ is a principally polarized abelian variety of dimension 5, the family $F(Y)$ of lines on Y is a smooth surface of irregularity 5, and the Abel-Jacobi map defines an isomorphism between $\text{Alb}F(Y)$ and $J(Y)$, see [CG].*

The most common situation when **(DM)** can be applied is the case when the symplectic variety $W = S$ is a K3 surface and $F = C$ is a smooth curve of genus g on S ; notice that any curve on a K3 surface S is a Lagrangian subvariety of S . In this lowest dimensional case, the base \mathcal{H}_s of the smooth Lagrangian deformation of C in S is an open subset of the complete linear system $|\mathcal{O}_S(C)| \cong \mathbb{P}^g$, and **(DM)** yields that the relative Jacobian \mathcal{J} is a Lagrangian fibration over \mathcal{H}_s . When $S = S_{2g-2}$ is a K3 surface with a primitive polarization of genus $g = g(C)$ this fibration can be extended to a Lagrangian fibration $\overline{\mathcal{J}} \rightarrow \mathbb{P}^g$ over the compactified

relative Jacobian $\overline{\mathcal{J}}$, which is a smooth compact complex symplectic variety birational to the g -th punctual Hilbert scheme $Hilb_g S$, see §2 in [B2]. In this trend, J. Sawon, in a collaboration with K. Yoshioka, has recently shown that for any $g, m \geq 2$ the g -th Hilbert power $Hilb_g T$ of the general primitive K3 surface T of degree $m^2(2g-2)$ can be represented as a torsor over the compactified relative Jacobian $\overline{\mathcal{J}} \rightarrow \mathbb{P}^g$ of a primitive K3 surface $S = S(T)$ of degree $2g-2$, thus proving the existence of a regular Lagrangian fibration over \mathbb{P}^g on the smooth compact complex symplectic variety $Hilb_g T$, see [Saw].

The above examples give rise to the question when the relative Picard fibrations from **(DM)** can be extended to Lagrangian fibrations including fibers over singular Lagrangian deformations of $F \subset W$. For general integrable systems of type relative Picard as in **(DM)**, a partial compactification is described in Theorem 8.18 of [DM2] but the proof remains unpublished.

At the end of the same Example 8.22 of [DM2] it is shown that the symplectic structure on $Pic^o(\mathcal{F}) \rightarrow \mathcal{H}_s$ can be extended to a still non-degenerate symplectic structure over the set $\mathcal{H}_n \supset \mathcal{H}_s$ of hyperplane sections Y of X that are allowed to be singular but with at most one node. For a general nodal cubic 3-fold $Y \in \partial\mathcal{H}_n := \mathcal{H}_n - \mathcal{H}_s$ the family of lines $\ell \subset Y$ that pass through the node of Y is parameterized by a smooth curve C of genus 4, and the generalized intermediate Jacobian $J(Y)$ of Y is a \mathbb{C}^* -extension of the Jacobian $J(C)$, see e.g. [CM]. Then the relative Jacobian fibration $\mathcal{J}(\mathcal{Y}) \rightarrow \mathcal{H}_n$ induces over the nodal boundary $\partial\mathcal{H}_n$ a boundary fibration $\mathcal{J}(C) \rightarrow \partial\mathcal{H}_n$ whose fibers are the Jacobians $J(C)$ of the genus 4 curves C . All this makes it possible to conclude that the boundary abelian fibration $\mathcal{J}(C) \rightarrow \partial\mathcal{H}_n$ of the algebraic integrable system $\mathcal{J}(\mathcal{Y}) \rightarrow \mathcal{H}_n$ is also an algebraic integrable system, at least over an open subset of $\partial\mathcal{H}_n$. Donagi and Markman call this system the *boundary integrable system* for $\mathcal{J}(\mathcal{Y})$, see the end of §8 in [DM2]. As communicated to us by Ron Donagi, this is an instance of the algebraic symplectic reduction, described in §2 of his later paper [DP] with E. Previato.

1.2 Summary of the results in the paper

In this paper we describe a new example of an algebraic integrable family of intermediate Jacobians in the context of the integrability conditions **(DM)**. It is an analog of the Example 8.22 from [DM2], where the fibers of the integrable system are the 5-dimensional intermediate Jacobians of smooth cubic 3-folds contained in a fixed cubic fourfold as hyperplane sections. In our case the fibers of the integrable system are the 21-dimensional intermediate Jacobians of the general cubic fivefolds containing the same cubic fourfold X as a hyperplane section. By Theorem 11, the first main conclusion in our paper:

The relative intermediate Jacobian is an algebraic integrable system over an open subset of the family of isomorphism classes of cubic fivefolds containing a fixed general cubic fourfold X as a hyperplane section.

To be more precise, the base of the integrable system is a quotient of an open subset of the family of cubic fivefolds containing X , by the subgroup of PGL_7 acting trivially on the hyperplane spanned by X .

Notice once again that the cubic fivefolds are the *unique* hypersurfaces of odd dimension bigger than three for which the intermediate Jacobian is a non trivial abelian variety. Next, we study the degeneration of this integrable system on the boundary parameterizing the nodal cubic fivefolds through X , and prove that the induced boundary abelian fibration is also an algebraic integrable system.

In our case the symplectic variety W from **(DM)** is still the same as in the example of Donagi and Markman: the 4-fold family $W = F(X)$ of lines on a fixed smooth cubic fourfold X . The difference is in the choice of the Lagrangian subvariety of departure $F \subset W$. At this point we should mention that discovering principally new or nontrivial examples of Lagrangian subvarieties of symplectic varieties, especially in a projective-geometric context, looks like a happy occurrence. In our case this occurrence is realized as the Fano surface $F_2(Z)$ of planes on the general cubic fivefold Z that contains X as a hyperplane section; the surface $F_2(Z)$ is regarded as a subvariety of $F(X)$ by the embedding given by the intersection map $i_Z : F_2(Z) \rightarrow F(X)$, $\mathbb{P}^2 \mapsto \mathbb{P}^2 \cap X$. We prove, see Proposition 4:

The intersection image $i_Z(F_2(Z))$ of the Fano surface $F_2(Z)$ of planes on the general cubic fivefold Z containing the cubic fourfold X as a hyperplane section, is a Lagrangian surface inside the 4-fold family $F(X)$ of lines in X (in fact this surface is not smooth, but has ordinary double points as singularities).

This relies on an explicit description of the symplectic form on $F(X)$, which is provided in §2.1. Recall that in [BD] the symplectic form on $F(X)$ is described only for the general Pfaffian cubic 4-folds X , which form a divisor in the space of all cubic fourfolds $X \subset \mathbb{P}^5$. For a general Pfaffian cubic fourfold X , the family $F(X)$ is known to be isomorphic to the Hilbert square $Hilb_2 S$ of a K3 surface S of genus 8, and the symplectic form on $F(X)$ is the symplectic form on $Hilb_2 S$ described earlier by Fujiki and Beauville, see [B1]. The existence of a symplectic form on $F(X)$ for the general cubic fourfold X then follows by a deformation argument, see [BD].

By Proposition 4 we enter in the conditions of **(DM)**, this time with the symplectic fourfold $F(X)$ and its Lagrangian surface $i_Z(F_2(Z))$. This gives rise to a Lagrangian fibration on the relative Picard $Pic^o \mathcal{F} \rightarrow \mathcal{H}_s$ over the scheme \mathcal{H}_o of smooth deformations of $i_Z(F_2(Z))$ in $F(X)$, see the beginning of Section 5. In this situation, the analog of **3** above is a result of Collino:

The family $F_2(Z)$ of lines on the general cubic 5-fold Z is a smooth surface of irregularity 21, and the Abel-Jacobi map induces an isomorphism between the Albanese variety $Alb F_2(Z)$ and the intermediate Jacobian $J(Z)$, see [Col].

In fact $h^{5,0}(Z)$ and $h^{4,1}(Z)$ vanish, while $h^{2,1}(Z) = 21$, which explains why $J(Z)$ is a p.p.a.v. of dimension 21. By duality, $J(Z)$ is also isomorphic with the Picard variety $Pic^o F_2(Z)$. This implies Theorem 11.

Next, we study a partial compactification of the relative Jacobian fibration from Theorem 11, by including in its base the codimension one boundary $\partial \mathcal{H}_n$ of cubic fivefolds with one node.

For a general nodal cubic 5-fold Z , the family of lines $\ell \subset Z$ which pass through the node of Z is parameterized by a general smooth prime Fano threefold Y of genus 4. This threefold Y is the analog of the genus 4 curve C from the boundary system in Example 8.22 of [DM2]. Since in this situation the generalized intermediate Jacobian $J(Z)$ is a \mathbb{C}^* -extension of $J(Y)$, the boundary fibration $\mathcal{J}(\mathcal{Y})$ has for base the set of nodal cubic 5-folds Z containing X and as general fibers the 20-dimensional intermediate Jacobians of their associated Fano 3-folds Y of genus 4. Such a Fano 3-fold Y is a complete intersection of a quadric and a cubic in \mathbb{P}^5 . While the cubic can be chosen to be X , the quadric (identified with the base of the projective tangent cone to the node of Z) can move.

In §2 we find the analog of **2** for a general nodal cubic fivefold Z . For this, we show in §2.2 that the Fano surface $F_2(Z)$ of planes on Z is almost the same as the Fano surface $F(Y)$ of

conics on Y . More precisely, the Fano surface $F_2(Z)$ has a double curve $\Gamma(Y)$, isomorphic to the family of lines on Y , and there is a natural map $\nu : F(Y) \rightarrow F_2(Z)$, which is a desingularization, see Proposition 9. We conclude that the surface $F = i_Z(F_2(Z))$ is a singular Lagrangian surface in $F(X)$, see Proposition 10.

In §4.1 we study in more detail the Fano surface $F(Y)$ of the general prime Fano 3-fold Y of genus 4. In particular we find its numerical invariants, see Corollary 15. Next, in §4.2 we prove the following analog of Collino's Abel-Jacobi isomorphism for planes on cubic fivefolds:

The family $F(Y)$ of conics on the general prime Fano 3-fold Y of genus 4 is a smooth surface of irregularity 20, and the Abel-Jacobi map induces an isomorphism between $\text{Alb}F(Y)$ and the intermediate Jacobian $J(Y)$, see Theorem 20.

To prove Theorem 20, we follow the same program as Clemens and Griffiths in their proof of the Abel-Jacobi isomorphism for the cubic threefold. This program was subsequently applied in [Let] to prove a similar Abel-Jacobi isomorphism for the Fano surface of conics on the general quartic hypersurface in \mathbb{P}^4 , then in [Col] for the Fano surface of planes on the general cubic 5-fold. In brief, this program consists in verifying that in a general Leftschetz pencil $\{Y_t : t \in \mathbb{P}^1\}$ of Fano 3-folds of genus 4, for a finite number of values of t either the Fano surface $F(Y_t)$ acquires isolated singular points but Y_t remains smooth, or when Y_t becomes singular then its singularity is a simple node and in this case $F(Y_t)$ has a smooth double curve for singular locus, see [Let] and §3.2. As above, the Abel-Jacobi isomorphism makes it possible to identify $\text{Pic}^0 F(Y)$ with $J(Y)$.

In §3.2 we use these results, together with the algebraic symplectic reduction procedure from [DP] to get Theorem 16, the second main conclusion of this paper:

The algebraic integrable system from Theorem 11 induces on the nodal boundary $\partial\mathcal{H}_n$ a fibration by intermediate Jacobians of Fano 3-folds of genus 4 which is generically an algebraic integrable system.

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2 Revisiting the family of lines on the cubic fourfold

2.1 The symplectic form

Let $X \subset \mathbb{P}V = \mathbb{P}^5$ be a general cubic hypersurface, with equation $P = 0$ for some general polynomial $P \in S^3V^*$. The Fano variety $F(X)$ of lines contained in X is a subvariety of the Grassmannian $G(2, V)$. It can be defined as the zero locus of the global section s_P of the vector bundle S^3T^* , naturally defined by P , where T denotes the rank two tautological bundle on the Grassmannian. Since s_P is a general section of the globally generated vector bundle S^3T^* , $F(X)$ is a smooth four dimensional variety.

