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# On polytopes associated to factorisations of prime-powers

Roland Bacher

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*Abstract*<sup>1</sup>: We study polytopes associated to factorisations of prime powers. These polytopes have explicit descriptions either in terms of their vertices or as intersections of closed halfspaces associated to their facets. We give formulae for their  $f$ -vectors.

## 1 Main results

Polytopes have two dual descriptions: They can be given either as convex hulls of finite sets or as compact sets of the form  $\cap_{f \in \mathcal{F}} f^{-1}(\mathbb{R}_+)$  where  $\mathcal{F}$  is a finite set of affine functions and where  $f^{-1}(\mathbb{R}_+)$  denotes the closed half-space on which the affine function  $f$  is non-negative.

It is difficult to construct families of polytopes where both descriptions are explicit. The aim of this paper is to study a new family of such examples. These polytopes are associated to vector-factorisations of prime-powers where a  $d$ -dimensional vector-factorisation of a prime-power  $p^e$  is an integral vector  $(v_1, v_2, \dots, v_d) \in \mathbb{N}^d$  such that  $p^e = v_1 \cdot v_2 \cdots v_d$ . Given a prime power  $p^e \in \mathbb{N}$  and a natural integer  $d \geq 1$ , we denote by  $\mathcal{P}(p^e, d)$  the convex hull of all  $d$ -dimensional vector-factorisations  $(v_1, v_2, \dots, v_d) \in \mathbb{N}^d$  of  $p^e$ . The case  $e = 0$  yields the unique vector-factorisation  $(1, 1, \dots, 1)$  and is without interest. For  $d = 2$  and  $e \geq 2$  the polytope  $\mathcal{P}(p^e, 2)$  is a 2-dimensional polygon with vertices  $(1, p^e), (p, p^{e-1}), \dots, (p^{e-1}, p), (1, p^e)$ . For  $e = 1$ , the polytope  $\mathcal{P}(p, d)$  is a  $(d - 1)$ -dimensional simplex with vertices  $(p, 1, \dots, 1), (1, p, 1, \dots, 1), \dots, (1, \dots, 1, p)$ .

The observation that the combinatorial properties of  $\mathcal{P}(p^e, d)$  are independent of the prime  $p$  in these examples is a general fact: The combinatorial properties of the polytope  $\mathcal{P}(p^e, d)$  are independent always independent of the prime number  $p$ . It is in fact possible to replace every occurrence of  $p$  by an arbitrary real constant which is strictly greater than 1. (The choice of a strictly positive real number which is strictly smaller than 1 leads to a

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combinatorially equivalent polytope with the opposite orientation.) Let us also mention that the polytopes  $\mathcal{P}(p^e, d)$  are invariant under permutations of coordinates.

In the sequel we suppose always  $e \geq 2$  and  $d \geq 2$ . This ensures that  $\mathcal{P}(p^e, d)$  is  $d$ -dimensional.

In order to state our first main result, the description of  $\mathcal{P}(p^e, d)$  in terms of inequalities, we consider for  $\lambda \in \{1, \dots, \min(e, d-1)\}$  the set  $\mathcal{R}_\lambda(d, e)$  consisting of all integral vectors  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$  such that  $\min(\alpha_1, \dots, \alpha_d) = 0$ ,

$$\max(\alpha_1, \dots, \alpha_d) d < e + \sum_{i=1}^d \alpha_i$$

and

$$e + \sum_{i=1}^d \alpha_i \equiv \lambda \pmod{d}.$$

We call  $\mathcal{R}_\lambda(d, e)$  the set of *regular vectors of type  $\lambda$* . The union

$$\mathcal{R}(d, e) = \bigcup_{\lambda=1}^{\min(e, d-1)} \mathcal{R}_\lambda(d, e)$$

is the set of *regular vectors*.

We denote by  $\mu$  the function on  $\mathcal{R}(d, e)$  defined by the equality

$$\mu(\alpha) d + \lambda = e + \sum_{i=1}^d \alpha_i$$

where  $\alpha = (\alpha_1, \dots, \alpha_d)$  is an element of  $\mathcal{R}_\lambda(d, e)$ . The formula

$$\mu(\alpha) = \left\lfloor \left( e + \sum_{i=1}^d \alpha_i \right) / d \right\rfloor = \left( e - \lambda + \sum_{i=1}^d \alpha_i \right) / d$$

shows that  $\mu(\alpha)$  is a natural integer.

**Theorem 1.1.** *Let  $e \geq 2$  and  $d \geq 2$  be two integers.*

*The polytope  $\mathcal{P}(p^e, d)$  is defined by the inequalities*

$$\begin{aligned} \sum_{i=1}^d x_i &\leq (d-1) + p^e, \\ x_i &\geq 1, \quad i = 1, \dots, d, \end{aligned}$$

$$\sum_{i=1}^d p^{\alpha_i} x_i \geq \lambda p^{\mu(\alpha)+1} + (d-\lambda) p^{\mu(\alpha)}, \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathcal{R}_\lambda(d, e)$$

for  $\lambda$  in  $\{1, \dots, \min(e, d-1)\}$ .

*This list of inequalities is minimal if  $d \geq 3$ . For  $d = 2$ , the minimal list is obtained by removing the two inequalities  $x_1 \geq 1$  and  $x_2 \geq 1$ .*

For  $\lambda \in \{1, \dots, d-1\}$ , we denote by

$$\Delta(\lambda) = \text{Conv}((\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d, \sum_{i=1}^d \epsilon_i = \lambda)$$

the  $(d-1)$ -dimensional hypersimplex of parameter  $\lambda$  (see for example page 19 of [4]).

