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# Optimal length estimates for stable CMC surfaces in 3-space-forms

Laurent Mazet

## Abstract

In this paper, we study stable constant mean curvature  $H$  surfaces in  $\mathbb{R}^3$ . We prove that, in such a surface, the distance from a point to the boundary is less than  $\pi/(2H)$ . This upper-bound is optimal and is extended to stable constant mean curvature surfaces in space forms.

## 1 Introduction

A constant mean curvature (cmc) surface  $\Sigma$  in a Riemannian 3-manifold  $\mathbb{M}^3$  is stable, if its stability operator  $L = -\Delta - \text{Ric}(n, n) - |A|^2$  is nonnegative. The nonnegativity of this operator means that  $\Sigma$  is a local minimizer of the area functional on surfaces regard to the infinitesimal deformations fixing its boundary.

The stability hypothesis was studied by several authors and has many consequences (see [6] for an overview). For example, D. Fischer-Colbrie and R. Schoen [4] studied the case of complete stable minimal surfaces when  $\mathbb{M}^3$  has non-negative scalar curvature. They obtain that the universal cover of  $\Sigma$  is not conformally equivalent to the disk and, as a consequence, prove that the plane is the only complete stable minimal surface in  $\mathbb{R}^3$ . From this, R. Schoen [8] has derived a curvature estimate for stable cmc surfaces.

In [2], T. H. Colding and W. P. Minicozzi introduced new technics and obtained area and curvature estimates for stable cmc surfaces. Afterward, these technics were used by P. Castillon [1] to answer a question asked in [4] about the consequences of the positivity of certain elliptic operators. Recently, the same ideas have been used by J. Espinar and H. Rosenberg [3] to obtain similar results.

In [7], A. Ros and H. Rosenberg study constant mean curvature  $H$  surfaces in  $\mathbb{R}^3$  with  $H \neq 0$  : they prove a maximum principle at infinity. One of their tools is a length estimate for stable cmc surface. In fact, they prove

that the intrinsic distance from a point  $p$  in a stable cmc surface  $\Sigma$  to the boundary of  $\Sigma$  is less than  $\pi/H$ . The aim of this paper is to improve this result. In fact, applying the ideas of [2], we prove that the distance is less than  $\pi/(2H)$ . This estimate is optimal since, for a hemisphere of radius  $1/H$ , the distance from the pole to the boundary is  $\pi/(2H)$ . Actually we prove that the hemisphere of radius  $1/H$  is the only stable cmc  $H$  surface where the distance  $\pi/(2H)$  is reached. We can generalize this result to stable cmc  $H$  surfaces in  $\mathbb{M}^3(\kappa)$ , where  $\mathbb{M}^3(\kappa)$  is the 3-space form of sectional curvature  $\kappa$ . We prove that when  $H^2 + \kappa > 0$  such an optimal estimate exists. In fact, it is already known that, when  $\kappa \leq 0$  and  $H^2 + \kappa \leq 0$ , there is no such estimate since there exist complete stable cmc  $H$  surfaces. But, in some sense, our results is an extension of the fact that the planes (resp. the horospheres) are the only stable complete constant mean curvature  $H$  surfaces in  $\mathbb{R}^3$  (resp.  $M^3(\kappa)$ ,  $\kappa < 0$ ) when  $H = 0$  (resp.  $H^2 + \kappa = 0$ ).

## 2 Definitions

On a constant mean curvature surface  $\Sigma$  in a Riemannian 3-manifold  $\mathbb{M}^3$ , the stability operator is defined by  $L = -\Delta - \text{Ric}(n, n) - |A|^2$ , where  $\Delta$  is the Laplace operator on  $\Sigma$ ,  $\text{Ric}$  is the Ricci tensor on  $\mathbb{M}^3$ ,  $n$  is the normal to  $\Sigma$  and  $A$  is the second fundamental form on  $\Sigma$ . When it is necessary, we will denote the stability operator by  $L_f$  to refer to the immersion  $f$  of  $\Sigma$  in  $\mathbb{M}^3$ .

