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Test vectors for trilinear forms : the case of two principal series

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1 Introduction

Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizing parameter π. Let V_1, V_2 and V_3 be three irreducible, admissible, infinite dimensional representations of $G = \mathrm{GL}_2(F)$ of central characters ω_1, ω_2 and ω_3 and conductors n_1, n_2 and n_3. Using the theory of Gelfand pairs, Diprenda Prasad proves in [P] that the space of G-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$ has dimension at most one and gives a precise criterion for this dimension to be one, that we will now explain.

Let D^* be the group of invertible elements of the unique quaternion division algebra D over F. When V_i is a discrete series representation of G, denote by V_i' the irreducible representation of D^* associated to V_i by the Jacquet-Langlands correspondence. Again, by the theory of Gelfand pairs, the space of D^*-invariant linear forms on $V_1' \otimes V_2' \otimes V_3'$ has dimension at most one.

A necessary condition for the existence on a non-zero G-invariant linear form on $V_1 \otimes V_2 \otimes V_3$ (resp. non-zero D^*-invariant linear form on $V_1' \otimes V_2' \otimes V_3'$), that we will *always assume*, is that

$$\omega_1 \omega_2 \omega_3 = 1.$$

Let σ_i be the two dimensional representations of the Weil-Deligne group of F associated to V_i. The triple tensor product $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ is an eight dimensional symplectic representation of the Weil-Deligne group having a local root number $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3)$ equal to 1 or -1. When $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$, one can prove that the V_i's are all discrete series representations of G.

Theorem 1. *(Prasad [P, Theorem 1.4]) If all the V_i's are supercuspidal, assume that the residue characteristic of F is not 2. Then*

- *$\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$ if, and only if, there exists a non-zero G-invariant linear form on $V_1 \otimes V_2 \otimes V_3$, and*
- *$\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ if, and only if, there exists a non-zero D^* invariant linear form on $V_1' \otimes V_2' \otimes V_3'$.*

Given a non zero G-invariant linear form ℓ on $V_1 \otimes V_2 \otimes V_3$, or a non-zero D^*-invariant linear form ℓ' on $V_1' \otimes V_2' \otimes V_3'$, the goal is to find a vector in $V_1 \otimes V_2 \otimes V_3$ which is not in the kernel of ℓ, or a vector in $V_1' \otimes V_2' \otimes V_3'$ which is not in the kernel of ℓ'. Such a vector is called a test vector. The following results of Prasad and Gross-Prasad show that new vectors can sometimes be used as test vectors. In what follows v_i denotes a new vector in V_i (see §2.2).

Theorem 2. *(Prasad [P, Theorem 1.3]) If all the V_i's are unramified principal series, then $v_1 \otimes v_2 \otimes v_3$ is a test vector.*

Theorem 3. *(Gross and Prasad [G-P, Proposition 6.3]) Suppose all the V_i's are unramified twists of the Steinberg representation.*

- *If $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$, then $v_1 \otimes v_2 \otimes v_3$ is a test vector.*
- *If $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ and if R is the the unique maximal order in D, then any vector belonging to the unique line in $V_1' \otimes V_2' \otimes V_3'$ fixed by $R^* \times R^* \times R^*$ is a test vector.*

Actually, the proof by Gross and Prasad of the first statement of the above theorem contains another result :

Theorem 4. *If two of the V_i's are unramified twists of the Steinberg representation and the third one is an unramified principal series, then $v_1 \otimes v_2 \otimes v_3$ is a test vector.*

However, as mentioned in [G-P], new vectors are not always test vectors. Let $K = \mathrm{GL}(\mathcal{O})$ be the maximal compact subgroup of G and suppose that V_1 and V_2 are unramified, but V_3 is ramified. Since v_1 and v_2 are K-invariant and ℓ is G-equivariant, $v \mapsto \ell(v_1 \otimes v_2 \otimes v)$ defines a K-invariant linear form on V_3. Since V_3 is ramified, so is its contragredient, and therefore the above linear form has to vanish. In particular $\ell(v_1 \otimes v_2 \otimes v_3) = 0$.

To go around this obstruction for new vectors to be test vectors, Gross and Prasad made the following suggestion : suppose that V_3 has conductor $n = n_3 \geq 1$; since V_3 has unramified central character, its contragredient representation has non-zero invariant vectors by the n-th standard Iwahori subgroup $I_n = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi^n \mathcal{O} & \mathcal{O}^\times \end{pmatrix}$ of G; put $\gamma = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and let $v_1^* \in V_1$ be a non-zero vector on the line fixed by the maximal compact subgroup $\gamma^n K \gamma^{-n}$ of G; since $K \cap \gamma^n K \gamma^{-n} = I_n$, the linear form on V_3 given by $v \mapsto \ell(v_1^* \otimes v_2 \otimes v)$ is not necessarily zero and there is still hope for $v_1^* \otimes v_2 \otimes v_3$ to be a test vector. This is the object of the following theorem

Theorem 5. *If V_1 and V_2 are unramified and V_3 has conductor n_3, then $v_1^* \otimes v_2 \otimes v_3$ is a test vector, where $v_1^* = \gamma^{n_3} \cdot v_1$.*

Theorem 5 for $n_3 = 1$, together with Theorems 2, 3 and 4, completes the study of test vectors when the V_i's have conductors 0 or 1 and unramified central characters.

Assume from now on that V_1 and V_2 are (ramified or unramified) principal series. Then for $i = 1, 2$ there exist quasi-characters μ_i and μ_i' of $F^\times$ such that $\mu_i' \mu_i^{-1} \neq |\cdot|^{\pm 1}$, and

$$V_i = \mathrm{Ind}_B^G \chi_i \text{ , with } \chi_i \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu_i(a) \mu_i'(d).$$

According to Theorem 1 there exists a non-zero G-invariant linear form ℓ on $V_1 \otimes V_2 \otimes V_3$, so we are looking for a test vector in $V_1 \otimes V_2 \otimes V_3$. The following theorem is our main result.

Theorem 6. *Suppose that V_1 and V_2 are principal series such that μ_1 and μ_2' are unramified. Put*

$$x = \max(n_2 - n_1, n_3 - n_1) \qquad \text{and} \qquad v_1^* = \gamma^x \cdot v_1.$$

Then $x \geq 0$ and, if $v_1^* \otimes v_2 \otimes v_3$ is not a test vector, then

- *either* $n_1 = 0$, $n_2 = n_3 > 0$ *and* $\gamma^{n_2-1}\cdot v_1 \otimes v_2 \otimes v_3$ *is a test vector,*
- *or* $n_2 = 0$, $n_1 = n_3 > 0$ *and* $v_1 \otimes \gamma\cdot v_2 \otimes v_3$ *is a test vector,*
- *or* $\widetilde{V_3}$ *is a quotient of* $\mathrm{Ind}_B^G(\chi_1\chi_2\delta^{\frac{1}{2}})$, $n_1 + n_2 = n_3$ *and* $v_1 \otimes \gamma^{n_1}\cdot v_2 \otimes v_3$ *is a test vector.*

The assumptions of the theorem imply in particular that V_1 and V_2 have minimal conductor among their twists. If V_1 and V_2 are two arbitrary principal series, then one can always find characters η_1, η_2 and η_3 of $F^\times$ with $\eta_1\eta_2\eta_3 = 1$, such that the above theorem applies to $(V_1 \otimes \eta_1) \otimes (V_2 \otimes \eta_2) \otimes (V_3 \otimes \eta_3)$. Nevertheless, we found also interesting to study the case when μ_1 or μ_2' is ramified. Then we are able to show that certain new vectors are *not* test vectors, while *a priori* this cannot be seen by a direct argument (the obstruction of Gross and Prasad described above does not apply to this case). Put $m_1 = \mathrm{cond}(\mu_1')$ and $m_2 = \mathrm{cond}(\mu_2')$

Theorem 7. *Suppose that* μ_1 *or* μ_2' *is ramified. Let* x, y *and* z *be integers such that*

- $x \geq m_1$,
- $y \geq m_2$,
- $x - n_3 \geq z \geq y$, *and*
- $x - y \geq \max(n_1 - m_1, n_2 - m_2, 1)$.