It was proved by Beauville and Donagi that $F(X)$ has a symplectic structure. Indeed they showed that for X a Pfaffian cubic hypersurface there exists a K3 surface S such that $F(X) =$

$Hilb_2S$, which is well known to inherit a symplectic structure from that of S . Then the general case follows by a deformation argument, see [BD]. Nevertheless the use of deformations makes that the symplectic form on $F(X)$ is not explicit. Markushevich and Tikhomirov [MT] showed how to deduce it from the Voisin's observation that the Fano varieties of lines in the hyperplane sections of X are Lagrangian surfaces in $F(X)$ (see section 3). We shall give below a completely explicit and down-to-earth description of this form. In fact it suffices to exhibit a non zero holomorphic two form on $F(X)$ – indeed since $h^{2,0}(F(X)) = 1$ we know that there exists a unique such form, so it must define the symplectic structure, that it, it will automatically be a non degenerate closed form. Consider the tangent exact sequence

$$0 \rightarrow TF(X) \rightarrow TG(2, V)|_{F(X)} \rightarrow S^3T^*_{|F(X)} \rightarrow 0.$$

Remember that $TG(2, V) \simeq Hom(T, Q)$, where $Q = V/T$ denote the quotient bundle on the Grassmannian. Therefore, the tangent space to $TF(X)$ at a line $\ell = \mathbb{P}T$ is

$$T_\ell F(X) = \{u \in Hom(T, Q), \quad P(x, x, \bar{u}(x)) = 0 \forall x \in T\},$$

where $P(x, y, z)$ is the polarization of $P(x)$ and $\bar{u}(x) \in V$ is any representative of $u(x) \in Q$.

Note that $P(x, x, \cdot) = 0$ is an equation of $T_x X$. If ℓ is a general line on X and x varies in ℓ , we get a quadratic pencil of hyperplanes in $\mathbb{P}V$, whose intersection is a plane $\Pi = \mathbb{P}S \supset \ell = \mathbb{P}T$. In particular, $T_\ell F(X) \supset Hom(T, S/T)$.

There is a natural skew-symmetric form on $T_\ell F(X)$, defined up to scalar as follows. Choose some vectors e, f, g, h of T and let, for $u, v \in T_\ell F(X)$,

$$\Psi(u, v; e, f, g, h) = P(e, e, u(g))P(f, f, v(h)) - P(e, e, v(g))P(f, f, u(h)).$$

This can be seen as defining a skew-symmetric bilinear map on $S^2T \otimes T$ with values in $\wedge^2 T_\ell F(X)^*$. Note that $S^2T \otimes T = S^3T \oplus T \otimes \wedge^2 T$. We get the component S^3T by letting $g = e$ in the formula above, which gives zero on $T_\ell F(X)^*$. So we need only consider the other component, which gives a map $(\wedge^2 T)^3 \rightarrow \wedge^2 T_\ell F(X)^*$ defined by the formula

$$\begin{aligned} \omega(u, v; e, f) &= P(e, e, u(f))P(f, f, v(e)) - P(e, e, v(f))P(f, f, u(e)) \\ &\quad + 2P(e, f, u(f))P(e, e, v(f)) - 2P(e, e, u(f))P(e, f, v(f)) \\ &\quad + 2P(f, f, u(e))P(e, f, v(e)) - 2P(e, f, u(e))P(f, f, v(e)). \end{aligned}$$

We can see ω as a form with values in the line bundle $(\wedge^2 T^*)^{\otimes 3} = \mathcal{O}(3)$. This form has rank two at the generic point of $F(X)$ and its radical is precisely $Hom(T, S/T)$.

There is also a natural quadratic form on $\wedge^2 T_\ell G(2, V)$, defined again up to scalar by the following formula: if $u, v, u', v' \in Hom(T, Q)$ and e, f is a basis of T , let

$$\begin{aligned} K(u \wedge v, u' \wedge v') &= u(e) \wedge u'(f) \wedge v(e) \wedge v'(f) - u(f) \wedge u'(f) \wedge v(e) \wedge v'(e) \\ &\quad + u(f) \wedge u'(e) \wedge v(f) \wedge v'(e) - u(e) \wedge u'(e) \wedge v(f) \wedge v'(f), \end{aligned}$$

seen as a element of the line $\wedge^4 Q$. More precisely, K is a quadratic form on $\wedge^2 T_\ell G(2, V)$ with values in the line bundle $Hom((\wedge^2 T), \wedge^4 Q) = \mathcal{O}(3) \otimes \det(V)$.

Observe that this form can be described in terms of the natural decomposition $\wedge^2 T_\ell G(2, V) = \wedge^2 T^* \otimes S^2 Q \oplus S^2 T^* \otimes \wedge^2 Q$, as the composition of the natural morphisms

$$S^2(\wedge^2 T_\ell G(2, V)) \rightarrow S^2(S^2 T^* \otimes \wedge^2 Q) \rightarrow S^2(S^2 T^*) \otimes S^2(\wedge^2 Q) \rightarrow (\wedge^2 T^*)^{\otimes 2} \otimes \wedge^4 Q.$$

Now we restrict the quadratic form K to $\wedge^2 T_\ell F(X)$. We claim that the restriction has rank at most two. Indeed, it is clear that K vanishes on $\text{Hom}(T, S/T) \wedge T_\ell F(X)$, a hyperplane in $\wedge^2 T_\ell F(X)$. In particular we can write

$$K(u \wedge v, u \wedge v) = \omega(u, v)\Omega(u, v)$$

for $u, v \in T_\ell F(X)$, where Ω is now a well-defined skew-symmetric form on $T_\ell F(X)$ (with values in $\det(V)$, to be precise), since K and ω both take values in $\mathcal{O}(3)$.

Theorem 1 *The form Ω defines a symplectic structure on the Fano variety $F(X)$.*

Proof. As explained above we just need to prove that Ω is non trivial at some point $\ell = \mathbb{P}T$ of $F(X)$. This reduces to an easy computation in coordinates: we chose a basis e_1, \dots, e_6 of V such that $T = \langle e_1, e_2 \rangle$. If z_1, \dots, z_6 are the corresponding coordinates on V , we may suppose that $P(e_1, e_1, z) = z_4$, $P(e_1, e_2, z) = z_5$, $P(e_2, e_2, z) = z_6$. In particular the intersection of the tangent spaces of X along ℓ is $S = \langle e_1, e_2, e_3 \rangle$.

An element of the tangent space $T_\ell F(X)$ is a homomorphism $u \in \text{Hom}(T, Q)$ such that

$$\begin{aligned} u(e_1) &= ae_3 + be_5 - 2ce_6, \\ u(e_2) &= de_3 - 2be_4 + ce_5 \end{aligned}$$

for some scalars a, b, c, d . Take another $u' \in T_\ell F(X)$ defined by the scalars a', b', c', d' . Then $K(u \wedge u', u \wedge u')$ is proportional to $u(e_1) \wedge u(e_2) \wedge u'(e_1) \wedge u'(e_2)$, that is, to the determinant

$$\left\| \begin{array}{cccc} a & d & a' & d' \\ 0 & -2b & 0 & -2b' \\ b & c & b' & c' \\ -2c & 0 & -2c' & 0 \end{array} \right\| = 4(bc' - b'c)(a'c - ac' + bd' - b'd).$$

The factor $bc' - b'c$ is proportional to $\omega(u, u')$. We conclude that $\Omega(u, u')$ is proportional to $a'c - ac' + bd' - b'd$, so that Ω has maximal rank. \square

Note that $\omega(u, u') = 0$ if and only if the tangent plane generated by u and u' contains an element $u'' \in \text{Hom}(T, Q)$ whose image is contained in S/T . If it is not the case, then this tangent plane is isotropic if and only if

$$u(T) + u'(T) \neq Q,$$

that is, there is a hyperplane $H \subset Q$ containing the image of any u'' in the tangent plane.

2.2 Lagrangian surfaces in $F(X)$

Our very explicit description of the symplectic structure of $F(X)$ will help us to identify Lagrangian surfaces.

2.2.1 Voisin's example

First we recover the following observation of Claire Voisin (see [Voi]).

Proposition 2 *Let $Y = X \cap H$ be a generic hyperplane section. Then the Fano surface $F(Y)$ of lines in Y is a Lagrangian surface in $F(X)$.*

Proof. For u, u' in $T_\ell F(Y) \subset \text{Hom}(T, Q)$, the images $u(T)$ and $u'(T)$ are contained in the hyperplane of Q defined by H . So clearly $K(u \wedge u', u \wedge u') = 0$. But $\omega(u, u')$ is non zero, at least generically, so $\Omega(u, u') = 0$. \square

Remark. As observed by the referee, one can also argue as follows. The universal families on $F(X)$ and $F(Y)$ allow to define a commutative diagram

$$\begin{array}{ccc} H^4(X) & \longrightarrow & H^2(F(X)) \\ \downarrow & & \downarrow \\ H^4(Y) & \longrightarrow & H^2(F(Y)). \end{array}$$

The symplectic form on $F(X)$ defines a class in $H^{2,0}(F(X))$ which can be lifted to $H^{3,1}(X)$. Therefore its restriction to $F(Y)$ is the image of a class in $H^{3,1}(Y)$. But this group is zero, so $F(Y)$ is Lagrangian.

2.2.2 Planes in cubic fivefolds

We now turn to the inverse situation where X is a hyperplane section of a cubic fivefold $Z \subset \mathbb{P}^6 = \mathbb{P}W$, say $X = Z \cap H$ with $H = \mathbb{P}V$. Generically, such a cubic fivefold contains projective planes, and the Fano variety $F_2(Z)$ of planes in Z is a smooth surface in the Grassmannian $G(3, W)$. Cutting such a plane Π with the hyperplane H we get a line in X , because X contains no plane. Hence a map

$$i_Z : F_2(Z) \longrightarrow F(X).$$

Lemma 3 *For general X and Z , the map i_Z is a closed immersion.*

Proof. Let $\Pi = \mathbb{P}S$ be a plane in Z , such that $\ell = P \cap H$ is a line in X . If $R = 0$ is an equation of Z , we have

$$T_\Pi F_2(Z) = \{u \in \text{Hom}(S, W/S), \quad R(x, x, u(x)) = 0 \forall x \in S\}.$$

The differential of i_Z at Π maps $u \in \text{Hom}(S, W/S)$ to its restriction $u|_T$ from T to $V/T \simeq W/S$. If this restriction is zero, then $u = e^* \otimes f$ has rank one and $R(x, x, f) = 0$ for all $x \in S$. This means that the intersection of the tangent hyperplanes to Z along Π intersect along a linear space strictly larger than Π . Call such a plane *special*. We claim that a general cubic fivefold contains no special plane. Indeed, choose linear coordinates x_0, \dots, x_6 on \mathbb{P}^6 , such that the plane Π is defined by $x_3 = x_4 = x_5 = x_6 = 0$. If Z contains Π , its equation is of the form $x_3Q_3 + x_4Q_4 + x_5Q_5 + x_6Q_6$ for some quadrics Q_3, \dots, Q_6 . The intersection of the tangent hyperplanes to Z along Π contains Π^+ of equations $x_4 = x_5 = x_6 = 0$ if and only if the quadric $Q_3 = 0$ contains Π . So Π is special if and only if it is special in the sense of [Col]. Collino proves that if Π is contained in the smooth locus of Z , then it defines a smooth point of $F_2(Z)$ if and only if it is non special. In particular a smooth Z such that $F_2(Z)$ is smooth contains no special plane, and i_Z is immersive. \square

Proposition 4 *The map i_Z maps $F_2(Z)$ to a Lagrangian surface in $F(X)$.*

Proof. If ω vanishes on $(i_Z)_*T_\Pi F_2(Z)$, then $T_\Pi F_2(Z)$ must contain a morphism $u \in \text{Hom}(S, W/S)$ whose restriction to T has rank one. But this cannot happen generically, so we just need to prove that our quadratic form K on $\wedge^2 T_\ell F(X)$ vanishes on the line $\wedge^2(i_Z)_*T_\Pi F_2(Z)$.