Faces of codimension 1 (or  $(d-1)$ -dimensional faces) of a  $d$ -dimensional polytope are called *facets*. The following result describes all facets of  $\mathcal{P}(p^e, d)$  in terms of their vertices.

**Theorem 1.2.** *The set of all facets of  $\mathcal{P}(p^e, d)$  (for  $e \geq 2$  and  $d \geq 3$ ) is given by the set of convex hulls of the following sets:*

$$\begin{aligned} & \{(p^e, 1, \dots, 1), (1, p^e, 1, \dots), \dots, (1, \dots, 1, p^e)\}, \\ & \{(p^{\beta_1}, \dots, p^{\beta_d}) \in \mathbb{N}^d \mid \beta_i = 0, \sum_{j=1}^d \beta_j = e, i = 1, \dots, d \\ & \{(p^{\beta_1+\epsilon_1}, \dots, p^{\beta_d+\epsilon_d}) \in \mathbb{N}^d \mid (\epsilon_1, \dots, \epsilon_d) \in \Delta(\lambda)\} \end{aligned}$$

with  $\lambda = 1, \dots, \min(e, d-1)$  and  $(p^{\beta_1}, \dots, p^{\beta_d})$  going through all  $d$ -dimensional vector-factorisations of  $p^{e-\lambda}$ .

Recall that the  $f$ -vector of a  $d$ -dimensional polytope  $\mathcal{P}$  counts the number  $f_k$  of  $k$ -dimensional faces contained in  $\mathcal{P}$ .

The following result describes the coefficients of the  $f$ -vector of  $\mathcal{P}(p^e, d)$ .

**Theorem 1.3.** *Let  $e \geq 2$  and  $d \geq 2$  be two integers.*

*The numbers  $f_0, \dots, f_k$  with  $f_k$  counting the number of  $k$ -dimensional faces of the polytope  $\mathcal{P}(p^e, d)$  are given by the formulae*

$$\begin{aligned} f_0 &= \binom{e+d-1}{d-1}, \\ f_1 &= \binom{d}{2} + \binom{d}{2} \binom{e+d-2}{d-1}, \\ f_k &= \binom{d+1}{k+1} + \binom{d}{k+1} \binom{e+d-1}{d} - \binom{d}{k+1} \binom{e-k+d-1}{d}, \quad 2 \leq k < d \\ f_d &= 1. \end{aligned}$$

The formula for the number  $f_0$  of vertices is easy: Identification of a vector-factorisation  $(p^{\beta_1}, p^{\beta_2}, \dots, p^{\beta_d})$  with the monomial  $x_1^{\beta_1} x_2^{\beta_2} \cdots x_d^{\beta_d}$  of  $\mathbb{Q}[x_1, \dots, x_d]$  shows that  $f_0$  is the dimension of the vector space spanned by homogeneous polynomials of degree  $e$  in  $d$  variables.

The plan of the paper is as follows:

Section 2 describes a few generalisations of the polytopes  $\mathcal{P}(p^e, d)$ .

The next three sections are devoted to the proof of Theorem 1.1 and Theorem 1.2. The idea for proving Theorem 1.1 is as follows: We show first that all inequalities of Theorem 1.1 hold and that they correspond to facets of  $\mathcal{P}(p^e, d)$ . Section 3 contains the details for facets associated to regular vectors, called *regular facets*. Section 4 is devoted to  $d + 1$  other obvious facets, called *exceptional facets*. The proofs contain explicit descriptions of all facets and imply easily Theorem 1.2 from Theorem 1.1. It remains to show that  $\mathcal{P}(p^e, d)$  has no “exotic” facets (neither regular nor exceptional). This is achieved in Section 5 by showing that any facet  $f$  of the  $(d - 1)$ -dimensional polytope defined by a (regular or exceptional) facet  $F$  of  $\mathcal{P}$  is also contained in a second (regular or exceptional) facet  $F'$  of  $\mathcal{P}$ . This implies the completeness of the list of facets.

Finally, Section 6 contains a proof of Theorem 1.3.

## 2 Generalisations

A straightforward generalisation of the polytope  $\mathcal{P}(p^e, d)$  is obtained by considering the polytopes with vertices given by vector-factorisations of an arbitrary natural integer  $N$ . The vertices of such a polytope  $\mathcal{P}(N, d)$  are the

$$\prod_{j=1}^h \binom{e_j + d - 1}{d - 1}$$

different  $d$ -dimensional vector-factorisations of  $N = p_1^{e_1} \dots p_h^{e_h}$  where  $p_1 < p_2 < \dots < p_h$  are all prime-divisors of  $N$ . A further generalisation is given by replacing  $N$  with a monomial  $\mathbf{T} = T_1^{e_1} \dots T_h^{e_h}$  and by considering real evaluations  $T_i = t_i > 0$  of all  $d$ -dimensional vector-factorisations of  $\mathbf{T}$  (defined in the obvious way). The combinatorial type of these polytopes depends on these evaluations (or on the primes  $p_1, \dots, p_h$  involved in  $N = p_1^{e_1} \dots p_h^{e_h}$ ). There should however exist a “limit-type” if the increasing sequence  $p_1 < p_2 < \dots < p_h$  formed by all prime-divisors of  $N$  grows extremely fast.