Put

$$v_1^* = \begin{cases} \gamma^{x-m_1}\cdot v_1 & \text{, if } \mu_1' \text{ is ramified,} \\ \gamma^{x}\cdot v_1 - \beta_1\gamma^{x-1}\cdot v_1 & \text{, if } \mu_1' \text{ is unramified.} \end{cases}$$
$$v_2^* = \begin{cases} \gamma^{y-m_2}\cdot v_2 & \text{, if } \mu_2 \text{ is ramified.} \\ \gamma^{y-n_2}\cdot v_2 - \alpha_2^{-1}\gamma^{y-n_2+1}\cdot v_2 & \text{, if } \mu_2 \text{ is unramified.} \end{cases} \tag{1}$$

Then

$$\ell(v_1^* \otimes v_2^* \otimes \gamma^z\cdot v_3) = 0.$$

We will prove theorems 6 and 7 by following the pattern of the proof of Theorem 2 in [P], with the necessary changes.

We believe that suitable generalization of the method of Gross and Prasad would give test vectors in the case where at least two of the V_i's are special representations, as well as in the case where one is a special representation and one is a principal series. On the other hand in order to find test vectors in the case where at least two of the V_i's are supercuspidal, one should use different techniques, involving probably computations in Kirillov models.

The search for test vectors in our setting is motivated by subconvexity problems for L-functions of triple products of automorphic forms on GL(2). Roughly speaking, one wants to bound the value of the L-function along the critical line $\Re(z) = \frac{1}{2}$. In [B-R 1] and [B-R 2] Joseph Bernstein and Andre Reznikov establish a *subconvexity bound* when the *eigenvalue* attached to one of the representations varies. Philippe Michel and Akshay Venkatesh considered the case when the *level* of one representation varies. More details about subconvexity and those related techniques can be found in [V] or [M-V]. Test vectors are key ingredients.

Bernstein and Reznikov use an explicit test vector. Venkatesh uses a theoretical one, but explains that the bounds would be better with an explicit one (see [V, §5]).

There is an extension of Prasad's result in [H-S], where Harris and Scholl prove that the dimension of the space of G-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$ is one when V_1, V_2 and V_3 are principal series representations (either irreducible or reducible, but with infinite dimensional irreducible subspace). They apply their result to the global setting to construct elements in the motivic cohomology of the product of two modular curves predicted by Beilinson.

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2 Background on induced admissible representations of $\mathrm{GL}(2)$.

2.1 About induced and contragredient representations.

Let (ρ, W) be a smooth representation of a closed subgroup H of G. Let Δ_H be the modular function on H. The induction of ρ from H to G, denoted $\mathrm{Ind}_H^G \rho$, is the space of functions f from G to W satisfying the two following conditions :

(1) $\forall h \in H, \quad \forall g \in G, \quad f(hg) = \Delta_H(h)^{-\frac{1}{2}} \rho(h) f(g)$,

(2) there exists an open compact subgroup K_f of G such that

$$\forall k \in K_f, \quad \forall g \in G, \quad f(gk) = f(g)$$

where G acts by right translation as follows :

$$\forall g, g' \in G, (g \cdot f)(g') = f(g'g).$$

With the additional condition that f must be compactly supported modulo H, one gets the *compact* induction denoted by ind_H^G. When G/H is compact, there is no difference between Ind_H^G and ind_H^G.

Let B the Borel subgroup of upper triangular matrices in G, and let T be the diagonal torus. The character Δ_T is trivial and we will use $\Delta_B = \delta^{-1}$ with $\delta \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |\frac{a}{d}|$ where $|\ |$ is the normalised valuation of F. The quotient $B \backslash G$ is compact and can be identified with $\mathbb{P}^1(F)$.

For a smooth representation V of G, the contragredient representation $\widetilde{V}$ is the space of smooth linear forms l on V, where G acts as follows :

$$\forall g \in G, \qquad \forall v \in V, \qquad (g \cdot l)(v) = l(g^{-1} \cdot v).$$

We refer the reader to [B-Z] for more details about induced and contragredient representations.

2.2 New vectors and ramification.

Let V be an irreducible, admissible, infinite dimensional representation of G with central character ω. Then $\widetilde{V} \cong V \otimes \omega^{-1}$. To the descending chain of compact subgroups of G

$$K = I_0 \supset I_1 \supset \cdots \supset I_n \supset I_{n+1} \cdots$$

one can associate an ascending chain of vector spaces

$$V^{I_0,\omega} = V^K \text{ , and for all } n \geq 1, \quad V^{I_n,\omega} = \left\{ v \in V \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot v = \omega(d)v \text{ , for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_n \right\}.$$

There exists a minimal n such that the vector space $V^{I_n,\omega}$ is non-zero. It is necessarily one dimensional and any non-zero vector in it is called a *new vector* of V. The integer n is the *conductor* of V. The representation V is said to be *unramified* if $n = 0$.

More information about new vectors can be found in [C].

2.3 New vectors as functions on G.

Let V be a principal series of G, with central character ω, and conductor n. There exist quasi-characters μ and μ' of $F^\times$ such that $\mu'\mu^{-1} \neq |\cdot|^{\pm 1}$, and

$$V = \mathrm{Ind}_B^G(\chi) \qquad \text{with} \qquad \chi \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} = \mu(a)\mu'(d).$$

Then $\omega = \mu\mu'$ and $n = \mathrm{cond}(\mu) + \mathrm{cond}(\mu')$. A new vector v in V is a non-zero function from G to $\mathbb{C}$ such that for all $b \in B$, $g \in G$ and $k = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in I_n$

$$v(bgk) = \chi(b)\delta(b)^{\frac{1}{2}}\omega(d)v(g).$$

Put

$$\alpha^{-1} = \mu(\pi)|\pi|^{\frac{1}{2}} \qquad \text{and} \qquad \beta^{-1} = \mu'(\pi)|\pi|^{-\frac{1}{2}}.$$

First, we assume that V is unramified, and we normalise v so that $v(1) = 1$.