By definition of K this means that two morphisms in $T_\Pi F_2(Z)$ cannot send T to two independent subspaces in W/S . So our claim follows from the following observation:

Lemma 5 *There exists a hyperplane h_Π in W/S such that for all $u \in T_\Pi F_2(Z) \subset \text{Hom}(S, W/S)$, we have $u(S) \subset h_\Pi$.*

Proof. Again we just need to prove this for a general cubic fivefold Z and a general plane $\Pi \in F_2(Z)$. The cubic equation R of Z defines a linear map $\rho : W/S \rightarrow \text{Sym}^2 S^*$, mapping \bar{w} to the quadratic form $x \mapsto R(x, x, w)$ on S , for any representative w of \bar{w} in W . Consider the diagram

$$\begin{array}{ccc} & & \wedge^2 S^* \otimes S^* \\ & & \downarrow \\ T_\Pi F_2(Z) \subset \text{Hom}(S, W/S) & \xrightarrow{id \otimes \rho} & S^* \otimes \text{Sym}^2 S^* \\ & & \downarrow \\ & & \text{Sym}^3 S^* \end{array}$$

where the vertical strand is part of a Koszul complex. Since $T_\Pi F_2(Z)$ maps to zero in $\text{Sym}^3 S^*$, its image by $id \otimes \rho$ lies in the image of $\wedge^2 S^* \otimes S^*$. Note that since S has dimension three, $\wedge^2 S^* \otimes S^* = \det S \otimes \text{End}(S)$. Once we chose a generator α of $\det S$, we thus conclude that for any $u \in T_\Pi F_2(Z)$, there exists some $\theta_u \in \text{End}(S)$ such that

$$R(x, x, u(y))\alpha = \theta_u(x) \wedge x \wedge y \quad \forall x, y \in S.$$

Generically, the endomorphism θ_u is semisimple, and for any $z \in u(S)$ the quadratic form $x \mapsto R(x, x, z)$ vanishes along the three eigenlines of θ_u .

We interpret this as follows. The pull-back by ρ of the discriminant defines a cubic surface Σ in $\mathbb{P}(W/S)$, and this surface has only isolated singularities by the genericity assumption. Since the image of the plane $\mathbb{P}u(S)$ by ρ is the net of conics passing through three general points, $\mathbb{P}u(S)$ is a tritangent plane to the surface Σ . But there are only a finite number of such tritangent planes, so $u(S)$ does not depend on u (as long as u has maximal rank). This is what we wanted to prove. \square

Remark. An alternative proof can also be given along the same lines as for Proposition 2, using again that $H^{3,1}(Z) = 0$.

Beware that i_Z is not injective, so that $i_Z(F_2(Z))$ is in fact a singular Lagrangian surface in $F(X)$. Indeed, a simple dimension count shows that the set of pairs (P, P') of (distinct) projective planes contained in a general cubic fivefold $Z \subset \mathbb{P}^6$, and meeting along a line, has expected dimension two. This implies that a finite number δ of lines $\ell = P \cap P'$ will be contained in a general hyperplane $H \subset \mathbb{P}^6$ (with $\delta > 0$ if this expected dimension is the effective one). And we can conclude that $i_Z(F_2(Z))$ has precisely δ double points.

As suggested by the referee, the number δ can be computed by noting that the self-intersection of a Lagrangian subvariety in a symplectic manifold is equal to its topological Euler characteristic. In order to compute the self-intersection of $i_Z(F_2(Z))$, we need the following Lemma. Let T denote the rank two tautological vector bundle on $G(2, 6)$, restricted to $F(X)$.

Lemma 6 *The fundamental class of $i_Z(F_2(Z))$ in the Chow ring of $F(X)$ is $63c_2(T)$.*

Proof. Consider the projective bundle $\pi : P = \mathbb{P}(W/T) \rightarrow F(X)$. Over P we have the rank-three tautological vector bundle S such that $S/T = \mathcal{O}_P(-1)$. The inclusion $S \subset V \otimes \mathcal{O}_P$ induces a map $P \rightarrow G(3, 7)$. Let K denote the kernel of the natural restriction map $Sym^3 S^* \rightarrow Sym^3 T^*$, a vector bundle of rank 6 on P . An equation of Z defines a global section of K , whose zero locus $F_P(Z)$ is mapped isomorphically to $F_2(Z)$ in $G(3, 7)$, in such a way that the restriction of π to $F_P(Z)$ gets identified with i_Z . We conclude that we can compute the fundamental classes of these surfaces as $[F_P(Z)] = c_6(K)$ and $[i_Z(F_2(Z))] = \pi_* c_6(K)$.

These computations are straightforward. First, K has a filtration with successive quotients $Sym^3(S/T)^*$, $Sym^2(S/T)^* \otimes T^*$, $(S/T)^* \otimes Sym^2 T^*$. Hence

$$\begin{aligned} c_6(K) &= 3hc_2(T^* \otimes \mathcal{O}_P(2))c_3(Sym^2 T^* \otimes \mathcal{O}_P(1)) \\ &= 3h(c_2 + 2hc_1 + 4h^2)(4c_1c_2 + (4c_2 + 2c_1^2)h + 3c_1h^2 + h^3), \end{aligned}$$

where $h = c_1(\mathcal{O}_P(1))$ and c_1, c_2 denote the Chern classes of T^* . Now recall that the Chow ring of P is a free module over that of $F(X)$, generated by the class h modulo the relation $h^5 + \sum_{k>0} c_k(W/T)h^{5-k} = 0$. In particular, up to lower order terms in h , we deduce that $h^5 = -c_1h^4$ and $h^6 = c_2h^4$ hence $c_6(K) = 63h^4c_2$. Since $\pi_* h^4 = 1$, while the lower order terms in h are killed by π_* , we deduce that $\pi_* c_6(K) = 63c_2$. \square

Observe that $c_2(T)$ is the fundamental class of $F(X \cap H) \subset F(X)$ for any general hyperplane $H \subset \mathbb{P}^5$. In particular $c_2(T)^2 = 27$, and therefore the self-intersection of $i_Z(F_2(Z))$ equals 27×63^2 . Now we can apply the double point formula ([Fu], Theorem 9.1). Observe that the normal bundle $N_{i_Z} = i_Z^* TF(X)/TF_2(Z)$ is identified through the symplectic form, since $i_Z(F_2(Z))$ is Lagrangian, with the cotangent bundle of $F_2(Z)$. We thus get

$$2\delta = 27 \times 63^2 - \chi_{top}(F_2(Z)) = 94122,$$

since the topological Euler characteristic $\chi_{top}(F_2(Z)) = 13041$ (see 4.1).

Proposition 7 *For a general cubic fivefold Z containing X , the image of $F_2(Z)$ in $F(X)$ is a singular Lagrangian surface with $\delta = 47061$ ordinary double points as only singularities.*

2.2.3 Conics on Fano threefolds of genus four

A Fano threefold of genus four is the complete intersection $Y = Q \cap X$ of a quadric Q and a cubic hypersurface X in \mathbb{P}^5 .

Proposition 8 *For Y general, the set of lines in Y is a smooth curve $\Gamma(Y)$ and the set of conics in Y is a smooth surface $F(Y)$.*

Proof. Choosing equations for Q and X we get sections σ_Q and σ_X of S^2T^* and S^3T^* on $G(2,6)$, and the zero locus of $\sigma_Q \oplus \sigma_X$, is precisely $\Gamma(Y)$. Hence the first assertion.

The Hilbert scheme \mathcal{K} of conics in \mathbb{P}^5 is nothing else than the total space of the projective bundle $\mathbb{P}(S^2T^*) \xrightarrow{\pi} G(3,6)$. On \mathcal{K} we have a tautological sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \pi^*S^2T^* \rightarrow \mathcal{Q} \rightarrow 0,$$

and the equation of Q defines a section τ_Q of \mathcal{Q} whose zero-locus is the set of conics contained in Q . In a similar way, the vector bundle \mathcal{M} defined by the exact sequence

$$0 \rightarrow \pi^*T^* \otimes \mathcal{O}(-1) \rightarrow \pi^*S^3T^* \rightarrow \mathcal{M} \rightarrow 0,$$

has a global section τ_X defined by the equation of X whose zero locus is the set of conics contained in C . Now $\tau_Q \oplus \tau_X$ is a general section of the globally generated vector bundle $\mathcal{Q} \oplus \mathcal{M}$, whose zero locus is $F(Y)$. So by Bertini, $F(Y)$ is a smooth surface. \square

Remark. The invariants of the curve $\Gamma(Y)$ were computed in [Mar], in particular its genus is $g(\Gamma(Y)) = 271$. We will compute the invariants of $F(Y)$ in the next section. As noticed in Ch. 2, §4 of [Isk], $F(Y)$ contains a special curve $B(Y) \subset F(Y)$ that consists of conics that lie in the ruling planes of Q . Since Q has two family of ruling planes, this curve has two connected components $B_+(Y)$ and $B_-(Y)$.

Now consider in \mathbb{P}^6 a general nodal cubic 5-fold Z , with equation $x_0p_Q + p_X$, where x_0 is an equation of the hyperplane $\mathbb{P}^5 \supset Q, X$, and p_Q, p_X are equations of Q, X , respectively. Denote its node by $e_o = (1 : 0 : \dots : 0)$. The projective tangent cone $K(Q)$ to Z at e_o is the cone in \mathbb{P}^6 with vertex e_o over the quadric Q . Its intersection with Z is the cone $K(Y)$ over the Fano 3-fold $Y = X \cap Q \subset \mathbb{P}^5$.

From another point of view, the cone $K(Y)$ is the union of all lines $\ell \subset Z$ which pass through the node of Z ; in other words, Y is the base of the family of lines on Z that pass through e_o . Denote by $F_2(Z)$ the Fano variety of planes in Z . We shall construct a natural map

$$\nu : F(Y) \rightarrow F_2(Z)$$

as follows. Let $q \subset Y$ be a conic. Then q , together with the node e_o span a 3-space P_q^3 , which intersects Z along a cubic surface $S_q = P_q^3 \cap Z$. This surface is not irreducible, since it contains the quadratic cone $K(q)$ with vertex e_o and base q . Therefore

$$S_q = K(q) + P_q^2,$$

where $P_q^2 \subset Z$ is a plane, and we let $q \mapsto \nu(q) := P_q^2$.

Note that $\ell_q := P_q^2 \cap X$ is a line that is 2-secant to q . We denote by j the map $F(Y) \rightarrow F(X)$ sending q to ℓ_q .

Next, we shall try to find where is the inverse of $\nu : F(Y) \rightarrow F_2(Z)$ defined.

First note that any plane Π in Z that passes through e_o evidently lies in $K(Y)$ and intersects on Y a line $\ell = \Pi \cap Y$. Conversely, Π is just the span of $\ell \cup e_o$. In other words, the curve of lines $\Gamma(Y)$ is the base of the family $\Gamma(Z)$ of planes on Z that pass through e_o . in particular $\Gamma(Z) \cong \Gamma(Y)$.

Let $\Pi \in F_2(Z) - \Gamma(Z)$, and let $\ell = \Pi \cap Y \in \Gamma(Y)$ be its corresponding line on Y . Since $e_o \notin \Pi$, the span of $\Pi \cup e_o$ is a 3-space, that intersects Z along a cubic surface $S_\Pi = \Pi \cup Q_\Pi$, where Q_Π is a quadric surface in Z .

