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Moshe Marcus, Laurent Veron

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# MAXIMAL SOLUTIONS FOR $-\Delta u + u^q = 0$ IN OPEN AND FINELY OPEN SETS

MOSHE MARCUS AND LAURENT VERON

## 1. INTRODUCTION

In this paper we study solutions of the equation

$$(1.1) \quad -\Delta u + |u|^{q-1}u = 0,$$

in  $\Omega \setminus F$ ,  $\Omega$  a smooth domain in  $\mathbb{R}^N$ ,  $N \geq 3$  and  $F \subset \Omega$ ,  $F$  compact or, more generally, a bounded set, closed in the  $C_{2,q'}$  fine topology. Here  $q > 1$  and  $C_{2,q'}$  refers to the Bessel capacity with the specified indexes. If  $1 < q < q_c = N/(N-2)$  then the fine topology is equivalent to the Euclidean topology. Therefore, throughout the paper we shall assume that  $q \geq q_c$ , in which case the two topologies are different.

If  $D$  is an open set and  $\mu$  is a Radon measure in  $D$ , a function  $u \in L_{\text{loc}}^q(D)$  is a solution of

$$(1.2) \quad -\Delta u + |u|^{q-1}u = \mu \text{ in } D$$

if the equation is satisfied in the distribution sense. It is known [6] that (1.2) possesses a solution if and only if  $\mu$  vanishes on sets of  $C_{2,q'}$  capacity zero. When this is the case we say that  $\mu$  satisfies the (B-P) $_q$  condition (i.e., the Baras-Pierre condition). If  $D = \mathbb{R}^N$  and  $\mu$  is a Radon measure satisfying this condition then (1.2) possesses a *unique* solution.

Further, if  $D$  is open, it is known that  $C_{2,q'}(\mathbb{R}^N \setminus D) = 0$  if and only if the only solution of (1.1) in  $D$  is the trivial solution. In view of the Keller – Osserman estimates, the set of solutions of (1.1) in  $D$  (denoted by  $\mathcal{U}_D$ ) is uniformly bounded in compact subsets of  $D$  and every sequence of solutions possesses a subsequence which converges to a solution  $u$ . Finally the compactness together with the maximum principle imply that  $\max \mathcal{U}_D$  is a solution in  $D$ . The *maximal solution* in  $D$  is denoted by  $U_F$ ,  $F = \mathbb{R}^N \setminus D$ .

Now suppose that  $F = \cup_{n=1}^{\infty} K_n$  where  $\{K_n\}$  is an increasing sequence of compact sets such that

$$C_{2,q'}(F \setminus K_j) \rightarrow 0.$$

Then  $\{U_{K_n}\}$  is non-decreasing and we denote  $V_F := \lim U_{K_n}$ . In this case  $F$  may not be closed; in fact, it may be dense in  $D = F^c$ , so that in general we cannot apply the Keller – Osserman estimates. Therefore, on this basis, it is not even clear whether  $V_F$  is finite a.e. in  $D$ . It will be shown in the course of this paper that this is actually the case.

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Naturally, further questions come up: Is  $V_F$ , in some sense, a generalized solution of (1.1) in  $D$  and, if so, is it the maximal solution? Is it possible to characterize  $V_F$  in terms of its behavior at the boundary?

The main objective of this paper is the study of properties of the maximal solution of (1.1) in  $F^c$ , *first* in the case that  $F$  is compact; *secondly* in the case that  $F$  is merely  $C_{2,q'}$ -finely closed. In the second case we introduce a new notion of solution which we call a *strong solution* (see Definition 7.1) and show that  $V_F$  is indeed a solution in this sense and that it is the maximal solution. We also show that many of the properties of the set of classical solutions are shared by the class of strong solutions.

For  $F$  compact, the properties of  $U_F$  have been intensively investigated, especially in the last twenty years. A question that received special attention was the existence, uniqueness and estimates of solutions of the boundary value problem

$$(1.3) \quad \begin{aligned} -\Delta u + |u|^{q-1}u &= 0 \quad \text{in } D = F^c, \\ \lim_{D \ni x \rightarrow y} u(x) &= \infty \quad \forall y \in \partial D. \end{aligned}$$

The question of existence reduces to the question whether  $U_F$  blows up everywhere on the boundary.

A solution of (1.3) is called a *large solution* of (1.1) in  $D$ . If  $D$  is a smooth domain with compact boundary, it is known that a large solution exists and is unique, (see [22], [2], [3], [32]). These results were extended in various ways, weakening the assumptions on the domain, extending it to more general classes of equations and obtaining more information on the asymptotic behavior of solutions at the boundary, (see [4], [23], [21], [5] and references therein).

In the present paper we also consider two related notions:

(a) A solution  $u$  is an *almost large solution* of (1.1) in  $D$  if

$$(1.4) \quad \lim_{D \ni x \rightarrow y} u(x) = \infty \quad C_{2,q'} \text{ a.e. } y \in \partial F.$$

This notion is, in a sense, more natural, because (as we shall show)  $U_F$  is invariable with respect to  $C_{2,q'}$  equivalence of sets. (Two Borel sets  $E, F$  are  $C_{2,q'}$  equivalent if  $C_{2,q'}(F \Delta E) = 0$ .)

(b) A solution  $u$  of (1.1) is a  $\partial_q$ -*large solution* in  $D$  if

$$(1.5) \quad \lim_{D \ni x \rightarrow y} u(x) = \infty \quad C_{2,q'} \text{ a.e. } y \in \partial_q F,$$

where  $\partial_q F$  denotes the boundary of  $F$  in the  $C_{2,q'}$ -fine topology.

Here is a quick review of results pertaining to the case  $F$  compact.

In the *subcritical case*, i.e.  $1 < q < q_c := N/(N-2)$ , the properties of  $U_F$  are well understood. In this case  $C_{2,q'}(F) > 0$  for any non-empty set and it is classical that positive solutions may have isolated point singularities of two types: weak and strong. This easily implies that the maximal solution  $U_F$  is always a large solution in  $F^c$ . Sharp estimates of the large solution

where obtained in [28]. In addition it is proved in [33] that the large solution is unique if  $\partial F^c \subset \overline{\partial F^c}$ .

In the subcritical case, solutions with point singularities served as building blocks for solutions with general singularities. In the *supercritical case*, i.e.  $q \geq q_c$ , the situation is much more complicated, because there are no solutions with point singularities.

Sharp estimates for  $U_F$  were obtained by Dhersin and Le Gall [10] in the case  $q = 2$ ,  $N \geq 4$ . These estimates were expressed in terms of the Bessel capacity  $C_{2,2}$  and were used to provide a Wiener type criterion – to which we refer as (WDL; 2) – for the pointwise blow up of  $U_F$ , i.e., given  $y \in F$ ,

$$(1.6) \quad \lim_{F^c \ni x \rightarrow y} U_F(x) = \infty \iff \text{the (WDL; 2) criterion is satisfied at } y.$$

These results were obtained by probabilistic tools; hence the restriction to  $q = 2$ .

Labutin [18] extended the results of [10] in the case  $q > q_c$ . Specifically, he obtained sharp estimates for  $U_F$  similar to those in [10], with  $C_{2,2}$  replaced by  $C_{2,q'}$ . These estimates were used to obtain a Wiener criterion involving  $C_{2,q'}$  (we refer to it as (WDL;q)) relative to which the following was proved:

$$(1.7) \quad U_F \text{ is a large solution} \iff \text{(WDL;q) holds everywhere in } F.$$

Of course this result is weaker than (1.6). However a careful examination of Labutin's proof reveals that, in the case  $q > q_c$ , his argument actually proves (1.6). In the case  $q = q_c$  Labutin's estimate was not sharp and it did not yield (1.6) although it was sufficient in order to obtain (1.7).