Lemma 2.1. *If V is unramified then for all $r \in \mathbb{N}$,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \beta^r & \text{, if } k \in K \backslash I, \\ \alpha^s \beta^{r-s} & \text{, if } k \in I_s \backslash I_{s+1} \text{ for } 1 \leq s \leq r-1, \\ \alpha^r & \text{, if } k \in I_r. \end{cases}$$

Similarly,

$$(\gamma^r \cdot v - \alpha^{-1}\gamma^{r+1} \cdot v)(k) = \begin{cases} \alpha^s \beta^{r-s} - \alpha^{s-1}\beta^{r+1-s} & \text{, if } k \in I_s \backslash I_{s+1} \text{ for } 0 \leq s \leq r, \\ 0 & \text{, if } k \in I_{r+1}. \end{cases}$$

Finally, for $r \geq 1$,

$$(\gamma^r \cdot v - \beta\gamma^{r-1} \cdot v)(k) = \begin{cases} \alpha^r(1 - \frac{\beta}{\alpha}) & \text{, if } k \in I_r, \\ 0 & \text{, if } k \in K \backslash I_r. \end{cases}$$

Proof : If $k \in I_r$, then $\gamma^{-r} k \gamma^r \in K$, so

$$(\gamma^r \cdot v)(k) = \alpha^r v(\gamma^{-r} k \gamma^r) = \alpha^r.$$

Suppose that $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_s \backslash I_{s+1}$ for some $0 \leq s \leq r-1$ (recall that $I_0 = K$). Then $\pi^{-s} c \in \mathcal{O}^\times$ and

$$(\gamma^r \cdot v)(k) = \alpha^r v \begin{pmatrix} a & \pi^r b \\ \pi^{-r} c & d \end{pmatrix} = \alpha^r v \begin{pmatrix} (ad-bc)\pi^{r-s} & a \\ 0 & \pi^{-r} c \end{pmatrix} = \alpha^s \beta^{r-s}.$$

The second part of the lemma follows by a direct computation. □

For the rest of this section we assume that V is ramified, that is $n \geq 1$. We put

$$m = \mathrm{cond}(\mu') \qquad \text{so that} \qquad n - m = \mathrm{cond}(\mu).$$

By Casselman [C, pp.305-306] the restriction to K of a new vector v is supported by the double coset of $\begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix}$ modulo I_n. In particular if μ' is unramified ($m = 0$), then v is supported by

$$I_n \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} I_n = I_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I_n = K \backslash I.$$

If $1 \leq m \leq n-1$, then v is supported by

$$I_n \begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix} I_n = I_m \backslash I_{m+1}.$$

If μ is unramified ($m = n$), then v is supported by I_n. We normalise v so that

$$v \begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix} = 1.$$

Lemma 2.2. *If μ and μ' are both ramified ($0 < m < n$), then for all $r \in \mathbb{N}$ and $k \in K$,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^r \mu\left(\frac{\det k}{\pi^{-(m+r)} c}\right) \mu'(d) & \text{, if } k = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in I_{m+r} \backslash I_{m+r+1}, \\ 0 & \text{, otherwise.} \end{cases}$$

Proof : For $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ we have

$$\alpha^{-r} (\gamma^r \cdot v)(k) = v(\gamma^{-r} k \gamma^r) = v \begin{pmatrix} a & \pi^r b \\ \pi^{-r} c & d \end{pmatrix}.$$

It is easy to check that for every $s \geq 1$,

$$K \cap B \gamma^r I_s \gamma^{-r} = I_{s+r}.$$

It follows that $\gamma^r \cdot v$ has its support in $I_{m+r} \backslash I_{m+r+1}$. If $k \in I_{m+r} \backslash I_{m+r+1}$ then $c \in \pi^{m+r}\mathcal{O}^\times$, $d \in \mathcal{O}^\times$ and we have the following decomposition :

$$\begin{pmatrix} a & \pi^r b \\ \pi^{-r} c & d \end{pmatrix} = \begin{pmatrix} \det k & \pi^{-m} cb \\ 0 & \pi^{-m-r} cd \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & \pi^{m+r} c^{-1} \end{pmatrix}. \tag{2}$$

Hence

$$\alpha^{-r}(\gamma^r \cdot v)(k) = \mu\Big(\det(k)\Big)\mu'(\pi^{-m-r}cd)(\mu\mu')(\pi^{m+r}c^{-1}) = \mu\Big(\frac{\det(k)}{\pi^{-(m+r)}c}\Big)\mu'(d).$$

□

Similarly we obtain :

Lemma 2.3. *Suppose that μ is unramified and μ' is ramified. Then, for all $r \in \mathbb{N}$ and $k \in K$,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^r \mu'(d) & \text{, if } k = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in I_{n+r}, \\ 0 & \text{, otherwise.} \end{cases}$$

$$\Big(\gamma^r \cdot v - \alpha^{-1}\gamma^{r+1} \cdot v\Big)(k) = \begin{cases} \alpha^r \mu'(d) & \text{, if } k = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in I_{n+r} \backslash I_{n+r+1}, \\ 0 & \text{, otherwise.} \end{cases}$$

Lemma 2.4. *Suppose that μ' is unramified and μ is ramified. Then for all $r \in \mathbb{N}$,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^s \beta^{r-s} \mu\left(\frac{\det(k)}{\pi^{-s}c}\right) & \text{, if } k = \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in I_s \backslash I_{s+1}, \text{ with } 0 \le s \le r, \\ 0 & \text{, if } k \in I_{r+1}. \end{cases}$$

Moreover, if $r \ge 1$, then

$$\Big(\gamma^r \cdot v - \beta\gamma^{r-1} \cdot v\Big)(k) = \begin{cases} \alpha^r \mu\left(\frac{\det(k)}{\pi^{-r}c}\right) & \text{, if } k = \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in I_r \backslash I_{r+1}, \\ 0 & \text{, otherwise.} \end{cases}$$

Proof : We follow the pattern of proof of lemma 2.2. The restriction of $\gamma^r \cdot v$ to K is zero outside

$$K \cap B\gamma^r(K \backslash I)\gamma^{-r} = K \backslash I_{r+1}.$$

For $0 \le s \le r$ and $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_s \backslash I_{s+1}$ we use the following decomposition :

$$\begin{pmatrix} a & \pi^r b \\ \pi^{-r} c & d \end{pmatrix} = \begin{pmatrix} -\frac{\det k}{\pi^{-r}c} & a + \frac{\det k}{\pi^{-r}c} \\ 0 & \pi^{-r}c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 + \frac{d}{\pi^{-r}c} \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Since $d \in \mathcal{O}$ and $\pi^r c^{-1} \in \mathcal{O}$ we deduce that :

$$\alpha^{-r}(\gamma^r \cdot v)(k) = \mu\Big(\frac{\det k}{\pi^{-r}c}\Big)\mu'(-\pi^{-r}c)\left|\pi^r c^{-1}\right| = \mu\Big(\frac{\det k}{\pi^{-s}c}\Big)\alpha^{s-r}\beta^{r-s}.$$

□

As direct consequence of these lemmas we obtain

Lemma 2.5. *Let v_1^* and v_2^* be as in Theorem 7. Then the support of v_1^* is*

$$\begin{cases} I_x \backslash I_{x+1} & \text{, if } \mu_1 \text{ is ramified,} \\ I_x & \text{, if } \mu_1 \text{ is unramified,} \end{cases}$$

and the support of v_2^* is

$$\begin{cases} I_y \backslash I_{y+1} & \text{, if } \mu_2' \text{ is ramified,} \\ K \backslash I_{y+1} & \text{, if } \mu_2' \text{ is unramified.} \end{cases}$$

3 Going down Prasad's exact sequence.

In this section we will explain how Prasad finds a non-zero $\ell \in \mathrm{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C})$ in the case where V_1 and V_2 are principal series representations.

3.1 Prasad's exact sequence.

The space $\mathrm{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C})$ is canonically isomorphic to $\mathrm{Hom}_G(V_1 \otimes V_2, \widetilde{V_3})$, hence finding ℓ it is the same as finding a non-zero element Ψ in it. We have

$$V_1 \otimes V_2 = \mathrm{Res}_G \, \mathrm{Ind}_{B \times B}^{G \times G} \Big(\chi_1 \times \chi_2 \Big)$$

where the restriction is taken with respect to the diagonal embedding of G in $G \times G$. The action of G on $(B \times B) \backslash (G \times G) \cong \mathbb{P}^1(F) \times \mathbb{P}^1(F)$ has precisely two orbits.