Since S_Π clearly passes through $e_o = \text{Sing}(Z)$, but Π does not, the quadric Q_Π must be singular at e_o . Therefore Q_Π must be a quadratic cone with vertex e_o , meeting \mathbb{P}^5 along a conic q on $Y = X \cap Q$. Clearly $q = \nu^{-1}(\Pi)$.

Thus ν is invertible outside $\Gamma(Z) \cong \Gamma(Y)$. There remains to find the pre-images on $F(Y)$ of the planes $\Pi \in \Gamma(Z) \subset F_2(Z)$. Let $\ell = \Pi \cap Y \in \Gamma(Y)$.

Let $q \in \nu^{-1}(\Pi)$. Then with the previous notations $S_q = P_q^3 \cap Z$ is a cone with vertex e_o , which can be defined as the set of points (x_0, y) , with y in the span P_q^2 of q , such that $x_0 p_Q(y) + p_X(y) = 0$. So Q must contain the plane P_q^2 , which itself contains ℓ . But a line in a four dimensional quadric is contained in exactly one plane in each of the two rulings of Q . Denote the two planes in Q that contain ℓ by P_+ and P_- . The intersection of X with P_\pm is the union of ℓ with a conic q_\pm , and we conclude that

$$\nu^{-1}(\Pi) = \{q_+, q_-\}.$$

We summarize our discussion.

Proposition 9 *The map $\nu : F(Y) \rightarrow F_2(Z)$, $q \mapsto P_q^2$ is an isomorphism outside the curve $B(Y) = B_+(Y) \cup B_-(Y) \subset F(Y)$, and the restriction of ν to $B_\pm(Y)$ is an isomorphism with $\Gamma(Z)$.*

In particular $\Gamma(Z)$ is the singular locus of $F_2(Z)$, whose normalization is ν .

Now, since the condition of being Lagrangian is closed, we can deduce from Propositions 4 and 9 the following result:

Proposition 10 *For any general cubic fourfold $X \subset \mathbb{P}^5$ and any general quadric $Q \subset \mathbb{P}^5$, the map sending a conic $q \subset Y = Q \cap X$ to the residual line ℓ_q in the intersection of the linear span P_q^2 of q , with X , maps the surface $F(Y)$ of conics in Y to a singular Lagrangian surface in $F(X)$.*

3 Two integrable systems

3.1 Cubic fivefolds containing a given fourfold

By a result of Z. Ran, deformations of smooth Lagrangian subvarieties are unobstructed. Moreover the same statement holds for deformations of Lagrangian immersions of smooth varieties in a fixed symplectic manifold. We will apply this result to the Lagrangian immersions $i_Z : F_2(Z) \rightarrow F(X)$, for Z a general cubic fivefold containing the fixed general cubic fourfold X as a hyperplane section.

The family of cubic fivefolds Z containing $X = Z \cap H$ as a hyperplane section is a parameterized by the linear system

$$|I_X(3)| = \langle c_X, x_0 | \mathcal{O}(2) \rangle,$$

where c_X is an equation of X in H and x_0 is an equation of H . This linear system has dimension 28 and admits a natural action of the seven-dimensional group G , consisting of automorphisms

of \mathbb{P}^6 whose restriction to H is trivial. Moreover the subvariety $i_Z(F_2(Z)) \subset F(X)$ does not change when Z moves in a G -orbit.

Remark. Note that the tangent space to the corresponding deformation is $H^1(Z, TZ \otimes I_X) = H^1(Z, TZ(-1))$, which can be computed from the exact sequence

$$0 \longrightarrow TZ(-1) \longrightarrow T\mathbb{P}^6(-1)|_Z \longrightarrow \mathcal{O}_Z(2) \longrightarrow 0.$$

The Euler exact sequence restricted to Z gives $H^0(Z, T\mathbb{P}^6(-1)|_Z) = H^0(\mathbb{P}^6, \mathcal{O}(1))^* := V$ and $H^1(Z, T\mathbb{P}^6(-1)|_Z) = 0$. Since $H^0(Z, \mathcal{O}_Z(2)) = H^0(\mathbb{P}^6, \mathcal{O}(2)) = S^2V^*$, we get that

$$H^1(Z, TZ \otimes I_X) = \text{Coker}(V \xrightarrow{c_Z} S^2V^*)$$

is the cokernel of the map c_Z defined as the differential of an equation of Z . This map is always injective for Z smooth, hence $h^1(Z, TZ \otimes I_X) = 21$. In particular we can easily obtain locally complete families for our deformation problem, parameterized by linear subsystems of $|I_X(3)|$ transverse to $\mathbb{P}c_X(V)$.

Let $\mathcal{Z} \rightarrow \mathcal{H} \simeq \mathbb{P}^{21}$ be such a locally complete family of cubic fivefolds Z containing X . Let \mathcal{H}_s be the open subset defined by the conditions that $F_2(Z)$ is a smooth surface and i_Z is immersive. Following [Ran], Corollary 3.4, for such a Z the space \mathcal{D} of deformations of i_Z with fixed target $F(X)$ is smooth at the corresponding point, with tangent space $H^0(N_{i_Z}) \simeq H^{1,0}(F_2(Z))$ (recall that the normal bundle N_{i_Z} is isomorphic with the cotangent bundle of $F_2(Z)$ since i_Z is a Lagrangian immersion). This tangent space has dimension 21, and the map $\mathcal{H}_s \rightarrow \mathcal{D}$ is étale. Moreover there is an obvious map from \mathcal{D} to the Hilbert scheme $\mathcal{H}_{F(X)}$ of $F(X)$. When $i_Z(F_2(Z))$ has only ordinary double points, it is easy to check that its normal sheaf $N \simeq (i_Z)_*\Omega_{F_2(Z)}^1$. In particular the tangent space to $\mathcal{H}_{F(X)}$ at the corresponding point is $H^0(N) \simeq H^{1,0}(F_2(Z))$, implying that the Hilbert scheme is smooth and the map $\mathcal{D} \rightarrow \mathcal{H}_{F(X)}$ is also étale at that point.

Consider the relative Picard bundle $\text{Pic}^0 F_2(\mathcal{Z}) \rightarrow \mathcal{H}_s$. By [DM2], Theorem 8.1, this is an algebraic completely integrable Hamiltonian system (ACIHS): the total space of the fibration admits a symplectic structure such that the fibers are Lagrangian subvarieties. In fact the result of Donagi and Markman is only stated for Lagrangian embeddings, but their proof adapts verbatim to the case of Lagrangian immersions.

It was proved by Collino that the Abel-Jacobi morphism

$$\text{Alb}F_2(Z) \longrightarrow J(Z)$$

to the intermediate Jacobian, is an isomorphism for a general cubic fivefold Z [Col]. In particular $\text{Alb}F_2(Z)$, like $J(Z)$, is self-dual, hence naturally isomorphic with the Picard variety $\text{Pic}^0 F_2(Z)$. Denote by $\mathcal{H}_a \subset \mathcal{H}_s$ the open subset over which the Abel-Jacobi theorem does hold. We conclude:

Theorem 11 *The relative intermediate Jacobian $J(\mathcal{H}_a)$ over the locally complete family \mathcal{H}_a of smooth cubic fivefolds containing X , is an algebraic completely integrable Hamiltonian system.*

In particular, the *cubic condition* of Donagi and Markman must hold (see [DM2], section 7.2). That is, the map

$$T_{J(Z)}\mathcal{A} = \text{Sym}^2 H^3(Z, \Omega_Z^2) \longrightarrow T_{[Z]}\mathcal{H}_a \simeq H^1(Z, TZ \otimes I_X),$$

where \mathcal{A} denotes the moduli space of polarized abelian varieties, must be completely symmetric. To be precise, note that $H^3(Z, \Omega_Z^2)$ can easily be computed with the help of the normal exact sequence for $Z \subset \mathbb{P}^6$ and the Euler sequence restricted to Z . We obtain

$$H^3(Z, \Omega_Z^2) = \text{Ker}(S^2V \xrightarrow{c_Z^t} V^*) = H^1(Z, TZ \otimes I_X)^*.$$

Let $H := H^3(Z, \Omega_Z^2)$. The cubic condition is that the map $S^2H \rightarrow H^*$ introduced above, which is just the differential of the induced map from \mathcal{H}_a to the moduli space \mathcal{A} , comes from a cubic form $\theta_Z \in S^3H^*$. This form can be described as the tautological cubic

$$\theta_Z : S^3H \hookrightarrow S^3(S^2V) \longrightarrow S^2(S^3V) \xrightarrow{c_Z^2} \mathbb{C}.$$

3.2 Fano threefolds of genus four contained in a given cubic fourfold

We want to extend our ACIHS to an open subset $\mathcal{H}_n \subset \mathcal{H}$, containing \mathcal{H}_a , by allowing Z to be a general cubic fivefold with a node. In this case we know that $F_2(Z)$ gets singular along a curve, and that its normalization is the Fano surface of conics $F(Y)$ of a general prime Fano threefold of genus 4.

We begin with a few observations. Let Z be a general nodal cubic fivefold containing X .

1. The singular surface $F_2(Z)$ is a flat deformation of the smooth $F_2(Z')$, with Z' smooth. This follows e.g. from Kollár's criterion (6.1.3) in [Kol], following which a family of relative dimension at least two on a smooth base, whose special fiber, with its reduced structure, is S_2 , must be flat. This criterion applies in our situation since $F_2(Z)$ is a locally complete intersection, hence has property S_2 .
2. The map $i_Z : F_2(Z) \rightarrow F(X)$ is again a closed immersion. Indeed, with the notations of Lemma 3, i_Z is the restriction of the natural projection $p : G(3, W) \dashrightarrow G(2, V)$. We observed in the proof of this lemma that the Zariski tangent space to $F_2(Z)$ at some Π is transverse to the tangent space to the fiber of p if and only if Π is non special. Then our claim follows, since a general nodal cubic fivefold contains no special plane by [Col], Lemma (1.6).
3. It follows that the singular surface $i_Z(F_2(Z))$ is a flat deformation of the singular surface $i_{Z'}(F_2(Z'))$, with Z' smooth. Indeed the effect of i_Z and $i_{Z'}$ is merely to create a certain number of double points, outside the double curve of $i_Z(F_2(Z))$, and these double points vary smoothly when Z deforms to a smooth cubic.

Proposition 12 *Let \mathcal{Z} denote the universal family over the \mathcal{H}_s . Then the relative Picard bundle, the ACIHS $\text{Pic}^0 F_2(\mathcal{Z}) \rightarrow \mathcal{H}_s$, extends to a dense open subset of \mathcal{H}_n .*

Proof. The strategy of proof will be the following. The Picard fibration extends to the general point of \mathcal{H}_n . Moreover, Donagi and Markman observed that the symplectic form which makes this fibration an ACIHS, extends to a natural skew-symmetric form, which will certainly remain closed. What we need to prove is that is also remains non-degenerate at the generic point of \mathcal{H}_n . For this we need a careful analysis of the Picard fibration.

We first describe, for Z a general nodal cubic, the relevant data in terms of the associated Fano threefold Y of genus 4. We keep the notations of 2.2.3. Recall that the map $\nu : F(Y) \rightarrow F_2(Z)$ restricts to an isomorphism

$$F(Y) - B_+(Y) \cup B_-(Y) \rightarrow F_2(Z) - \Gamma(Z).$$

We have seen that the curves $B_+(Y)$ and $B_-(Y)$ are mapped isomorphically to $\Gamma(Z)$. Moreover the two branches of $F_2(Z)$ along $\Gamma(Z)$ intersect transversely at every point, and this remains true for $i_Z(F_2(Z))$. We denote by Γ_0 the set of double points on $i_Z(F_2(Z))$. It does not meet the image of $\Gamma(Z)$. We let $\mu := i_Z \circ \nu : F(Y) \rightarrow F(X)$. For simplicity we denote $\Gamma = i_Z(\Gamma(Z))$ and $F = i_Z(F_2(Z)) = \mu(F(Y)) \subset F(X)$.