Uniqueness was not discussed in the above papers. Necessary and sufficient conditions are not yet known. Sufficient conditions for uniqueness of large solutions, for arbitrary  $q > 1$ , can be found in [23], [27] and references therein. Uniqueness will also be one of the main subjects of the present work.

The first part of the present paper (Sections 2-4) is devoted to the study of the maximal solution  $U_F$  when  $F$  is compact and of the almost large solution in bounded open sets. Here is the list of main results obtained in this part of the paper:

**I.** Sharp capacity estimates of  $U_F$  in the full supercritical range  $q \geq q_c$ ,  $N \geq 3$ . As a result, we show that a variant of (1.6) holds in the entire supercritical range. Specifically, we show that, for  $y \in F$ ,

$$(1.8) \quad \lim_{F^c \ni x \rightarrow y} U_F(x) = \infty \iff W_F(y) = \infty,$$

where  $W_F : \mathbb{R}^N \rightarrow [0, \infty]$  is the *capacity potential* of  $F$ , (see (2.2) for its definition).

For  $q > q_c$  the condition  $W_F(y) = \infty$  is equivalent to the (WDL;q) criterion mentioned before. However our proof does not require separate treatment of the border case  $q = q_c$  and is simpler than the proof in [18] even for

$q > q_c$ .

**II.** For every compact set  $F$ ,  $U_F$  is an almost large solution in  $F^c$  and  $U_F$  is  $\sigma$ -moderate.

The statement ' $U_F$  is  $\sigma$ -moderate' means that there exists a monotone increasing sequence of bounded, positive measures concentrated in  $F$ ,  $\{\mu_n\}$ , satisfying the (B-P) $_q$  condition, such that  $u_{\mu_n} \uparrow U_F$ .

Finally we establish an existence and uniqueness result; for its statement we need some additional notation. For any set  $E \subset \mathbb{R}^N$

$$\tilde{E} = \text{closure of } E \text{ in the } C_{2,q'}\text{-fine topology, } \partial_q E := \tilde{E} \cap \tilde{E}^c.$$

**III.** Let  $\Omega = \cup \Omega_n$ , where  $\{\Omega_n\}$  is an increasing sequence of open sets, and put  $D_n = \mathbb{R}^N \setminus \overline{\Omega}_n$ . Assume that

$$(1.9) \quad C_{2,q'}(\Omega \setminus \Omega_n) \rightarrow 0 \quad \text{and} \quad C_{2,q'}(\partial \Omega_n \setminus \tilde{D}_n) \rightarrow 0.$$

Then the boundary value problem

$$(1.10) \quad -\Delta u + u^q = 0 \quad \text{in } \Omega, \quad \lim_{\Omega \ni x \rightarrow y} u(x) = \infty \quad \text{for } C_{2,q'} \text{ a.e. } y \in \partial_q \Omega$$

possesses exactly one solution.

In other words, an open set  $\Omega$  as above, possesses exactly one  $\partial_q$ -large solution. If  $\partial \Omega$  is compact then, this solution is an almost large solution. Indeed, by **II**, the maximal solution  $U_{\partial \Omega}$  is an almost large solution in  $\Omega$ . Since  $\partial_q \Omega \subset \partial \Omega$ , this implies that  $U_{\partial \Omega}$  is a  $\partial_q$ -large solution. By **III**,  $U_{\partial \Omega}$  is the unique such solution in  $\Omega$ .

In the second part of the paper (Sections 5-7) we extend our investigation to the case where  $F$  is  $C_{2,q'}$  finely closed. We introduce the notion of strong solution in  $D = \mathbb{R}^N \setminus F$ , which is now merely  $C_{2,q'}$ -finely open, and prove that  $V_F$  is a strong solution. By definition a strong solution belongs to a certain type of local Lebesgue space described in Section 6 below. Further we derive integral a-priori estimates which serve to replace the Keller-Osserman estimate in this case. Using them we prove removability and compactness results. In addition we show that the capacity estimates **I** and the Wiener criterion for pointwise blowup, namely (1.8), persist for  $V_F$ . We also establish the following version of **II**:

**II'**. For every  $C_{2,q'}$ -finely closed set  $F$ ,  $V_F$  is the maximal strong solution in  $F^c$ .  $V_F$  is a  $\partial_q$ -large solution and it is  $\sigma$ -moderate.

Finally, we have the following existence and uniqueness result:

**III'**. Let  $\Omega$  be a  $C_{2,q'}$ -finely open set. Let  $\{G_n\}$  be a sequence of open sets such that

$$(1.11) \quad C_{2,q'}(G_n \Delta \Omega) \rightarrow 0, \quad C_{2,q'}(\partial G_n \setminus \partial_q \tilde{G}_n) \rightarrow 0.$$

Then (1.10) possesses exactly one solution.

Note that here we do not assume that  $G_n$  is contained in  $\Omega$  or contains  $\Omega$ . If  $\Omega \subset G_n$  for every  $n \in \mathbb{N}$  then (1.11) implies (1.9).

This seems to be the first study of the subject in the setting of the  $C_{2,q'}$  fine topology, introducing a notion of solution in sets where the classical distribution derivative is not applicable. However the concept of 'finely harmonic functions' has been studied for a long time (see e.g. [16]).

The framework presented here is natural for the study of (1.1) and (1.2) because problems in a  $C_{2,q'}$  finely open set appear as limits of problems in open domains.

Partial list of notations

- $[a < f < b]$  means  $\{x : a < f(x) < b\}$ .
- $A\Delta B = (A \cup B) \setminus (A \cap B)$ .
- If  $f, g$  are non-negative functions with domain  $D$  then  $f \sim g$  means that there exists a constant  $C$  such that  $C^{-1}f \leq g \leq Cf$ .
- $A \stackrel{q}{\sim} B$  means  $C_{2,q'}(A\Delta B) = 0$ ,  $A \stackrel{q}{\subset} B$  means  $C_{2,q'}(A \setminus B) = 0$ .
- $\tilde{A}$  means 'the closure of  $A$  in the  $C_{2,q'}$  fine topology'.
- $\partial_q A$  means 'the boundary of  $A$  in the  $C_{2,q'}$  fine topology'.
- $\text{int}_q A$  means 'the interior of  $A$  in the  $C_{2,q'}$  fine topology'.
- $A \Subset B$  means ' $A$  bounded and  $\bar{A} \subset B$ '.
- $B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$ .
- $\chi_A$  denotes the characteristic function of the set  $A$ .
- (B-P) $_q$  condition: A measure  $\mu$  satisfies this condition if  $|\mu|(E) = 0$  for every Borel set  $E$  such that  $C_{2,q'}(E) = 0$ .
- $u_\mu$  denotes the solution of (1.2) in  $\mathbb{R}^N$  when  $\mu$  is a Radon measure satisfying the (B-P) $_q$  condition.