**Remark 2.1.** *One can also consider polytopes defined as the convex hull of vector-factorisations (of a given integer) subject to various restrictions. A perhaps interesting case is given by considering only factorisations with decreasing coordinates. The number of such decreasing  $d$ -dimensional vector-factorisations of  $p^e$  equals the number of partitions of  $e$  having at most  $d$  parts if  $N = p^e$  is the  $e$ -th power of a prime  $p$ .*

The polytopes  $\mathcal{P}(N, d)$  have the following natural generalisation: Consider an integral symmetric matrix  $A$  of size  $d \times d$ . For a given natural number  $N$ , consider the set  $\mathcal{D}_A(N)$  of all integral diagonal matrices  $D$  of size  $d \times d$  such that  $A + D$  is positive definite and has determinant  $N$ . One

can show that  $\mathcal{D}_A(N)$  is always finite. Brunn-Minkowski's inequality for mixed volumes, see eg. Theorem 6.2 in [3], states that

$$\det \left( A + \sum_{D \in \mathcal{D}_A(N)} \lambda_D D \right)^{1/d} \geq \sum_{D \in \mathcal{D}_A(N)} \lambda_D \det(A + D)^{1/d} = N^{1/d}$$

if  $\sum_{D \in \mathcal{D}_A(N)} \lambda_D = 1$  with  $\lambda_D \geq 0$ . This inequality is strict except in the obvious case where  $\lambda_D$  is equal to 1 for a unique matrix  $D \in \mathcal{D}_A(N)$ . This implies that  $\mathcal{D}_A(N)$  is the set of vertices of the polytope  $\mathcal{P}_A(N)$  defined as the convex hull of  $\mathcal{D}_A(N)$ .

The polytope  $\mathcal{P}(N, d)$  discussed above correspond to the case where  $A$  is the zero matrix of size  $d \times d$ .

The choice of a Dynkin matrix of size  $d \times d$  for a root system of type  $A$  and of  $N = 1$  leads to polytopes having  $\binom{2d}{d}/(d+1)$  vertices, see for example the solution of problem (18) in [1] or [2]. These polytopes are different from the Stasheff polytopes (or associahedra) since they are of dimension  $d$  for  $d \geq 3$ .

Another natural and perhaps interesting choice is given by considering for  $A$  (an integral multiple of) the all one matrix.

The determination of the number of vertices of  $\mathcal{P}_A(N)$  (or even of  $\mathcal{P}_A(1)$ ) is perhaps a non-trivial problem, say, for  $A$  the adjacency matrix of a connected finite simple graph.

### 3 Regular facets

We show first that all inequalities of Theorem 1.1 associated to regular vectors in  $\mathcal{R}_\lambda(d, e)$  are satisfied on  $\mathcal{P}(p^e, d)$ .

We show then that every such inequality is sharp on a subset of vertices in  $\mathcal{P}(p^e, d)$  spanning a polytope affinely equivalent to a  $(d-1)$ -dimensional hypersimplex. All these inequalities define thus facets and are necessary. We call a facet  $F_\alpha$  associated to such a regular vector  $\alpha \in \mathcal{R}_\lambda(p^e, d)$  a *regular facet*.

**Proposition 3.1.** *Let  $\alpha \in \mathcal{R}_\lambda(d, e)$  be a regular vector. Setting  $\mu = \mu(\alpha)$ , we have*

$$\sum_{i=1}^d p^{\alpha_i} x_i \geq \lambda p^{\mu+1} + (d - \lambda) p^\mu$$

for every element  $(x_1, \dots, x_d)$  of  $\mathcal{P}(p^e, d)$ .

**Proof** Given  $\alpha = (\alpha_1, \dots, \alpha_d)$  in  $\mathcal{A}_\lambda = \mathcal{R}_\lambda(d, e)$ , we denote by  $l_\alpha$  the linear form defined by

$$l_\alpha(x_1, \dots, x_d) = \sum_{i=1}^d p^{\alpha_i} x_i .$$

We have to show that  $l_\alpha(x) \geq \lambda p^{\mu+1} + (d - \lambda)p^\mu$  for  $x$  in  $\mathcal{P} = \mathcal{P}(p^e, d)$  and  $\mu = \mu(\alpha)$ . Since  $l_\alpha$  is linear, it is enough to establish the inequality for all vertices of  $\mathcal{P}$ .

Let  $v = (p^{\beta_1}, \dots, p^{\beta_d})$  be a vertex of  $\mathcal{P}$  realising the minimum  $l_\alpha(v) = \min l_\alpha(\mathcal{P})$ . Set

$$\begin{aligned} a &= \min(\alpha_1 + \beta_1, \dots, \alpha_d + \beta_d) , \\ A &= \max(\alpha_1 + \beta_1, \dots, \alpha_d + \beta_d) . \end{aligned}$$

Choose indices  $i, j$  in  $\{1, \dots, d\}$  such that  $a = \alpha_i + \beta_i$  and  $A = \alpha_j + \beta_j$ .

If  $a = A$ , the equality

$$dA = \sum_{i=1}^d \alpha_i + \beta_i = e + \sum_{i=1}^d \alpha_i \equiv \lambda \pmod{d}$$

shows  $\lambda = 0$  in contradiction with  $\lambda \in \{1, \dots, d-1\}$ .

