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FORGETTING OF THE INITIAL DISTRIBUTION FOR NON-ERGODIC HIDDEN MARKOV CHAINS

BY ELISABETH GASSIAT

Université Paris-Sud 11, CNRS UMR 8628

BY BENOIT LANDELLE

Thales Optronique

BY ERIC MOULINES

Institut Télécom / Télécom ParisTech, UMR CNRS 5181

Abstract

In this paper, the forgetting of the initial distribution for a non-ergodic Hidden Markov Models (HMM) is studied. A new set of conditions is proposed to establish the forgetting property of the filter, which significantly extends all the existing results. Both a pathwise-type convergence of the total variation distance of the filter started from two different initial distributions, and a convergence in expectation are considered. The results are illustrated using generic models of non-ergodic HMM and extend all the results known so far.

1. Introduction and notations. A Hidden Markov Model (HMM) is a doubly stochastic process with an underlying Markov chain that is not directly observable. More specifically, let X and Y be two spaces equipped with countably generated σ -fields \mathcal{X} and \mathcal{Y} ; denote by Q and G respectively, a Markov transition kernel on $(\mathsf{X}, \mathcal{X})$ and a transition kernel from $(\mathsf{X}, \mathcal{X})$ to $(\mathsf{Y}, \mathcal{Y})$. Consider the Markov transition kernel defined for any $(x, y) \in \mathsf{X} \times \mathsf{Y}$ and $C \in \mathcal{X} \otimes \mathcal{Y}$ by

$$(1) \quad T[(x, y), C] \stackrel{\text{def}}{=} Q \otimes G[(x, y), C] = \iint Q(x, dx') G(x', dy') \mathbb{1}_C(x', y').$$

We consider $\{X_k, Y_k\}_{k \geq 0}$ the Markov chain with transition kernel T and initial distribution $\nu \otimes G(C) \stackrel{\text{def}}{=} \iint \nu(dx) G(x, dy) \mathbb{1}_C(x, y)$, where ν is a probability measure on $(\mathsf{X}, \mathcal{X})$. We assume that the chain $\{X_k\}_{k \geq 0}$ is not observable (hence the name *hidden*). In addition, we assume that there exists a measure

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μ on (Y, \mathcal{Y}) such that for all $x \in X$, $G(x, \cdot)$ is absolutely continuous with respect to μ ; under these assumptions, the joint transition kernel T may be expressed as

$$(2) \quad T[(x, y), C] = \iint Q(x, dx') g(x', y') \mathbb{1}_C(x', y') \mu(dy'), \quad C \in \mathcal{X} \otimes \mathcal{Y},$$

where $g(x, \cdot) = \frac{dG(x, \cdot)}{d\mu}$ denotes the Radon-Nikodym derivative of $G(x, \cdot)$ with respect to μ ; $g(x, \cdot)$ is referred to as the *likelihood* of the observation. We denote by $\phi_{\nu, n}[y_{0:n}]$ the distribution of the hidden state X_n conditionally on the observations $y_{0:n} \stackrel{\text{def}}{=} [y_0, \dots, y_n]$, which is given by

$$(3) \quad \begin{aligned} \phi_{\nu, n}[y_{0:n}](A) &= \frac{\nu[g(\cdot, y_0)Qg(\cdot, y_1)Q \dots Qg(\cdot, y_n)\mathbb{1}_A]}{\nu[g(\cdot, y_0)Qg(\cdot, y_1)Q \dots Qg(\cdot, y_n)]} \\ &= \frac{\int_{\mathcal{X}_{n+1}} \nu(dx_0)g(x_0, y_0) \prod_{i=1}^n Q(x_{i-1}, dx_i)g(x_i, y_i)\mathbb{1}_A(x_n)}{\int_{\mathcal{X}_{n+1}} \nu(dx_0)g(x_0, y_0) \prod_{i=1}^n Q(x_{i-1}, dx_i)g(x_i, y_i)}, \end{aligned}$$

where $Qf(x) = Q(x, f) \stackrel{\text{def}}{=} \int Q(x, dx')f(x')$, for any function $f \in \mathbb{B}_+(X)$ the set of non-negative functions $f : X \rightarrow \mathbb{R}$, such that f is $\mathcal{X}/\mathcal{B}(\mathbb{R})$ measurable, with $\mathcal{B}(\mathbb{R})$ the Borel σ -algebra. Let $(\Omega, \mathcal{F}, \mathbb{P}_\star)$ be a probability space and $\{Y_k\}_{k \geq 0}$ be a Y -valued stochastic process defined on (Ω, \mathcal{F}) .

A typical question is under which conditions the distance between the filtering measures $\phi_{\nu, n}$ and $\phi_{\nu', n}$ for two different choices of the initial distribution ν and ν' vanishes, *i.e.*

$$\lim_{n \rightarrow \infty} \|\phi_{\nu, n}[Y_{0:n}] - \phi_{\nu', n}[Y_{0:n}]\|_{\text{TV}} = 0 \quad \mathbb{P}_\star - \text{a.s.},$$

where $\|\cdot\|_{\text{TV}}$ denotes the total variation norm. We stress that $\{Y_k\}_{k \geq 0}$ is not necessarily itself the observation sequence associated to the HMM used to define the sequence of filtering distribution, which means that we are interested in studying the forgetting property of the initial condition even when the model is mis-specified, which happens to be often the case in practical settings. The forgetting property of the initial condition of the optimal filter in nonlinear state space models has attracted many research efforts; see for example the in-depth tutorial of [5]. The brief overview below is mainly intended to allow comparison of assumptions and results presented in this contributions with respect to those previously reported in the literature.

The filtering equation can be seen as a positive random non-linear operator acting on the space of probability measure; the forgetting property can be investigated using tools from the theory of positive operators, namely the Birkhoff contraction inequality for the Hilbert projective metric (see [1],

[14], [13]). The results obtained using this approach require stringent *mixing* conditions for the transition kernels; these conditions state that there exist positive constants ϵ_- and ϵ_+ and a probability measure λ on $(\mathsf{X}, \mathcal{X})$ such that for $f \in \mathbb{B}^+(\mathsf{X})$,

$$(4) \quad \epsilon_- \lambda(f) \leq Q(x, f) \leq \epsilon_+ \lambda(f), \quad \text{for any } x \in \mathsf{X}.$$

This condition in particular implies that the chain is uniformly geometrically ergodic. Similar results were obtained independently by [6] using the Dobrushin ergodicity coefficient (see [7] for further refinements under this assumption). The mixing condition has later been weakened by [4], under the assumption that the kernel Q is positive recurrent and is dominated by some reference measure λ :

$$\sup_{(x, x') \in \mathsf{X} \times \mathsf{X}} q(x, x') < \infty \quad \text{and} \quad \int \text{essinf} q(x, x') \pi(x) \lambda(dx) > 0,$$

where $q(x, \cdot) = \frac{dQ(x, \cdot)}{d\lambda}$, essinf is the essential infimum with respect to λ and $\pi d\lambda$ is the stationary distribution of the chain Q . If the upper bound is reasonable, the lower bound is restrictive in many applications and fails to be satisfied e.g. for the linear state space Gaussian model.

In [14], the stability of the optimal filter is studied for a class of kernels referred to as *pseudo-mixing*. The definition of pseudo-mixing kernel is adapted to the case where the state space is $\mathsf{X} = \mathbb{R}^d$, equipped with the Borel sigma-field \mathcal{X} . A kernel Q on $(\mathsf{X}, \mathcal{X})$ is *pseudo-mixing* if for any compact set C with a diameter d large enough, there exist positive constants $\epsilon_-(d) > 0$ and $\epsilon_+(d) > 0$ and a measure λ_{C} (which may be chosen to be finite without loss of generality) such that

$$(5) \quad \epsilon_-(d) \lambda_{\mathsf{C}}(A) \leq Q(x, A) \leq \epsilon_+(d) \lambda_{\mathsf{C}}(A), \quad \text{for any } x \in \mathsf{C}, A \in \mathcal{X}$$

This condition implies that for any $(x', x'') \in \mathsf{C} \times \mathsf{C}$,

$$\frac{\epsilon_-(d)}{\epsilon_+(d)} < \text{essinf}_{x \in \mathsf{X}} q(x', x)/q(x'', x) \leq \text{esssup}_{x \in \mathsf{X}} q(x', x)/q(x'', x) \leq \frac{\epsilon_+(d)}{\epsilon_-(d)},$$

where $q(x, \cdot) \stackrel{\text{def}}{=} dQ(x, \cdot)/d\lambda_{\mathsf{C}}$, and esssup and essinf denote the essential supremum and infimum with respect to λ_{C} . This condition is obviously more general than (4), but still it is not satisfied in the linear Gaussian case (see [14, Example 4.3]).

Several attempts have been made to establish the stability conditions under the so-called *small* noise condition. The first result in this direction

has been obtained by [1] (in continuous time) who considered an ergodic diffusion process with constant diffusion coefficient and linear observations: when the variance of the observation noise is sufficiently small, [1] established that the filter is exponentially stable. Small noise conditions also appeared (in a discrete time setting) in [3] and [15]. These results do not allow to consider the linear gaussian state space model with arbitrary noise variance.

A very significant step has been achieved by [12], who considered the filtering problem of Markov chain $\{X_k\}_{k \geq 0}$ with values in $\mathsf{X} = \mathbb{R}^d$ filtered from observations $\{Y_k\}_{k \geq 0}$ in $\mathsf{Y} = \mathbb{R}^\ell$,

$$(6) \quad \begin{cases} X_{k+1} = X_k + b(X_k) + \sigma(X_k)\zeta_k, \\ Y_k = h(X_k) + \beta\varepsilon_k. \end{cases}$$

Here $\{(\zeta_k, \varepsilon_k)\}_{k \geq 0}$ is a i.i.d. sequence of standard Gaussian random vectors in $\mathbb{R}^{d+\ell}$, $b(\cdot)$ is a d -dimensional vector function, $\sigma(\cdot)$ a $d \times d$ -matrix function, $h(\cdot)$ is a ℓ -dimensional vector-function and $\beta > 0$. The authors established, under appropriate conditions on b , h and σ , that the optimal filter forgets the initial conditions; these conditions cover (with some restrictions) the linear gaussian state space model.

A new approach for ergodic HMM using the so-called *Local Doeblin property* is proposed in [8]. Both almost sure convergence and convergence in expectation for the distance in total variation norm for two filters with different initial distributions are proven. The results hold under weaker conditions than those appearing under other mixing assumptions and, in particular, cover the linear Gaussian state-space model. Moreover, assumptions on observations are relaxed and convergence theorems apply for sequences which are not necessarily HMM.

The works mentioned above mainly deal with ergodic HMM, *i.e.* the situations in which the hidden Markov chain is ergodic. Non-ergodic HMM models are routinely used in the non-linear filtering literature, many models used for example in tracking or financial econometrics being simply random walks (see [9] and [16] and the references therein). Non-ergodic HMM have been considered much less frequently in the literature. The main references in this direction are [3] and [15]. In [3], the observation process is the signal (state) corrupted by an additive white noise of sufficiently small noise variance. In [15], the authors also assumed that the observation is a possibly non-linear function of the signal (satisfying some additional technical conditions) and that this function of the signal is also observed in an additive noise model of sufficiently small variance. The authors propose to truncate the Markov kernels on random sets depending on the observation sequences, which are chosen in such a way that the truncated kernels satisfy mixing

conditions. The authors establish the convergence of the first-order moment of the difference under signal-to-noise ratio condition.

In this contribution, we propose a new set of conditions to establish the forgetting property of the filter, which are more general than those proposed in [3] and [15]. In Theorem 5, the convergence of the total variation distance of the filter started from two different initial distributions is established, and is shown to hold almost surely w.r.t. the probability distribution of the observation process $\{Y_k\}_{k \geq 0}$. Then, in Theorem 6, a bound for the expectation of this total variation distance is obtained and used in Section 3 for nonlinear state-space models to obtain a geometric rate. The results are shown to hold under rather weak conditions on the observation process $\{Y_k\}_{k \geq 0}$ which do not necessarily entail that the observations are produced by the filtering model.