2. UPPER ESTIMATE OF THE MAXIMAL SOLUTION.

In this section  $F$  denotes a non-empty compact set in  $\mathbb{R}^N$  and the maximal solution of (1.1) in  $\mathbb{R}^N \setminus F$  is denoted by  $U_F$ . Further, for  $x \in \mathbb{R}^N$ , we denote

$$(2.1) \quad \begin{aligned} T_m(x) &= \{y \in \mathbb{R}^N : 2^{-(m+1)} \leq |y - x| \leq 2^{-m}\}, \\ F_m(x) &= F \cap T_m(x), \quad F_m^*(x) = F \cap \bar{B}_{2^{-m}}(x), \end{aligned}$$

$$(2.2) \quad \begin{aligned} W_F(x) &= \sum_{-\infty}^{\infty} 2^{\frac{2m}{q-1}} C_{2,q'}(2^m F_m(x)), \\ W_F^*(x) &= \sum_{-\infty}^{\infty} 2^{\frac{2m}{q-1}} C_{2,q'}(2^m F_m^*(x)). \end{aligned}$$

We call  $W_F$  the  $C_{2,q'}$ -capacitary potential of  $F$ . It is known that the two functions in (2.2) are equivalent, i.e., there exists a constant  $C$  depending only on  $q, N$  such that

$$(2.3) \quad W_F(x) \leq W_F^*(x) \leq CW_F(x)$$

see e.g. [29].

If  $K$  is a compact subset of a domain  $\Omega$  put,

$$(2.4) \quad X_K(\Omega) := \{\eta \in C_c^2(\Omega) : 0 \leq \eta \leq 1, \eta = 1 \text{ on } N_\eta^K\},$$

where  $N_\eta^K$  denotes an open neighborhood of  $K$  depending on  $\eta$ .

The following theorem is due to Labutin [18]:

**Theorem 2.1.** *Let  $q \geq q_c$ . There exists a constant  $C$  depending only on  $q, N$  such that, for every compact set  $F$ ,*

$$(2.5) \quad U_F(x) \leq CW_F(x) \quad \forall x \in D.$$

For the convenience of the reader we provide a concise proof; components of this proof will also be used later on in the paper. The main ingredient in the proof is contained in the lemma stated below.

**Lemma 2.2.** *Let  $R > 1$  and denote by  $\varphi_R$  the solution of*

$$(2.6) \quad -\Delta\varphi = \chi_{B_R(0)} \quad \text{in } \mathbb{R}^N, \quad \lim_{|x| \rightarrow \infty} \varphi(x) = 0.$$

Given  $\eta \in W^{2,q'}(\mathbb{R}^N)$ ,  $0 \leq \eta \leq 1$ , put

$$\zeta_\eta = \varphi_R(1 - \eta)^{2q'}.$$

There exists a constant  $\bar{c}(N, q, R)$  such that, for every compact set  $K \subset B_1(0)$ ,

$$(2.7) \quad \int_{\mathbb{R}^N \setminus K} U_K^q \zeta_\eta dx \leq \bar{c} \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}^{q'} \quad \forall \eta \in X_K(\mathbb{R}^N),$$

$$(2.8) \quad \int_{B_R(0) \setminus K} U_K(1 - \eta)^{2q'} dx \leq \bar{c} \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}^{q'} \quad \forall \eta \in X_K(\mathbb{R}^N).$$

*Proof.* For  $|x| \geq R + 2$ ,

$$(2.9) \quad 0 < \varphi_R(x) + U_K(x) \leq c|x|^{2-N}, \quad |\nabla\varphi_R(x)| + |\nabla U_K(x)| \leq c|x|^{1-N}$$

where  $c = c(N, q, R)$ . For every  $R' > R$  and  $\eta \in W^{2,q'}(\mathbb{R}^N)$ ,

$$(2.10) \quad \int_{B_{R'}(0) \setminus K} (-U_K \Delta \zeta_\eta + U_K^q \zeta_\eta) dx = -\frac{1}{R'} \int_{\partial B_{R'}} (U_K \nabla \zeta_\eta - \zeta_\eta \nabla U_K) \cdot x dS.$$

By (2.9), the right hand side of (2.10) tends to zero as  $R' \rightarrow \infty$  and we obtain,

$$(2.11) \quad \int_D (-U_K \Delta \zeta_\eta + U_K^q \zeta_\eta) dx = 0,$$

where  $D := \mathbb{R}^N \setminus K$ . Further,

$$\Delta \zeta_\eta = \varphi_R \Delta(1 - \eta)^{2q'} - (1 - \eta)^{2q'} \chi_{B_2} + 2\nabla\varphi_R \cdot \nabla(1 - \eta)^{2q'}$$

so that,

$$(2.12) \quad \int_D U_K^q \zeta_\eta dx + \int_{B_R(0) \setminus K} U_K (1 - \eta)^{2q'} dx = \int_D U_K (\varphi_R \Delta((1 - \eta)^{2q'}) + 2 \nabla \varphi_R \cdot \nabla((1 - \eta)^{2q'})) dx.$$

Now,

$$\Delta((1 - \eta)^{2q'}) = -2q'(1 - \eta)^{2q'-1} \Delta \eta + 2q'(2q' - 1)(1 - \eta)^{2q'-2} |\nabla \eta|^2,$$

so that

$$(2.13) \quad \int_D U_K \varphi_R \Delta((1 - \eta)^{2q'}) dx \leq c(I_1 + I_2),$$

where

$$I_1 := \int_D U_K \varphi_R (1 - \eta)^{2q'-1} |\Delta \eta| dx, \quad I_2 := \int_D U_K \varphi_R (1 - \eta)^{2q'-2} |\nabla \eta|^2 dx.$$

The estimate of  $I_1$  is standard.

$$(2.14) \quad \begin{aligned} I_1 &\leq \left( \int_D U_K^q \zeta_\eta dx \right)^{1/q} \left( \int_D \varphi_R (1 - \eta) |\Delta \eta|^{q'} dx \right)^{1/q'} \\ &\leq c \left( \int_D U_K^q \zeta_\eta dx \right)^{1/q} \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}. \end{aligned}$$

To estimate  $I_2$  we consider  $\eta \in X_K(B_R(0))$  and use the interpolation inequality

$$(2.15) \quad \|\nabla \eta\|_{L^{q'}(D)} \leq c(q, N, R) \|\eta\|_{L^\infty(D)} \|D^2 \eta\|_{L^{q'}(D)}.$$

We obtain,

$$(2.16) \quad \begin{aligned} I_2 &\leq \left( \int_D U_K^q \zeta_\eta dx \right)^{1/q} \left( \int_D \varphi_R |\nabla \eta|^{2q'} dx \right)^{1/q'} \\ &\leq c \left( \int_D U_K^q \zeta_\eta dx \right)^{1/q} \|\nabla \eta\|_{L^{q'}(D)}^2 \\ &\leq c \left( \int_D U_K^q \zeta_\eta dx \right)^{1/q} \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}. \end{aligned}$$

for  $\eta \in X_K(B_R(0))$ . Next

$$(2.17) \quad \begin{aligned} \int_D U_K \nabla \varphi_R \cdot \nabla((1 - \eta)^{2q'}) dx &\leq 2q' \int_D U_K |\nabla \varphi_R| |\nabla \eta| (1 - \eta)^{2q'-1} dx \\ &\leq c \left( \int_D U_K^q \zeta_\eta dx \right)^{1/q} \left( \int_D \varphi_R^{-\frac{q'}{q}} (|\nabla \varphi_R| |\nabla \eta|)^{q'} dx \right)^{1/q'}. \end{aligned}$$

In view of the fact that, for  $|x| \geq R + 2$ ,  $\varphi_R(x) \geq c|x|^{2-N}$ , (2.9) implies

$$\varphi_R^{-\frac{q'}{q}} |\nabla \varphi_R|^{q'} \leq c(N, q, R).$$

Hence

$$(2.18) \quad \int_D U_K \nabla \varphi_R \cdot \nabla ((1-\eta)^{2q'}) dx \leq c \left( \int_D U_K^q \zeta \eta dx \right)^{1/q} \|\eta\|_{W^{1,q'}(\mathbb{R}^N)}$$

Combining (2.12)–(2.18) we obtain (2.7) and (2.8) for  $\eta \in X_K(B_R(0))$ .