The first is the diagonal $\Delta_{B \backslash G}$, which is closed and can be identified with $B \backslash G$. The second is its complement which is open and can be identified with $T \backslash G$ via the bijection :

$$\begin{array}{rcl} T \backslash G & \longrightarrow & \Big(B \backslash G \times B \backslash G \Big) \setminus \Delta_{B \backslash G} \\ Tg & \longmapsto & \Big(Bg, B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \Big) \end{array}$$

Hence, there is a short exact sequence of G-modules :

$$0 \to \mathrm{ind}_T^G \Big(\chi_1 \chi_2' \Big) \xrightarrow{\textbf{ext}} V_1 \otimes V_2 \xrightarrow{\textbf{res}} \mathrm{Ind}_B^G \Big(\chi_1 \chi_2 \delta^{\frac{1}{2}} \Big) \to 0, \tag{4}$$

where $\chi_2' \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu_2'(a) \mu_2(d)$. The surjection **res** is given by the restriction to the diagonal. The injection **ext** takes a function $f \in \mathrm{ind}_T^G \Big(\chi_1 \chi_2' \Big)$ to a function $F \in \mathrm{Ind}_{B \times B}^{G \times G} \Big(\chi_1 \times \chi_2 \Big)$ vanishing on $\Delta_{B \backslash G}$, such that for all $g \in G$

$$F \Big(g, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \Big) = f(g).$$

Applying the functor $\mathrm{Hom}_G \Big(\bullet, \widetilde{V_3} \Big)$ yields a long exact sequence :

$$\begin{aligned} 0 \to \mathrm{Hom}_G \Big(\mathrm{Ind}_B^G \Big(\chi_1 \chi_2 \delta^{\frac{1}{2}} \Big), \widetilde{V_3} \Big) \to \mathrm{Hom}_G \Big(V_1 \otimes V_2, \widetilde{V_3} \Big) \to \mathrm{Hom}_G \Big(\mathrm{ind}_T^G \Big(\chi_1 \chi_2' \Big), \widetilde{V_3} \Big) \\ \downarrow \\ \cdots \leftarrow \mathrm{Ext}_G^1 \Big(\mathrm{Ind}_B^G \Big(\chi_1 \chi_2 \delta^{\frac{1}{2}} \Big), \widetilde{V_3} \Big) \end{aligned} \tag{5}$$

3.2 The simple case.

The situation is easier if V_3 occurs in $\mathrm{Ind}_B^G(\chi_1^{-1}\chi_2^{-1}\delta^{-\frac{1}{2}})$. Then $\chi_1\chi_2$ does not factor through the determinant and there is a natural surjection

$$\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big) \twoheadrightarrow \widetilde{V_3}.$$

This surjection is an isomorphism, unless there exists a quasi-character η of $F^\times$ such that $\chi_1\chi_2\delta = \eta\circ\det$ in which case the kernel is a line generated by the function $\eta\circ\det$. From (4) we obtain a surjective homomorphism Ψ completing the following commutative diagram :

$$\begin{array}{ccc} V_1\otimes V_2 & \xrightarrow{\mathbf{res}} & \mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big) \\ {\scriptstyle\Psi}\searrow & & \swarrow \\ & \widetilde{V_3} & \end{array} \tag{6}$$

Finding a test vector is then reduced to finding an element of $V_1\otimes V_2$ whose image by **res** is not zero (resp. not a multiple of $\eta\circ\det$), if V_3 is principal series (resp. special representation).

Following the notations of paragraph 2.3 put, for $i=1$ and $i=2$

$$m_i = \mathrm{cond}(\mu_i') \qquad \alpha_i^{-1} = \mu_i(\pi)|\pi|^{\frac{1}{2}} \quad \text{and} \qquad \beta_i^{-1} = \mu_i'(\pi)|\pi|^{-\frac{1}{2}}.$$

3.2.1 Proof of theorem 7 in the simple case.

To prove theorem 7, suppose that μ_1 or μ_2' is ramified. By our assumptions $x>y$, hence $I_x\cap(K\backslash I_{y+1})=\varnothing$. Therefore the supports of v_1^* and v_2^* are disjoint and

$$\mathbf{res}(v_1^*\otimes v_2^*)=0.$$

Using the diagram (6) we see that for any $v\in V_3$:

$$\ell(v_1^*\otimes v_2^*\otimes v) = \Psi(v_1^*\otimes v_2^*)(v) = 0.$$

In particular $\ell(v_1^*\otimes v_2^*\otimes\gamma^z\cdot v_3)=0$ which proves Theorem 7 in the simple case.

The rest of section 3.2 will be devoted to the proof of Theorems 5 and 6 in the simple case. Consequently, we will suppose that μ_1 and μ_2' are unramified, that is $m_1-n_1=m_2=0$.

3.2.2 Proof of Theorem 5 in the simple case.

Since V_1 and V_2 are unramified, by theorem 2 we may assume that V_3 is ramified. Then necessarily

$$\widetilde{V_3} = \eta\otimes\mathrm{St},$$

where St is the Steinberg representation and η is an unramified character. Hence $n_3=1$ and we will prove that $\gamma\cdot v_1\otimes v_2\otimes v_3$ is a test vector.

The function

$$\begin{cases} G & \longrightarrow & \mathbb{C} \\ g & \mapsto & \eta\Big(\det(g)\Big)^{-1}\mathbf{res}(\gamma\cdot v_1\otimes v_2)(g) \end{cases}$$

is not constant, since according to lemma 2.1

$$\eta\Big(\det(1)\Big)^{-1}(\gamma\cdot v_1\otimes v_2)(1)=v_1(\gamma)v_2(1)=\alpha_1$$

and

$$\eta\Big(\det\begin{pmatrix}0&1\\1&0\end{pmatrix}\Big)^{-1}(\gamma\cdot v_1\otimes v_2)\begin{pmatrix}0&1\\1&0\end{pmatrix}=\eta(-1)v_1\begin{pmatrix}1&0\\0&\pi^{-1}\end{pmatrix}=\beta_1,$$

and $\alpha_1\neq\beta_1$ because V_1 is a principal series.

Hence $\Psi(\gamma\cdot v_1\otimes v_2)\neq 0$. Moreover, since

$$\gamma K\gamma^{-1}\cap K=I$$

we deduce that

$$\Psi(\gamma\cdot v_1\otimes v_2)\in\widetilde{V_3}^{I,\omega_3{}^{-1}}.$$

Hence $\Psi(\gamma\cdot v_1\otimes v_2)$ cannot vanish on the line $V_3{}^{I,\omega_3}$, which is generated by v_3, and therefore $\gamma\cdot v_1\otimes v_2\otimes v_3$ is a test vector.

This completes the proof of Theorem 5 in the simple case.

3.2.3 Proof of Theorem 6 in the simple case, when $\widetilde{V_3}$ is a special representation.

Assume now that

$$\widetilde{V_3}=\eta\otimes\mathrm{St},$$

where St is the Steinberg representation and η is a character. Since

$$\eta=\mu_1\mu_2|\cdot|=\mu_1'\mu_2'|\cdot|^{-1}$$

and μ_1 and μ_2' are unramified, it follows that η is unramified if, and only if, both V_1 and V_2 are unramified. Since this case was taken care of in the previous paragraph, we can assume for the rest of this paragraph that η is ramified. Then

$$n_1=n_2=\mathrm{cond}(\eta)\geq 1\qquad\text{and}\qquad n_3=2n_1=n_1+n_2.$$

We will prove that $v_1\otimes\gamma^{n_1}\cdot v_2\otimes v_3$ is a test vector.