The paper is organized as followed. In section 2, we introduce the assumptions and state the main results. In section 3, nonlinear state-space models are considered with different kind of dependence for the state noise and with observations not necessarily produced by the model defining the filter. The proofs are given in sections 4, 5, 6.

2. Main results. In this section, we present two theorems on the forgetting properties of the optimal filter. These results require the choice of a set-valued function, referred to as *Local Doeblin set function*, which extends the so-called local Doeblin sets introduced in [19] and later exploited in [12]. The difference between LD-sets of [19] and LD-set functions lies in the dependence on the successive observations.

DEFINITION 1 (LD-set function). *A set-valued function $C : Y \mapsto C(y)$ from Y to \mathcal{X} is called a Local Doeblin set function (LD-set function) if there exists a map $(y, y') \mapsto (\varepsilon_C^-(y, y'), \varepsilon_C^+(y, y'))$ from $Y \times Y$ to $(0, \infty)^2$ such that, for all $(y, y') \in Y \times Y$, there exists a measure $\lambda_{y, y'}$ on (X, \mathcal{X}) satisfying*

$$(7) \quad \varepsilon_C^-(y, y') \lambda_{y, y'}[A \cap C(y')] \leq Q[x, A \cap C(y')] \leq \varepsilon_C^+(y, y') \lambda_{y, y'}[A \cap C(y')]$$

for all $x \in C(y)$ and $A \in \mathcal{X}$.

Some general conditions on the Local Doeblin set function involving the distributions of the observations ensure the forgetting property of the optimal filter. The case of nonlinear state-space models is studied in Section 3. Roughly speaking, inequality (7) means that the transition of the hidden chain, when the state is in a given subset $C(y)$ does not depend too much on the current state.

We denote, for a set $A \in \mathcal{X}$ and an observation $y \in \mathcal{Y}$, the supremum of the likelihood over A ,

$$(8) \quad \Upsilon_A(y) \stackrel{\text{def}}{=} \sup_{x \in A} g(x, y) .$$

Consider the following assumptions on the likelihood of the observations :

(H1) For all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $g(x, y) > 0$.

(H2) For all $\eta > 0$, there exists an LD-set function C_η satisfying, for all $y \in \mathcal{Y}$,

$$(9) \quad \Upsilon_{C_\eta(y)}(y) \leq \eta \Upsilon_{\mathcal{X}}(y) .$$

The first condition states that the likelihood is everywhere positive. This excludes the case of additive noise with bounded support; see for example [2]. When $\mathcal{X} = \mathbb{R}^d$, the second assumption is typically satisfied when, for any given y , the likelihood goes to zero as the state $|x|$ goes to infinity: $\lim_{|x| \rightarrow \infty} g(x, y) = 0$. This assumption is satisfied in many models of practical interest, and roughly implies that the observation effectively provides information on the state range of value.

For a given LD-set function C , we set

$$(10) \quad \Phi_{\nu, C}(y, y') \stackrel{\text{def}}{=} \nu \left[g(\cdot, y) Q g(\cdot, y') \mathbf{1}_{C(y')}(\cdot) \right] ,$$

$$(11) \quad \Psi_C(y, y') \stackrel{\text{def}}{=} \lambda_{y, y'} \left[g(\cdot, y') \mathbf{1}_{C(y')} \right] .$$

The main idea of the proof is that the states belong very often to the LD-sets. Every time the state is in a LD set and jumps to another LD set, the forgetting mechanism comes into play. From now on, for all $(x, x') \in \mathcal{X}^2$, denote by $\bar{x} = (x, x')$ the product $\bar{g}(\bar{x}, y) = g(x, y)g(x', y)$. Similarly, for all $A \in \mathcal{X}$, denote $\bar{A} = A \times A$, for all LD-set function C , \bar{C} the set-valued function $\bar{C}(y) = C(y) \times C(y)$. Finally, for all $(x, x') \in \mathcal{X}^2$, and $A, B \in \mathcal{X}$, set $\bar{Q}(x, x', A \times B) = Q(x, A)Q(x', B)$. Then, for any $A \in \mathcal{X}$ and ν and ν' two probability measures on $(\mathcal{X}, \mathcal{X})$, the difference $\phi_{\nu, n}[y_{0:n}](A) - \phi_{\nu', n}[y_{0:n}](A)$

may be expressed as

(12)

$$\begin{aligned}
& \phi_{\nu,n}[y_{0:n}](A) - \phi_{\nu',n}[y_{0:n}](A) \\
&= \frac{\mathbb{E}_{\nu}^Q [\prod_{i=0}^n g(X_i, y_i) \mathbb{1}_A(X_n)]}{\mathbb{E}_{\nu}^Q [\prod_{i=0}^n g(X_i, y_i)]} - \frac{\mathbb{E}_{\nu'}^Q [\prod_{i=0}^n g(X_i, y_i) \mathbb{1}_A(X_n)]}{\mathbb{E}_{\nu'}^Q [\prod_{i=0}^n g(X_i, y_i)]}, \\
&= \frac{\mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} [\prod_{i=0}^n \bar{g}(\bar{X}_i, y_i) \mathbb{1}_A(X_n)] - \mathbb{E}_{\nu' \otimes \nu}^{\bar{Q}} [\prod_{i=0}^n \bar{g}(\bar{X}_i, y_i) \mathbb{1}_A(X_n)]}{\mathbb{E}_{\nu}^Q [\prod_{i=0}^n g(X_i, y_i)] \mathbb{E}_{\nu'}^Q [\prod_{i=0}^n g(X_i, y_i)]}, \\
(13) \quad &= \frac{\mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} [\prod_{i=0}^n \bar{g}(\bar{X}_i, y_i) \{ \mathbb{1}_A(X_n) - \mathbb{1}_A(X'_n) \}]}{\mathbb{E}_{\nu}^Q [\prod_{i=0}^n g(X_i, y_i)] \mathbb{E}_{\nu'}^Q [\prod_{i=0}^n g(X_i, y_i)]},
\end{aligned}$$

We compute bounds for the numerator and the denominator of the previous expression. Such bounds are given in the two following Propositions (proofs are postponed to Section 4). For an LD -set function C denotes:

$$(14) \quad \rho_C(y, y') \stackrel{\text{def}}{=} 1 - (\varepsilon_C^- / \varepsilon_C^+)^2(y, y').$$

PROPOSITION 2. *Let C be an LD -set function and ν and ν' be two probability measures on (X, \mathcal{X}) . For any integer n and any sequence $\{y_i\}_{i=0}^n$ in Y , let us define*

$$(15) \quad \Delta_n(\nu, \nu', y_{0:n}) = \sup_{A \in \mathcal{X}} \left| \mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left[\prod_{i=0}^n \bar{g}(\bar{X}_i, y_i) \mathbb{1}_A(X_n) \right] - \mathbb{E}_{\nu' \otimes \nu}^{\bar{Q}} \left[\prod_{i=0}^n \bar{g}(\bar{X}_i, y_i) \mathbb{1}_A(X_n) \right] \right|.$$

Then,

$$\Delta_n(\nu, \nu', y_{0:n}) \leq \mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left\{ \bar{g}(\bar{X}_0, y_0) \prod_{i=1}^n \bar{g}(\bar{X}_i, y_i) \rho_{\bar{C}(y_{i-1}, y_i)}^{\delta_i} \right\},$$

where $\delta_i = \mathbb{1}_{\bar{C}(y_{i-1}) \times \bar{C}(y_i)}(\bar{X}_{i-1}, \bar{X}_i)$.

PROPOSITION 3. *Let C be an LD -set function and $\{y_i\}_{i=0}^n$ a sequence in Y . We have for all $n \in \mathbb{N}$*

$$\mathbb{E}_{\nu}^Q \left[\prod_{i=0}^n g(X_i, y_i) \right] \geq \left(\prod_{i=2}^n \varepsilon_C^-(y_{i-1}, y_i) \right) \Phi_{\nu, C}(y_0, y_1) \prod_{i=2}^n \Psi_C(y_{i-1}, y_i).$$

By combining these two Propositions, we obtain an explicit bound for the total variation distance $\|\phi_{\nu,n}[y_{0:n}] - \phi_{\nu',n}[y_{0:n}]\|_{TV}$. It is worthwhile to

note that the bound we obtain is valid for any sequence $y_{0:n}$ and any initial distributions ν and ν' . To state the result, some additional notations are required. Under assumption (H2), for a fixed $\eta > 0$ and a corresponding LD-set function C_η , let us define, for $\alpha \in (0, 1)$ and a sequence $y_{0:n} = \{y_i\}_{i=0}^n$ in \mathcal{Y} ,

$$(16) \quad \Lambda_\eta(y_{0:n}, \alpha) \stackrel{\text{def}}{=} \max \left\{ \prod_{k=1}^n \rho_\eta^{\delta_k}(y_{k-1}, y_k), \{\delta_k\}_{k=1}^n \in \{0, 1\}^n : \sum_{k=1}^n \delta_k \geq \alpha n \right\},$$

where ρ_η is a shorthand notation for ρ_{C_η} (see (14))

THEOREM 4. *Let α be some number in $(0, 1)$, ν and ν' some probability measures on $(\mathcal{X}, \mathcal{X})$ and $\{y_i\}_{i=0}^n$ a sequence in \mathcal{Y} . Then,*

$$(17) \quad \|\phi_{\nu,n}[y_{0:n}] - \phi_{\nu',n}[y_{0:n}]\|_{\text{TV}} \leq \Lambda_\eta(y_{0:n}, \alpha) + \eta^{a_n} \prod_{i=2}^n \left(\varepsilon_{\bar{C}}^-(y_{i-1}, y_i) \Psi_{\bar{C}}(y_{i-1}, y_i) \right)^{-2} \prod_{i=0}^n \Upsilon_{\bar{X}}^2(y_i) \Phi_{\nu, \bar{C}}^{-1}(y_0, y_1) \Phi_{\nu', \bar{C}}^{-1}(y_0, y_1),$$

with $a_n = \lfloor \frac{(1-\alpha)n}{2} \rfloor$.