Pick  $\omega \in C_c^\infty(B_R(0))$  such that  $0 \leq \omega \leq 1$  and  $\omega = 1$  in  $B_1(0)$ . Given  $\eta \in X_K(\mathbb{R}^N)$ , (2.7) and (2.8) are valid if  $\eta$  is replaced by  $\omega\eta$ . However  $(1-\eta) \leq (1-\omega\eta)$  and

$$\|\omega\eta\|_{W^{2,q'}(\mathbb{R}^N)} \leq c(N, q, \omega) \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}.$$

Therefore (2.7) and (2.8) are valid for every  $\eta \in X_K(\mathbb{R}^N)$ .  $\square$

**Corollary 2.3.** *Assume that  $R > 3/2$ . There exists a constant  $c_1 = c_1(N, q, R)$  such that, for every compact set  $K \subset B_1(0)$*

$$(2.19) \quad \int_{[3/2 < |x|]} U_K^q \varphi_R dx + \int_{[3/2 < |x| < R]} U_K dx \leq c_1 C_{2,q'}(K)$$

and

$$(2.20) \quad \sup_{[3/2 < |x| < R]} U_K \leq c_1 C_{2,q'}(K).$$

*Proof.* Recall that

$$(2.21) \quad C_{2,q'}(K) = \inf \{ \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}^{q'} : \eta \in X_K(\mathbb{R}^N) \}.$$

Let  $\omega \in C_c^\infty(B_{3/2}(0))$  be a function such that  $0 \leq \omega \leq 1$  and  $\omega = 1$  on  $B_1(0)$ . For every compact set  $K \subset B_1(0)$  put

$$(2.22) \quad C_{2,q'}^\omega(K) = \inf \{ \|\omega\eta\|_{W^{2,q'}(\mathbb{R}^N)}^{q'} : \eta \in X_K(\mathbb{R}^N) \}.$$

Clearly  $C_{2,q'}(K) \leq C_{2,q'}^\omega(K)$  and since

$$\|\omega\eta\|_{W^{2,q'}(\mathbb{R}^N)}^{q'} \leq c(N, q, \omega) \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}^{q'}$$

we have

$$(2.23) \quad C_{2,q'}(K) \leq C_{2,q'}^\omega(K) \leq c(N, q, \omega) C_{2,q'}(K).$$

Let  $\{\eta_n\}$  be a sequence in  $X_K(\mathbb{R}^N)$  such that

$$\|\omega\eta_n\|_{W^{2,q'}(\mathbb{R}^N)}^{q'} \rightarrow C_{2,q'}^\omega(K).$$

For  $K \subset B_1(0)$  (2.7) implies that,

$$(2.24) \quad \int_{\mathbb{R}^N \setminus B_{3/2}} U_K^q \varphi_R dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus K} U_K^q \varphi_R (1 - \omega\eta_n)^{2q'} dx \\ \leq c(N, q, \omega) C_{2,q'}(K).$$

This proves (2.19). Inequality (2.20) (with the supremum over a slightly smaller annulus, say,  $[3/2 + \epsilon < |x| < R - \epsilon]$  with  $\epsilon > 0$  such that  $R > 3/2 + 2\epsilon$ ) follows from (2.19) and Harnack's inequality applied as in [28].  $\square$

*Proof of Theorem 2.1.* Inequality (2.20) implies,

$$(2.25) \quad U_F(x) \leq c(N, q) \rho_F(x)^{-2/(q-1)} C_{2, q'}(F/\rho_F(x))$$

for every every compact set  $F \subset \mathbb{R}^N$  and every  $x \in \mathbb{R}^N \setminus F$  such that  $\rho_F(x) \geq (3/2)\text{diam } F$ . Recall that  $\rho_F(x) := \text{dist}(x, F)$ .

The implication relies on the similarity transformation associated with (1.1). For any  $a > 0$ , we have

$$(2.26) \quad U_F(x) = a^{-2/(q-1)} U_{F/a}(x/a) \quad \forall x \in \mathbb{R}^N \setminus F.$$

Assume, as we may, that  $F \subset B_R(0)$ ,  $R = \text{diam } F$ . Fix a point  $\bar{x} \in \mathbb{R}^N \setminus F$  such that  $a := \rho_F(\bar{x}) \geq R$ . Applying (2.20) to the set  $K = 3F/2a$ , we obtain

$$\begin{aligned} U_F(\bar{x}) &= (2a/3)^{-2/(q-1)} U_K(3\bar{x}/2a) \\ &\leq c(N, q) a^{-2/(q-1)} C_{2, q'}(K) \leq c'(N, q) a^{-2/(q-1)} C_{2, q'}(F/a). \end{aligned}$$

Next we show that (2.25) is equivalent to (2.5). Let  $x \in D$  and put

$$(2.27) \quad M(x) := \min\{m \in \mathbb{N} : 2^{-m} < \rho_F(x)\}.$$

Then  $F_k(x) = \emptyset$  for all  $k \geq M(x)$  and consequently

$$W_F(x) = \sum_{k=-\infty}^{M(x)} 2^{\frac{2k}{q-1}} C_{2, q'}(2^k F_k(x)) \leq C 2^{\frac{2M(x)}{q-1}} \sup_{k \leq M(x)} C_{2, q'}(2^k F_k(x)).$$

However it is known that there exists a constant  $A$  depending only on  $q, N$  such that

$$(2.28) \quad C_{2, q'}(aE) \leq A a^{N - \frac{2}{q-1}} C_{2, q'}(E) \quad \forall a \in (0, 1),$$

(see e.g. [29]). In addition, for every  $\ell > 1$  there exists a constant  $A$ , depending on  $q, N, \ell$ , such that

$$(2.29) \quad C_{2, q'}(aE) \leq A a^{N - \frac{2}{q-1}} C_{2, q'}(E) \quad \forall a \in (1, \ell).$$

Inequality (2.28) implies that

$$\begin{aligned} W_F(x) &\leq C_1 2^{\frac{2M(x)}{q-1}} C_{2, q'}(2^{M(x)} F) \\ &\leq C_2 \rho_F(x)^{\frac{-2}{q-1}} C_{2, q'}(2F/\rho_F(x)) \leq C_3 \rho_F(x)^{\frac{-2}{q-1}} C_{2, q'}(F/\rho_F(x)), \end{aligned}$$

where  $C_i$  are constants depending only on  $q, N$ . Thus (2.5) implies (2.25).

To prove the implication in the opposite direction we use the following facts:

For every compact set  $F$  there exists a sequence of bounded domains  $\{D_n\}$  such that

$$(2.30) \quad \text{(i) } \cup D_n = D := F^c, \text{ (ii) } \overline{D}_n \subset D_{n+1}, \text{ (iii) } \partial D_n \text{ is Lipschitz.}$$

Such a sequence is called a *Lipschitz exhaustion of  $D$* .

If  $u_n$  denotes the maximal solution of (1.1) in  $D_n$  then  $u_n$  is the unique large solution of (1.1) in  $D_n$  (see [27]),  $u_n > u_{n+1}$  in  $D_n$  and  $U_F = \lim u_n$ .