The function

$$\begin{cases}G&\longrightarrow&\mathbb{C}\\ g&\mapsto&\eta\Big(\det(g)\Big)^{-1}\mathbf{res}(v_1\otimes\gamma^{n_1}\cdot v_2)(g)\end{cases}$$

is not constant, since according to lemmas 2.3 and 2.4

$$\eta\Big(\det(1)\Big)^{-1}(v_1\otimes\gamma^{n_1}\cdot v_2)(1)=0$$

whereas

$$\eta\Big(\det\begin{pmatrix}1&0\\\pi^{n_1}&1\end{pmatrix}\Big)^{-1}(v_1\otimes\gamma^{n_1}\cdot v_2)\begin{pmatrix}1&0\\\pi^{n_1}&1\end{pmatrix}=\alpha_2^{n_1}\neq 0.$$

Hence $\Psi(v_1\otimes\gamma^{n_1}\cdot v_2)\neq 0$. Moreover, since

$$I_{n_1}\cap\gamma^{n_1}I_{n_2}\gamma^{-n_1}=I_{n_1+n_2}=I_{n_3}$$

we deduce that

$$\Psi(v_1\otimes\gamma^{n_1}\cdot v_2)\in\widetilde{V_3}^{I_{n_3},\omega_3{}^{-1}}.$$

Hence $\Psi(v_1\otimes\gamma^{n_1}\cdot v_2)$ cannot vanish on the line $V_3{}^{I_{n_3},\omega_3}$, which is generated by v_3, and therefore $v_1\otimes\gamma^{n_1}\cdot v_2\otimes v_3$ is a test vector.

3.2.4 Proof of Theorem 6 in the simple case, when $\widetilde{V_3}$ is a principal series.

Finally, we consider the case where $\widetilde{V_3}$ is a principal series representation. Then

$$\widetilde{V_3} = \mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big)$$

and

$$n_3 = \mathrm{cond}(\mu_1\mu_2) + \mathrm{cond}(\mu_1'\mu_2') = n_2 + n_1.$$

We will prove that $v_1 \otimes \gamma^{n_1} \cdot v_2 \otimes v_3$ is a test vector.

According to lemmas 2.1, 2.3 and 2.4 we have

$$(v_1 \otimes \gamma^{n_1} \cdot v_2)\begin{pmatrix} 1 & 0 \\ \pi^{n_1} & 1 \end{pmatrix} = \alpha_2^{n_1} \neq 0,$$

hence $\mathbf{res}(v_1 \otimes \gamma^{n_1} \cdot v_2) \neq 0$.

Therefore $\Psi(v_1 \otimes \gamma^{n_1} v_2) \neq 0$. Moreover, since

$$I_{n_1} \cap \gamma^{n_1} I_{n_2} \gamma^{-n_1} = I_{n_1+n_2} = I_{n_3}$$

we deduce that

$$\Psi(v_1 \otimes \gamma^{n_1} v_2) \in (\widetilde{V_3})^{I_{n_3}, {\omega_3}^{-1}}.$$

Hence $\Psi(v_1 \otimes \gamma^{n_1} v_2)$ cannot vanish on the line $V_3^{I_{n_3},\omega_3}$, which is generated by v_3. Thus $v_1 \otimes \gamma^{n_1} \cdot v_2 \otimes v_3$ is a test vector.

This completes the proof of Theorem 6 in the simple case.

3.3 The other case.

The situation is more complicated if $\mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi_1\chi_2\delta^{\frac{1}{2}}), \widetilde{V_3}) = 0$. By [P, Corollary 5.9] we have $\mathrm{Ext}_G^1(\mathrm{Ind}_B^G(\chi_1\chi_2\delta^{\frac{1}{2}}), \widetilde{V_3}) = 0$, hence the long exact sequence (5) yields the following isomorphism :

$$\mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) \simeq \mathrm{Hom}_G\Big(\mathrm{ind}_T^G(\chi_1\chi_2'), \widetilde{V_3}\Big).$$

Finally, by Frobenius reciprocity

$$\mathrm{Hom}_G\Big(\mathrm{ind}_T^G(\chi_1\chi_2'), \widetilde{V_3}\Big) \simeq \mathrm{Hom}_T\Big(\chi_1\chi_2', \widetilde{V_{3|T}}\Big).$$

By [W, Lemmes 8-9] the latter space is one dimensional, since the restriction of $\chi_1\chi_2'$ to the center equals ω_3^{-1} (recall that $\omega_1\omega_2\omega_3 = 1$). Thus, we have four canonically isomorphic lines with corresponding bases :

$$\begin{array}{rcc} 0 \neq \ell & \in & \mathrm{Hom}_G\Big(V_1 \otimes V_2 \otimes V_3, \mathbb{C}\Big) \\ & & \downarrow \wr \\ 0 \neq \Psi & \in & \mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) \\ & & \downarrow \wr \\ 0 \neq \Phi & \in & \mathrm{Hom}_G\Big(\mathrm{ind}_T^G(\chi_1\chi_2'), \widetilde{V_3}\Big) \\ & & \downarrow \wr \\ 0 \neq \varphi & \in & \mathrm{Hom}_T\Big(\chi_1\chi_2', \widetilde{V_{3|T}}\Big) \end{array} \tag{7}$$

Observe that φ can be seen as a linear form on V_3 satisfying :

$$\forall t \in T, \qquad \forall v \in V_3, \qquad \varphi(t \cdot v) = (\chi_1 \chi_2')(t)^{-1} \varphi(v). \tag{8}$$

Lemma 3.1. *$\varphi(v_3) \neq 0$ if, and only if, $\mu_1 \mu_2'$ is unramified.*

Proof : Suppose $\varphi(v_3) \neq 0$. Since $v_3 \in V_3$ is a new vector, for all $a, d \in \mathcal{O}^\times$ we have

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot v_3 = \omega_3(d) v_3 = (\mu_1 \mu_1' \mu_2 \mu_2')(d)^{-1} v_3.$$

Comparing it with (8) forces $\mu_1 \mu_2'$ to be unramified.

Conversely, assume that $\mu_1 \mu_2'$ is unramified. Take any $v \in V_3$ such that $\varphi(v) \neq 0$. By smoothness v is fixed by the principal congruence subgroup $\ker(K \to \mathrm{GL}_2(\mathcal{O}/\pi^s))$, for some $s \geq 0$. Then $\varphi(\gamma^s \cdot v) = (\mu_1 \mu_2')(\pi^s) \varphi(v) \neq 0$ and $\gamma^s \cdot v$ is fixed by the congruence subgroup

$$I_{2s}^1 := \left\{ k \in K \middle| k \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\pi^{2s}} \right\}.$$

By replacing $\gamma^s \cdot v$ by v and $2s$ by s, we may assume that $v \in V_3^{I_s^1}$ for some $s \geq 0$. Since I_s / I_s^1 is a finite abelian group, $V_3^{I_s^1}$ decomposes as a direct sum of spaces indexed by the characters of I_s / I_s^1. Then φ has to be non-zero on $V_3^{I_s, \omega_3}$ (defined in paragraph 2.2) since by (8), φ vanishes on all other summands of $V_3^{I_s^1}$.