PROOF. The expression (12) together with Proposition 2 imply

$$\|\phi_{\nu,n}[y_{0:n}] - \phi_{\nu',n}[y_{0:n}]\|_{\text{TV}} \leq \frac{\Delta_n(\nu, \nu', y_{0:n})}{\mathbb{E}_{\nu'}^Q [\prod_{i=0}^n g(X_i, y_i)] \mathbb{E}_{\nu}^Q [\prod_{i=0}^n g(X_i, y_i)]},$$

where $\Delta_n(\nu, \nu', y_{0:n})$ is defined by (15). Set

$$N_{\bar{C},n} = \sum_{i=1}^n \mathbb{1}\{\bar{X}_{i-1} \in \bar{C}(y_{i-1})\} \mathbb{1}\{\bar{X}_i \in \bar{C}(y_i)\}, \quad M_{\bar{C}^c,n} = \sum_{i=0}^{n-1} \mathbb{1}_{\bar{C}^c(y_i)}(\bar{X}_i).$$

For any sequence $\{u_j\}$, such that $u_j \in \{0, 1\}$ for $j \in \{0, \dots, n\}$ and $u_j = 0$ for $j \geq n$,

$$n \geq \sum_{i=0}^{n-1} u_i \vee u_{i+1} = \sum_{i=0}^{n-1} (u_i + u_{i+1} - u_i u_{i+1}) \geq \sum_{i=0}^n u_i - 1 - \sum_{i=0}^{n-1} u_i u_{i+1},$$

which implies that $\sum_{i=0}^{n-1} u_i \leq (n+1)/2 + \sum_{i=0}^{n-1} u_i u_{i+1}$. Using this inequality with $u_i = \mathbb{1}\{\bar{X}_i \in \bar{C}(y_i)\}$ for $i \in \{0, \dots, n\}$ shows that $N_{\bar{C},n} < \alpha n$ implies

that $M_{\bar{C},n} \geq a_n$. Therefore, using Proposition 2, we obtain

$$(18) \quad \begin{aligned} \Delta_n(\nu, \nu', y_{0:n}) &\leq \mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left[\bar{g}(\bar{X}_0, y_0) \prod_{i=1}^n \bar{g}(\bar{X}_i, y_i) \rho_\eta^{\delta_i}(y_{i-1}, y_i) \mathbb{1}\{N_{\bar{C},n} \geq \alpha n\} \right] \\ &\quad + \mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left[\bar{g}(\bar{X}_0, y_0) \prod_{i=1}^n \bar{g}(\bar{X}_i, y_i) \rho_\eta^{\delta_i}(y_{i-1}, y_i) \mathbb{1}\{N_{\bar{C},n} < \alpha n\} \right], \end{aligned}$$

with $\delta_i = \mathbb{1}_{\bar{C}(y_{i-1}) \times \bar{C}(y_i)}(\bar{X}_{i-1}, \bar{X}_i)$. The last term in the right-hand side of (18) satisfies

$$\begin{aligned} \mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left[\bar{g}(\bar{X}_0, y_0) \prod_{i=1}^n \bar{g}(\bar{X}_i, y_i) \rho_\eta^{\delta_i}(y_{i-1}, y_i) \mathbb{1}\{N_{\bar{C},n} < \alpha n\} \right] \\ \leq \mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left[\prod_{i=0}^n \bar{g}(\bar{X}_i, y_i) \mathbb{1}\{M_{\bar{C},n} \geq a_n\} \right]. \end{aligned}$$

By splitting this last product, we obtain

$$\begin{aligned} &\prod_{i=0}^n \bar{g}(\bar{X}_i, y_i) \mathbb{1}\{M_{\bar{C},n} \geq a_n\} \\ &= \prod_{\substack{0 \leq i \leq n, \\ \bar{X}_i \in \bar{C}(y_i)^c}} \bar{g}(\bar{X}_i, y_i) \mathbb{1}\{M_{\bar{C},n} \geq a_n\} \times \prod_{\substack{0 \leq i \leq n, \\ \bar{X}_i \in \bar{C}(y_i)}} \bar{g}(\bar{X}_i, y_i) \mathbb{1}\{M_{\bar{C},n} \geq a_n\}, \\ &\leq \eta^{a_n} \times \prod_{\substack{0 \leq i \leq n, \\ \bar{X}_i \in \bar{C}(y_i)^c}} \Upsilon_{\bar{X}}^2(y_i) \times \prod_{\substack{0 \leq i \leq n, \\ \bar{X}_i \in \bar{C}(y_i)}} \Upsilon_{\bar{X}}^2(y_i), \end{aligned}$$

which implies $\mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left[\prod_{i=0}^n \bar{g}(\bar{X}_i, y_i) \mathbb{1}\{M_{\bar{C},n} \geq a_n\} \right] \leq \eta^{a_n} \prod_{i=0}^n \Upsilon_{\bar{X}}^2(y_i)$. The first term in the right-hand side expression of (18) satisfies

$$\begin{aligned} \mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left[\bar{g}(\bar{X}_0, y_0) \prod_{i=1}^n \bar{g}(\bar{X}_i, y_i) \prod_{i=1}^n \rho_\eta^{\delta_i}(y_{i-1}, y_i) \mathbb{1}\{N_{\bar{C},n} \geq \alpha n\} \right] \\ \leq \mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left[\prod_{i=0}^n \bar{g}(\bar{X}_i, y_i) \right] \Lambda_\eta(y_{0:n}, \alpha). \end{aligned}$$

By combining the above relations, the result follows. \square

The last step consists in finding conditions upon which the bound in the right hand side of (17) is small. This bound depends explicitly on the observations Y 's; it is therefore not difficult to state general conditions upon

which this quantity is small. Let $\{Y_k\}_{k \geq 0}$ be a stochastic process with probability distribution \mathbb{P}_\star in $(\mathcal{Y}, \mathcal{Y})$, which is not necessarily related to the model under which the filter is computed. We first formulate an almost sure convergence on the total variation distance of the filter initialized with two different probability measures ν and ν' and then later establish a convergence of the expectation.

THEOREM 5. *Assume (H1) and (H2). Assume moreover that there exists some LD-set function C such that*

$$(19) \quad \liminf_{n \rightarrow \infty} n^{-1} \sum_{k=2}^n \log \varepsilon_C^-(Y_{k-1}, Y_k) > -M, \quad \mathbb{P}_\star - \text{a.s.}$$

$$(20) \quad \limsup_{n \rightarrow \infty} n^{-1} \sum_{k=0}^n \log \Upsilon_X(Y_k) < M, \quad \mathbb{P}_\star - \text{a.s.}$$

$$(21) \quad \liminf_{n \rightarrow \infty} n^{-1} \sum_{k=2}^n \log \Psi_C(Y_{k-1}, Y_k) > -M, \quad \mathbb{P}_\star - \text{a.s.}$$

for some constant $M > 0$. Assume in addition that, for all $\eta > 0$ and $\alpha \in (0, 1)$,

$$(22) \quad \limsup_{n \rightarrow \infty} n^{-1} \log \Lambda_\eta(Y_{0:n}, \alpha) < 0, \quad \mathbb{P}_\star - \text{a.s.}$$

Then, for any initial probability distributions ν and ν' on $(\mathcal{X}, \mathcal{X})$ such that

$$\nu Q \mathbb{1}_{C(Y_1)} > 0, \quad \mathbb{P}_\star - \text{a.s.} \quad \nu' Q \mathbb{1}_{C(Y_1)} > 0, \quad \mathbb{P}_\star - \text{a.s.}$$

we have

$$\limsup_{n \rightarrow \infty} n^{-1} \log \|\phi_{\nu,n}[Y_{0:n}] - \phi_{\nu',n}[Y_{0:n}]\|_{\text{TV}} < 0, \quad \mathbb{P}_\star - \text{a.s.}$$

PROOF OF THEOREM 5. Under the stated assumptions, there exists a LD-set function C and some constant $M > 0$ such that

$$\limsup_{n \rightarrow \infty} \exp(-2Mn) \prod_{i=2}^n \left(\varepsilon_C^-(Y_{i-1}, Y_i) \right)^{-2} \leq 1, \quad \mathbb{P}_\star - \text{a.s.}$$

$$\limsup_{n \rightarrow \infty} \exp(-2Mn) \prod_{i=0}^n \Upsilon_X^2(Y_i) \leq 1, \quad \mathbb{P}_\star - \text{a.s.}$$

$$\limsup_{n \rightarrow \infty} \exp(-2Mn) \prod_{i=2}^n \Psi_C^{-2}(Y_{i-1}, Y_i) \leq 1, \quad \mathbb{P}_\star - \text{a.s.}$$

Let α be some number in $(0, 1)$. Since $a_n = \frac{(1-\alpha)n}{2} + o(n)$, by choosing η small enough, it follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \eta^{a_n} \prod_{i=2}^n \left[\varepsilon_{\mathcal{C}}^-(Y_{i-1}, Y_i) \Psi_{\mathcal{C}}(Y_{i-1}, Y_i) \right]^{-2} \prod_{i=0}^n \Upsilon_{\mathcal{X}}^2(Y_i) \\ \leq \limsup_{n \rightarrow \infty} \eta^{a_n} e^{6Mn} \leq \limsup_{n \rightarrow \infty} e^{-cn} \end{aligned}$$

for some $c > 0$. The proof is concluded by using inequality and (22). \square

The assumptions linking the LD-set function and the observations make this theorem quite abstract. With a filtering model defined by specific equations, assumptions can be directly formulated on the model and on the observations. Such situations will be described through examples presented in Section 3.

Compared to [8, Theorem 1] in the ergodic case, the conditions (19) and (22) are specific to the non-ergodic case, since they involve the functions $\varepsilon_{\mathcal{C}}^-$ and $\varepsilon_{\mathcal{C}}^+$. In the ergodic case, these functions are constant and assumptions (19) and (22) are trivially satisfied.

THEOREM 6. *Assume (H1) and (H2). Let \mathcal{C} be a LD-set function. Then, for any $M_i > 0$, $i = 0, \dots, 3$, $\delta > 0$ and $\alpha \in (0, 1)$, there exist constants $\eta > 0$ and $\beta \in (0, 1)$ such that, for all $n \in \mathbb{N}$,*

$$(23) \quad \mathbb{E}_{\star} \left[\left\| \phi_{\nu, n}[Y_{0:n}] - \phi_{\nu', n}[Y_{0:n}] \right\|_{\text{TV}} \right] \leq 2\beta^n + r_0(\nu, n) + r_0(\nu', n) + \sum_{i=1}^4 r_i(n)$$

where the sequences in the right-hand side of (23) are defined by

$$(24) \quad r_0(\nu, n) \stackrel{\text{def}}{=} \mathbb{P}_{\star} (\log \Phi_{\nu, \mathcal{C}}(Y_0, Y_1) \leq -M_0 n),$$

$$(25) \quad r_1(n) \stackrel{\text{def}}{=} \mathbb{P}_{\star} \left(\sum_{k=2}^n \log \varepsilon_{\mathcal{C}}^-(Y_{k-1}, Y_k) \leq -M_1 n \right),$$

$$(26) \quad r_2(n) \stackrel{\text{def}}{=} \mathbb{P}_{\star} \left(\sum_{k=0}^n \log \Upsilon_{\mathcal{X}}(Y_k) \geq M_2 n \right),$$

$$(27) \quad r_3(n) \stackrel{\text{def}}{=} \mathbb{P}_{\star} \left(\sum_{k=2}^n \log \Psi_{\mathcal{C}}(Y_{k-1}, Y_k) \leq -M_3 n \right),$$

$$(28) \quad r_4(n) \stackrel{\text{def}}{=} \mathbb{P}_{\star} (\log \Lambda_{\eta}(Y_{0:n}, \alpha) \geq -\delta n).$$

The proof is along the same lines as above and left to the reader. This result does not provide directly a rate of convergence. Indeed, only the first

term of the right-hand side of equation (23) gives a geometric rate. In Section 3, for given filtering equations, explicit majorations of the other terms will be obtained with geometric rates. Like for the pathwise convergence, the terms r_1 and r_4 which involve the functions ε_C^- and ε_C^+ are specific to the non-ergodic case.

3. Nonlinear state-space models. Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^p$ with $p \leq n$, endowed with the Borel σ -algebra \mathcal{X} and \mathcal{Y} . We consider the model:

$$(29) \quad \begin{cases} X_k = f(X_{k-1}) + \zeta_k, \\ Y_k = h(X_k) + \varepsilon_k, \end{cases}$$

where f and h denote some measurable functions. The observation noise $\{\varepsilon_k\}_{k \geq 0}$ is a sequence of i.i.d. random variables with positive density ν with respect to the Lebesgue measure λ^{Leb} on Y . We consider the following assumptions:

(E1) f is a -Lipschitz, i.e. $|f(x) - f(y)| \leq a|x - y|$ and h is uniformly continuous and surjective and, for all $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ in the preimage of y_1 and y_2 , there exist constant b_0 and b such that,

$$|x_1 - x_2| \leq b_0 + b|y_1 - y_2|.$$

(E2) The density ν is bounded, and $\lim_{|u| \rightarrow \infty} \nu(u) = 0$. Moreover, for all compact set $K \subset Y$, the quantity $\inf_{y \in K} \nu(y)$ is positive.