Let  $E_i$ ,  $i = 1, \dots, k$  be compact sets and  $E := \cup_1^k E_i$ . One can choose a Lipschitz exhaustion  $\{D_{i,n}\}_{n=1}^\infty$  of  $D_i := E_i^c$ ,  $i = 1, \dots, k$ , such that the sequence  $\{D_n\}$ ,  $D_n = \cap_{i=1}^k D_{i,n}$ , is a Lipschitz exhaustion of  $D$ . Let  $u_{i,n}$  be the large solution in  $D_{i,n}$ . Then  $v_n = \max(u_{1,n}, \dots, u_{k,n})$  is a subsolution while  $w_n = \sum_{i=1}^k u_{i,n}$  is a supersolution of (1.1) in  $D_n$ . Hence  $u_n$ , the unique large solution of (1.1) in  $D_n$ , satisfies  $v_n \leq u_n \leq w_n$ . Consequently

$$(2.31) \quad \max(U_{E_1}, \dots, U_{E_k}) \leq U_E \leq \sum_{i=1}^k U_{E_i}.$$

Returning to the notation of Theorem 2.1, fix  $\bar{x} \in D$  and put

$$i(\bar{x}) = \max\{i \in \mathbb{Z} : F \subset \overline{B}_{2^{-i}}(\bar{x})\}.$$

Then  $F = \cup_{i=i(\bar{x})}^{M(\bar{x})} F_m(\bar{x})$  and, by (2.31) and (2.5),

$$U_F \leq \sum_{m=i(\bar{x})}^{M(\bar{x})} U_{F_m(\bar{x})} \leq C \sum_{m=i(\bar{x})}^{M(\bar{x})} 2^{\frac{2m}{q-1}} C_{2,q'} (2^m F_m(\bar{x})).$$

In particular,  $U_F(\bar{x}) \leq CW_F(\bar{x})$ . Thus (2.25) implies (2.5).

### 3. LOWER ESTIMATE OF THE MAXIMAL SOLUTION

We need the following well-known result:

**Proposition 3.1.** *Let  $\mu$  be a positive measure in  $W_{\text{loc}}^{-2,q}(\mathbb{R}^N)$  and let  $\Omega$  be a smooth domain with compact boundary. Then there exists a unique solution of each of the problems*

$$(3.1) \quad -\Delta u + u^q = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

and

$$(3.2) \quad -\Delta u + u^q = \mu \quad \text{in } \Omega, \quad u = \infty \quad \text{on } \partial\Omega.$$

If  $\Omega$  is the whole space then there exists a unique solution  $u_\mu$  of the equation

$$(3.3) \quad -\Delta u + u^q = \mu \quad \text{in } \mathbb{R}^N.$$

In each case the solution increases monotonically with  $\mu$ . Finally

$$u_\mu = \lim_{R \rightarrow \infty} u_{\mu,0}^R = \lim_{R \rightarrow \infty} u_{\mu,\infty}^R$$

where  $u_{\mu,0}^R$  and  $u_{\mu,\infty}^R$  are the solutions of (3.1) and (3.2) respectively, when  $\Omega = B_R(0)$ .

When  $\mu \in L^1_{loc}(\mathbb{R}^N)$  the result is due to Brezis [8] and Brezis-Strauss [9]. In the case of a smooth bounded domain  $\Omega$ , with  $\mu \in W^{-2,q}(\Omega)$ , the result is due to Baras and Pierre [6]. The final observation is easily verified.

If  $D$  is a smooth bounded domain and  $\mu$  is a positive measure in  $W^{-2,q}(D)$  we denote by  $u_{\mu,D}$  the solution of (1.2) in  $D$  vanishing on the boundary  $\partial D$ . If  $F$  is a compact subset of  $\mathbb{R}^N$ , we define the maximal  $\sigma$ -moderate solution  $\underline{U}_F$  of (1.1) in  $F^c := \mathbb{R}^N \setminus F$  by

$$(3.4) \quad V_F := \sup\{u_\mu : \mu \in \mathfrak{M}_+(\mathbb{R}^N) \cap W^{-2,q}(\mathbb{R}^N), \mu(F^c) = 0\}.$$

Obviously,

$$(3.5) \quad \underline{U}_F \leq U_F.$$

We derive a lower estimate for  $\underline{U}_F$ , equivalent to the upper estimate for  $U_F$  obtained in the previous section. More precisely:

**Theorem 3.2.** *Assume that  $F$  is a compact subset of  $B_a(0)$  and let  $D$  be a bounded smooth domain such that  $B_{6a}(0) \subset D$ . Then, for every  $x \in B_{2a}(0) \setminus F$ , there exists a positive measure  $\mu^x \in W^{-2,q}(\mathbb{R}^N)$ , supported in  $F$ , such that*

$$(3.6) \quad cW_F(x) \leq u_{\mu^x,D}(x) \leq V_F(x),$$

where  $c$  is a positive constant depending only on  $N, q$ . In particular,

$$(3.7) \quad c(N, q)W_F(x) \leq V_F(x) \quad \forall x \in \mathbb{R}^N \setminus F.$$

*Proof.* Let  $\lambda$  be a bounded Borel measure supported in  $D$ . We denote by  $\mathbb{G}_D[\lambda]$  the Green potential of the measure in  $D$ :

$$(3.8) \quad \mathbb{G}_D[\lambda](\cdot) := \int_D g_D(\cdot, \xi) d\lambda(\xi),$$

where  $g_D$  denotes Green's function in  $D$ .

If  $\mu$  is a positive measure in  $W^{-2,q}(D)$  then,

$$u_{\mu,D} \leq \mathbb{G}_D[\mu]$$

and consequently

$$(3.9) \quad u_{\mu,D} = \mathbb{G}_D[\mu] - \mathbb{G}_D[u_{\mu,D}^q] \geq \mathbb{G}_D[\mu] - \mathbb{G}_D[(\mathbb{G}_D[\mu])^q].$$

Given  $x_0 \in B_{2a} \setminus F$  we construct a measure  $\mu^{x_0} \in W^{-2,q}(\mathbb{R}^N)$ , concentrated on  $F$  such that (3.6) holds. By shifting the origin to  $x_0$  we may assume that  $x_0 = 0$ . We observe that (3.6) is invariant with respect to dilation. Therefore we may assume that  $a = 1/2$ . Following the shift and the dilation we have

$$(3.10) \quad F \subset B_1(0), \quad B_2(0) \subset D, \quad 0 \in F^c,$$

and we have to prove (3.6), with an appropriate measure  $\mu^0$ , at  $x = 0$ . The right inequality in (3.6) is trivial. Therefore we have to prove only that, for some non-negative measure  $\mu^0 \in W^{-2,q}(\mathbb{R}^N)$  supported in  $F$ ,

$$(3.11) \quad c(N, q)W_F(0) \leq u_{\mu^0,D}(0).$$

In view of (3.10),

$$u_{\mu^0, B_2(0)} \leq u_{\mu^0, D}.$$

Therefore it is enough to prove (3.11) for  $D = B_2(0)$  which we assume in the rest of the proof.