Outside Γ_0 , the surface F is a locally complete intersection in $F(X)$, so its normal sheaf $N = \text{Hom}(\mathcal{I}_F/\mathcal{I}_F^2, \mathcal{O}_F)$ is a locally free \mathcal{O}_F -module of rank two. Since the two branches of F along Γ are Lagrangian, their tangent spaces generate in $TF(X)$ the orthogonal $T\Gamma^\perp$ to $T\Gamma$ with respect to the symplectic form. The quotient of $T\Gamma^\perp$ by $T\Gamma$ is the direct sum of the normal bundles N_+ and N_- of $B_+(Y)$ and $B_-(Y)$ in $F(Y)$. In particular the symplectic form restricts to a non degenerate pairing $N_+ \otimes N_- \rightarrow \mathcal{O}_\Gamma$.

Note that the natural map $TF(X) \otimes \mathcal{O}_F \rightarrow N$ is not surjective along Γ . Let us choose local coordinates x, y, z, t on $F(X)$ at some point $p \in \Gamma$, such that F be given by the equations $xy = z = 0$, and Γ by $x = y = z = 0$. Then the image of $TF(X) \otimes \mathcal{O}_F$ in $N = \text{Hom}(\mathcal{I}_F/\mathcal{I}_F^2, \mathcal{O}_F)$ is defined by the condition that $xy \in \mathcal{I}_F$ be sent to \mathcal{I}_Γ . If we denote by δ the \mathcal{O}_F -module map sending xy to 1 and z to zero, the image of $TF(X) \otimes \mathcal{O}_F$ is $\mathcal{I}_\Gamma\delta + \mathcal{O}_F\partial/\partial z$. On the other hand, there is a globally defined map of \mathcal{O}_F -modules $N \rightarrow N_+ \otimes N_-$, sending δ to $\partial/\partial x \otimes \partial/\partial y$ and $\partial/\partial z$ to zero. The kernel of this map is precisely the image of $TF(X) \otimes \mathcal{O}_F$.

Now consider the tangent map $TF(Y) \rightarrow \mu^*TF(X) \simeq \mu^*\Omega_{F(X)}^1$. Dualizing and pushing forward, we get a map $TF(X) \otimes \mathcal{O}_F \rightarrow \mu_*\mu^*TF(X) \rightarrow \mu_*\Omega_{F(Y)}^1$, whose image we denote by $N_0 \subset \mu_*\Omega_{F(Y)}^1$. Clearly N_0 and $\mu_*\Omega_{F(Y)}^1$ are isomorphic outside Γ . At our point $p \in \Gamma$, an element of the fiber of $\mu_*\Omega_{F(Y)}^1$ consists in pairs of differential forms on the two branches of $F_2(Z)$ at that point. Such a pair is in the image of $TF(X) \otimes \mathcal{O}_F \simeq \Omega_{F(X)}^1 \otimes \mathcal{O}_F$ if and only if the restrictions of the two forms to $T_p\Gamma$ coincide. So the quotient of $\mu_*\Omega_{F(Y)}^1$ by N_0 is naturally identified with Ω_Γ^1 .

Using the fact that the two branches of $F_2(Z)$ at p are Lagrangian, one easily sees that the maps from $TF(X) \otimes \mathcal{O}_F$ to N and N_0 have exactly the same kernels. An elementary computation shows that this remains true over double points of F . So there is an induced map $N_0 \rightarrow N$. We get the following diagram:

$$\begin{array}{ccccccc}
& & 0 & & & & \\
& & \downarrow & & & & \\
0 & \longrightarrow & N_0 & \longrightarrow & \mu_*\Omega_{F(Y)}^1 & \longrightarrow & \Omega_\Gamma^1 \longrightarrow 0 \\
& & \downarrow & & & & \\
& & N & & & & \\
& & \downarrow & & & & \\
& & N_+ \otimes N_- \simeq \mathcal{O}_\Gamma & & & & \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array}$$

Taking global sections, we get that $h^0(N_0) \leq h^0(\mu_*\Omega_{F(Y)}^1) = h^0(\Omega_{F(Y)}^1)$ and $h^0(N) \leq 1 + h^0(N_0) \leq 1 + h^0(\Omega_{F(Y)}^1) = 21$. But $H^0(N)$ is the Zariski tangent space to the Hilbert scheme of $F(X)$ at the point defined by F . This Hilbert scheme has dimension at least 21 at this point, because we are on the boundary of its open subset considered in the previous subsection. Thus in fact $h^0(N) = 21$ and the Hilbert scheme of $F(X)$ is smooth at the point defined by F . Moreover we get the exact sequence

$$0 \longrightarrow H^0(N_0) \simeq H^0(\Omega_{F(Y)}^1) \longrightarrow H^0(N) \longrightarrow H^0(\mathcal{O}_\Gamma) \longrightarrow 0.$$

This must be interpreted as follows: $H^0(N)$ is the tangent space to the Hilbert scheme of $F(X)$ at the point defined by F ; its hyperplane $H^0(N_0) \simeq H^0(\Omega_{F(Y)}^1)$ is the tangent space to the hypersurface parameterizing the surfaces $i_Z(F_2(Z'))$ for Z' a nodal cubic, or equivalently the images of the Fano surfaces $F(Y)$ for Y a Fano threefold of genus 4 contained in X . We have proved:

Lemma 13 *Let Z be a general nodal cubic fivefold containing X . Let $F = i_Z(F_2(Z)) \subset F(X)$. Then the Hilbert scheme $\mathcal{H}_{F(X)}$ of $F(X)$ is smooth at the corresponding point, as well as the hypersurface defined by the nodal deformations of Z .*

After these preliminaries, we consider the relative Picard fibration $Pic^0(\mathcal{F}) \rightarrow \mathcal{H}_n$. As we mentioned above, the symplectic form defined over \mathcal{H}_s has a natural extension over \mathcal{H}_n : this is explained in [DM2], section 8.4, for the Picard fibration over the Hilbert scheme of deformations of smooth Lagrangian subvarieties. In the case we consider, the component of $\mathcal{H}_{F(X)}$ that we deal with parametrizes Lagrangian subvarieties with ordinary double points, and we pull-back the form by the étale map $\mathcal{H}_n \rightarrow \mathcal{H}_{F(X)}$. Nevertheless the existence of double points will only require minor modifications.

The extension of the skew-symmetric form over the Picard fibration goes as follows. If L is an invertible sheaf over $F_2(Z)$, we can consider the torsion sheaf $L_F := i_{Z*}L$ on $F(X)$. This embeds the total space of the Picard fibration into Simpson's moduli space of simple sheaves on $F(X)$, whose tangent space at the corresponding point is $Ext_{\mathcal{O}_{F(X)}}^1(L_F, L_F)$. When $F_2(Z)$ is smooth, and F has only ordinary double points as singularities, a local computation shows that

$$\mathcal{E}xt_{\mathcal{O}_{F(X)}}^1(L_F, L_F) \simeq N \simeq i_{Z*}N_{i_Z}$$

(recall that N_{i_Z} is the normal bundle to i_Z , defined as the quotient $i_Z^*TF(X)/TF_2(Z)$; the role of the sheaf $i_{Z*}N_{i_Z}$ is considered in [Ca]; since i_Z is a Lagrangian immersion and $F_2(Z)$ is smooth, we simply have $N_{i_Z} \simeq \Omega_{F_2(Z)}^1$). The local to global spectral sequence for Ext 's yields the exact sequence

$$0 \longrightarrow H^1(\mathcal{O}_{F_2(Z)}) \longrightarrow Ext_{\mathcal{O}_{F(X)}}^1(L_F, L_F) \longrightarrow H^0(i_{Z*}N_{i_Z}) = H^0(\Omega_{F_2(Z)}^1) \longrightarrow 0.$$

which identifies with the tangent sequence of the Picard fibration.

Donagi and Markman observe that the skew-symmetric form on this space, which defines the symplectic structure, can be defined by the composition

$$\wedge^2 Ext_{\mathcal{O}_{F(X)}}^1(L_F, L_F) \longrightarrow Ext_{\mathcal{O}_{F(X)}}^2(L_F, L_F) \xrightarrow{c_1(\mathcal{O}(1))} Ext_{\mathcal{O}_{F(X)}}^4(L_F, L_F) \simeq Hom(L_F, L_F)^* = \mathbb{C},$$

where we have used Serre duality and the fact that the canonical sheaf of $F(X)$ is trivial. The point is that this description extends to the boundary points where Z is a general nodal cubic. What we need to check is that the extended form remains non degenerate at the generic point of the boundary.

For a general nodal cubic Z , we still have $\mathcal{E}xt_{\mathcal{O}_{F(X)}}^1(L_F, L_F) \simeq N$. Again we consider the induced exact sequence

$$0 \longrightarrow H^1(\mathcal{O}_{F_2(Z)}) \longrightarrow \mathcal{E}xt_{\mathcal{O}_{F(X)}}^1(L_F, L_F) \longrightarrow H^0(N) \longrightarrow 0,$$

which is again the tangent sequence to the Picard fibration. Since this fibration is Lagrangian over \mathcal{H}_s , by continuity $H^1(\mathcal{O}_{F_2(Z)})$ remains isotropic on the boundary. So the skew-symmetric form that we defined on $\mathcal{E}xt_{\mathcal{O}_{F(X)}}^1(L_F, L_F)$ remains non degenerate if and only if the induced pairing

$$H^1(\mathcal{O}_{F_2(Z)}) \otimes H^0(N) \longrightarrow H^1(N) \xrightarrow{c_1(\mathcal{O}(1))} H^2(N \otimes \Omega_{F(X)|F}^1) \longrightarrow H^2(N \otimes N) \longrightarrow \mathbb{C}$$

is non degenerate. (Note that to define the rightmost arrows, we first used the natural map $\Omega_{F(X)|F}^1 \simeq TF(X)|_F \rightarrow N$, and then, since $N \simeq \mathcal{E}xt_{\mathcal{O}_{F(X)}}^1(L_F, L_F)$, the induced Yoneda map $N \otimes N \rightarrow \mathcal{E}xt_{\mathcal{O}_{F(X)}}^2(L_F, L_F)$; finally we used the fact that, again by the local to global spectral sequence for $\mathcal{E}xt$'s, $H^2(\mathcal{E}xt_{\mathcal{O}_{F(X)}}^2(L_F, L_F)) \simeq \mathcal{E}xt_{\mathcal{O}_{F(X)}}^4(L_F, L_F) \simeq \mathbb{C}$.)

Over \mathcal{H}_s , N is identified with $i_{Z*}\Omega_{F_2(Z)}^1$ and the non degeneracy follows from the Hard Lefschetz theorem on the smooth surface $F_2(Z)$. Over $\partial\mathcal{H}_n := \mathcal{H}_n - \mathcal{H}_s$ we consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathcal{O}_\Gamma) & \longrightarrow & H^1(\mathcal{O}_{F_2(Z)}) & \longrightarrow & H^1(\mathcal{O}_{F(Y)}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1(\Omega_\Gamma^1) & \longrightarrow & H^0(N)^* & \longrightarrow & H^0(\Omega_{F(Y)}^1)^* & \longrightarrow & 0 \end{array}$$

The rightmost vertical map

$$H^1(\mathcal{O}_{F(Y)}) \longrightarrow H^0(\Omega_{F(Y)}^1)^* \simeq H^2(\Omega_{F(Y)}^1)$$

is defined by the cup product with $c_1(\mu^*\mathcal{O}(1))$. Since μ is finite $\mu^*\mathcal{O}(1)$ is still ample on the smooth surface $F(Y)$, so Hard Lefschetz applies and this map is an isomorphism. On the other hand, the leftmost vertical map $H^0(\mathcal{O}_\Gamma) \longrightarrow H^1(\Omega_\Gamma^1)$ is given by $c_1(\mathcal{O}(1)|_\Gamma)$, and again it is an isomorphism. So the middle vertical map is an isomorphism, and this concludes the proof of the Proposition. \square

Now we will obtain a new ACIHS by symplectic reduction on the boundary. We need two easy observations.