Notice that f is not necessarily contracting so that the model is possibly non-ergodic. The assumption (E1) has been first considered in [15]. A function f satisfying (E1) can be viewed as a perturbation of a bijective function whose inverse is b -Lipschitz. The rationale for considering such assumption is the following. For two successive observations $y_1, y_2 \in Y$, the distance between inverse images of y_1, y_2 can not be arbitrarily large. Even if h is not bijective, the distance $|y_1 - y_2|$ gives information on the distance of two successive preimage states. The assumption (E2) is more classical and is satisfied, for example, by Gaussian densities.

We first consider the simplest situation where the state noise is a sequence of i.i.d. random variables independent of the observation noise $\{\varepsilon_k\}_{k \geq 0}$ and the observations are distributed according to the model. Then, we study more general dependence structure of the state noise distribution and the case where the observations do not necessarily follow the model.

3.1. Nonlinear state-space model with i.i.d. state noise. In this section, we assume that the state noise $\{\zeta_k\}_{k \geq 0}$ is a sequence of i.i.d. random variables with positive density γ with respect to the Lebesgue measure denoted

λ^{Leb} and independent of the observation noise $\{\varepsilon_k\}_{k \geq 0}$. Then, for any $A \in \mathcal{X}$,

$$(30) \quad Q(x, A) = \int_A \gamma[x' - f(x)] \lambda^{\text{Leb}}(dx').$$

For any $\Delta \in (0, \infty)$, let us define the following set-valued function from Y to \mathcal{X} by

$$(31) \quad y \mapsto \mathsf{C}(y, \Delta) \stackrel{\text{def}}{=} \{x \in \mathsf{X} : |h(x) - y| \leq \Delta\}.$$

For any $y \in \mathsf{Y}$, $\mathsf{C}(y, \Delta)$ is included in a neighborhood of the preimage of y . Indeed, under assumption (E1), for any $z \in \mathsf{X}$ in the preimage of y , and any $x \in \mathsf{C}(y, \Delta)$,

$$|x - z| \leq b_0 + b\Delta.$$

Let $(y, y') \in \mathsf{Y}^2$. By the condition (E1), h is surjective so the preimage of y and y' by h is non empty. We choose arbitrarily z and z' in these preimages: $y = f(z)$ and $y' = f(z')$. By the triangle inequality and the condition (E1), it follows that, for all $(x, x') \in \mathsf{C}(y, \Delta) \times \mathsf{C}(y', \Delta)$,

$$(32) \quad |f(x) - x'| \leq |f(x) - f(z)| + |f(z) - z'| + |z' - x'| \leq a(b_0 + b\Delta) + D(y, y') + b_0 + b\Delta,$$

where D is defined by

$$(33) \quad D(y, y') \stackrel{\text{def}}{=} \sup \{|f(z) - z'| : (z, z') \in \mathsf{X}^2 \text{ with } h(z) = y, h(z') = y'\}.$$

For any $r > 0$, we consider the minimum and the maximum of the state noise density over a ball of radius r :

$$(34) \quad \gamma^-(r) \stackrel{\text{def}}{=} \inf_{|s| \leq r} \gamma(s), \quad \gamma^+(r) \stackrel{\text{def}}{=} \sup_{|s| \leq r} \gamma(s),$$

It follows from (30) and (32) that, for all $A \in \mathcal{X}$ and $x \in \mathsf{C}(y, \Delta)$,

$$(35) \quad \varepsilon_{\Delta}^-(y, y') \lambda^{\text{Leb}}[A \cap \mathsf{C}(y', \Delta)] \leq Q[x, A \cap \mathsf{C}(y', \Delta)] \leq \varepsilon_{\Delta}^+(y, y') \lambda^{\text{Leb}}[A \cap \mathsf{C}(y', \Delta)],$$

where,

$$\begin{aligned} \varepsilon_{\Delta}^-(y, y') &\stackrel{\text{def}}{=} \gamma^-[(a+1)b_0 + (a+1)b\Delta + D(y, y')], \\ \varepsilon_{\Delta}^+(y, y') &\stackrel{\text{def}}{=} \gamma^+[(a+1)b_0 + (a+1)bd\Delta + D(y, y')]. \end{aligned}$$

Since γ is a positive density, it follows by (35) that the application defined by (31) is a LD-set function. By assumption (E2), for all $\eta > 0$, we may

choose Δ large enough so that $\sup_{|s|>\Delta} v(s) \leq \eta \sup_{s \in \mathcal{X}} v(s)$, which implies that assumption (H2)

$$(36) \quad \Upsilon_{C^c(y,\Delta)}(y) \leq \eta \Upsilon_{\mathcal{X}}(y) ,$$

is satisfied. The positiveness of v implies assumption (H1).

To check assumptions (19) and (22), it is required to compute an upper bound for $\{D(Y_{k-1}, Y_k)\}_{k \geq 1}$. For $z, z' \in \mathcal{X}$ such that $h(z) = Y_{k-1}$, $h(z') = Y_k$, it follows from the triangle inequality and assumption (E1) that

$$\begin{aligned} |f(z) - z'| &\leq |f(z) - f(X_{k-1})| + |f(X_{k-1}) - X_k| + |X_k - z'| , \\ &\leq a(b_0 + b|\varepsilon_{k-1}|) + |\zeta_k| + b_0 + b|\varepsilon_k| . \end{aligned}$$

Therefore, for all integer $k \geq 1$,

$$(37) \quad D(Y_{k-1}, Y_k) \leq (a+1)b_0 + ab|\varepsilon_{k-1}| + |\zeta_k| + b|\varepsilon_k| .$$

Thanks to this bound, assumptions (19) and (22) are satisfied by applying the Law of Large Numbers, see Propositions 7 and 9 and their proofs. Since γ^- is a non increasing function, it follows by (37) that, for all integer $k \geq 1$, $\log \varepsilon_{\Delta}^-(Y_{k-1}, Y_k) \leq -Z_k^{\Delta}$ where for all $\Delta > 0$ and all integer $k \geq 1$,

$$(38) \quad Z_k^{\Delta} \stackrel{\text{def}}{=} -\log \gamma^- [2(a+1)b_0 + (a+1)b\Delta + ab|\varepsilon_{k-1}| + |\zeta_k| + b|\varepsilon_k]] .$$

PROPOSITION 7. *Let us consider the filtering model defined by (29). Assume (E1), (E2) and, for all $\Delta > 0$,*

$$(39) \quad \mathbb{E}|Z_1^{\Delta}| < \infty .$$

Let $\{Y_k\}_{k \geq 0}$ be the sequence of observations produced by the filtering equations (29) and let C be the LD-set function defined by (31). Then, for any initial probability distributions ν and ν' on $(\mathcal{X}, \mathcal{X})$ and $\Delta > 0$ such that

$$\nu Q \mathbb{1}_{C(Y_1, \Delta)} > 0 , \quad \mathbb{P}_{\star} - \text{a.s.} \quad \nu' Q \mathbb{1}_{C(Y_1, \Delta)} > 0 , \quad \mathbb{P}_{\star} - \text{a.s.}$$

we have

$$\limsup_{n \rightarrow \infty} n^{-1} \log \|\phi_{\nu, n}[Y_{0:n}] - \phi_{\nu', n}[Y_{0:n}]\|_{\text{TV}} < 0 , \quad \mathbb{P}_{\star} - \text{a.s.}$$

The condition (39), is not very restrictive. For example, let us assume that γ is a centered Gaussian density and that $\{\zeta_k\}_{k \geq 0}$ and $\{\varepsilon_k\}_{k \geq 0}$ are sequences of Gaussian random variables. It follows, that $\gamma^-(r) = \gamma(r)$ for

all $r \geq 0$. The condition (39) holds if $\mathbb{E}(|\varepsilon_1|^2) < \infty$ and $\mathbb{E}(|\zeta_1|^2) < \infty$ which are trivially satisfied.

With more stringent conditions for initial laws, geometric rates hold for the convergence of the expected value of the total variation. Let us recall the definition of the log-moment generating function that will be used in the sequel.

DEFINITION 8. *The log-moment generating function $\psi_Z(\lambda)$ of the random variable Z is defined on the set $\{\lambda \geq 0 : \mathbb{E}[e^{\lambda Z}] < \infty\}$ by $\psi_Z(\lambda) \stackrel{\text{def}}{=} \log \mathbb{E}[e^{\lambda Z}]$.*

PROPOSITION 9. *Let us consider the filtering model defined by (29) and satisfying (E1), (E2) and, for all $\Delta > 0$, there exists $\tau > 0$ such that*

$$(40) \quad \psi_{Z_\Delta} \text{ is finite on } [0, \tau).$$

Let $\{Y_k\}_{k \geq 0}$ be the sequence produced by the filtering equations (29) and let \mathbf{C} denotes the LD-set function defined by (31). Then, for ν and ν' two probability measures on $(\mathbf{X}, \mathcal{X})$ and $\Delta > 0$ such that, for some $\lambda > 0$,

$$(41) \quad \mathbb{E}_\star \left\{ \exp \left(\lambda [\log \nu g(\cdot, Y_0) Q \mathbb{1}_{\mathbf{C}(Y_1, \Delta)}]_- \right) \right\} < \infty , \\ \mathbb{E}_\star \left\{ \exp \left(\lambda [\log \nu' g(\cdot, Y_0) Q \mathbb{1}_{\mathbf{C}(Y_1, \Delta)}]_- \right) \right\} < \infty ,$$

we have

$$\limsup_{n \rightarrow \infty} n^{-1} \log \mathbb{E}_\star \left[\|\phi_{\nu, n}[Y_{0:n}] - \phi_{\nu', n}[Y_{0:n}]\|_{\text{TV}} \right] < 0 .$$

Assume that γ is the density of a standard Gaussian random variable. The condition $\mathbb{E}[e^{\lambda Z_1^\Delta}] < \infty$ is equivalent to

$$\int_{\mathbb{R}^{n+2p}} \exp \left[(\lambda - \varsigma) |x|^2 \right] dx < \infty ,$$

where ς denotes some positive constant. Therefore, for $\lambda > 0$ small enough, the condition (40) is satisfied.

The conditions (41) can be interpreted as non-degenerative conditions. Indeed, they forbid that $\nu g(\cdot, Y_0) Q \mathbb{1}_{\mathbf{C}(Y_1, \Delta)}$ is null almost everywhere and the same for ν' . Intuitively, it means that the distribution of the random variable $\nu g(\cdot, Y_0) Q \mathbb{1}_{\mathbf{C}(Y_1, \Delta)}$ is not concentrated close to zero. For example, if there exists a constant $c > 0$ such that

$$\nu g(\cdot, Y_0) Q \mathbb{1}_{\mathbf{C}(Y_1, \Delta)} \geq c , \quad \mathbb{P}_\star - a.s \quad \nu' g(\cdot, Y_0) Q \mathbb{1}_{\mathbf{C}(Y_1, \Delta)} \geq c , \quad \mathbb{P}_\star - a.s$$

then the conditions of (41) are satisfied. Proofs of Propositions 7 and 9 are given in Section 5.

3.2. *Nonlinear state-space model with dependent state noise.* We now consider the case where the state noise $\{\zeta_k\}_{k \geq 0}$ can depend on previous states. This model has been introduced in [15, Section 3] and is important because it covers the case of partially observed discretely sampled diffusion, as well as partially observed stochastic volatility models [3, Section 2]. This example illustrates that the forgetting property is kept even when the distributions of the observations differ from the model.