In what follows we shall freely use the notation introduced in the previous section and write simply  $F_n, T_n$  instead of  $F_n(0), T_n(0)$  etc.. Observe that in the present case  $F_n = \emptyset$  for  $n \leq -1$  and  $F_n^* = F$  for  $n \leq 0$ . For every non-negative integer  $n$ , let  $\nu_n$  denote the capacitary measure of  $2^n F_n$ . Thus,  $\nu_n$  is a positive measure in  $W^{-2,q}(\mathbb{R}^N)$  supported in  $2^n F_n$  which satisfies

$$(3.12) \quad \nu_n(2^n F_n) = C_{2,q'}(2^n F_n) = \|\nu_n\|_{W^{-2,q}}^q.$$

Let  $\mu_n, \mu$  be the Borel measures in  $\mathbb{R}^N$  given by

$$(3.13) \quad \mu_n(A) = 2^{-n(N-2q')} \nu_n(2^n A) \quad n = 0, 1, 2, \dots \quad \mu = \sum_0^\infty \mu_n.$$

Thus

$$(3.14) \quad \text{supp } \mu_n \subset F_n, \quad \text{supp } \mu \subset F,$$

$$(3.15) \quad \mu_n(F_n) = 2^{-n(N-2q')} C_{2,q'}(2^n F_n), \quad \mu \in W^{-2,q}(\mathbb{R}^N).$$

Observe also that, for  $x, \xi \in B_1(0)$ ,

$$(3.16) \quad g_D(x, \xi) \approx |x - \xi|^{2-N}.$$

The notation  $f \approx h$  means that there exists a positive constant  $c$  depending only on  $N, q$  such that  $c^{-1}h \leq f \leq ch$ .

The remaining part of the proof consists of a series of estimates of the terms on the right hand side of (3.9) for  $\mu$  as above.

*Lower estimate of  $\mathbb{G}_D[\mu]$ .* Using (3.15) and (3.16) we obtain,

$$c_N 2^{-(n+1)(2-N)} \leq g(0, \xi) \quad \forall \xi \in B_1(0),$$

$$(3.17) \quad \begin{aligned} \mathbb{G}_D[\mu](0) &= \sum_{n \geq 0} \int_{F_n} g(0, \xi) d\mu_n(\xi) \geq c \sum_{n \geq 0} \int_{F_n} 2^{n(N-2)} d\mu_n(\xi) \\ &= \sum_{n \geq 0} c 2^{-2n/(q-1)} C_{2,q'}(2^n F_n) = c W_F(0). \end{aligned}$$

*Upper estimate of  $\mathbb{G}_D[(\mathbb{G}_D[\mu])^q](0)$ .* We prove that

$$(3.18) \quad \begin{aligned} \mathbb{G}_D[(\mathbb{G}_D[\mu])^q](0) &= \int_D g_D(0, \xi) \mathbb{G}_D[\mu]^q(\xi) d\xi \\ &= \sum_{-1}^\infty \int_{T_k} g_D(0, \xi) \left( \sum_{n \geq 0} \mathbb{G}_D[\mu_n](\xi) \right)^q d\xi \leq c(N, q) W_F(0). \end{aligned}$$

This estimate requires several steps. Denote

$$(3.19) \quad I_1 = \sum_{k=3}^{\infty} \int_{T_k} g_D(0, \xi) \left( \sum_{n=0}^{k-3} \mathbb{G}_D[\mu_n](\xi) \right)^q d\xi$$

$$(3.20) \quad I_2 = \sum_{-1}^{\infty} \int_{T_k} g_D(0, \xi) \left( \sum_{n>k+2} \mathbb{G}_D[\mu_n](\xi) \right)^q d\xi$$

$$(3.21) \quad I_3 = \sum_{-1}^{\infty} \int_{T_k} g_D(0, \xi) \left( \sum_{n=(k-2)_+}^{k+2} \mathbb{G}_D[\mu_n](\xi) \right)^q d\xi$$

Then

$$(3.22) \quad \mathbb{G}_D[(\mathbb{G}_D[\mu])^q](0) \leq 3^q (I_1 + I_2 + I_3)$$

and we estimate each of the terms on the right hand side separately.

*Estimate of  $I_1$ .* We start with the following facts:

$$g_D(0, \xi) \leq c_N 2^{k(N-2)} \quad \forall \xi \in T_k$$

and

$$g_D(\xi, z) \leq c_N 2^{-n(2-N)} \quad \forall (\xi, z) \in T_k \times F_n.$$

These inequalities and (3.15) imply, for every  $\xi \in T_k$ ,

$$\begin{aligned} \mathbb{G}_D[\mu_n](\xi) &= \int_{F_n} g_D(\xi, z) d\mu_n(z) \leq c_N 2^{n(N-2)} \mu_n(F_n) \\ &= c_N 2^{n(N-2)} 2^{-n(N-2q')} C_{2,q'}(2^n F_n) = c_N 2^{2n/(q-1)} C_{2,q'}(2^n F_n). \end{aligned}$$

Hence

$$\begin{aligned} (3.23) \quad I_1 &\leq c(N, q) \sum_{k=3}^{\infty} 2^{k(N-2)} \int_{T_k} \left( \sum_{n=0}^{k-3} 2^{2n/(q-1)} C_{2,q'}(2^n F_n) \right)^q d\xi \\ &\leq \sum_{k=3}^{\infty} 2^{k(N-2)} 2^{-kN} \left( \sum_{n=0}^{k-3} 2^{2n/(q-1)} C_{2,q'}(2^n F_n) \right)^q \\ &\leq \sum_{k=3}^{M+1} 2^{-2k} \left( \sum_{n=0}^{k-3} 2^{2n/(q-1)} C_{2,q'}(2^n F_n^*) \right)^q. \end{aligned}$$

where  $M = M(0)$  is defined as in (2.27). Further, we claim that,

$$(3.24) \quad \begin{aligned} I_1' &:= \sum_{k=3}^{M+1} 2^{-2k} \left( \sum_{n=0}^{k-3} 2^{2n/(q-1)} C_{2,q'}(2^n F_n^*) \right)^q \leq \\ &c(N, q) \sum_{n=0}^{M+1} 2^{2n/(q-1)} C_{2,q'}(2^n F_n^*). \end{aligned}$$

This inequality is a consequence of the following statement proved in [29, App. B]:

**Lemma 3.3.** *Let  $K$  be a compact set in  $\mathbb{R}^N$  and let  $\alpha > 0$  and  $p > 1$  be such that  $\alpha p \leq N$ . Put*

$$(3.25) \quad \phi(t) = C_{\alpha,p}\left(\frac{1}{t}(K \cap B_t)\right) = C_{\alpha,p}\left(\frac{1}{t}K \cap B_1\right), \quad \forall t > 0.$$

Put  $r_m = 2^{-m}$ . Then, for every  $\gamma \in \mathbb{R}$  and every  $k \in \mathbb{N}$ ,

$$(3.26) \quad \frac{1}{c} \sum_{m=i+1}^k r_m^\gamma \phi(r_m) \leq \int_{r_k}^{r_i} t^\gamma \phi(t) \frac{dt}{t} \leq c \sum_{m=i+1}^k r_{m-1}^\gamma \phi(r_{m-1}).$$

where  $c$  is a constant depending only on  $\gamma, q, N$ .

Actually, in [29] this result was proved in the case  $\alpha = 2/q, p = q'$ , in  $\mathbb{R}^{N-1}$  assuming  $2/(q-1) \leq N-1$ . However the proof applies to any  $\alpha, p$  such that  $\alpha p \leq N$ . In particular it applies to the present case, namely,  $\alpha = 2, p = q'$  with  $2q' \leq N$ .