Proposition 14 *Let Z be a general nodal cubic and Y the associated prime Fano threefold of genus 4. There is an exact sequence*

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \text{Pic}^0 F_2(Z) \xrightarrow{\nu^*} \text{Pic}^0 F(Y) \simeq J(Y) \longrightarrow 0.$$

Proof. The fact that $Pic^0 F(Y) \simeq J(Y)$ will be proved in §4, see Theorem 20. Now consider the exact sequence $0 \rightarrow \mathcal{O}_{F_2(Z)}^* \rightarrow \nu_* \mathcal{O}_{F(Y)}^* \rightarrow \mathcal{O}_{\Gamma(Y)}^* \rightarrow 0$, and the associated long exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_{\Gamma(Y)}^*) = \mathbb{C}^* \longrightarrow Pic F_2(Z) \xrightarrow{\nu^*} Pic F(Y).$$

Since $H^0(\mathcal{O}_{\Gamma(Y)}^*)$ is connected, it certainly maps to $Pic^0 F_2(Z)$. We thus have the same exact sequence with Pic^0 instead of Pic . Finally, the rightmost arrow must be surjective since its differential is. Indeed, the additive version $0 \rightarrow \mathcal{O}_{F_2(Z)} \rightarrow \nu_* \mathcal{O}_{F(Y)} \rightarrow \mathcal{O}_{\Gamma(Y)} \rightarrow 0$ of the previous exact sequence yields the exact sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_\Gamma) \longrightarrow H^1(\mathcal{O}_F) \longrightarrow H^1(\mathcal{O}_{F(Y)}) \longrightarrow 0,$$

and the righthand map is the differential in question. \square

Recall that the Fano threefold Y has been defined as the intersection in $H = \mathbb{P}^5$ of the fixed cubic fourfold X , with the base Q of the tangent cone to Z at its unique node p_Z . The map $Z \mapsto Q$ is constant along the orbits of the group G , the group of projective transformations acting trivially on H .

Proposition 15 *There is a birational isomorphism*

$$|\mathcal{O}_{\mathbb{P}^5}(2)| \longrightarrow \partial\mathcal{H}_n = \mathcal{H}_n - \mathcal{H}_s.$$

Proof. Let $\Delta_X \subset |I_X(3)|$ denote the discriminant hypersurface, parameterizing the singular cubics containing X . For X general Δ_X is irreducible. Let Δ_X^n denote the open subset parameterizing cubics with a single node. The map $\Delta_X^n \rightarrow |\mathcal{O}_{\mathbb{P}^5}(2)|$ sending the nodal cubic Z to the trace on H of its tangent cone at the node, is constant on the G -orbits. More precisely, it is easy to check that its fibers are precisely the G -orbits. This implies that there exists a G -stable open subset $\Delta_X^0 \subset \Delta_X^n$ such that the restriction map $\Delta_X^0 \rightarrow |\mathcal{O}_{\mathbb{P}^5}(2)|$ is a good quotient of the G -action. In particular, there is an induced rational map $|\mathcal{O}_{\mathbb{P}^5}(2)| \rightarrow \partial\mathcal{H}_n$, which is clearly dominant.

So we just need to check that this map is generically bijective, that is, a general Fano threefold $Y = X \cap Q$ is uniquely determined by the singular Lagrangian surface $i_Z(F_2(Z)) \subset F(X)$ associated to the nodal cubic Z whose equation is $P + x_0 Q = 0$ for some equation x_0 of $H \subset \mathbb{P}^6$. By construction, the singular locus of $i_Z(F_2(Z))$ is nothing else than the curve $\Gamma(Y) \subset F(X) \subset G(2, 6)$ of lines in Y . So we just need to check that $\Gamma(Y)$ defines uniquely the quadric Q such that $Y = X \cap Q$. That is, we must prove that

$$H^0(G(2, 6), I_{\Gamma(Y)} \otimes S^2 T^*) = \mathbb{C}.$$

For Y generic, $I_{\Gamma(Y)}$ can be resolved by the Koszul complex of the section of the vector bundle $E = S^2 T^* \oplus S^3 T^*$ that defines $\Gamma(Y)$. The claim then follows from the identities $H^0(G(2, 6), E^* \otimes S^2 T^*) = \mathbb{C}$ and $H^k(G(2, 6), \wedge^{k+1} E^* \otimes S^2 T^*) = 0$ for $k > 0$. Both facts are easy consequences of Bott's theorem on the Grassmannian. \square

Now we are exactly in the situation considered in [DP]: an ACIHS $\mathcal{F} \rightarrow B$ is defined over some smooth base \mathcal{B} , and there is a smooth hypersurface $\Delta \subset \mathcal{B}$ over which the fibers degenerate to extensions

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{F} \longrightarrow J(\mathcal{F}) \longrightarrow 0,$$

where $J(\mathcal{F})$ is a family of abelian varieties over Δ . If h is a local equation of Δ , suppose that the corresponding Hamiltonian vector field, which is a vertical vector field for the fibration, is tangent to the \mathbb{C}^* direction. Then symplectic reduction applies, and there is an induced ACIHS $J(\mathcal{F}) \rightarrow \Delta$.

We apply this result to our setting: $\mathcal{F} \rightarrow B$ is our relative Picard fibration $Pic^0(F_2(\mathcal{Z})) \rightarrow \mathcal{H}_n$. Over the hypersurface $\Delta = \partial\mathcal{H}_n$ defined by nodal cubics, the fibers $Pic^0(F_2(\mathcal{Z}))$ are \mathbb{C}^* -extensions of the intermediate Jacobians $J(Y)$ of the associated Fano threefolds Y of genus 4. Moreover we have seen that the \mathbb{C}^* factor is orthogonal, with respect to the extended symplectic form, to the tangent hyperplane to Δ . So symplectic reduction applies and we conclude:

Theorem 16 *Let $X \subset \mathbb{P}^5$ be a general cubic fourfold. Consider inside the 20-dimensional linear system $|\mathcal{O}_{\mathbb{P}^5}(2)|$, the open subset $|\mathcal{O}_{\mathbb{P}^5}(2)|_X$ of smooth quadrics Q transverse to X . Denote by*

$$J(\mathcal{Y}_X) \longrightarrow |\mathcal{O}_{\mathbb{P}^5}(2)|_X$$

the family of intermediate Jacobians of the genus 4 prime Fano threefolds $Y = X \cap Q$, where $Q \in |\mathcal{O}_{\mathbb{P}^5}(2)|_X$. Then $J(\mathcal{Y}_X)$ is an algebraic completely integrable Hamiltonian system over a dense open subset of the base.

Here again the cubic form can easily be identified. Let $W = H^0(\mathcal{O}_{\mathbb{P}^5}(1))^*$. From the normal sequence of Y we can easily deduce an exact sequence

$$H^0(Y, T\mathbb{P}^6(-1)|_Y) = W \xrightarrow{c_Q, c_X} W^* \oplus S^2W^*/Q \longrightarrow H^1(Y, TY(-1)) \simeq H^1(Y, \Omega_Y^2) \longrightarrow 0.$$

Since Q smooth, c_Q is an isomorphism and we get an isomorphism $S^2W^*/Q \simeq H^1(Y, \Omega_Y^2)$. The dual space $H^1(Y, \Omega_Y^2)$ can therefore be identified with the space $A_Q \subset S^2W$ corresponding to quadrics in the dual projective space which are apolar to Q . The cubic form θ_Q on A_Q is then simply given by the composition

$$\theta_Q : S^3A_Q \hookrightarrow S^3(S^2W) \longrightarrow S^2(S^3W) \xrightarrow{c_X^2} \mathbb{C}.$$

4 The family of conics on the Fano threefold of genus four

In this section we consider the Fano surface of conics on the general Fano threefold Y of genus four. We first compute its numerical invariants – these results will not be used in the sequel, but they complete the computations made in [Mar]. Then we prove the Abel-Jacobi type isomorphism theorem that we used to prove Proposition 14.

4.1 Invariants of the Fano surface $F(Y)$

Recall from §2.2.3 that we denoted by \mathcal{K} the Hilbert scheme of conics in \mathbb{P}^5 , which is the total space of the projective bundle $\mathbb{P}(S^2T^*) \xrightarrow{\pi} G(3, 6)$. On \mathcal{K} we defined vector bundles \mathcal{Q} and \mathcal{M} by the natural exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{O}(-1) \rightarrow \pi^*S^2T^* \rightarrow \mathcal{Q} \rightarrow 0, \\ 0 &\rightarrow \pi^*T^* \otimes \mathcal{O}(-1) \rightarrow \pi^*S^3T^* \rightarrow \mathcal{M} \rightarrow 0, \end{aligned}$$

and we observed that for a general Y , the Fano surface of conics $F(Y) \subset \mathcal{K}$ is defined as the zero-locus of a general section of the vector bundle $\mathcal{Q} \oplus \mathcal{M}$. Its fundamental class is therefore given by the Thom-Porteous formula, that is

$$[F(Y)] = c_{12}(\mathcal{Q} \oplus \mathcal{M}) = c_5(\mathcal{Q})c_7(\mathcal{M}) \in A^{12}(\mathcal{K}).$$

The Chow ring $A(\mathcal{K})$ is generated over $A(G)$ by the hyperplane class $h = c_1(\mathcal{O}(1))$, modulo the relation $c_6(\mathcal{Q}) = 0$. In particular $A(\mathcal{K})$ is a free $A(G)$ -module with basis $1, \dots, h^5$. We say that a class in $A(\mathcal{K})$ is written in normal form when it is expressed in that basis. We have

$$c(\mathcal{Q}) = \frac{c(S^2T^*)}{1-h}, \quad c(\mathcal{M}) = \frac{c(S^3T^*)}{c(T^*(-1))}.$$

In particular the first identity gives the relation $hc_5(\mathcal{Q}) = -c_6(S^2T^*)$.

The Chern classes of S^2E and S^3E , for E a vector bundle of rank three, are given by universal formulas in terms of the Chern classes of E , that we denote by c_1, c_2, c_3 . The splitting principle easily gives

$$c(S^2E) = (1 + 2c_1 + 4c_2 + 8c_3)(1 + 2c_1 + c_1^2 + c_2 + c_1c_2 - c_3).$$

The computation for S^3E is much more complicated. With the help of Maple we get the following formulas:

$$\begin{aligned} c_1(S^3E) &= 10c_1 \\ c_2(S^3E) &= 40c_1^2 + 15c_2 \\ c_3(S^3E) &= 82c_1^3 + 111c_1c_2 + 27c_3 \\ c_4(S^3E) &= 91c_1^4 + 315c_1^2c_2 + 189c_1c_3 + 63c_2^2 \\ c_5(S^3E) &= 52c_1^5 + 429c_1^3c_2 + 513c_1^2c_3 + 324c_1c_2^2 + 162c_2c_3 \\ c_6(S^3E) &= 12c_1^6 + 282c_1^4c_2 + 679c_1^3c_3 + 593c_1^2c_2^2 + 792c_1c_2c_3 + 85c_2^3 + 27c_3^2 \\ c_7(S^3E) &= 72c_1^5c_2 + 448c_1^4c_3 + 464c_1^3c_2^2 + 1386c_1^2c_2c_3 + 259c_1c_2^3 + 108c_1c_3^2 + 243c_2^2c_3 \\ c_8(S^3E) &= 120c_1^5c_3 + 132c_1^4c_2^2 + 1116c_1^3c_2c_3 + 246c_1^2c_2^3 + 81c_1^2c_3^2 + 567c_1c_2^2c_3 + 36c_2^4 + 243c_2c_3^2 \\ c_9(S^3E) &= 360c_1^4c_2c_3 + 72c_1^3c_2^3 + 108c_1^3c_3^2 + 540c_1^2c_2^2c_3 + 36c_1c_2^4 - 243c_1c_2c_3^2 + 108c_2^3c_3 + 729c_3^3 \\ c_{10}(S^3E) &= 108c_1^4c_3^2 + 216c_1^3c_2^2c_3 - 486c_1^2c_2c_3^2 + 108c_1c_2^3c_3 + 729c_1c_3^3 \end{aligned}$$

If x_1, x_2, x_3 are the Chern roots of E , and if L is a line bundle with first Chern class h , we can also compute

$$\begin{aligned} c(E \otimes L^*)^{-1} &= \prod_i (1 + x_i - h)^{-1} = (1 - h)^{-3} \prod_i (1 + \frac{x_i}{1-h})^{-1} \\ &= \sum_{k \geq 0} (-1)^k s_k(E) (1 - h)^{-k-3}, \end{aligned}$$

where s_k denotes the k -th Segre class of E .