(G) $\{\zeta_k\}_{k \geq 0}$ is a sequence of random variables such that, for all integer k , ζ_k is independent of ε_k and for all $A \in \mathcal{X}$,

$$\mathbb{P}(\zeta_k \in A | X_{k-1} = x) = \int q(x, u) \mathbb{1}_A(u) \lambda^{\text{Leb}}(du) .$$

Moreover, there exist a positive probability density ψ and positive constants μ^- , μ^+ such that, for all $x, u \in \mathbf{X}$,

$$\mu^- \psi(u) \leq q(x, u) \leq \mu^+ \psi(u) .$$

A first example of state equation satisfying (G) is considered in [3]. A signal takes its values in \mathbf{X} and follows the equation

$$(42) \quad X_k = f(X_{k-1}) + \sigma(X_{k-1})\xi_k ,$$

where $\{\xi_k\}_{k \geq 0}$ is a sequence of i.i.d random variables and where $\sigma : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$ is a measurable function that satisfies, for all $x, u \in \mathbf{X}$, the following hypoellipticity condition:

$$(43) \quad \sigma^- |u|^2 \leq \langle u, \sigma(x)\sigma^T(x)u \rangle \leq \sigma^+ |u|^2 ,$$

where σ^-, σ^+ are positive constants and the superscript T denotes the transposition. Another important example where (G) is satisfied is the case of certain discretely sampled diffusions. Let $(X_t)_{t \geq 0}$ be the unique solution of the following stochastic differential equation

$$dX_t = \rho(X_t)dt + \sigma(X_t)dB_t ,$$

where B is the n -dimensional Brownian motion and the functions $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ are respectively of class C^1 and C^3 . Then, the sequence $\{X_k\}_{k \geq 0}$ satisfies assumption (G) if the function σ is hypoelliptic (condition (43)); see [15]. The assumptions (E1), (E2) and (G) are a bit more stringent than those made in [15]. Indeed, in [15], the function h is not necessarily uniformly continuous and no restrictions are made on v . This allows to establish the forgetting of the initial condition with probability

one without restriction on the signal-to-noise ratio and for sequences of observations which are not necessarily distributed according to the model used to compute the filtering distribution. Let us denote by Q the transition kernel for $\{X_k\}_{k \geq 0}$. Then, for all $A \in \mathcal{X}$ and for all $x \in \mathbf{X}$,

$$Q(x, A) = \int_A q[x, x' - f(x)] \lambda^{\text{Leb}}(dx') .$$

For the same reasons as above, we consider the same set-valued function C (31) as before. Let $(y, y') \in \mathbf{Y}^2$. Like in (32), it follows by (E1) and the triangle inequality that, for all $(x, x') \in C(y, \Delta) \times C(y', \Delta)$,

$$|f(x) - x'| \leq c + d\Delta + D(y, y') ,$$

where D is defined in (33), $c = (a + 1)b_0$ and $d = (a + 1)b$. By setting

$$(44) \quad q^-(r) \stackrel{\text{def}}{=} \mu^- \times \inf_{|v| \leq r} \psi(v) , \quad q^+(r) \stackrel{\text{def}}{=} \mu^+ \times \sup_{|v| \leq r} \psi(v) ,$$

it follows from condition (G) that, for all $A \in \mathcal{X}$ and $x \in C(y, \Delta)$,

$$(45) \quad \varepsilon_{\Delta}^-(y, y') \lambda^{\text{Leb}}[A \cap C(y', \Delta)] \leq Q[x, A \cap C(y', \Delta)] \leq \varepsilon_{\Delta}^+(y, y') \lambda^{\text{Leb}}[A \cap C(y', \Delta)] ,$$

where

$$\varepsilon_{\Delta}^-(y, y') \stackrel{\text{def}}{=} q^-[c + d\Delta + D(y, y')] , \quad \varepsilon_{\Delta}^+(y, y') \stackrel{\text{def}}{=} q^+[c + d\Delta + D(y, y')] .$$

Since ψ is a positive density, the application defined by (31) is a LD-set function. As in Section 3.1, assumptions (H1) and (H2) are satisfied. Assume now that the process $\{Y_k^*\}_{k \geq 0}$ is generated by the following non-linear state-space observations

$$(46) \quad \begin{cases} X_k^* = f^*(X_{k-1}^*) + \zeta_k^* , \\ Y_k^* = h^*(X_k^*) + \varepsilon_k^* , \end{cases}$$

where $\{\varepsilon_k^*\}_{k \geq 0}$ is a sequence of i.i.d random variables, f^* is a^* -Lipschitz, h^* is surjective and for all $x_1, x_2 \in \mathbf{X}$,

$$|x_1 - x_2| \leq b_0^* + b^* |h^*(x_1) - h^*(x_2)| ,$$

for some positive constants b_0^* , b^* . For all integer $k \geq 1$, ζ_k^* is independent of ε_k^* and, for all $A \in \mathcal{X}$,

$$\mathbb{P}(\zeta_k^* \in A | X_{k-1}^* = x) = \int q^*(x, u) \mathbb{1}_A(u) \lambda^{\text{Leb}}(du) .$$

There exists probability densities ψ^* and positive constants μ_-^* , μ_+^* such that, for all $x, u \in \mathsf{X}$,

$$(47) \quad \mu_-^* \psi^*(u) \leq q^*(x, u) \leq \mu_+^* \psi^*(u) .$$

We assume that

$$(O1) \quad f^* \text{ and } h^* \text{ are such that } \|f - f^*\|_\infty < \infty \text{ and } \|h - h^*\|_\infty < \infty .$$

LEMMA 10. *Let $\{Y_k^*\}_{k \geq 0}$ be the sequence following (46). Under (O1), for all integer $k \geq 1$,*

$$D(Y_{k-1}^*, Y_k^*) \leq \kappa + 2a^*b^* + a^*b^*|\varepsilon_{k-1}^*| + b^*|\varepsilon_k^*| + |\zeta_k^*| ,$$

where

$$\kappa = \|f - f^*\|_\infty + (b_0 + b\|h^* - h\|_\infty)(1 + a^*)$$

PROOF OF LEMMA 10. For all integer $k \geq 1$, for $z, z' \in \mathsf{X}$ such that $h(z) = Y_{k-1}^*$, $h(z') = Y_k^*$ and for $u, u' \in \mathsf{X}$ such that $h^*(u) = Y_{k-1}^*$, $h^*(u') = Y_k^*$, it follows by the triangle inequality that

$$(48) \quad \begin{aligned} |f(z) - z'| &\leq |f(z) - f^*(z)| + |f^*(z) - f^*(u)| + |f^*(u) - u'| + |u' - z'| , \\ &\leq \|f - f^*\|_\infty + a^*|z - u| + |f^*(u) - u'| + |u' - z'| . \end{aligned}$$

Let us notice that

$$|z - u| \leq b_0 + b|h(z) - h(u)| \leq b_0 + b \underbrace{|h(z) - h^*(u)|}_{=0} + b|h^*(u) - h(u)| .$$

Then, by denoting $K = b_0 + b\|h^* - h\|_\infty$, it follows that $|z - u| \leq K$ and, for the same reasons, $|z' - u'| \leq K$. Combining these two majorations with (48) leads to

$$\begin{aligned} |f(z) - z'| &\leq \kappa + |f^*(u) - f^*(X_{k-1})| + |f^*(X_{k-1}) - X_k| + |X_k - u'| , \\ &\leq \kappa + a^*[b_0^* + b^*|h^*(z) - h^*(X_{k-1})|] + |\zeta_k^*| + b_0^* + b^*|h^*(X_k) - h^*(u')| , \end{aligned}$$

where $\kappa = \|f - f^*\|_\infty + K(1 + a^*)$. Thus, it is proven that, for all integer $k \geq 1$,

$$D(Y_{k-1}^*, Y_k^*) \leq K' + 2a^*b^* + a^*b^*|\varepsilon_{k-1}^*| + b^*|\varepsilon_k^*| + |\zeta_k^*| .$$

□

Let us define for all $\Delta > 0$

$$(49) \quad V_+^{*\Delta} = \log q^- [c + d\Delta + \kappa + 2a^*b^* + a^*b^*|\varepsilon_0^*| + b^*|\varepsilon_1^*| + |\zeta_+^*|] ,$$

where ζ_+^* is a random variable independent of $\{\varepsilon_k^*\}_{k \geq 0}$ with density ψ^* .

PROPOSITION 11. *Let us consider the filtering model defined by (29) and satisfying (E1), (E2) and (G). Let \mathcal{C} be the LD-set function defined by (31) and let $\{Y_k^*\}_{k \geq 0}$ be the sequence following (46) such that (O1) holds and, for all $\Delta > 0$,*

$$(50) \quad \mathbb{E} \left(|V_+^{*\Delta}| \log_+ |V_+^{*\Delta}| \right) < \infty .$$

Then, for any initial probability distributions ν and ν' on $(\mathsf{X}, \mathcal{X})$ and $\Delta > 0$ satisfying

$$\nu Q \mathbb{1}_{\mathcal{C}(Y_1^*, \Delta)} > 0 , \quad \mathbb{P}_* - \text{a.s.} \quad \nu' Q \mathbb{1}_{\mathcal{C}(Y_1^*, \Delta)} > 0 , \quad \mathbb{P}_* - \text{a.s.}$$

we have

$$\limsup_{n \rightarrow \infty} n^{-1} \log \|\phi_{\nu, n}[Y_{0:n}^*] - \phi_{\nu', n}[Y_{0:n}^*]\|_{\text{TV}} < 0, \quad \mathbb{P}_* - \text{a.s.}$$

This proposition has important consequences. Observations issued from equations (29) under conditions (E1), (E2) and (G) are of the observations produced by (46) under (O1). It is only needed that $\|f - f^*\|_\infty$ and $\|h - h^*\|_\infty$ are bounded to ensure the w.p.1 convergence.

Let us write $\zeta_k^* = g^*(X_{k-1}^*, A_k^*)$ where g^* denotes a measurable function and $\{A_k^*\}_{k \geq 0}$ a sequence of i.i.d. random variables with uniform law on $(0, 1)$. We make the following assumptions

- (O3) there exists a measurable function g_+^* such that, for all $x \in \mathsf{X}$ and $a \in (0, 1)$, $|g^*(x, a)| \leq g_+^*(a)$;
- (O4) Let $\{Z_k^{*\Delta}\}_{k \geq 0}$ be the sequence defined by, for all $\Delta > 0$ and for all integer $k \geq 1$,

$$Z_k^{*\Delta} = -\log q^- [c + d\Delta + \kappa + 2a^*b^* + a^*b^*|\varepsilon_{k-1}^*| + b^*|\varepsilon_k^*| + g_+^*(U_k^*)] ,$$

For all $\Delta > 0$, there exists $\tau > 0$ such that the log-moment generating function $\Psi_{Z_1^{*\Delta}}$ is finite on $[0, \tau)$.

PROPOSITION 12. *Let us consider the filtering model defined by (29) and satisfying (E1), (E2) and (G). Let $\{Y_k^*\}_{k \geq 0}$ be the sequence following (46) such that (O1), (O3) and (O4) hold and let \mathcal{C} be the LD-set function defined*

by (31). Then, for ν and ν' two probability measures on (X, \mathcal{X}) and $\Delta > 0$ such that, for some $\lambda > 0$,

$$\begin{aligned} \mathbb{E}_* \left\{ \exp \left(\lambda [\log \nu g(\cdot, Y_0^*) Q \mathbb{1}_{C(Y_1^*, \Delta)}]_- \right) \right\} &< \infty, \\ \mathbb{E}_* \left\{ \exp \left(\lambda [\log \nu' g(\cdot, Y_0^*) Q \mathbb{1}_{C(Y_1^*, \Delta)}]_- \right) \right\} &< \infty, \end{aligned}$$

we have

$$\limsup_{n \rightarrow \infty} n^{-1} \log \mathbb{E}_* \left[\|\phi_{\nu, n}[Y_{0:n}^*] - \phi_{\nu', n}[Y_{0:n}^*]\|_{\text{TV}} \right] < 0.$$

For the convergence in expectation, the restrictive assumption (O3) has to be made. Let us precise that, for the case considered in [3], this condition is satisfied since the function σ in (42) is bounded. The case of [15] is not covered by this condition. It seems quite difficult to get the same results as in [15] with observations not necessarily from an HMM without strengthening the assumptions on $\{\zeta_k^*\}_{k \geq 0}$. Let us precise that the convergence theorem of [15] is proved for observations issued from the filtering equations. The assumption (O4) is of the same type as (40).