We proceed to derive (3.24) from the above lemma. Put  $r_m = 2^{-m}$ ,  $\gamma = -\frac{2}{q-1}$  and define  $\phi$  and  $\varphi$  by

$$(3.27) \quad \phi(r_m) := C_{2,q'}(r_m^{-1}F_m^*), \quad \varphi(r, s) := \int_r^s t^\gamma \phi(t) \frac{dt}{t} \quad 0 < r < s.$$

By Lemma 3.3,

$$(3.28) \quad \frac{1}{c} \sum_{m=i+1}^k r_m^\gamma C_{2,q'}(r_m^{-1}F_m^*) \leq \varphi(r_k, r_i) \leq c \sum_{m=i+1}^k r_{m-1}^\gamma C_{2,q'}(r_{m-1}^{-1}F_{m-1}^*),$$

for every  $i, k \in \mathbb{N}, i < k$ . The constant  $c$  depends only on  $q, N, Q$ . Hence (taking into account that  $F_m^* = \emptyset$  for  $m > M+1$ )

$$(3.29) \quad \begin{aligned} \varphi(0, r_i) &:= \lim_{r \downarrow 0} \varphi(r, r_i) \leq c \sum_{m=i+1}^{\infty} r_{m-1}^\gamma C_{2,q'}(r_{m-1}^{-1}F_{m-1}^*) \\ &\leq c \sum_{m=i}^{M+1} r_m^\gamma C_{2,q'}(r_m^{-1}F_m^*). \end{aligned}$$

Further, by (3.28),

$$(3.30) \quad I_1' = \sum_{k=3}^{M+1} r_k^2 \left( \sum_{n=0}^{k-3} r_n^\gamma C_{2,q'}(r_n^{-1}F_n^*) \right)^q \leq \sum_{k=3}^{M+1} r_k^2 \varphi^q(r_{k-3}, 1).$$

Since  $\varphi(\cdot, s)$  is non-increasing,

$$(3.31) \quad \sum_{k=3}^{M+1} r_k^2 \varphi^q(r_{k-3}, 1) \leq c \int_{r_{M-2}}^1 t^2 \varphi^q(t, 1) \frac{dt}{t} \leq c \int_0^1 t \varphi^q(t, 1) dt.$$

By (3.29)

$$(3.32) \quad \begin{aligned} \int_0^1 t\varphi^q(t, 1) dt &\leq -c \int_0^1 t^2\varphi^{q-1}(t, 1)\dot{\varphi}(t, 1)dt \\ &\leq -c \int_0^1 \dot{\varphi}(t, 1)dt \leq c\varphi(0, 1) \leq c \left( \sum_{m=0}^{M+1} r_m^\gamma C_{2,q'}(r_m^{-1}F_m^*) \right) \end{aligned}$$

Finally (3.30)–(3.32) imply (3.24). In turn, (3.23), (3.24) and (2.3) imply,

$$(3.33) \quad I_1 \leq c(N, q)W_F(0).$$

*Estimate of  $I_2$ .* Let  $\sigma > 0$  and  $\{a_n\}$  be a sequence of positive numbers. Then,

$$\sum_{n=k}^{\infty} a_n \leq 2^{-\sigma k} \left( \frac{1}{1 - 2^{-\sigma q'}} \right)^{\frac{1}{q'}} \left( \sum_{n=k}^{\infty} 2^{\sigma n q} a_n^q \right)^{\frac{1}{q}}.$$

Applying this inequality with  $a_n = \mathbb{G}_D[\mu_n](\xi)$  we obtain

$$(3.34) \quad \begin{aligned} I_2 &\leq c(N, q, \sigma) \sum_{k=-1}^{\infty} \int_{T_k} g_D(0, \xi) 2^{-\sigma q k} \sum_{n=k+2}^{\infty} 2^{\sigma n q} \mathbb{G}_D[\mu_n](\xi)^q d\xi \\ &\leq c \sum_{n \geq 1} 2^{\sigma n q} \sum_{1 \leq k < n-2} \int_{T_k} 2^{-\sigma k q} g_D(0, \xi) \mathbb{G}_D[\mu_n](\xi)^q d\xi \\ &\leq c \sum_{n \geq 1} 2^{\sigma n q} \sum_{1 \leq k < n-2} \int_{T_k} 2^{-\sigma k q} 2^{k(N-2)} \mathbb{G}_D[\mu_n](\xi)^q d\xi, \end{aligned}$$

where, in the last inequality, we used the fact that

$$g_D(0, \xi) \leq c_N 2^{k(N-2)} \quad \forall \xi \in T_k.$$

Choosing  $\sigma = (N-1)/q$  we obtain,

$$(3.35) \quad I_2 \leq c(N, q) \sum_{n \geq 1} 2^{n(N-1)} \sum_{1 \leq k < n-2} \int_{T_k} 2^{-k} \mathbb{G}_D[\mu_n](\xi)^q d\xi.$$

Next we estimate the term

$$(3.36) \quad J_{k,n} := \int_{T_k} \mathbb{G}_D[\mu_n](\xi)^q d\xi,$$

in the case  $1 \leq k < n-2$ . In view of (3.13) we have

$$\mathbb{G}_D[\mu_n](\xi) = \int_{F_n} g_D(\xi, z) d\mu_n(z) = 2^{-n(N-2q')} \int_{2^n F_n} \tilde{g}(\xi', z') d\nu_n(z')$$

where

$$\xi' = 2^n \xi, \quad \tilde{g}(\xi', z') = g_D(2^{-n} \xi', 2^{-n} z').$$

Observe that, if  $\xi \in T_k$  then  $\xi' \in T_{k-n}$ . Thus

$$J_{k,n} = 2^{-nN} \int_{T_{k-n}} \left( 2^{-n(N-2q')} \int_{2^n F_n} \tilde{g}(\xi', z') d\nu_n(z') \right)^q d\xi'.$$

Since

$$\tilde{g}(\xi', z') \leq c_N 2^{-n(2-N)} |\xi' - z'|^{2-N}$$

we obtain

$$(3.37) \quad J_{k,n} \leq c(N, q) 2^{-n(N-2q')} \int_{T_{k-n}} \left( \int_{2^n F_n} |\xi' - z'|^{2-N} d\nu_n(z') \right)^q d\xi'.$$

Since  $z' \in 2^n F_n \subset B_1(0)$  while  $\xi' \in T_{k-n}$ ,  $k-n < -2$  it follows that  $|\xi'| \geq 2$  and consequently

$$|\xi' - z'| \geq \frac{1}{2} |\xi'|.$$

Therefore

$$\begin{aligned} & \int_{T_{k-n}} \left( \int_{2^n F_n} |\xi' - z'|^{2-N} d\nu_n(z') \right)^q d\xi' \leq c \nu_n(2^n F_n)^q \int_{T_{k-n}} |\xi'|^{(2-N)q} d\xi' \\ & \leq c(N, q) C_{2,q'}(2^n F_n)^q \int_{2^{n-k-1}}^{2^{n-k}} r^{(2-N)q+N-1} dr \leq c(N, q) C_{2,q'}(2^n F_n) A(q, N) \end{aligned}$$

where we used the fact that  $C_{2,q'}(2^n F_n) \leq C_{2,q'}(B_1)$  and

$$A(q, N) = \begin{cases} 2^{(2-N)q+N} & \text{if } q > q_c \\ \ln 2 & \text{if } q = q_c. \end{cases}$$

Thus, for  $k \geq n-2 \geq -2$ ,

$$(3.38) \quad \begin{aligned} J_{k,n} & \leq c(N, q) 2^{-n(N-2q')} \|\nu_n\|_{W^{-2,q}(\mathbb{R}^N)}^q \\ & = c(N, q) 2^{-n(N-2q')} C_{2,q'}(2^n F_n). \end{aligned}$$

By (3.35) and (3.38),

$$(3.39) \quad \begin{aligned} I_2 & \leq c(N, q) \sum_{n \geq 0} 2^{n(N-1)} \sum_{k \geq n-2} 2^{-k} 2^{-n(N-2q')} C_{2,q'}(2^n F_n) \\ & \leq c(N, q) \sum_{n \geq 0} 2^{n(-2+2q')} C_{2,q'}(2^n F_n) \\ & = c(N, q) \sum_{n \geq 0} 2^{\frac{2n}{q-1}} C_{2,q'}(2^n F_n) = c(N, q) W_F(0). \end{aligned}$$