The surface  $\Sigma$  is called *stable* if the operator  $L$  is nonnegative *i.e.*, for every compactly supported function  $u$ , we have

$$0 \leq \int_{\Sigma} uL(u)d\sigma = \int_{\Sigma} \|\nabla u\|^2 - (\text{Ric}(n, n) + |A|^2)u^2d\sigma$$

We remark that this property is sometimes called strong stability since it means that the second derivatives of the area functional is nonnegative with respect to any compactly supported infinitesimal deformations  $u$  whereas  $\Sigma$  is critical for this functional only for compactly supported infinitesimal deformations with vanishing mean value *i.e.*  $\int_{\Sigma} u d\sigma = 0$ .

In the following, on a cmc surface, the normal  $n$  is always chosen such that  $H$  is non-negative.

We will denote by  $d_{\Sigma}$  the intrinsic distance on  $\Sigma$  and by  $K$  the sectional curvature of the surface.

### 3 Results

The main result of this paper is the following theorem.

**Theorem 1.** *Let  $H$  be positive. Let  $\Sigma$  be a stable constant mean curvature  $H$  surface in  $\mathbb{R}^3$ . Then, for  $p \in \Sigma$ , we have :*

$$d_{\Sigma}(p, \partial\Sigma) \leq \frac{\pi}{2H} \quad (1)$$

Moreover, if the equality is satisfied,  $\Sigma$  is a hemisphere.

In  $\mathbb{R}^3$ , the stability operator can be written  $L = -\Delta - 4H^2 + 2K$ .

*Proof.* We denote by  $R_0$  the distance  $d_{\Sigma}(p, \partial\Sigma)$  and assume that  $R_0 \geq \pi/(2H)$ . If  $R_0 < \pi/H$  we denote by  $I$  the segment  $[\pi/(2H), R_0]$ , otherwise  $I = [\pi/(2H), \pi/H)$ . In fact, because of the work of Ros and Rosenberg [7], we already know that  $R_0 \leq \pi/H$ . Let  $R$  be in  $I$ .

The surface  $\Sigma$  has constant mean curvature  $H$  thus its sectional curvature is less than  $H^2$ . So the exponential map  $\exp_p$  is a local diffeomorphism on the disk  $D(0, R) \subset T_p\Sigma$  of center 0 and radius  $R$ . On this disk, we consider the induced metric and the operator  $\mathcal{L} = -\Delta - 4H^2 + 2K$ . The surface  $\Sigma$  is stable so it exists a positive function  $g$  on  $\Sigma$  such that  $L(g) = 0$  (see Theorem 1 in [4]). On  $D(0, R)$ , the function  $\tilde{g} = g \circ \exp_p$  is then positive and satisfies  $\mathcal{L}(\tilde{g}) = 0$  since  $D(0, R)$  and  $\Sigma$  are locally isometric. The operator  $\mathcal{L}$  is thus nonnegative on  $D(0, R)$  [4].

For  $r \in [0, R]$ , we define  $l(r)$  as the length of the circle  $\{v, |v| = r\} \subset D(0, R)$  and  $\mathcal{K}(r) = \int_{D(0,r)} K d\sigma$ . Since  $D(0, R)$  and  $\Sigma$  are locally isometric, the sectional curvature  $K$  of  $D(0, R)$  is less than  $H^2$ . Then

$$l(r) \geq \frac{2\pi}{H} \sin Hr \quad (2)$$

By Gauss-Bonnet, we have:

$$\mathcal{K}(r) = 2\pi - l'(r) \quad (3)$$

Let us consider a function  $\eta : [0, R] \rightarrow [0, 1]$  with  $\eta(0) = 1$  and  $\eta(R) = 0$ . Let us write the nonnegativity of  $\mathcal{L}$  for the radial function  $u = \eta(r)$ .

$$0 \leq \int_0^R (\eta'(r))^2 l(r) dr - 4H^2 \int_0^R \eta^2(r) l(r) dr + 2 \int_0^R \mathcal{K}'(r) \eta^2(r) dr$$