We claim next that  $\beta_j \geq 1$ . Indeed, we have otherwise  $A = \alpha_j$  and

$$e + \sum_{k=1}^d \alpha_k = \sum_{k=1}^d (\alpha_k + \beta_k) \leq Ad = \max(\alpha_1, \dots, \alpha_d) d$$

in contradiction with the inequality  $\max(\alpha_1, \dots, \alpha_d) d < e + \sum_{k=1}^d \alpha_k$  satisfied by  $\alpha \in \mathcal{R}_\lambda$ .

We consider now the vertex

$$\tilde{v} = (p^{\tilde{\beta}_1}, \dots, p^{\tilde{\beta}_d})$$

where  $\tilde{\beta}_k = \beta_k$  if  $k \notin \{i, j\}$ ,  $\tilde{\beta}_i = \beta_i + 1 = a + 1$  and  $\tilde{\beta}_j = \beta_j - 1 = A - 1$ . We have

$$l_\alpha(v) - l_\alpha(\tilde{v}) = \sum_{k=1}^d p^{\alpha_k + \beta_k} - \sum_{k=1}^d p^{\alpha_k + \tilde{\beta}_k} = p^A + p^a - (p^{A-1} + p^{a+1}) .$$

Since  $p > 1$  and  $A > a$  we have

$$p^A + p^a - (p^{A-1} + p^{a+1}) = (p^{A-1} - p^a)(p - 1) \geq 0 .$$

This shows that  $A - 1 = a$  by minimality of  $l_\alpha(v)$ .

In order to compute the value of  $l_\alpha(v)$ , we use the equalities

$$\sum_{i=1}^d \alpha_i + \beta_i = e + \sum_{i=1}^d \alpha_i = \lambda + \mu d .$$

Since the vector  $w = (\alpha_1 + \beta_1, \dots, \alpha_d + \beta_d)$  has all its coefficients in  $\{a, a+1 = A\}$ , we get  $a = \mu$  and  $w$  takes the value  $\mu$  with multiplicity  $d - \lambda$  and  $\mu + 1$  with multiplicity  $\lambda$ . This shows  $l_\alpha(v) = \lambda p^{\mu+1} + (d - \lambda)p^\mu$ .  $\square$

**Proposition 3.2.** *The map  $(\alpha_1, \dots, \alpha_d) \mapsto (p^{\mu-\alpha_1}, \dots, p^{\mu-\alpha_d})$  (where  $\mu = \lfloor (e + \sum_{i=1}^d \alpha_i)/d \rfloor = (e - \lambda + \sum_{i=1}^d \alpha_i)/d$ ) is a one-to-one map from the set  $\mathcal{R}_\lambda(d, e)$  of regular vectors of type  $\lambda$  onto the set of  $d$ -dimensional vector-factorisations of  $p^{e-\lambda}$ . The inverse map is given by  $(p^{\beta_1}, \dots, p^{\beta_d}) \mapsto (B - \beta_1, \dots, B - \beta_d)$  where  $B = \max(\beta_1, \dots, \beta_d)$ .*

**Proof** We define  $\mathcal{F}_\lambda = \mathcal{F}_\lambda(d, e)$  as the finite set of all integral vectors  $(\beta_1, \dots, \beta_d) \in \mathbb{N}^d$  such that  $\sum_{i=1}^d \beta_i = e - \lambda$ . The map

$$\mathcal{F}_\lambda \ni (\beta_1, \dots, \beta_d) \mapsto (p^{\beta_1}, \dots, p^{\beta_d})$$

yields a bijection between  $\mathcal{F}_\lambda$  and the set of  $d$ -dimensional vector-factorisations of  $p^{e-\lambda}$ .

We show first the inclusions  $\varphi(\mathcal{R}_\lambda) \subset \mathcal{F}_\lambda$  and  $\psi(\mathcal{F}_\lambda) \subset \mathcal{R}_\lambda$  where

$$\varphi(\alpha_1, \dots, \alpha_d) = (\mu - \alpha_1, \dots, \mu - \alpha_d)$$

with  $\mu = \left( e - \lambda + \sum_{i=1}^d \alpha_i \right) / d$  and

$$\psi(\beta_1, \dots, \beta_d) = (B - \beta_1, \dots, B - \beta_d)$$

with  $(\beta_1, \dots, \beta_d) \in \mathbb{N}^d$  such that  $\sum_{i=1}^d \beta_i = e - \lambda$  and  $B = \max(\beta_1, \dots, \beta_d)$ .

The inclusion  $\varphi(\mathcal{R}_\lambda) \subset \mathcal{F}_\lambda$  follows from

$$\max(\alpha_1, \dots, \alpha_d) \leq \lfloor (e + \sum_{i=1}^d \alpha_i)/d \rfloor = \mu = (e - \lambda + \sum_{i=1}^d \alpha_i)/d$$

showing  $\varphi(\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$  and from

$$\sum_{i=1}^d (\mu - \alpha_i) = \mu d - \sum_{i=1}^d \alpha_i = e - \lambda.$$

Consider now  $(\beta_1, \dots, \beta_d) \in \mathcal{F}_\lambda$ . We have  $(B - \beta_1, \dots, B - \beta_d) \in \mathbb{N}^d$  and  $\min(B - \beta_1, \dots, B - \beta_d) = 0$  where  $B = \max(\beta_1, \dots, \beta_d)$ . We have moreover the inequalities

$$e + \sum_{i=1}^d (B - \beta_i) = \lambda + Bd > Bd \geq \max(B - \beta_1, \dots, B - \beta_d) d$$

since  $\lambda > 0$ . Finally, the computation

$$e + \sum_{i=1}^d (B - \beta_i) = \lambda + Bd \equiv \lambda \pmod{d}$$

proves the inclusion of  $\psi(\beta_1, \dots, \beta_d) = (B - \beta_1, \dots, B - \beta_d)$  in  $\mathcal{R}_\lambda$ .