Proofs of Propositions 11 and 12 are given in Section 6.

4. Proofs of Propositions 2 and 3.

PROOF OF PROPOSITION 2. For convenience, we write $C_i = C(y_i)$, $\varepsilon_i^- = \varepsilon_{\bar{C}}^-(y_{i-1}, y_i)$, $\varepsilon_i^+ = \varepsilon_{\bar{C}}^+(y_{i-1}, y_i)$, $g_i(x) = g(x, y_i)$, $\lambda_i = \lambda_{y_{i-1}, y_i}$ and $\rho_i = 1 - (\varepsilon_i^- / \varepsilon_i^+)^2$. Let us define $\bar{\lambda}_i \stackrel{\text{def}}{=} \lambda_i \otimes \lambda_i$. Since C is an LD-set function, for all $i = 1, \dots, n$, $\bar{x} \in \bar{C}_{i-1}$, and \bar{f} a non-negative function on $X \times X$,

$$(51) \quad (\varepsilon_i^-)^2 \bar{\lambda}_i(\mathbb{1}_{\bar{C}_i} \bar{f}) \leq \bar{Q}(\bar{x}, \mathbb{1}_{\bar{C}_i} \bar{f}) \leq (\varepsilon_i^+)^2 \bar{\lambda}_i(\mathbb{1}_{\bar{C}_i} \bar{f}).$$

Let us define the sequence of unnormalized kernels \bar{Q}_i^0 and \bar{Q}_i^1 by, for all $\bar{x} \in X^2$, and \bar{f} a non-negative function on $X \times X$,

$$\begin{aligned} \bar{Q}_i^0(\bar{x}, \bar{f}) &= (\varepsilon_i^-)^2 \mathbb{1}_{\bar{C}_{i-1}} \bar{\lambda}_i(\mathbb{1}_{\bar{C}_i} \bar{f}), \\ \bar{Q}_i^1(\bar{x}, \bar{f}) &= \bar{Q}(\bar{x}, \bar{f}) - (\varepsilon_i^-)^2 \mathbb{1}_{\bar{C}_{i-1}} \bar{\lambda}_i(\mathbb{1}_{\bar{C}_i} \bar{f}). \end{aligned}$$

It follows from (51) that, for all \bar{x} in \bar{C}_{i-1} , $0 \leq \bar{Q}_i^1(\bar{x}, \mathbb{1}_{\bar{C}_i} \bar{f}) \leq \rho_i \bar{Q}(\bar{x}, \mathbb{1}_{\bar{C}_i} \bar{f})$ which implies that, for all $\bar{x} \in X^2$,

$$\begin{aligned} \bar{Q}_i^1(\bar{x}, \bar{f}) &= \mathbb{1}_{\bar{C}_{i-1}}(\bar{x}) \bar{Q}_i^1(\bar{x}, \mathbb{1}_{\bar{C}_i} \bar{f}) + \mathbb{1}_{\bar{C}_{i-1}}(\bar{x}) \bar{Q}_i^1(\bar{x}, \mathbb{1}_{\bar{C}_i^c} \bar{f}) + \mathbb{1}_{\bar{C}_{i-1}^c}(\bar{x}) \bar{Q}_i^1(\bar{x}, \bar{f}), \\ &\leq \rho_i \mathbb{1}_{\bar{C}_{i-1}}(\bar{x}) \bar{Q}(\bar{x}, \mathbb{1}_{\bar{C}_i} \bar{f}) + \mathbb{1}_{\bar{C}_{i-1}}(\bar{x}) \bar{Q}_i^1(\bar{x}, \mathbb{1}_{\bar{C}_i^c} \bar{f}) + \mathbb{1}_{\bar{C}_{i-1}^c}(\bar{x}) \bar{Q}_i^1(\bar{x}, \bar{f}), \\ &\leq \bar{Q} \left(\bar{x}, \rho_i \mathbb{1}_{\bar{C}_{i-1}}(\bar{x}) \mathbb{1}_{\bar{C}_i} \bar{f} \right). \end{aligned}$$

We write $\Delta_n(\nu, \nu', y_{0:n}) = \sup_{A \in \mathcal{X}} |\Delta_n(A)|$, where

$$\Delta_n(A) \stackrel{\text{def}}{=} \nu \otimes \nu'(\bar{g}_0 \bar{Q}_0 \bar{g}_1 \dots \bar{Q}_{n-1} \bar{g}_n \mathbb{1}_{A \times X}) - \nu' \otimes \nu(\bar{g}_0 \bar{Q}_0 \bar{g}_1 \dots \bar{Q}_{n-1} \bar{g}_n \mathbb{1}_{A \times X}).$$

We decompose $\Delta_n(A)$ into $\Delta_n(A) = \sum_{t_{0:n-1} \in \{0,1\}^n} \Delta(A, t_{0:n-1})$, where

$$\begin{aligned} \Delta_n(A, t_{0:n-1}) \stackrel{\text{def}}{=} & \nu \otimes \nu'(\bar{g}_0 \bar{Q}_0^{t_0} \bar{g}_1 \dots \bar{Q}_{n-1}^{t_{n-1}} \bar{g}_n \mathbb{1}_{A \times X}) \\ & - \nu' \otimes \nu(\bar{g}_0 \bar{Q}_0^{t_0} \bar{g}_1 \dots \bar{Q}_{n-1}^{t_{n-1}} \bar{g}_n \mathbb{1}_{A \times X}). \end{aligned}$$

Note that, for any $t_{0:n-1} \in \{0,1\}^n$ and any sets $A, B \in \mathcal{X}$,

$$\nu \otimes \nu'(\bar{g}_0 \bar{Q}_0^{t_0} \bar{g}_1 \dots \bar{Q}_{n-1}^{t_{n-1}} \bar{g}_n \mathbb{1}_{A \times B}) = \nu' \otimes \nu(\bar{g}_0 \bar{Q}_0^{t_0} \bar{g}_1 \dots \bar{Q}_{n-1}^{t_{n-1}} \bar{g}_n \mathbb{1}_{B \times A}).$$

If there is an index $i \in \{0, \dots, n-1\}$ such that $t_i = 0$, then

$$\begin{aligned} & \nu \otimes \nu'(\bar{g}_0 \bar{Q}_0^{t_0} \bar{g}_1 \dots \bar{Q}_{n-1}^{t_{n-1}} \bar{g}_n \mathbb{1}_{A \times X}) \\ &= \nu \otimes \nu'(\bar{g}_0 \bar{Q}_0^{t_0} \bar{g}_1 \dots \bar{Q}_{i-1}^{t_{i-1}} \bar{g}_i \mathbb{1}_{\bar{C}_i}) \times (\varepsilon_{i+1}^-)^2 \bar{\lambda}_i(\mathbb{1}_{\bar{C}_{i+1}} \bar{g}_{i+1} \bar{Q}_{i+1}^{t_{i+1}} \dots \bar{Q}_{n-1}^{t_{n-1}} \bar{g}_n \mathbb{1}_{A \times X}), \\ &= \nu' \otimes \nu(\bar{g}_0 \bar{Q}_0^{t_0} \bar{g}_1 \dots \bar{Q}_{i-1}^{t_{i-1}} \bar{g}_i \mathbb{1}_{\bar{C}_i}) \times (\varepsilon_{i+1}^-)^2 \bar{\lambda}_i(\mathbb{1}_{\bar{C}_{i+1}} \bar{g}_{i+1} \bar{Q}_{i+1}^{t_{i+1}} \dots \bar{Q}_{n-1}^{t_{n-1}} \bar{g}_n \mathbb{1}_{A \times X}). \end{aligned}$$

Thus, $\Delta_n(A, t_{0:n-1}) = 0$ except if for all $i \in \{0, \dots, n-1\}$, $t_i = 1$, and we obtain

$$\Delta_n(A) = \nu \otimes \nu'[\bar{g}_0 \bar{Q}_0^1 \bar{g}_1 \dots \bar{Q}_{n-1}^1 \bar{g}_n (\mathbb{1}_{A \times X} - \mathbb{1}_{X \times A})].$$

It then follows

$$\Delta_n(\nu, \nu', y_{0:n}) \leq \nu \otimes \nu'(\bar{g}_0 \bar{Q}_0^1 \bar{g}_1 \dots \bar{Q}_{n-1}^1 \bar{g}_n) \leq \mathbb{E}_{\nu \otimes \nu'}^{\bar{Q}} \left[\bar{g}(\bar{X}_0, y_0) \prod_{i=1}^n \bar{g}(\bar{X}_i, y_i) \rho_i^{\delta_i} \right],$$

with $\delta_i = \mathbb{1}_{\bar{C}_{i-1} \times \bar{C}_i}(\bar{X}_{i-1}, \bar{X}_i)$. \square

PROOF OF PROPOSITION 3. Since \mathbf{C} is an LD-set function, there exist some applications $\varepsilon_{\bar{C}}^-, \varepsilon_{\bar{C}}^+$ such that, for all $i = 1, \dots, n$, for all $x \in \mathbf{C}(y_{i-1})$ and for all $A \in \mathcal{X}$ with $A \subset \mathbf{C}(y_i)$,

$$(52) \quad \varepsilon_{\bar{C}}^-(y_{i-1}, y_i) \lambda_{y_{i-1}, y_i}(A) \leq Q(x, A) \leq \varepsilon_{\bar{C}}^+(y_{i-1}, y_i) \lambda_{y_{i-1}, y_i}(A).$$

Let us write the obvious inequality

$$\mathbb{E}_{\nu}^Q \left[\prod_{i=0}^n g(X_i, y_i) \right] \geq \mathbb{E}_{\nu}^Q \left[g(X_0, y_0) \prod_{i=1}^n g(X_i, y_i) \mathbb{1}_{\mathbf{C}(y_i)}(X_i) \right].$$

Then, for the right-hand side of this expression, by (52) we have

$$\begin{aligned} & \mathbb{E}_\nu^Q \left[g(X_0, y_0) \prod_{i=1}^n g(X_i, y_i) \mathbb{1}_{\mathbf{C}(y_i)}(X_i) \right] \\ &= \mathbb{E}_\nu^Q \left[g(X_0, y_0) g(X_1, y_1) \mathbb{1}_{\mathbf{C}(y_1)}(X_1) \prod_{i=2}^n g(X_i, y_i) \mathbb{1}_{\mathbf{C}(y_{i-1}) \times \mathbf{C}(y_i)}(X_{i-1}, X_i) \right], \\ &\geq \nu[g(\cdot, y_0) Q g(\cdot, y_1) \mathbb{1}_{\mathbf{C}(y_1)}(\cdot)] \prod_{i=2}^n \varepsilon_{\mathbf{C}}^-(y_{i-1}, y_i) \lambda_{y_{i-1}, y_i} [g(\cdot, y_i) \mathbb{1}_{\mathbf{C}(y_i)}]. \end{aligned}$$

□

5. Proofs of Propositions 7 and 9 .

PROOF OF PROPOSITION 7. Since, by definition (34), γ^- is a decreasing function, the inequality (37) leads to

$$(53) \quad n^{-1} \sum_{k=2}^n \log \varepsilon_{\Delta}^-(Y_{k-1}, Y_k) \geq -n^{-1} \sum_{k=2}^n Z_k^{\Delta},$$

where Z_k^{Δ} is defined in (38). Since the process $\{ab|\varepsilon_{k-1}| + |\zeta_k| + b|\varepsilon_k|\}_{k \geq 1}$ is stationary 2-dependent, the strong law of large numbers for m -dependent sequences and the integrability condition (39) yield

$$(54) \quad \lim_{n \rightarrow \infty} n^{-1} \sum_{k=2}^n Z_k^{\Delta} = \mathbb{E}(Z_1^{\Delta}) < \infty, \quad \mathbb{P}_{\star} - \text{a.s.}$$