Now we specialize to $E = \pi^*T^*$ (for simplicity we omit the symbol π^* in the sequel) and $L = \mathcal{O}(1)$, and we deduce that

$$c_7(\mathcal{M}) = \sum_{j,k \geq 0} (-1)^k \binom{j+k+2}{j} c_{7-j-k}(S^3T^*) s_k(T^*) h^j.$$

We use the basis of $A(G)$ given by the Schubert classes σ_{ijk} , $3 \geq i \geq j \geq k \geq 0$. Among these are the Chern classes $c_i(T^*) = \sigma_i$ and the Segre classes $s_k(T^*) = \sigma_{1^k}$ of T^* . In the Schubert basis the total Chern class of S^2T^* is

$$c(S^2T^*) = 1 + 4\sigma_1 + 5\sigma_2 + 5\sigma_{11} + 20\sigma_3 + 15\sigma_{21} + 30\sigma_{31} + 10\sigma_{22} + 6\sigma_{211} + 20\sigma_{32} + 12\sigma_{311} + 4\sigma_{221} + 8\sigma_{321}.$$

Denote by $c_7(\mathcal{M})_{(j)}$ the coefficient of h^j in that sum, for $0 \leq j \leq 7$. Using the relation $hc_5(\mathcal{Q}) = -c_6(S^2T^*) = -8\sigma_{321}$, we get

$$[F(Y)] = c_5(\mathcal{Q})c_7(\mathcal{M})_{(0)} - 8\sigma_{321} \sum_{j=1}^7 c_7(\mathcal{M})_{(j)}h^{j-1}.$$

This almost expresses $[F(Y)]$ in normal form, except for the term with $j = 7$ in the sum. Since $c_7(\mathcal{M})_{(7)} = 36$, the normal form is

$$[F(Y)] = c_5(\mathcal{Q})c_7(\mathcal{M})_{(0)} - 8\sigma_{321} \sum_{j=1}^6 c_7(\mathcal{M})_{(j)}h^{j-1} + 288\sigma_{321} \sum_{i=0}^5 c_{6-i}(S^2T^*)h^i.$$

The computations that remain are to be done in $A(G)$, which is zero in degree greater than nine. Since $[F(Y)]$ has degree 12, it must be a combination of the four classes $\sigma_{333}h^3$, $\sigma_{332}h^4$, $\sigma_{331}h^5$ and $\sigma_{322}h^5$.

First we compute $c_7(\mathcal{M})_{(0)} = c_7(S^3T^*) - c_6(S^3T^*)\sigma_1 + c_5(S^3T^*)\sigma_{11} - c_4(S^3T^*)\sigma_{111}$. The formulas for the Chern classes of a third symmetric power give

$$\begin{aligned} c_7(S^3T^*) &= 8820\sigma_{331} + 6342\sigma_{322}, \\ c_6(S^3T^*) &= 3675\sigma_{33} + 7140\sigma_{321} + 1302\sigma_{221}, \\ c_5(S^3T^*) &= 2870\sigma_{32} + 2436\sigma_{311} + 1442\sigma_{211}, \\ c_4(S^3T^*) &= 1155\sigma_{31} + 560\sigma_{22} + 588\sigma_{211}, \end{aligned}$$

hence $c_7(\mathcal{M})_{(0)} = 1757\sigma_{331} + 1190\sigma_{332}$. The other relevant classes for \mathcal{M} are

$$\begin{aligned} c_7(\mathcal{M})_{(6)} &= 196\sigma_1 \\ c_7(\mathcal{M})_{(5)} &= 595\sigma_2 + 406\sigma_{11} \\ c_7(\mathcal{M})_{(4)} &= 1375\sigma_3 + 1500\sigma_{21} + 404\sigma_{111} \end{aligned}$$

The final result of the computation is:

Proposition 17 *The fundamental class of the surface $F(Y) \subset \mathcal{K}$ is*

$$[F(Y)] = 15840\sigma_{333}h^3 + 8100\sigma_{332}h^4 + (1341\sigma_{331} + 774\sigma_{322})h^5.$$

In particular, we have the following intersection numbers in the Chow ring of $F(Y)$:

$$h^2 = 0, \quad h\sigma_1 = -360, \quad \sigma_2 = 1341, \quad \sigma_1^2 = 2115.$$

Now we can compute the Chern numbers of $F(Y)$. From the exact sequences

$$\begin{aligned} 0 \longrightarrow TF(Y) \longrightarrow T\mathcal{K}|_F \longrightarrow \mathcal{Q}|_F \oplus \mathcal{M}|_F \longrightarrow 0, \\ 0 \longrightarrow T^v\mathcal{K} \longrightarrow T\mathcal{K} \longrightarrow \pi^*TG \longrightarrow 0 \end{aligned}$$

and the observation that the vertical tangent bundle of the fibration π is nothing else than $\mathcal{Q}(1)$, we deduce that

$$\begin{aligned} c_1(F(Y)) &= 2h - 3\sigma_1, \\ c_2(F(Y)) &= -3h^2 - 9h\sigma_1 + 13\sigma_{11} - \sigma_2. \end{aligned}$$

Corollary 18 *The Chern numbers and the arithmetical genus of the Fano surface $F(Y)$ are*

$$c_1(F(Y))^2 = 23355, \quad c_2(F(Y)) = 11961, \quad p_a(F(Y)) = 2942.$$

Remark. We can compare with the invariants of the Fano surface $F_2(Z)$ of planes in a cubic fivefold $Z \subset \mathbb{P}^6$. As a subvariety of $G(3, 7)$, $F_2(Z)$ is defined as the zero locus of the global section of the rank ten vector bundle S^3T^* , defined by an equation of Z . Therefore

$$[F_2(Z)] = c_{10}(S^3T^*) = 1134\sigma_{442} + 1701\sigma_{433},$$

as we can easily deduce from the expression given above for the Chern classes of a third symmetric power. On $F_2(Z)$ we thus have $\sigma_2 = 1134$ and $\sigma_1^2 = 2835$. Since $c(TF) = 1 - 3\sigma_1 + 13\sigma_{11} - \sigma_2$, we conclude that

$$c_1(F_2(Z))^2 = 25515, \quad c_2(F_2(Z)) = 13041, \quad p_a(F_2(Z)) = 3212.$$

In particular we have the relation $p_a(F(Y)) = p_a(F_2(Z)) + 1 - g(\Gamma(Y))$, as expected.

We can also deduce the number ν of conics in Y passing through a general point. For this consider the universal conic $\mathcal{C} \rightarrow \mathcal{K}$. This is a subscheme of the universal supporting plane $\mathcal{P} = \mathbb{P}(\pi^*T) \xrightarrow{\rho} \mathcal{K}$. Let $\gamma = \pi \circ \rho$. On \mathcal{P} we have a tautological line bundle $\mathcal{O}_{\mathcal{P}}(-1) \subset \gamma^*T$, and a tautological section τ of $\rho^*\mathcal{O}_{\mathcal{K}}(1) \otimes \mathcal{O}_{\mathcal{P}}(2)$ defined by the composition

$$\rho^*\mathcal{O}_{\mathcal{K}}(-1) \hookrightarrow \gamma^*S^2T^* \longrightarrow \mathcal{O}_{\mathcal{P}}(2);$$

the zero locus of τ is precisely \mathcal{C} .

Let $V \subset \mathbb{C}^6$ be a general three dimensional subspace. The zero locus \mathcal{P}_V of the induced morphism

$$\mathcal{O}_{\mathcal{P}}(-1) \subset \gamma^*T \longrightarrow (\mathbb{C}^6/V) \otimes \mathcal{O}_{\mathcal{P}}$$

is the set of points in \mathcal{P} mapped to $\mathbb{P}V$ by the natural morphism $\mathcal{P} \rightarrow \mathbb{P}^5$. In particular the class of \mathcal{P}_V is H^3 , if H denotes the first Chern class of $\mathcal{O}_{\mathcal{P}}(1)$, and the class of its intersection \mathcal{C}_V with \mathcal{C} is $(2H + h)H^3$.

Now the intersection of the cycle \mathcal{C}_V with $\rho^{-1}(F(Y))$ is the number of conics in Y meeting $\mathbb{P}V$. Since Y has degree 6 this number is equal to 6ν , and therefore

$$6\nu = (2H + h)H^3\rho^*[F(Y)] = \rho_*(2H^4 + hH^3)[F(Y)].$$

But $\rho_*H^3 = -\sigma_1$ and $\rho_*H^4 = \sigma_{11}$, so using Proposition 17 we obtain:

Proposition 19 *There are $\nu = 318$ conics in Y passing through a given general point.*

Along the same line of ideas, one can consider in $F(Y)$ the curve Δ of degenerate conics, and the Steiner map $s : \Delta \rightarrow \mathbb{P}^5$ mapping each such conic to its vertex. The same arguments as in [CMW] yield

$$\deg s(\Delta) = (2h + 2\sigma_1)(3h + 2\sigma_1) = 4860.$$

4.2 The Abel-Jacobi isomorphism for $F(Y)$

Since Clemens and Griffiths proved that the Abel-Jacobi mapping $Alb F(X) \rightarrow J(X)$ is an isomorphism for X a general cubic threefold and $F(X)$ is Fano variety of lines, similar statements have been obtain for several other Fano manifolds. We fill in a gap in the literature by proving the following statement.

Theorem 20 *Let Y be a general prime Fano threefold of genus 4. Then the Abel-Jacobi mapping*

$$Alb F(Y) \longrightarrow J(Y)$$

is an isomorphism; and so by duality $J(Y) = J(Y)^ \simeq Alb F(Y)^* = Pic^0 F(Y)$.*

We adopt the strategy developed by Clemens and Griffiths, and used in [Let] for the quartic threefold, in [CV] for the sextic double solid and in [Col] for the cubic fivefold. The claim is that the theorem will follow from the existence of a Lefschetz pencil $(Y_t)_{t \in \mathbb{P}^1}$ such that

1. $Y_t = Q \cap C_t$ is contained in a fixed smooth quadric Q ,
2. Y_t and $F(Y_t)$ are smooth for general t ,
3. if Y_t is smooth but not $F(Y_t)$, then the later only has isolated singularities,
4. if Y_t is singular, then the singular locus of $F(Y_t)$ is a smooth curve, along which $F(Y_t)$ has two smooth branches intersecting transversely.

Consider a generic quadric hypersurface $Q \subset \mathbb{P}^5$, a generic pencil of cubics $C_t \subset \mathbb{P}^5$, $t \in \mathbb{P}^1$, and then the pencil of complete intersections $Y_t = Q \cap C_t$. Condition (2) clearly holds, and (3) follows from a simple dimension count. We focus on condition (4). The pencil (Y_t) meets the discriminant hypersurface Δ parameterizing singular complete intersections of type $(2, 3)$ at a finite number of points. These points must belong to the dense open subset of Δ parameterizing complete intersections with a single node. So we are reduced to proving that for a general nodal complete intersection Y , the Fano surface of conics $F(Y)$ has the type of singularities allowed by condition (4).

This involves the following steps, which we only sketch since the arguments are similar to those of [Let, CV, Col, PB].