The computation of

$$\mu = \left( e - \lambda + \sum_{i=1}^d (B - \beta_i) \right) / d = B$$

shows that  $\varphi \circ \psi$  is the identity map of  $\mathcal{F}_\lambda$ .

Consider  $(\alpha_1, \dots, \alpha_d) \in \mathcal{R}_\lambda$ . Since  $\min(\alpha_1, \dots, \alpha_d) = 0$ , we have  $\max(\mu - \alpha_1, \dots, \mu - \alpha_d) = \mu$ . This implies that

$$\psi \circ \varphi(\alpha_1, \dots, \alpha_d) = (\mu - (\mu - \alpha_1), \dots, \mu - (\mu - \alpha_d))$$

is the identity map of the set  $\mathcal{R}_\lambda$ . □

**Proposition 3.3.** (i) Consider an integral vector  $(\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ , an integer  $\lambda \in \{1, \dots, d-1\}$  and a prime  $p$ . The affine isomorphism

$$(x_1, \dots, x_d) \mapsto \left( \frac{x_1 - p^{\beta_1}}{p^{\beta_1+1} - p^{\beta_1}}, \dots, \frac{x_d - p^{\beta_d}}{p^{\beta_d+1} - p^{\beta_d}} \right)$$

of  $\mathbb{R}^d$  induces a one-to-one map between the set

$$S = \{(p^{\beta_1+\epsilon_1}, \dots, p^{\beta_d+\epsilon_d}) \in \mathbb{N}^d \mid (\epsilon_1, \dots, \epsilon_d) \in \Delta(\lambda)\}$$

and the set

$$\left\{ (\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d, \sum_{i=1}^d \epsilon_i = \lambda \right\}$$

of vertices of the  $(d-1)$ -dimensional hypersimplex  $\Delta(\lambda)$ .

(ii) The facets of the convex hull of  $S$  are of the form  $x_i = p^{\beta_i+\epsilon}$  for  $i = 1, \dots, d$  and  $\epsilon \in \{0, 1\}$  except if  $\lambda = 1$  or  $\lambda = d-1$  where all facets are of the form  $x_i = p^{\beta_i}$  respectively  $x_i = p^{\beta_i+1}$ .

**Proof** We leave the easy proof of assertion (i) to the reader.

Assertion (ii) follows from the peculiar form of the affine isomorphism introduced in assertion (i) and from the observation that facets of  $\Delta(\lambda)$  are given as intersections of the hyperplane defined by the equation  $\sum_{i=1}^d x_i = \lambda$  with one of the  $2d$  facets of the  $d$ -dimensional cube  $[0, 1]^d$ . □

**Corollary 3.4.** The inequality

$$\sum_{i=1}^d p^{\alpha_i} x_i \geq \lambda p^{\mu+1} + (d-\lambda)p\mu$$

associated to a regular vector  $(\alpha_1, \dots, \alpha_d)$  in  $\mathcal{R}_\lambda(d, e)$  is sharp on the set

$$S_\alpha = \{(p^{\mu-\alpha_1+\epsilon_1}, \dots, p^{\mu-\alpha_d+\epsilon_d}) \mid (\epsilon_1, \dots, \epsilon_d) \in \Delta(\lambda)\}$$

of vertices of  $\mathcal{P}(p^e, d)$ . The convex hull of  $S_\alpha$  is a facet of  $\mathcal{P}(p^e, d)$  which is affinely equivalent to the  $(d-1)$ -dimensional hypersimplex  $\Delta(\lambda)$ .

**Proof** The obvious equalities

$$\begin{aligned}\mu &= \min(\mu - \alpha_1 + \epsilon_1 + \alpha_1, \dots, \mu - \alpha_d + \epsilon_d + \alpha_d) \\ \mu + 1 &= \max(\mu - \alpha_1 + \epsilon_1 + \alpha_1, \dots, \mu - \alpha_d + \epsilon_d + \alpha_d)\end{aligned}$$

and the definition of  $\Delta(\lambda)$  show that  $S_\alpha$  consists exactly of all vertices of  $\mathcal{P}(p^e, d)$  such that  $A = a + 1$  with  $a, A$  as in the proof of Proposition 3.1. The arguments of the proof of Proposition 3.1 imply thus that  $S_\alpha$  is the subset of vertices of  $\mathcal{P}(p^e, d)$  on which the linear form

$$x = (x_1, \dots, x_d) \mapsto l_\alpha(x) = \sum_{i=1}^d p^{\alpha_i} x_i$$

is minimal. This shows that the convex hull  $F_\alpha$  of the set  $S_\alpha$  defines a  $k$ -dimensional face of  $\mathcal{P}(p^e, d)$  for some integer  $k \in \{0, \dots, d-1\}$ . Assertion (i) of Proposition 3.3 implies now that  $F_\alpha$  is affinely equivalent to a hypersimplex  $\Delta(\lambda)$  of dimension  $d-1$ . In particular, the convex hull  $F_\alpha$  of  $S_\alpha$  is a facet of  $\mathcal{P}(p^e, d)$ .  $\square$

## 4 Exceptional facets

We leave it the reader to check that we have

$$\sum_{i=1}^d x_i \leq d - 1 + p^e$$

for  $(x_1, \dots, x_d) \in \mathcal{P}(p^e, d)$ . The details are straightforward and involve computations similar to those used for proving Proposition 3.1. Equality holds for the elements of the exceptional facet  $F_\infty$  given by the  $(d-1)$ -dimensional simplex with vertices  $(p^e, 1, \dots, 1), \dots, (1, \dots, 1, p^e)$ .