Hence, following the ideas in [2] and using (3) and the boundary values of  $\eta$ , we have:

$$\begin{aligned}
\int_0^R (4H^2\eta^2 - \eta'^2)l dr &\leq 2 \left( [\mathcal{K}(r)\eta^2(r)]_0^R - \int_0^R \mathcal{K}(r)(\eta^2(r))' dr \right) \\
&\leq -2 \int_0^R \mathcal{K}(r)(\eta^2(r))' dr \\
&\leq -2 \int_0^R (2\pi - l'(r))(\eta^2(r))' dr \\
&\leq 4\pi + 2 \int_0^R (\eta^2(r))' l'(r) dr \\
&\leq 4\pi + [2(\eta^2(r))' l(r)]_0^R - 2 \int_0^R (\eta^2(r))'' l(r) dr \\
&\leq 4\pi - 2 \int_0^R (\eta^2(r))'' l(r) dr
\end{aligned}$$

Thus we obtain

$$\int_0^R \left( 4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' \right) l dr \leq 4\pi \quad (4)$$

We shall apply this equation to the function  $\eta(r) = \cos \frac{\pi r}{2R}$ . In this case we have

$$\begin{aligned}
\eta'^2 &= \frac{\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R} \\
(\eta^2)'' &= -\frac{\pi^2}{2R^2} \left( \cos^2 \frac{\pi r}{2R} - \sin^2 \frac{\pi r}{2R} \right)
\end{aligned}$$

Thus

$$4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' = \left( 4H^2 - \frac{\pi^2}{R^2} \right) \cos^2 \frac{\pi r}{2R} + \frac{3\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R}$$

As  $R \geq \frac{\pi}{2H}$ ,  $4H^2\eta^2 - \eta'^2 + 2(\eta^2)''$  is non-negative and, by (2),

$$\begin{aligned}
(4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' ) l &\geq \left( \left( 4H^2 - \frac{\pi^2}{R^2} \right) \cos^2 \frac{\pi r}{2R} + \frac{3\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R} \right) \frac{2\pi}{H} \sin Hr \\
&\geq \frac{\pi}{H} \left( \left( 4H^2 - \frac{\pi^2}{4R^2} \right) \sin Hr + \left( 4H^2 - \frac{7\pi^2}{4R^2} \right) \frac{1}{2} \left( \sin \left( \frac{\pi}{R} + H \right) r - \sin \left( \frac{\pi}{R} - H \right) r \right) \right)
\end{aligned}$$

Thus integrating in (4), we obtain (we recall that  $R < \pi/H$ )

$$4\pi \geq \frac{\pi}{H} \left( (4H^2 - \frac{\pi^2}{4R^2}) \frac{1}{H} (1 - \cos HR) \right. \\ \left. + (4H^2 - \frac{7\pi^2}{4R^2}) \frac{1}{2} \left( \frac{R}{\pi + HR} (1 - \cos(\pi + HR)) - \frac{R}{\pi - HR} (1 - \cos(\pi - HR)) \right) \right)$$

After some simplifications in the above expression, we obtain

$$4\pi \geq \pi \frac{(-32H^2R^4 + 24\pi^2H^2R^2 - \pi^4) - (10\pi^2H^2R^2 - \pi^4) \cos HR}{4H^2R^2(\pi^2 - H^2R^2)}$$

Now, passing  $4\pi$  on the right-hand side of the above inequality and simplifying by  $\pi$ , we obtain:

$$0 \geq \frac{-(4H^2R^2 - \pi^2)^2 - (10\pi^2H^2R^2 - \pi^4) \cos HR}{4H^2R^2(\pi^2 - H^2R^2)}$$

We denote by  $F(R)$  the right-hand term of the above inequality. Hence we have proved that, for every  $R$  in  $I$ ,  $F(R) \leq 0$ . If we write  $R = \pi/(2H) + x$ , we compute the Taylor expansion of  $F$  and obtain

$$F\left(\frac{\pi}{2H} + x\right) = 2Hx + o(x)$$

which is positive if  $x > 0$ . Thus, if  $R_0 > \pi/(2H)$ , we get a contradiction and the inequality (1) is proved.