By Casselman [C, Theorem 1] the space $V_3^{I_s, \omega_3}$ has dimension $n_3 - s + 1$ and has a basis

$$\left(\quad v_3 \quad , \quad \gamma \cdot v_3 \quad , \ldots , \quad \gamma^{n_3 - s} \cdot v_3 \quad \right)$$

(recall that n_3 denotes the conductor of V_3). Again by (8), $\varphi(\gamma^i v_3) \neq 0$ for some i is equivalent to $\varphi(v_3) \neq 0$. □

Notice that, when $\mu_1 \mu_2'$ and $\mu_1' \mu_2$ are both unramified, the claim follows from the first case in [G-P, Proposition 2.6] applied to the split torus T of G.

4 Going up Prasad's exact sequence.

In this section we take as a starting point lemma 3.1 and follow the isomorphisms (7).

4.1 From φ to Φ.

Let x, y and z be integers such that

$$x - n_3 \geq z \geq y \geq 0 \qquad \text{and} \qquad x - y \geq 1.$$

For the proof of Theorem 6 we will take

$$x = \max(n_1, n_3) \geq 1 \qquad \text{and} \qquad y = z = 0.$$

Given a quasi-character μ of $F^\times$ define :

$$\mathcal{O}^\mu = \begin{cases} \mathcal{O} & \text{, if } \mu \text{ is unramified,} \\ \mathcal{O}^\times & \text{, if } \mu \text{ is ramified.} \end{cases}$$

Put

$$I_f = \begin{pmatrix} 1 & \pi^{-y}\mathcal{O}^{\mu_2'} \\ \pi^x\mathcal{O}^{\mu_1} & 1 \end{pmatrix},$$

and consider the unique function $f \in \mathrm{ind}_T^G(\chi_1\chi_2')$ which is zero outside the open compact subset TI_f of $T\backslash G$ and such that for all $b_0 \in \pi^{-y}\mathcal{O}^{\mu_2'}$ and $c_0 \in \pi^x\mathcal{O}^{\mu_1}$ we have :

$$f\begin{pmatrix} 1 & b_0 \\ c_0 & 1 \end{pmatrix} = \begin{cases} \mu_1(\frac{\pi^x}{c_0})\mu_2'(b_0\pi^y) & \text{, if } \mu_1 \text{ and } \mu_2' \text{ are ramified ;} \\ \mu_2'(b_0\pi^y) & \text{, if } \mu_1 \text{ is unramified and } \mu_2' \text{ is ramified ;} \\ \mu_1(\frac{\pi^x}{c_0}) & \text{, if } \mu_1 \text{ is ramified and } \mu_2' \text{ is unramified ;} \\ 1 & \text{, if } \mu_1 \text{ and } \mu_2' \text{ are unramified.} \end{cases} \tag{9}$$

Since $x - n_3 \geq z \geq y \geq 0$ and $x - y \geq 1$ we have

$$I_f \subset \gamma^z I_{n_3}^1 \gamma^{-z}$$

and so every $k_0 \in I_f$ fixes $\gamma^z \cdot v_3$.

By definition, the function $g \mapsto f(g)\varphi(g\gamma^z \cdot v_3)$ on G factors through $T\backslash G$ and

$$\Big(\Phi(f)\Big)(\gamma^z \cdot v_3) = \int_{T\backslash G} f(g)\,\varphi(g\gamma^z \cdot v_3)dg = \varphi(\gamma^z \cdot v_3)\int_{I_f} f(k_0)dk_0.$$

If we write $k_0 = \begin{pmatrix} 1 & b_0 \\ c_0 & 1 \end{pmatrix} \in I_f$, then by separating the variables b_0 and c_0 we obtain

$$\int_{I_f} f(k_0)dk_0 = \begin{cases} |\pi|^{x-y} & \text{, if } \mu_1 \text{ and } \mu_2' \text{ are unramified,} \\ 0 & \text{, otherwise.} \end{cases}$$

From this and from lemma 3.1 we deduce :

Lemma 4.1. *$\Phi(f)(\gamma^z \cdot v_3) \neq 0$ if, and only if, μ_1 and μ_2' are both unramified.*

4.2 From Φ to Ψ.

Now, we are going to compute $F = \mathbf{ext}(f)$ as a function on $G \times G$. Recall that $F : G \times G \to \mathbb{C}$ is a function such that :

- for all $b_1, b_2 \in B$, $g_1, g_2 \in G$, $F(b_1g_1, b_2g_2) = \chi_1(b_1)\chi_2(b_2)\delta^{\frac{1}{2}}(b_1b_2)F(g_1, g_2)$,
- for all $g \in G$, $F(g, g) = 0$ and $F(g, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g) = f(g)$.

Since $G = BK$, F is uniquely determined by its restriction to $K \times K$. Following the notations of paragraph 2.3 put

$$\alpha_i^{-1} = \mu_i(\pi)|\pi|^{\frac{1}{2}} \quad \text{and} \quad \beta_i^{-1} = \mu_i'(\pi)|\pi|^{-\frac{1}{2}}.$$

Lemma 4.2. *Suppose that $x - n_3 \geq z \geq y \geq 0$ and $x - y \geq \max(n_1 - m_1, n_2 - m_2, 1)$. Then for all $k_1 = \begin{pmatrix} * & * \\ c_1 & d_2 \end{pmatrix}$ and $k_2 = \begin{pmatrix} * & * \\ c_2 & d_2 \end{pmatrix}$ in K we have $F(k_1, k_2) = 0$ unless*

$$d_1c_2 \neq 0, \qquad \frac{c_1}{d_1} \in \pi^x\mathcal{O}^{\mu_1} \qquad \text{and} \qquad \frac{d_2}{c_2} \in \pi^{-y}\mathcal{O}^{\mu_2'},$$

in which case, if we denote by s the valuation of c_2, we have

$$F(k_1,k_2)=\begin{cases}\mu_1\left(\frac{\det(k_1)}{\pi^{-x}c_1}\right)\mu_1'(d_1)\mu_2\left(\frac{-\det(k_2)}{\pi^{-s}c_2}\right)\mu_2'(d_2)\left(\frac{\alpha_2}{\beta_2}\right)^s & \text{, if } \mu_1 \text{ and } \mu_2' \text{ are ramified ;}\\ \mu_1'(d_1)\mu_2\left(\frac{-\det(k_2)}{\pi^{-s}c_2}\right)\mu_2'(d_2)\left(\frac{\alpha_2}{\beta_2}\right)^s & \text{, if } \mu_1 \text{ is unramified and } \mu_2' \text{ is ramified ;}\\ \mu_1\left(\frac{\det(k_1)}{\pi^{-x}c_1}\right)\mu_1'(d_1)\mu_2\left(\frac{-\det(k_2)}{\pi^{-s}c_2}\right)\left(\frac{\alpha_2}{\beta_2}\right)^s & \text{, if } \mu_1 \text{ is ramified and } \mu_2' \text{ is unramified ;}\\ \mu_1'(d_1)\mu_2\left(\frac{-\det(k_2)}{\pi^{-s}c_2}\right)\left(\frac{\alpha_2}{\beta_2}\right)^s & \text{, if } \mu_1 \text{ and } \mu_2' \text{ are unramified.}\end{cases}$$