1. A conic $q \subset Y$ not passing through the vertex v of Y , defines a smooth point of $F(Y)$. To check this we need to characterize, for a conic $q \subset Y_{reg}$ (Y having arbitrary singularities), the fact that it defines a smooth point in $F(Y)$. This goes as follows. Choose coordinates in \mathbb{P}^5 such that q be defined by the equations $x_3 = x_4 = x_5 = 0$ and $\bar{q}(x_0, x_1, x_2) = 0$. If $Y = Q \cap C$, write the equations of Q and C as

$$\begin{aligned} \bar{Q}(x) &= \alpha \bar{q}(x_0, x_1, x_2) + x_3 l_3 + x_4 l_4 + x_5 l_5, \\ \bar{C}(x) &= \lambda \bar{q}(x_0, x_1, x_2) + x_3 q_3 + x_4 q_4 + x_5 q_5. \end{aligned}$$

Suppose that $\alpha \neq 0$, which means that the supporting plane of q is not contained in Q . We say in that case that q is a *non isotropic conic*. Then there is a unique cubic hypersurface $\alpha C - \lambda Q$ containing Y and this supporting plane. Replacing C by this

cubic we may suppose that $\lambda = 0$. Then we consider the exact sequence of normal bundles $0 \rightarrow N_{q/X} \rightarrow N_{q/\mathbb{P}^5} \rightarrow N_{X/\mathbb{P}^5|_q} \rightarrow 0$. We have $N_{q/\mathbb{P}^5} = \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$ and $N_{X/\mathbb{P}^5} = \mathcal{O}(2) \oplus \mathcal{O}(3)$. Moreover, $F(Y)$ is smooth at q if and only if $h^0(N_{q/X}) = 2$, which is equivalent to the surjectivity of the map $H^0(N_{q/\mathbb{P}^5}) \rightarrow H^0(N_{X/\mathbb{P}^5|_q})$. This map is easy to identify in terms of $\alpha, l_3, l_4, l_5, q_3, q_4, q_5$, and we come to the following conclusion: q defines a smooth point of $F(Y)$ if and only if q_3, q_4, q_5 define linearly independent sections of $\mathcal{O}(2)|_q$.

Now suppose that $\alpha \neq 0$, that is q is an *isotropic conic*. We checked that the map $H^0(N_{q/\mathbb{P}^5}) \rightarrow H^0(N_{X/\mathbb{P}^5|_q})$ is always surjective under the hypothesis that $q \subset Y_{reg}$.

2. For Y nodal but general, $F(Y)$ is smooth at any point defined by some conic $q \subset Y_{reg}$. Indeed we can call a conic in some (arbitrary) Y *special*, either if it is non isotropic but does not verify the condition above, or if it is isotropic and meets the singular locus of Y . Then the space of complete intersections Y containing a special conic has at most two irreducible components, none of which coincides with the discriminant hypersurface.
3. The set of conics in Y passing through the vertex v , is the union of a complete curve D parameterizing conics which are all smooth at v , and of 66 conics $q(l, l') = l + l'$, where l and l' are lines in Y passing through v . Moreover the conics $q(l, l')$ define smooth points of $F(Y)$. Indeed Y contains 12 lines passing through v ; hence the 66 reducible conics. The fact that for Y general (among nodal complete intersections), a conic of type $q(l, l')$ defines a smooth point of $F(Y)$ is a boring computation. This computation also shows that the (linear) condition for a conic to pass through v is transverse to the tangent space to $F(Y)$ at $q(l, l')$, which is thus isolated among the conics in Y passing through v .
4. Let $\mathbb{P}^+ \rightarrow \mathbb{P}^5$ be the blow-up of v , and $Y^+ \rightarrow Y$ the strict transform of Y . Let $F(Y^+) \xrightarrow{\pi} F(Y)$ be the Fano surface of conics in Y^+ . Then $\pi^{-1}(D) = D_+ \cup D_-$ is the union of two curves, and the restriction of π to D_{\pm} is a bijection with D . More precisely, $F(Y^+)$ is the component of the Hilbert scheme of Y^+ containing the strict transforms of the conics in Y not passing through v . If $\tilde{q} \in \pi^{-1}(q)$, where q is a conic passing through v but smooth at that point, then \tilde{q} must be the union of the strict transform \bar{q} of q , with a line ℓ in the exceptional divisor. Moreover ℓ must meet \bar{q} at the point defined by the tangent line to q at v . But $S = E \cap Y^+$ is a smooth quadric surface, hence through that point pass exactly two lines ℓ_+ and ℓ_- , one from each of the two rulings of S . This is why $\pi^{-1}(q) = \{\bar{q} + \ell_+, \bar{q} + \ell_-\}$, and $\pi^{-1}(D) = D_+ \cup D_-$.
5. The Fano surface $F(Y^+)$ of conics in Y^+ is smooth along D_{\pm} , and the differential of the birational morphism $F(Y^+) \rightarrow F(Y)$ maps the tangent planes to $F(Y^+)$ along D_{\pm} to planes in the Zariski tangent space to $F(Y)$, meeting exactly along the tangent space to D . The smoothness assertion follows from a computation with normal bundles, using the fact that Y^+ is a complete intersection in $\mathbb{P}^+ \subset \mathbb{P}^5 \times \mathbb{P}^4$, of type $(1, 1), (1, 2)$. Moreover the conic $\bar{q} + \ell_{\pm}$ is also a complete intersection, of type $(1, 0), (0, 1), (0, 1), (1, 1)$. If $p = \bar{q} \cap \ell_{\pm}$,

we have a commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(N) & \longrightarrow & H^0(N(\ell_{\pm})) \oplus H^0(N(\bar{q})) & \longrightarrow & N_p & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(E) & \longrightarrow & H^0(E(\ell_{\pm})) \oplus H^0(E(\bar{q})) & \longrightarrow & E_p & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(F) & \longrightarrow & H^0(F(\ell_{\pm})) \oplus H^0(F(\bar{q})) & \longrightarrow & F_p & \longrightarrow & 0
\end{array}$$

where N denotes the normal bundle of \tilde{q} in Y^+ (recall that Y^+ is smooth and \tilde{q} is a locally complete intersection), E denotes the normal bundle of \tilde{q} in \mathbb{P}^+ , and F the normal bundle of Y^+ in \mathbb{P}^+ , restricted to \tilde{q} . Moreover $N(\ell_{\pm})$ denotes the restriction of N to ℓ_{\pm} , and so on.

Denote by $\phi(\ell_{\pm})$ the map $H^0(E(\ell_{\pm})) \rightarrow H^0(F(\ell_{\pm}))$. The smoothness of the tangent cone to C at the vertex v is enough to imply that $\phi(\ell_{\pm})$ is surjective, hence $h^0(N(\ell_{\pm})) = 2$. Moreover there is no non trivial section vanishing at p , and the evaluation map $H^0(N(\ell_{\pm})) \rightarrow N_p$ is an isomorphism. Therefore $H^0(N) \simeq H^0(N(\bar{q}))$.

Now denote by $\phi(\bar{q})$ the map $H^0(E(\bar{q})) \rightarrow H^0(F(\bar{q}))$. The fact that q is non special implies that $\phi(\bar{q})$ is surjective. So $h^0(N) = h^0(N(\bar{q})) = 2$ and $F(Y^+)$ is smooth at \tilde{q}_{\pm} .

All this also holds for the finite number of isotropic conics passing through the vertex. Indeed one can check that if it did not hold for such an isotropic conic, then the conic would have to meet another singular point of Y , which is impossible.

Finally, we check that the images of the differentials of π at $\bar{q} + \ell_{\pm}$ meet along a line, which must be the Zariski tangent space to D . So D is smooth, and isomorphic with D_{\pm} .

This concludes the proof. □

References

- [AS] P.-L. del Angel, S. Müller-Stach, *Picard-Fuchs equations, integrable systems and higher algebraic K-theory*, Yui, Noriko (ed.) et al., Calabi-Yau varieties and mirror symmetry; Providence, RI: American Mathematical Society (AMS); Fields Inst. Commun. **38** (2003) 43-55.
- [B1] A. Beauville, *variétés Kaehleriennes dont la première classe de Chern est nulle*, J. Differ. Geom. **18** (1983), 755-782
- [B2] A. Beauville, *Systèmes hamiltoniens complètement intégrables associés aux surfaces K3*, Symposia Mathematica **32** (1991) 25-31.
- [BD] A. Beauville, R. Donagi, *La variété des droites d'une hypersurface cubique de dimension 4*, C. R. Acad. Sci. Paris **301**, 703-706 (1985).
- [Ca] F. Catanese, *Moduli of algebraic surfaces*, Lecture Notes in Math. **1337** (1988), 1-83.

- [CG] C. H. Clemens, Ph. A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Annals of Math. **95** (1972), 281-356.
- [Col] A. Collino, *The Abel-Jacobi isomorphism for the cubic fivefold*, Pacific J. of Math. **122** No. 1, 43-55 (1986).
- [CM] A. Collino, J. P. Murre, *The intermediate Jacobian of a cubic threefold with one ordinary double point; an algebraic-geometric approach*, Indag. Math. **40** (1978), part I: 43-55, part II: 56-71.
- [CMW] A. Collino, J. P. Murre, G. E. Welters, *On the family of conics lying on a quartic threefold*, Rend. Sem. Mat. Univ. Politec. Torino **38** (1980), no. 1, 151-181.
- [CV] G. Ceresa, A. Verra, *The Abel-Jacobi isomorphism for the sextic double solid*, Pacific J. Math. **124** (1986), no. 1, 85-105.
- [DM1] R. Donagi, E. Markman, *Cubics, integrable systems, and Calabi-Yau threefolds*, Isr. Math. Conf. Proc. **9** (1996), 199-221.
- [DM2] R. Donagi, E. Markman, *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, Lecture Notes in Math. **1620** (1996), 1-119.
- [DP] R. Donagi, E. Previato, *Abelian solitons*, in Nonlinear waves: computation and theory (Athens, GA, 1999), Math. Comput. Simulation **55** (2001), 407-418.
- [FM] Fantechi B., Manetti M., *On the T^1 -lifting theorem*, J. Algebraic Geom. **8** (1999), 31-39.
- [Fr] D. Freed, *Special Kahler manifolds*, Commun. Math. Phys. **203** No.1 (1999), 31-52.
- [Fu] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete **3**, Springer-Verlag, Berlin 1998.
- [GH] Ph. A. Griffiths, J. Harris, *Principles of Algebraic geometry*, John Willey & Sons, 1978.
- [Isk] V.A. Iskovskikh, *Birational automorphisms of three-dimensional algebraic varieties*, J. Sov. Math. **13**, 815-868 (1980).
- [Kaw] Kawamata Y., *Unobstructed deformations. A remark on a paper of Z. Ran*, J. Algebraic Geom. **1** (1992), no. 2, 183-190.
- [Kol] Kollár J., *Flatness criteria*, J. Algebra **175** (1995), no. 2, 715-727.
- [Let] M. Letizia, *The Abel-Jacobi mapping for the quartic threefold*, Invent. Math. **75** (1984), no. 3, 477-492.
- [Mar] D. Markushevich, *Numerical invariants of families of lines on some Fano varieties*, Mat. USSR Sb. **44**, 239-260 (1983).
- [MT] D. Markushevich, A. Tikhomirov, *Symplectic structure on a moduli space of sheaves on the cubic threefold*, Izv. Math. **67** No.1, 121-144 (2003).

- [PB] Picco Botta L., *On the intersection of three quadrics*, J. Reine Angew. Math. **399**, 188–207 (1989).
- [Ran] Z. Ran, *Hodge theory and deformations of maps*, Compos. Math. **97** No.3, 309–328 (1995).
- [Saw] J. Sawon, *Lagrangian fibrations on Hilbert schemes of points on K3 surfaces*, e-print: math.AG/0509224, to appear in: J. Algebr. Geom.
- [Voi] C. Voisin, *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, Lond. Math. Soc. Lect. Note Ser. **179**, 294–303 (1992).

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