By combining (53) and (54), the first condition (19) of Theorem 5 is satisfied. By assumption (E2), the density ν is bounded which implies that $\sup_{y \in \mathbf{Y}} \Upsilon_{\mathbf{X}}(y) \leq \sup \nu$. Hence, the second condition (20) of Theorem 5 is satisfied. We now consider the third condition (21). Since the measure appearing in the definition of the LD-set function does not depend on y, y' , the function $(y, y') \mapsto \Psi_{\mathbf{C}(y', \Delta)}(y, y')$, defined in (11), does not depend on y and is given by

$$\Psi_{\mathbf{C}(y', \Delta)}(y, y') = \int_{\mathbf{C}(y', \Delta)} \nu[y' - h(x)] \lambda^{\text{Leb}}(dx) \geq \lambda^{\text{Leb}}[\mathbf{C}(y', \Delta)] \times \inf_{|s| \leq \Delta} \nu(s).$$

Since the function h is uniformly continuous, for any fixed $\Delta > 0$, there exist $\delta > 0$ such that, for all $x, x' \in \mathbf{X}$ satisfying $|x - x'| \leq \delta$, we have $|h(x) - h(x')| \leq \Delta$, showing that $\lambda^{\text{Leb}}[\mathbf{C}(y', \Delta)] \geq \delta$. Thus, we have, for all $y, y' \in \mathbf{Y}$,

$$(55) \quad \Psi_{\mathbf{C}(y', \Delta)}(y, y') \geq \varrho_{\Delta},$$

for some $\varrho_\Delta > 0$, depending only on Δ . The third condition (21) of Theorem 5 follows. Since assumption (H2) is satisfied, for any fixed $\eta > 0$, we choose $\Delta > 0$ such that inequality (36) holds. Let us write

$$(56) \quad R_\Delta(x) \stackrel{\text{def}}{=} \log [1 - (\gamma^-/\gamma^+)^2(2c + d\Delta + x)] .$$

We will repeatedly use the following representation of the so-called L -statistic (see [17, Chapter 8]):

LEMMA 13. *Let $\{U_1, \dots, U_n\}$ be a sequence and $U_{n,1} \leq U_{n,2} \leq \dots \leq U_{n,n}$ the upper ordered statistic. Then,*

$$n^{-1} \sum_{k=j}^n U_{n,k} = \int_{j/n}^1 F_{n,U}^{-1}(s) ds$$

where $F_{n,U}^{-1}(s) \stackrel{\text{def}}{=} \inf\{t \in \mathbb{R}, F_{n,U}(t) \geq s\}$ is the empirical quantile function, i.e. the generalized inverse of the empirical distribution function $F_{n,U}(t) \stackrel{\text{def}}{=} n^{-1} \sum_{k=1}^n \mathbb{1}_{\{U_k \leq t\}}$.

Applying this representation yields

$$(57) \quad n^{-1} \log \Lambda_\eta(Y_{0:n}, \alpha) \leq \int_0^1 \mathbb{1}\{u \geq 1 - r_n\} F_n^{-1}(u) du ,$$

where $r_n = (\lceil n\alpha \rceil - 1)/n$, $F_n(t) = n^{-1} \sum_{k=1}^n \mathbb{1}\{R_\Delta(ab|\varepsilon_{k-1}| + |\zeta_k| + b|\varepsilon_k|) \leq t\}$ and F_n^{-1} its generalized inverse. The function R_Δ defined by (56) is negative and then, $F_n(0) = 1$ which implies that $F_n^{-1}(u) \geq 0$ for all $u \in (0, 1)$. Thus, by Fatou's lemma,

$$(58) \quad \limsup_{n \rightarrow \infty} \int_0^1 \mathbb{1}\{u \geq 1 - r_n\} F_n^{-1}(u) du \\ \leq \int_0^1 \limsup_{n \rightarrow \infty} \mathbb{1}\{u \geq 1 - r_n\} F_n^{-1}(u) du \quad \mathbb{P}_* - \text{a.s.}$$

The following lemma is a generalization of [18, Lemma 21.2].

LEMMA 14. *Let $\{\Psi_n\}_{n \geq 0}$ be a sequence of nondecreasing functions and Ψ a bounded nondecreasing function such that for all $x \in \mathbb{X}$, $\lim_{n \rightarrow \infty} \Psi_n(x) = \Psi(x)$. Then, Ψ^{-1} has at most a countable number of discontinuity points and at any point u where Ψ^{-1} is continuous,*

$$\lim_{n \rightarrow \infty} \Psi_n^{-1}(u) = \Psi^{-1}(u) .$$

Let us denote $F(t) = \mathbb{P}[R_\Delta(ab|\varepsilon_0| + |\zeta_1| + b|\varepsilon_1|) \leq t]$ and notice that $F(0) = 1$. Then, combining (57), (58) and Lemma 14 leads to

$$\limsup_{n \rightarrow \infty} n^{-1} \log \Lambda_\eta(Y_{0:n}, \alpha) \leq \int_{1-\alpha}^1 F^{-1}(u) du < 0, \quad \mathbb{P}_* - \text{a.s.} .$$

This shows that the fourth condition (22) is satisfied and finally, Theorem 5 applies. \square

Let us recall that ψ_Z denotes the log-moment generating function of the random variable Z defined by $\psi_Z(\lambda) \stackrel{\text{def}}{=} \log \mathbb{E}[e^{\lambda Z}]$ and we define its Legendre's transformation by

$$\psi_Z^*(x) = \sup_{\lambda \geq 0} \{x\lambda - \psi_Z(\lambda)\} .$$

PROOF OF PROPOSITION 9. We start by giving an exponential inequality for m -dependent variables.

LEMMA 15. *Let $\{Z_k\}_{k \geq 0}$ be a sequence of m -dependent stationary random variables. There exists some constant $C > 0$ such that, for all $M \geq 0$,*

$$\mathbb{P} \left(\sum_{k=1}^n Z_k \geq Mn \right) \leq C \exp[-n\psi_{Z_1}^*(2Mm)/(2m)] .$$

The proof is elementary and left to the reader. It follows by equation (53) that

$$\mathbb{P} \left(n^{-1} \sum_{k=2}^n \log \varepsilon_\Delta^-(Y_{k-1}, Y_k) \leq -M_1 n \right) \leq \mathbb{P} \left(\sum_{k=2}^n Z_k^\Delta \geq M_1 n \right) .$$

Thanks to (40), by applying Lemma 15, there exist some constant $c_1, \delta_1 > 0$ such that $r_1(n) \leq c_1 e^{-\delta_1 n}$. Since v is bounded, we can choose M_2 large enough such that $r_2(n) = 0$. By (55), for all $(y, y') \in \mathcal{Y}^2$, $\Psi_{C(y', \Delta)}(y, y') \geq \varrho_\Delta$, for some $\varrho_\Delta > 0$. Then, by choosing M_3 large enough, we have $r_3(n) = 0$. For $r_4(n)$, we need an exponential inequality for L -statistics based on m -dependent variables.

LEMMA 16. *Let $\{U_k\}_{k \geq 0}$ be a sequence of m -dependent stationary negative random variables. For all $\alpha \in (0, 1)$, there exists a real $r > 0$ such that*

$$\lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P} \left(\sum_{k=n-\lceil \alpha n \rceil + 1}^n U_{n,k} \geq -rn \right) < 0 .$$

PROOF OF LEMMA 16. For $j \in \{1, \dots, m\}$, define $I_j = \{j, j + m, j + 2m, \dots\}$ and let $n_j = |I_j|$ the cardinal of I_j . For any $j \in \{1, \dots, m\}$, the sequence $\{U_k, k \in I_j\}$ is i.i.d.. Denote $\{U_k^{(j)}\}_{1 \leq k \leq n_j}$ the sequence $\{U_k, k \in I_j\}$. Since $U_k < 0$ for all integer k , it then follows that

$$\sum_{k=n-\lceil \alpha n \rceil + 1}^n U_{n,k} \leq \sum_{j=1}^m \sum_{k=(n_j - \lceil \alpha n \rceil + 1) \vee 0}^{n_j} U_{n_j,k}^{(j)},$$

and

$$\mathbb{P} \left(\sum_{k=n-\lceil \alpha n \rceil + 1}^n U_{n,k} \geq -rn \right) \leq \sum_{j=1}^m \mathbb{P} \left(\sum_{k=(n_j - \lceil \alpha n \rceil + 1) \vee 0}^{n_j} U_{n_j,k}^{(j)} \geq -rn/m \right),$$

for all $n \geq N$ larger than some integer N . The sequence $\{U_k^{(j)}\}_{1 \leq k \leq n_j}$ is a sequence of i.i.d. random variable. Then, using [10, Theorem 6.1], we have

$$\lim_{n \rightarrow \infty} n_j^{-1} \log \mathbb{P} \left(n_j^{-1} \sum_{k=(n_j - \lceil \alpha n \rceil + 1) \vee 0}^{n_j} U_{n_j,k}^{(j)} \geq -\delta \right) < 0,$$

for some positive δ and the result follows since $n_j/n = 1/m + o(1)$. \square

Define by $U_k = R_\Delta [ab|\varepsilon_{k-1}| + |\zeta_k| + b|\varepsilon_k|]$ for all integer $k \geq 1$. By the definition (16) of Λ_η ,

$$(59) \quad n^{-1} \log \Lambda_\eta(Y_{0:n}, \alpha) \leq n^{-1} \sum_{k=n-\lceil \alpha n \rceil + 1}^n U_{n,k}.$$

Then, by equation (59) and by applying Lemma 16, there exist some constants $c_4, \delta_4 > 0$ such that $r_4(n) \leq c_4 e^{-\delta_4 n}$. Finally, under assumptions (E1), (E2) and (40), Theorem 6 applies and provides a geometric rate. \square

6. Proofs of Propositions 11 and 12 .

PROOF OF PROPOSITION 11. Let us define, for all $\Delta > 0$ and for all integer $k \geq 1$,

$$(60) \quad V_k^{*\Delta} = \log q^- [c + d\Delta + \kappa + 2a^*b^* + a^*b^*|\varepsilon_{k-1}^*| + b^*|\varepsilon_k^*| + |\zeta_k^*|].$$

Using the definitions (44), (45) of q^- and ε_Δ^- , Lemma 10 shows that

$$(61) \quad n^{-1} \sum_{k=2}^n \varepsilon_\Delta^-(Y_{k-1}^*, Y_k^*) \geq n^{-1} \sum_{k=2}^n V_k^{*\Delta}.$$

Thus, to check (19), it suffices to control the asymptotic behavior of right-hand side of this inequality. We use the following result [11, Chapter 2, Section 6].