The inequalities  $x_i \geq 1$ ,  $i = 1, \dots, d$  hold obviously for  $(x_1, \dots, x_d) \in \mathcal{P}(p^e, d)$ . For  $d \geq 3$ , these inequalities define  $d$  exceptional facets  $F_1, \dots, F_d$  which are all affinely equivalent to the  $(d-1)$ -dimensional polytope  $\mathcal{P}(p^e, d-1)$ .

**Remark 4.1.** *The  $d+1$  inequalities associated to exceptional facets define a  $d$ -dimensional simplex with vertices  $(1, 1, \dots, 1, 1), (p^e, 1, \dots, 1), \dots, (1, \dots, 1, p^e)$ . This simplex contains  $\mathcal{P}(p^e, d)$ .*

## 5 Proof of Theorem 1.1 and 1.2

The main tool for proving Theorem 1.1 is the following obvious and well-known result.

**Proposition 5.1.** *Let  $\mathcal{F}$  be a non-empty set of facets of a polytope  $\mathcal{P}$ . The set  $\mathcal{F}$  contains all facets of  $\mathcal{P}$  if and only if for every element  $F \in \mathcal{F}$  and for every facet  $f$  of  $F$ , there exists a distinct element  $F' \neq F$  in  $\mathcal{F}$  such that  $f$  is also a facet of  $F'$ .*

**Proof** Call two facets of a  $d$ -dimensional polytope  $\mathcal{P}$  *adjacent* if they intersect in a common  $(d-2)$ -face of  $\mathcal{P}$ . Consider the graph with vertices formed by all facets of  $\mathcal{P}$  and edges given by adjacent pairs of facets. This graph is connected and its edges are in bijection with  $(d-2)$ -faces of  $\mathcal{P}$  since the intersection of three distinct facets is of dimension  $\leq d-3$ . Proposition 5.1 boils now down to the trivial observation that a non-empty subset  $\mathcal{V}'$  of vertices of a connected graph  $\Gamma$  coincides with the set of vertices of  $\Gamma$  if and only if for every vertex  $v$  of  $\mathcal{V}'$ , the set  $\mathcal{V}'$  contains also all vertices of  $\Gamma$  which are adjacent to  $v$ .  $\square$

Given a subset  $S$  of vertices of  $\mathcal{P}(p^e, d)$ , we consider

$$m_i = \min_{(p^{\beta_1}, \dots, p^{\beta_d}) \in S} (\beta_i)$$

and

$$M_i = \max_{(p^{\beta_1}, \dots, p^{\beta_d}) \in S} (\beta_i).$$

We have thus

$$m_i \leq \beta_i \leq M_i$$

for every element  $(p^{\beta_1}, \dots, p^{\beta_d})$  of  $S$  and these inequalities are sharp. We set  $m(S) = (m_1, \dots, m_d)$  and  $M(S) = (M_1, \dots, M_d)$ .

An important ingredient of all proofs is the following result.

**Lemma 5.2.** *Let  $S$  be a subset of vertices of  $\mathcal{P}(p^e, d)$  such that  $M - m \in \{0, 1\}^d$  where  $m = m(S)$  and  $M = M(S)$  are as above. Then  $S$  is contained in the set of vertices of a regular facet of  $\mathcal{P}(p^e, d)$ .*

**Proof** Set  $\lambda = e - \sum_{i=1}^d m_i$ . Suppose first  $\lambda = 0$ . This implies that  $S$  is reduced to a unique element  $v = (p^{m_1}, \dots, p^{m_d})$ . Choose two distinct indices  $i, j$  in  $\{1, \dots, d\}$  such that  $m_i > 0$  in order to construct the element  $\tilde{v} = (p^{\tilde{m}_1}, \dots, p^{\tilde{m}_d})$  where  $\tilde{m}_k = m_k$  if  $k \neq i, j$ ,  $\tilde{m}_i = m_i - 1$ ,  $\tilde{m}_j = m_j + 1$ . The set  $\tilde{S} = \{v, \tilde{v}\}$  contains  $S$  and satisfies the conditions of Lemma 5.2 with  $\lambda = 1$ . We may thus assume  $\lambda \geq 1$ .

The obvious identity  $\prod_{i=1}^d p^{\beta_i} = p^e$  shows the equality  $\sum_{i=1}^d \beta_i = \lambda + \sum_{i=1}^d m_i$  for every element  $(p^{\beta_1}, \dots, p^{\beta_d})$  of  $S$ . The inclusion  $M - m \in \{0, 1\}^d$  shows  $m_i \leq \beta_i \leq M_i \leq m_i + 1$  and implies  $\lambda \leq d$ . Moreover, if  $\lambda = d$  then  $\beta_i = m_i + 1$  for every element  $(p^{\beta_1}, \dots, p^{\beta_d})$  of  $S$ . This shows that  $S$  is reduced to the unique element  $(p^{m_1+1}, \dots, p^{m_d+1})$  and contradicts the definition of  $m = (m_1, \dots, m_d)$ . We have thus  $\lambda \in \{1, \dots, d-1\}$ .