Now if  $R_0 = \pi/(2H)$ , we have in fact equality all along the computation, so  $l(r) = 2\pi/H \sin Hr$  and  $\mathcal{K}(r) = 2\pi - l'(r) = 2\pi(1 - \cos Hr)$ . But we also know that the sectional curvature is less than  $H^2$  thus  $\mathcal{K}(r) \leq H^2 \int_0^r l(u) du = 2\pi(1 - \cos Hr)$ . Since this inequality is in fact an equality, the sectional curvature is in fact  $H^2$  at every point. Thus the principal curvatures of a point in  $\Sigma$  are  $H$  and  $H$  *i.e.* there are only umbilical points. Hence  $\Sigma$  is a piece of a sphere of radius  $1/H$  and, since  $d_\Sigma(p, \partial\Sigma) = \frac{\pi}{2H}$ , it contains the hemisphere of pole  $p$ . A hemisphere can not be strictly contained in a stable subdomain of the sphere, so  $\Sigma$  is a hemisphere.  $\square$

With this result we have an important corollary.

**Corollary 2.** *Let  $H \geq 0$  and  $\kappa \in \mathbb{R}$  such that  $H^2 + \kappa > 0$ . Let  $\Sigma$  be a stable constant mean curvature  $H$  surface in  $\mathbb{M}^3(\kappa)$ . Then for  $p \in \Sigma$ , we have :*

$$d_\Sigma(p, \partial\Sigma) \leq \frac{\pi}{2\sqrt{H^2 + \kappa}}$$

*Moreover, if the equality is satisfied,  $\Sigma$  is a geodesical hemisphere of  $\mathbb{M}^3(\kappa)$ .*

The proof is based on the Lawson's correspondence between constant mean curvature surfaces in space forms (see [5]).

*Proof.* First, the case  $\kappa = 0$  is Theorem 1.

Let  $\Pi : \tilde{\Sigma} \rightarrow \Sigma$  be the universal cover of  $\Sigma$ . We then have a constant mean curvature immersion of  $\tilde{\Sigma}$  in  $\mathbb{M}^3(\kappa)$ , let  $\mathcal{L} = -\Delta - 2\kappa - |A|^2$  be the stability operator on  $\tilde{\Sigma}$ .  $\Sigma$  is stable, so there exists a positive function  $g$  on  $\Sigma$  such that  $L(g) = -\Delta g - (2\kappa + |A|^2)g = 0$ . Thus the function  $\tilde{g} = g \circ \Pi$  is a positive function on  $\tilde{\Sigma}$  satisfying  $\mathcal{L}(\tilde{g}) = 0$ . Hence  $\tilde{\Sigma}$  is stable. Let  $I$  and  $S$  be respectively the first fundamental form and the shape operator on  $\tilde{\Sigma}$ . They satisfy the Gauss and Codazzi equations for  $\mathbb{M}^3(\kappa)$ .

We define  $S' = S + (-H + \sqrt{H^2 + \kappa})\text{id}$  on  $\tilde{\Sigma}$ . Then  $I$  and  $S'$  satisfy the Gauss and Codazzi equations for  $\mathbb{M}^3(0) = \mathbb{R}^3$  (see [5]). Hence there exists an immersion  $f$  of  $\tilde{\Sigma}$  in  $\mathbb{R}^3$  with first fundamental form  $I$  and shape operator  $S'$  (we notice that the induced metric is the same). Its mean curvature is then  $H + (-H + \sqrt{H^2 + \kappa}) = \sqrt{H^2 + \kappa}$  *i.e.* the immersion has constant mean curvature. The stability operator is

$$\begin{aligned} L_f &= -\Delta - \|S'\|^2 \\ &= -\Delta - (\|S\|^2 + 4H(-H + \sqrt{H^2 + \kappa}) + 2(-H + \sqrt{H^2 + \kappa})^2) \\ &= -\Delta - (\|S\|^2 + 2\kappa) \\ &= \mathcal{L} \end{aligned}$$

Hence the surface  $f(\tilde{\Sigma})$  is stable. So, from Theorem 1, we have

$$d_{\Sigma}(p, \partial\Sigma) = d_{\tilde{\Sigma}}(\tilde{p}, \partial\tilde{\Sigma}) \leq \frac{\pi}{2\sqrt{H^2 + \kappa}}$$

where  $\Pi(\tilde{p}) = p$ .

The equality case comes from the equality case in Theorem 1 and since the Lawson's correspondence sends spheres into spheres.  $\square$

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