LEMMA 17. *Let us denote by $\{\mathcal{H}_k\}_{k \geq 0}$ a filtration and consider a sequence $\{U_k\}_{k \geq 0}$ of random variable adapted to $\{\mathcal{H}_k\}_{k \geq 0}$. Let us assume that there exists a random variable U such that $\mathbb{E}(|U| \log_+ |U|) < \infty$ and $\mathbb{P}(|U_k| > x) \leq c \mathbb{P}(|U| > x)$ for all $x > 0$ and some $c > 0$. Then*

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n [U_k - \mathbb{E}(U_k | \mathcal{H}_{k-1})] = 0, \quad \mathbb{P} - \text{a.s.}$$

Define the filtration $\{\mathcal{F}_k^*\}_{k \geq 0}$ where $\mathcal{F}_k^* = \sigma(\{X_j^*\}_{0 \leq j \leq k}, \{\zeta_j^*\}_{0 \leq j \leq k}, \{\varepsilon_j^*\}_{j \geq 0})$. Since q_- defined in (44) is non-increasing, there exists $c > 0$ such that, for all $x > 0$, $\mathbb{P}(|V_k^{*\Delta}| > x) \leq c \mathbb{P}(|V_+^{*\Delta}| > x)$, where $V_+^{*\Delta}$ is defined in (49). Hence, we may apply Lemma 17 which yields, for any $\Delta > 0$,

$$(62) \quad \liminf_{n \rightarrow \infty} n^{-1} \sum_{k=2}^n V_k^{*\Delta} = \liminf_{n \rightarrow \infty} n^{-1} \sum_{k=2}^n \mathbb{E}\{V_k^{*\Delta} | \mathcal{F}_{k-1}^*\}, \quad \mathbb{P}_* - \text{a.s.}$$

By (47), since for all $x > 0$, $\log x \geq -\log_- x$, then, by the strong law of large numbers,

$$(63) \quad \liminf_{n \rightarrow \infty} n^{-1} \sum_{k=2}^n \mathbb{E}\{V_k^{*\Delta} | \mathcal{F}_{k-1}^*\} \geq -\mathbb{E}[H_\Delta(a^* b^* | \varepsilon_0^* + b^* | \varepsilon_1^*)], \quad \mathbb{P}_* - \text{a.s.}$$

where $H_\Delta(x) = \mu_+^* \times \int \log_- q^- [c + d\Delta + \kappa + 2a^* b^* + x + |w|] \psi^*(w) dw$. By (50), $\mathbb{E}[H_\Delta(a^* b^* | \varepsilon_0^* + b^* | \varepsilon_1^*)] < \infty$, it then follows by (61), (62) and (63) that

$$\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=2}^n \log \varepsilon_\Delta^-(Y_{k-1}^*, Y_k^*) \geq \liminf_{n \rightarrow \infty} n^{-1} \sum_{k=2}^n V_k^{*\Delta} > -\infty, \quad \mathbb{P}_* - \text{a.s.}$$

and the condition (19) is satisfied. The proof of assumptions (20) and (21) can be checked as in Proposition 7. Since (H2) is satisfied, for a fixed $\eta > 0$,

we choose $\Delta > 0$ such that $\Upsilon_{C^c(y,\Delta)}(y) \leq \eta \Upsilon_X(y)$. Applying Lemma 13 yields

$$n^{-1} \log \Lambda_\eta(Y_{0:n}^*, \alpha) \leq \int_0^1 \mathbb{1}\{1 - r_n \leq u\} F_n^{*-1}(u) du, \quad \mathbb{P}_* - \text{a.s.}$$

where $r_n = (\lceil n\alpha \rceil - 1)/n$ and F_n^{*-1} is the generalized inverse of the distribution function:

$$(64) \quad F_n^*(t) = n^{-1} \sum_{k=1}^n \mathbb{1}\{R_\Delta[\kappa + 2a^*b^* + a^*b^*|\varepsilon_{k-1}^*| + b^*|\varepsilon_k^*| + |\zeta_k^*|] \leq t\},$$

with R_Δ is defined in (56). For convenience, let us write $G(\varepsilon_{k-1}^*, \varepsilon_k^*, \zeta_k^*) = R_\Delta[\kappa + 2a^*b^* + a^*b^*|\varepsilon_{k-1}^*| + b^*|\varepsilon_k^*| + |\zeta_k^*|]$. Setting

$$(65) \quad H_n^*(t) = n^{-1} \sum_{k=1}^n \mathbb{P}\{G(\varepsilon_{k-1}^*, \varepsilon_k^*, \zeta_k^*) \leq t | \mathcal{F}_{k-1}^*\},$$

it follows from Lemma 17 that, for a fixed $t \in \mathbb{R}$,

$$(66) \quad \lim_{n \rightarrow \infty} \{F_n^*(t) - H_n^*(t)\} = 0, \quad \mathbb{P}_* - \text{a.s.}$$

The convergence in (66) may be shown to hold uniformly in t :

LEMMA 18. *Let us consider the stochastic functions F_n^* and H_n^* defined by (64), (65). Then,*

$$(67) \quad \lim_{n \rightarrow \infty} \|F_n^* - H_n^*\|_\infty = 0, \quad \mathbb{P}_* - \text{a.s.}$$

PROOF. Let us define

$$(68) \quad J_n^*(t) = n^{-1} \sum_{k=1}^n \int \mathbb{P}\{G(\varepsilon_{k-1}^*, \varepsilon_k^*, w) \leq t | \mathcal{F}_{k-1}^*\} \psi^*(w) dw,$$

$$(69) \quad J^*(t) = \mathbb{E} \left[\int \mathbb{1}\{G(\varepsilon_0^*, \varepsilon_1^*, w) \leq t\} \psi^*(w) dw \right].$$

By the Glivenko-Cantelli Theorem, $\lim_{n \rightarrow \infty} \|J_n^* - J^*\|_\infty = 0$, \mathbb{P}_* -a.s. Set $\varepsilon > 0$ and a sequence $-\infty = t_0 \leq t_1 \dots \leq t_N = \infty$ such that $J^*(t_i^-) - J^*(t_{i-1}) < \varepsilon/\mu_+^*$ for every i . By (47), for all real numbers $t < t'$, \mathbb{P}_* -a.s.

$$H_n^*(t') - H_n^*(t) = n^{-1} \sum_{k=1}^n \mathbb{P}(t < G(\varepsilon_{k-1}^*, \varepsilon_k^*, \zeta_k^*) \leq t' | \mathcal{F}_{k-1}^*) \leq \mu_+^* [J_n^*(t') - J_n^*(t)],$$

and then

$$\limsup_{n \rightarrow \infty} |H_n^*(t') - H_n^*(t)| \leq \mu_+^* |J^*(t') - J^*(t)|, \quad \mathbb{P}_\star - \text{a.s.}$$

For all $t \in \mathbb{R}$, there exists an index i such that $t_{i-1} \leq t < t_i$. Since F_n^* and H_n^* are increasing functions, it follows that

$$F_n^*(t_{i-1}) \leq F_n^*(t) \leq F_n^*(t_i^-), \quad H_n^*(t_{i-1}) \leq H_n^*(t) \leq H_n^*(t_i^-).$$

These inequalities imply

$$\sup_{t \in \mathbb{R}} |F_n^*(t) - H_n^*(t)| \leq \max_{0 \leq i \leq N} |F_n^*(t_i^-) - H_n^*(t_i^-)| + \max_{1 \leq i \leq N} |H_n^*(t_i^-) - H_n^*(t_{i-1})|,$$

and then

$$\limsup_{n \rightarrow \infty} \sup_{t \in \mathbb{R}} |F_n^*(t) - H_n^*(t)| \leq \varepsilon, \quad \mathbb{P}_\star - \text{a.s.}$$

□

By (47), for all $t \in \mathbb{R}$,

$$F_n^*(t) = F_n^*(t) - H_n^*(t) + H_n^*(t) \geq F_n^*(t) - H_n^*(t) + \mu_-^* J_n^*(t), \quad \mathbb{P}_\star - \text{a.s.}$$

Hence, using the limit (67), for a given $\delta > 0$, there exists an integer l such that, for all $n \geq l$ and $t \in \mathbb{R}$,

$$(70) \quad F_n^*(t) \geq \mu_-^* J_n^*(t) - \delta, \quad \mathbb{P}_\star - \text{a.s.}$$

Let us notice that J_n^* is an increasing function with $\lim_{t \rightarrow -\infty} J_n^*(t) = 0$ and $\lim_{t \rightarrow +\infty} J_n^*(t) = 1$. Then, we can define its generalized inverse denoted by J_n^{*-1} . By (70), it follows that, for all $u \in [0, (\mu_-^* - \delta) \wedge 0]$,

$$F_n^{*-1}(u) \leq J_n^{*-1}[(u + \delta)/\mu_-^*], \quad \mathbb{P}_\star - \text{a.s.}$$

By choosing $\delta > 0$ such that $\mu_-^* - \delta > 1 - \alpha$, there exists an integer $i \geq l$ such that, for all $n \geq i$, we have

$$\begin{aligned} & \int_0^1 \mathbb{1}\{1 - r_n \leq u\} F_n^{*-1}(u) du \\ & \leq \int_0^1 \mathbb{1}\{1 - r_n \leq u \leq \mu_-^* - \delta\} J_n^{*-1}[(u + \delta)/\mu_-^*] du, \quad \mathbb{P}_\star - \text{a.s.} \end{aligned}$$

By Fatou's lemma,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_0^1 \mathbb{1}\{1 - r_n \leq u\} F_n^{*-1}(u) du \\ & \leq \int_0^1 \limsup_{n \rightarrow \infty} \mathbb{1}\{1 - r_n \leq u \leq \mu_-^* - \delta\} J_n^{*-1}[(u + \delta)/\mu_-^*] du, \quad \mathbb{P}_* - \text{a.s.} \end{aligned}$$

It follows by Lemma 14 that

$$\limsup_{n \rightarrow \infty} n^{-1} \log \Lambda_\eta(Y_{0:n}^*, \alpha) \leq \int_{1-\alpha}^{\mu_-^* - \delta} J^{*-1}[(u + \delta)/\mu_-^*] du < 0, \quad \mathbb{P}_* - \text{a.s.}$$

Thus, condition (22) is satisfied and Theorem 5 applies. \square

PROOF OF PROPOSITION 12. It follows, by definition of r_1 , Lemma 10 and (O3), that

$$\begin{aligned} r_1(n) &= \mathbb{P}_* \left(n^{-1} \sum_{k=2}^n \log q^- [c + d\Delta + D(Y_{k-1}^*, Y_k^*)] \leq -M_1 n \right) \leq \\ & \mathbb{P}_* \left(n^{-1} \sum_{k=2}^n \log q^- [c_0 + a^* b^* |\varepsilon_{k-1}^*| + b^* |\varepsilon_k^*| + g_+^*(A_k^*)] \leq -M_1 n \right). \end{aligned}$$

with $c_0 = c + d\Delta + \kappa + 2a^*b^*$. Then, by (O4) and applying Lemma 15, there exist some constants $c_1, \delta_1 > 0$ such that $r_1(n) \leq c_1 e^{-\delta_1 n}$. By the same arguments as in proof of Proposition 9, the real numbers M_2 and M_3 can be chosen large enough such that $r_2(n) = 0$ and $r_3(n) = 0$. Let us denote by $\{U_k^+\}_{k \geq 0}$ the sequence defined by $U_k^+ = R_\Delta[\kappa + 2a^*b^* + a^*b^*|\varepsilon_{k-1}^*| + b^*|\varepsilon_k^*| + g_+^*(A_k^*)]$, for all integer $k \geq 1$. By definition of Λ_η ,

$$(71) \quad n^{-1} \log \Lambda_\eta(Y_{0:n}^*, \alpha) \leq n^{-1} \sum_{k=n-\lceil \alpha n \rceil + 1}^n U_{n,k}^+.$$

By applying Lemma 16, there exist some constants $c_4, \delta_4 > 0$ such that $r_4(n) \leq c_4 e^{-\delta_4 n}$. Finally, Theorem 6 applies and provides a geometric rate. \square

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LABORATOIRE DE MATHÉMATIQUES D'ORSAY,
BÂTIMENT 425, 15, RUE GEORGES CLEMENCEAU,
91405 ORSAY CEDEX ORSAY, FRANCE.
E-MAIL: gassiat@math.u-psud.fr

SYSTÈMES AÉROPORTES,
2 AV GAY LUSSAC,
78990 ELANCOURT, FRANCE
E-MAIL: benoit.landelle@math.u-psud.fr

TÉLÉCOM PARISTECH,
46 RUE BARRAULT, 75634, PARIS, FRANCE.
E-MAIL: eric.moulines@telecom-paristech.fr