Up to enlarging the set  $S$ , we can assume

$$S = \{(p^{m_1+\epsilon_1}, \dots, p^{m_d+\epsilon_d}) \in \mathbb{N}^d \mid (\epsilon_1, \dots, \epsilon_d) \in \Delta(\lambda)\}$$

for some integer  $\lambda \in \{1, \dots, d-1\}$ .

Consider the function  $\psi$  defined by  $\psi(m_1, \dots, m_d) = (\mu - m_1, \dots, \mu - m_d)$  where  $\mu = \max(m_1, \dots, m_d)$ , see the proof of Proposition 3.2. Proposition 3.2 shows that we have  $\psi(m_1, \dots, m_d) = (\mu - m_1, \dots, \mu - m_d) \in \mathcal{R}_\lambda(p^e, d)$ . Corollary 3.4 shows now that  $\mathcal{S}$  is the vertex set of the regular facet defined by the regular vector  $\psi(m_1, \dots, m_d) = (\mu - m_1, \dots, \mu - m_d)$  of  $\mathcal{R}_\lambda(p^e, d)$ .  $\square$

**Proof of Theorem 1.1** Corollary 3.4 and Section 4 show that all inequalities of Theorem 1.1 are satisfied and necessary if  $d \geq 3$ . We have thus to show that every facet of  $\mathcal{P}(p^e, d)$  is either in  $\{F_\infty, F_1, F_2, \dots, F_d\}$  or is among the set  $\{F_\alpha\}_{\alpha \in \mathcal{R}}$  of regular facets indexed by the set  $\mathcal{R} = \bigcup_{\lambda=1}^{\min(e, d-1)} \mathcal{R}_\lambda(d, e)$  of regular vectors. We show this using Proposition 5.1 with respect to the set of facets

$$\mathcal{F} = F_\infty \cup \bigcup_{i=1}^d F_i \cup \bigcup_{\alpha \in \mathcal{R}} F_\alpha .$$

We consider first the exceptional facet  $F_\infty$ . A facet  $f$  of  $F_\infty$  is defined by an additional equality  $x_i = 1$  for some  $i \in \{1, \dots, d\}$  and we have thus  $f \in F_\infty \cap F_i$  where  $F_i$  is the exceptional facet defined by  $x_i = 1$ .

Consider next an exceptional facet  $F_i$  defined by  $x_i = 1$  for some  $i \in \{1, \dots, d\}$ . Such a facet  $F_i$  coincides with the polytope  $\mathcal{P}' = \mathcal{P}(p^e, d-1)$  of all  $(d-1)$ -dimensional vector-factorisations of  $p^e$ . Facets of  $F_i$  are thus in bijection with facets of  $\mathcal{P}'$ . Using induction on  $d$  (the initial case  $d=2$  is easy), we know thus complete list of facets of  $F_i$ . Consider first a facet  $f$  of  $F_i$  corresponding to an ordinary facet of  $\mathcal{P}'$ . Its vertices satisfy the conditions of Lemma 5.2 and are thus also contained in a regular facet of  $\mathcal{P}$ . A facet  $f$  of  $F_i$  corresponding to the exceptional facet  $F'_\infty$  of  $\mathcal{P}'$  is also contained in the exceptional facet  $F_\infty$  of  $\mathcal{P}$ . All other exceptional facets of  $\mathcal{P}'$  are given by  $x_i = x_j = 1$  for some  $j \neq i$  and are thus contained in  $F_i \cap F_j$ . (Remark that the last case does never arise for  $d=3$ .)

We consider now a regular facet  $F$  of type  $\lambda$  with vertices

$$\{(p^{\beta_1 + \epsilon_1}, \dots, p^{\beta_d + \epsilon_d}) \mid (\epsilon_1, \dots, \epsilon_d) \in \Delta(\lambda)\} .$$

Since  $F$  is affinely equivalent after multiplication by a diagonal matrix and a translation to the hypersimplex  $\Delta_{d-1}(\lambda)$ , a facet  $f$  of  $F$  is given by an additional equality  $x_i = c$  with  $i$  in  $\{1, \dots, d\}$  and  $c$  in  $\{p^{\beta_i}, p^{\beta_i+1}\}$ .

If  $c = 1$  then  $f$  is also contained in the exceptional facet  $F_i$ .

If  $c = p^{\beta_i} > 1$  and  $\lambda < d-1$  then  $f$  belongs also to the regular facet of type  $\lambda+1$  with vertices

$$\{(t^{\beta_1 + \epsilon_1}, \dots, p^{\beta_{i-1} + \epsilon_{i-1}}, p^{\beta_i - 1 + \epsilon_i}, p^{\beta_{i+1} + \epsilon_{i+1}}, \dots, p^{\beta_d + \epsilon_d}) \mid (\epsilon_1, \dots, \epsilon_d) \in \Delta(\lambda+1)\} .$$

The case  $\gamma_i = \beta_i$  and  $\lambda = d-1$  implies that the set of vertices of  $f$  is reduced to  $(p^{\beta_1+1}, \dots, p^{\beta_{i-1}+1}, p^{\beta_i}, p^{\beta_{i+1}+1}, \dots, p^{\beta_d+1})$ . This is impossible for  $d \geq 3$ .