Proof : By definition $F(k_1,k_2)=0$ unless there exist $k_0=\begin{pmatrix}1 & b_0\\ c_0 & 1\end{pmatrix}\in I_f$ such that

$$k_1k_0^{-1}\in B \qquad \text{and} \qquad k_2k_0^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\in B,$$

in which case

$$F(k_1,k_2)=\chi_1(k_1k_0^{-1})\chi_2\Big(k_2k_0^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Big)\delta^{\frac{1}{2}}\Big(k_1k_0^{-1}k_2k_0^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Big)f(k_0).$$

From $k_1k_0^{-1}\in B$, we deduce that $c_1=c_0d_1$. From $k_2k_0^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\in B$ we deduce that $d_2=b_0c_2$. Hence

$$d_1\in\mathcal{O}^\times, \qquad \frac{c_1}{d_1}\in\pi^x\mathcal{O}^{\mu_1}, \qquad c_2\neq 0 \qquad \text{and} \qquad \frac{d_2}{c_2}\in\pi^{-y}\mathcal{O}^{\mu_2'}.$$

Moreover

$$k_1k_0^{-1}=\begin{pmatrix}\frac{\det k_1}{d_1\det k_0} & *\\ 0 & d_1\end{pmatrix} \text{ and } k_2k_0^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}\frac{-\det k_2}{c_2\det k_0} & *\\ 0 & c_2\end{pmatrix}.$$

Since $x-y\geq n_1-m_1$, $x-y\geq n_2-m_2$ and $x-y\geq 1$ we have

$$\mu_1(\det k_0)=\mu_2(\det k_0)=1.$$

Hence

$$F(k_1,k_2)=\mu_1(\frac{\det k_1}{d_1})\mu_1'(d_1)\mu_2(\frac{-\det k_2}{c_2})\mu_2'(c_2)\left|\frac{1}{c_2}\right|f\begin{pmatrix}1 & \frac{d_2}{c_2}\\ \frac{c_1}{d_1} & 1\end{pmatrix}.$$

From here and (9) follows the desired formula for F.

Conversely, if k_1 and k_2 are such that $\frac{c_1}{d_1}\in\pi^x\mathcal{O}^{\mu_1}$ and $\frac{d_2}{c_2}\in\pi^{-y}\mathcal{O}^{\mu_2'}$ one can take

$$k_0=\begin{pmatrix}1 & d_2c_2^{-1}\\ c_1d_1^{-1} & 1\end{pmatrix}.$$

□

Remark 4.3. One can compute F without the assumption $x-y\geq\max(n_1-m_1,n_2-m_2,1)$. However, F needs not decompose as a product of functions of one variable as in the above lemma.

For example, if $x=n_3=0$ and $n_1=n_2$, then for all $k_1\in K$ and $k_2\in K$

$$F(k_1,k_2)=\begin{cases}\omega_1(\frac{c_1d_2-d_1c_2}{\det k_2}) & \text{, if } d_1\in\mathcal{O}^\times,\ c_2\in\mathcal{O}^\times \text{ and } c_1d_2\neq d_1c_2\\ 0 & \text{, otherwise.}\end{cases}$$

4.3 From Ψ to ℓ

Now, we want to express $F \in V_1 \otimes V_2$ in terms of the new vectors v_1 and v_2.

From now on we suppose that x, y and z are integers as in theorem 7. We may also suppose that $x \geq 1$, because otherwise V_1, V_2 and V_3 are all unramified and this case is covered in Theorem 2. Observe also that if $y = 0$, then μ_2' is unramified and therefore $\mathcal{O}^{\mu_2'} = \mathcal{O}$.

For $i = 1, 2$, since $k_i \in K$, both c_i and d_i are in $\mathcal{O}$, and one of them is in $\mathcal{O}^\times$. Hence

- $\frac{c_1}{d_1} \in \pi^x \mathcal{O}^\times$ if, and only if $k_1 \in I_x \backslash I_{x+1}$,
- $\frac{c_1}{d_1} \in \pi^x \mathcal{O}$ if, and only if $k_1 \in I_x$,
- $\frac{d_2}{c_2} \in \pi^{-y} \mathcal{O}^\times$ with $y \geq 1$ if, and only if $k_2 \in I_y \backslash I_{y+1}$,
- $\frac{d_2}{c_2} \in \pi^{-y} \mathcal{O}$ with $y \geq 0$ if, and only if $k_2 \in K \backslash I_{y+1}$.

Lemma 4.4. *With the notations of (1), F is a non-zero multiple of $v_1^* \otimes v_2^*$.*

Proof : Both F and $v_1^* \otimes v_2^*$ are elements in $\mathrm{Ind}_{B \times B}^{G \times G}\Big(\chi_1 \times \chi_2\Big)$, hence it is enough to compare their restrictions to $K \times K$. By the above discussion together with lemmas 4.2 and 2.5 the two restrictions are supported by

$$\begin{cases} (I_x \backslash I_{x+1}) \times (I_y \backslash I_{y+1}) & \text{, if } \mu_1 \text{ and } \mu_2' \text{ are ramified ;} \\ I_x \times (I_y \backslash I_{y+1}) & \text{, if } \mu_1 \text{ is unramified and } \mu_2' \text{ is ramified ;} \\ (I_x \backslash I_{x+1}) \times (K \backslash I_{y+1}) & \text{, if } \mu_1 \text{ is ramified and } \mu_2' \text{ is unramified ;} \\ I_x \times (K \backslash I_{y+1}) & \text{, if } \mu_1 \text{ and } \mu_2' \text{ are unramified.} \end{cases}$$

There are 16 different cases depending on whether each one among μ_1, μ_1', μ_2 and μ_2' is ramified or unramified. Since it is a straightforward verification from lemmas 2.1, 2.2, 2.3 and 2.4, in order to avoid repetitions or cumbersome notations, we will only give the final result :

$$\begin{aligned} F &= \lambda_1 \lambda_2 \mu_2(-1) \alpha_1^{m_1 - x} \alpha_2^{m_2} \beta_2^{-y} (v_1^* \otimes v_2^*) \text{ , where} \\ \lambda_i &= \begin{cases} \left(1 - \frac{\beta_i}{\alpha_i}\right)^{-1} & \text{, if } V_i \text{ is unramified,} \\ 1 & \text{, if } V_i \text{ is ramified.} \end{cases} \end{aligned} \tag{10}$$

In all cases F is a non-zero multiple of $v_1^* \otimes v_2^*$. □

Since by definition $\ell(F \otimes \bullet) = \Psi(F) = \Phi(f)$, the above lemma together with lemma 4.1 imply theorem 7.

4.4 Proof of Theorems 5 and 6.

We assume henceforth that μ_1 and μ_2' are *both* unramified ($n_1 - m_1 = m_2 = 0$). We put $y = z = 0$ and $x = \max(n_1, n_3) \geq 1$. Since $\omega_1 \omega_2 \omega_3 = 1$, $\max(n_1, n_3) = \max(n_1, n_2, n_3) \geq 1$.