If  $c = p^{\beta_i+1}$  and  $\lambda > 1$  then  $f$  belongs also to the regular facet of type  $\lambda - 1$  with vertices

$$\{(t^{\beta_1+\epsilon_1}, \dots, p^{\beta_{i-1}+\epsilon_{i-1}}, p^{\beta_i+1+\epsilon_i}, p^{\beta_{i+1}+\epsilon_{i+1}}, \dots, p^{\beta_d+\epsilon_d}) \mid (\epsilon_1, \dots, \epsilon_d) \in \Delta(\lambda-1)\}.$$

In the case  $c = p^{\beta_i+1}$  and  $\lambda = 1$  the set of vertices of  $f$  is reduced to  $(p^{\beta_1}, \dots, p^{\beta_{i-1}}, p^{\beta_i+1}, p^{\beta_{i+1}}, \dots, p^{\beta_d})$  and this is impossible for  $d \geq 3$ .

Proposition 5.1 shows that  $\mathcal{F}$  is the complete list of facets for  $\mathcal{P}$ . This ends the proof of Theorem 1.1.  $\square$

**Proof of Theorem 1.2** Theorem 1.2 follows easily from Theorem 1.1 and from the explicit descriptions of regular facets given by Proposition 3.2 and Corollary 3.4.

## 6 Proof of Theorem 1.3

Theorem 1.3 is easily checked in the case  $d = 2$  where  $\mathcal{P}(p^e, 2)$  is the polygon defined by the  $e + 1$  vertices  $(p^e, 1), (p^{e-1}, p), \dots, (p, p^{e-1}), (1, p^e)$ .

We suppose henceforth  $e \geq 2$  and  $d \geq 3$  and we consider  $\mathcal{P} = \mathcal{P}(p^e, d)$  (where  $p$  is a prime).

The formula for the number  $f_0$  of vertices of  $\mathcal{P}$  certainly holds. Indeed,  $f_0$  is equal to the number of  $d$ -dimensional vector-factorisations of  $p^e$ . Such vector-factorisations are in one-to-one correspondence with homogeneous monomials of degree  $e$  in  $d$  commuting variables. The vector space spanned by these monomials is of dimension  $\binom{e+d-1}{d-1}$ .

We call a  $k$ -dimensional face of  $\mathcal{P}$  *regular* if it is contained in a regular facet of  $\mathcal{P}$ . A  $k$ -dimensional face of  $\mathcal{P}$  is *exceptional* otherwise.

A 1-dimensional face is exceptional if and only if it is contained in the exceptional facet  $F_\infty$ . There are thus  $\binom{d}{2}$  exceptional 1-dimensional faces.

For  $k \geq 2$ , a  $k$ -dimensional face  $f$  which is exceptional is either contained in the exceptional facet  $F_\infty$  and there are  $\binom{d}{k+1}$  such faces, or it is the intersection of  $d - k$  distinct exceptional facets in  $\{F_1, \dots, F_d\}$ . For  $k$  in  $\{2, \dots, d - 1\}$ , the polytope  $\mathcal{P}$  contains thus

$$\binom{d}{k+1} + \binom{d}{d-k} = \binom{d+1}{k+1}$$

exceptional  $k$ -dimensional faces.

A regular face  $f$  of dimension  $k \geq 1$  has a type  $\lambda = e - \sum_{i=1}^d m_i \in \{1, \dots, k\}$  where  $m(S) = (m_1, \dots, m_d)$  is associated to the vertex set  $S$  of  $f$  as in Lemma 5.2. The face  $f$  is then defined by the support consisting of the  $k + 1$  non-zero coordinates of  $M(S) - m(s) \in \{0, 1\}^d$  and by the regular facet with vertices

$$\{(p^{m_1+\epsilon_1}, \dots, p^{m_d+\epsilon_d}) \mid (\epsilon_1, \dots, \epsilon_d) \in \Delta(\lambda)\}.$$

The number of regular  $k$ -dimensional faces contained in  $\mathcal{P}$  is thus given by

$$\binom{d}{k+1} \sum_{\lambda=1}^{\min(k,e)} \#(A_\lambda) = \binom{d}{k+1} \sum_{\lambda=1}^{\min(k,e)} \binom{e-\lambda+d-1}{d-1}$$

where the first factor corresponds to the choice of a support for  $M(S) - m(S)$ , the sum corresponds to all possibilities for  $\lambda$  and the factor  $\#(A_\lambda) = \binom{e-\lambda+d-1}{d-1}$  corresponds to all possibilities for the “minimal” regular facet with vertices

$$\{(p^{m_1+\epsilon_1}, \dots, p^{m_d+\epsilon_d}) \mid (\epsilon_1, \dots, \epsilon_d) \in \Delta(\lambda)\}$$

which contains  $f$ .

Iterated application of the identity  $\binom{a-1}{b-1} + \binom{a-1}{b} = \binom{a}{b}$  shows

$$\sum_{\lambda=1}^{\min(k,e)} \binom{e-\lambda+d-1}{d-1} = \binom{e-1+d}{d} - \binom{e-\min(e,k)+d-1}{d}.$$

This yields the closed expression

$$f_k = \binom{d+1}{k+1} + \binom{d}{k+1} \binom{e+d-1}{d} - \binom{d}{k+1} \binom{e-k+d-1}{d}$$

for  $f_2, \dots, f_{d-1}$  and ends the proof of Theorem 1.3.  $\square$

**Remark 6.1.** *The proof of Theorem 1.3 contains the detailed description in terms of vertices of all faces of  $\mathcal{P}$ . It is thus easy to work out the face-lattice of  $\mathcal{P}(p^e, d)$ .*

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