Then lemma 4.1 yields :

$$\ell(F \otimes v_3) = \Psi(F)(v_3) = \Phi(f)(v_3) \neq 0. \tag{11}$$

From this and lemma 4.4 we deduce :

Lemma 4.5. *We have* $\ell(v_1^* \otimes v_2^* \otimes v_3) \neq 0$ *where*

$$v_1^* = \begin{cases} \gamma^{x-n_1} \cdot v_1 & \text{, if } \mu_1' \text{ is ramified,} \\ \gamma^x \cdot v_1 - \beta_1 \gamma^{x-1} \cdot v_1 & \text{, if } \mu_1' \text{ is unramified.} \end{cases}$$

$$v_2^* = \begin{cases} v_2 & \text{, if } \mu_2 \text{ is ramified.} \\ v_2 - \alpha_2^{-1} \gamma \cdot v_2 & \text{, if } \mu_2 \text{ is unramified.} \end{cases}$$

4.4.1 The case of two unramified representations.

Suppose that $n_1 = n_2 = 0$, so that $x = n_3$. Then lemma 4.5 yields :

$$\ell\Big((\gamma^{n_3} \cdot v_1 - \beta_1 \gamma^{n_3-1} \cdot v_1) \otimes (\gamma \cdot v_2 - \alpha_2 v_2) \otimes v_3\Big) \neq 0.$$

This expression can be simplified as follows. Consider for $m \geq 0$ the linear form :

$$\psi_m(\bullet) = \ell(\gamma^m \cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V_3}.$$

As observed in the introduction, ψ_m is invariant by $\gamma^m K \gamma^{-m} \cap K = I_m$, hence vanishes if $m < n_3 = \mathrm{cond}(\widetilde{V_3})$. Therefore, for $n_3 \geq 2$:

$$\begin{aligned} &\ell\Big((\gamma^{n_3} \cdot v_1 - \beta_1 \gamma^{n_3-1} \cdot v_1) \otimes (\gamma \cdot v_2 - \alpha_2 v_2) \otimes v_3\Big) \\ &\quad = -\alpha_2 \psi_{n_3}(v_3) + \beta_1 \alpha_2 \psi_{n_3-1}(v_3) + \psi_{n_3-1}(\gamma^{-1} \cdot v_3) - \beta_1 \psi_{n_3-2}(\gamma^{-1} \cdot v_3) \\ &\quad = -\alpha_2 \psi_{n_3}(v_3) \\ &\quad = -\alpha_2 \ell(\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3) \neq 0. \end{aligned}$$

If $n_3 = 1$, only the two terms in the middle vanish and we obtain

$$\alpha_2 \ell(\gamma \cdot v_1 \otimes v_2 \otimes v_3) + \beta_1 \ell(v_1 \otimes \gamma \cdot v_2 \otimes v_3) \neq 0.$$

Put $g = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. Then $g\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K$ and $\gamma^{-1} g = \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix} \in \pi K$. Hence :

$$\begin{aligned} \beta_1 \ell(v_1 \otimes \gamma \cdot v_2 \otimes v_3) &= \beta_1 \ell(\gamma\gamma^{-1} g \cdot v_1 \otimes g\gamma \cdot v_2 \otimes g \cdot v_3) \\ &= \beta_1 \omega_1(\pi) \ell\big(\gamma \cdot v_1 \otimes v_2 \otimes g \cdot v_3\big) \\ &= \alpha_1^{-1} \ell(\gamma \cdot v_1 \otimes v_2 \otimes g \cdot v_3). \end{aligned}$$

Therefore

$$\ell\Big(\gamma \cdot v_1 \otimes v_2 \otimes (g \cdot v_3 + \alpha_1 \alpha_2 v_3)\Big) \neq 0,$$

in particular

$$\Psi(\gamma \cdot v_1 \otimes v_2) \neq 0.$$

By the same argument as in paragraph 3.2.4 we conclude that

$$\ell(\gamma \cdot v_1 \otimes v_2 \otimes v_3) = \Psi(\gamma \cdot v_1 \otimes v_2)(v_3) \neq 0.$$

Hence, if $n_3 \geq 1$, $\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3$ is a test vector. This completes the proof of Theorem 5.

4.4.2 The case of two ramified principal series.

Suppose that V_1 and V_2 are both ramified ($m_1 > 0$, $n_1 - m_1 = 0$, $m_2 = 0$, $n_2 > 0$) and put $n = x - n_1 = \max(n_2 - n_1, n_3 - n_1)$. Then lemma 4.5 yields :

$$\ell(\gamma^n \cdot v_1 \otimes v_2 \otimes v_3) \neq 0,$$

hence $\gamma^n \cdot v_1 \otimes v_2 \otimes v_3$ is a test vector.

4.4.3 The case where V_1 is unramified and V_2 is ramified.

Suppose that $n_1 = 0$, but $n_2 > 0$. Then $x = n_3 \geq n_2$ and lemma 4.5 yields :

$$\ell\Big((\gamma^{n_3} \cdot v_1 - \beta_1 \gamma^{n_3 - 1} \cdot v_1) \otimes v_2 \otimes v_3\Big) \neq 0.$$

If $n_2 < n_3$, then

$$\gamma^{n_3-1} K \gamma^{1-n_3} \cap I_{n_2} \supset I_{n_3-1},$$

and therefore

$$\ell(\gamma^{n_3-1} \cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V_3}^{I_{n_3-1}, \omega_3^{-1}} = \{0\}.$$

Hence

$$\ell(\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3) \neq 0,$$

that is $\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3$ is a test vector.

If $n_2 = n_3$, the condition on the central character forces V_3 and ω_3 to have the same conductor. Hence V_3 is also a principal series. In this case we do not see *a priori* a reason for either $\ell(\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3)$ or $\ell(\gamma^{n_3-1} \cdot v_1 \otimes v_2 \otimes v_3)$ to vanish. But we can notice that the two linear forms

$$\ell(\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes \bullet) \quad \text{and} \quad \ell(\gamma^{n_3-1} \cdot v_1 \otimes v_2 \otimes \bullet)$$

belong both to the new line $\widetilde{V_3}^{I_n, \omega_3^{-1}}$ of $\widetilde{V_3}$, hence they are proportionals.

4.4.4 The case where V_1 is ramified and V_2 is unramified.

Suppose that $n_1 > 0$ and $n_2 = 0$. Then $x = n_3 \geq n_1$ and lemma 4.5 yields :

$$\ell\Big(\gamma^{n_3 - n_1} \cdot v_1 \otimes (\gamma \cdot v_2 - \alpha_2 v_2) \otimes v_3\Big) \neq 0.$$

If $n_1 < n_3$, then

$$\ell(\gamma^{n_3 - n_1 - 1} \cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V_3}^{I_{n_3-1}, \omega_3^{-1}} = \{0\}.$$

Then

$$\ell(\gamma^{n_3 - n_1} \cdot v_1 \otimes \gamma \cdot v_2 \otimes v_3) = \ell(\gamma^{n_3 - n_1 - 1} \cdot v_1 \otimes v_2 \otimes \gamma^{-1} \cdot v_3) = 0.$$

Hence

$$\ell(\gamma^{n_3 - n_1} \cdot v_1 \otimes v_2 \otimes v_3) \neq 0,$$

that is $\gamma^{n_3 - n_1} \cdot v_1 \otimes v_2 \otimes v_3$ is a test vector.

If $n_1 = n_3$, the condition on the central character forces V_3 to be also a principal series. In this case we do not see *a priori* a reason for either $\ell(v_1 \otimes v_2 \otimes v_3)$ or $\ell(v_1 \otimes \gamma \cdot v_2 \otimes v_3)$ to vanish. But we can once again notice that the two linear forms

$$\ell(v_1 \otimes v_2 \otimes \bullet) \quad \text{and} \quad \ell(v_1 \otimes \gamma \cdot v_2 \otimes \bullet)$$

belong to the line generated by a new vector in $\widetilde{V_3}$, hence are proportionals.

The proof of Theorem 6 is now complete.

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