*Estimate of  $I_3$*  By (3.21) and the notation (3.36) we have

$$(3.40) \quad \begin{aligned} I_3 & \leq 5^q \sum_{k=-1}^{\infty} \int_{T_k} g_D(0, \xi) \sum_{n=(k-2)_+}^{k+2} \mathbb{G}_D[\mu_n](\xi)^q d\xi \\ & \leq \sum_{k=-1}^{\infty} 2^{k(N-2)} \sum_{n=(k-2)_+}^{k+2} J_{k,n}. \end{aligned}$$

By (3.37)

$$J_{k,n} \leq c(N, q) 2^{-n(N-2q')} \int_{T_{k-n}} \left( \int_{2^n F_n} |\xi' - z'|^{2-N} d\nu_n(z') \right)$$

and, in the present case  $-2 \leq n - k \leq 2$ . Therefore  $T_{k-n} \subset B_4(0)$  and consequently, for  $(\xi', z')$  in the domain of integration of the integral above,

$$|\xi' - z'|^{2-N} \approx \mathcal{B}_2(\xi', z')$$

where  $\mathcal{B}_2$  denotes the Bessel kernel with index 2. Hence,

$$\begin{aligned} & \int_{T_{k-n}} \left( \int_{2^n F_n} |\xi' - z'|^{2-N} d\nu_n(z') \right)^q d\xi' \leq \\ & c(N, q) \|\nu_n\|_{W^{-2,q}(\mathbb{R}^N)}^q = c(N, q) C_{2,q'}(2^n F_n). \end{aligned}$$

Therefore,

$$\begin{aligned} (3.41) \quad I_3 & \leq c(N, q) \sum_{k=-1}^{\infty} 2^{k(N-2)} \sum_{n=(k-2)_+}^{k+2} 2^{-n(N-2q')} C_{2,q'}(2^n F_n) \\ & \leq c(N, q) \sum_{k=-1}^{\infty} 2^{k(N-2)} 2^{-k(N-2q')} C_{2,q'}(2^k F_k) \\ & = c(N, q) \sum_{k=-1}^{\infty} 2^{2k/(q-1)} C_{2,q'}(2^k F_k) \leq c(N, q) W_F(0) \end{aligned}$$

Combining (3.22) with the inequalities (3.33), (3.39) and (3.41) we obtain

$$(3.42) \quad \mathbb{G}_D[(\mathbb{G}_D[\mu])^q](0) \leq c(N, q) W_F(0).$$

Finally, we combine (3.9) with (3.17) and (3.42) and replace  $\mu$  by  $\epsilon\mu$ ,  $\epsilon > 0$ , to obtain

$$(3.43) \quad u^{\epsilon\mu}(0) \geq (c_1(N, q)\epsilon - c_2(N, q)\epsilon^q) W_F(0).$$

Choosing  $\epsilon := (c_1(N, q)/2c_2(N, q))^{1/(q-1)}$  we obtain (3.11) with  $c(N, q) = c_1(N, q)\epsilon/2$ .  $\square$

#### 4. PROPERTIES OF $U_F$ FOR $F$ COMPACT

As before we assume that  $F$  is a compact set. Combining the capacity estimates contained in Theorems 2.1, 3.2 and (2.3) we have

$$(4.1) \quad U_F \sim W_F \sim W_F^* \quad \text{in } D = \mathbb{R}^N \setminus F$$

In the present section we use this result in order to establish several properties of the maximal solution.

4.1. *The maximal solution is  $\sigma$ -moderate.*

**Theorem 4.1.**  $U_F = V_F$ ; consequently  $U_F$  is  $\sigma$ -moderate.

*Proof.* By (4.1) there exists a constant  $c = c(N, q)$  such that

$$(4.2) \quad U_F \leq cV_F.$$

If the two solutions are not identical we have

$$(4.3) \quad V_F(x) < U_F(x) \quad \forall x \in D.$$

Let  $\alpha = \frac{1}{2c}$  and put  $v = (1 + \alpha)V_F(x) - \alpha U_F$ . Then  $\alpha V_F(x) < v < U_F$  and (as  $0 < \alpha < 1$ )  $\alpha V_F(x)$  is a subsolution of (1.1) in  $D$ . As in [24] we find that  $v$  is a supersolution. It follows that there exists a solution  $w$  such that  $\alpha V_F(x) \leq w \leq v < U_F(x)$ . But, by the definition of  $V_F$  (see (3.4)), it is easy to see that the smallest solution of (1.1) dominating  $\alpha V_F(x)$  is  $V_F(x)$ . Therefore  $w = V_F(x)$ . This contradicts (4.3).

By a standard argument, the definition of  $V_F(x)$  implies that it is  $\sigma$ -moderate.  $\square$

4.2. *A continuity property of  $\int U_K$  relative to capacity.*

**Lemma 4.2.** *There exists a positive constant  $c$  depending only on  $N, q$  such that, for every compact set  $K \subset B_1(0)$ , there exists an open neighborhood  $N_K$  of  $K$  such that*

$$(4.4) \quad C_{2,q'}(N_K) \leq 4C_{2,q'}(K) \quad \text{and} \quad \int_{B_1(0) \setminus N_K} U_K dx \leq cC_{2,q'}(K).$$

*Note.* In general  $\int_{B_1(0) \setminus K} U_K dx$  may be infinite. Of course, (4.4) is meaningful only if  $4C_{2,q'}(K) < C_{2,q'}(B_1(0))$ .

*Proof.* Let  $\bar{c}$  be the constant in (2.7) with  $R = 2$ . Assume that

$$(4.5) \quad C_{2,q'}(K) \leq a := C_{2,q'}(B_1)/8$$

and pick  $\gamma_1$  so that

$$(4.6) \quad 0 < \gamma_1 \leq C_{2,q'}(K).$$

By Lemma 2.2 and (2.21) there exists  $\eta \in X_K(\mathbb{R}^N)$  such that

$$(4.7) \quad \begin{aligned} \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}^{q'} &\leq C_{2,q'}(K) + \gamma_1, \\ \int_{B_2(0) \setminus K} U_K (1 - \eta)^{2q'} dx &\leq \bar{c}(C_{2,q'}(K) + \gamma_1). \end{aligned}$$

Fix  $\eta$  and denote,

$$K_{(\alpha)} = \{x \in B_1(0) : (1 - \alpha) \leq \eta\} \quad \forall \alpha \in (0, 1).$$

Then  $K \subset K_{(\alpha)}$  and

$$\begin{aligned} C_{2,q'}(K_{(\alpha)}) &\leq (1 - \alpha)^{-q'} \|\eta\|_{W^{2,q'}(\mathbb{R}^N)}^{q'} \\ &\leq (1 - \alpha)^{-q'} (C_{2,q'}(K) + \gamma_1) \leq \frac{2C_{2,q'}(K)}{(1 - \alpha)^{q'}}. \end{aligned}$$

Therefore, using (4.5), we obtain

$$(1 - \alpha)^{-q'} = 2 \implies C_{2,q'}(K_{(\alpha)}) \leq 4C_{2,q'}(K) \leq C_{2,q'}(B_1)/2.$$

Hence, by (4.7),

$$(4.8) \quad \int_{B_2(0) \setminus K(\alpha)} U_K dx \leq \bar{c} \alpha^{-2q'} (C_{2,q'}(K) + \gamma_1) \leq (4\bar{c}) C_{2,q'}(K)$$

where  $\alpha = 1 - 2^{-1/q'}$ .  $\square$

4.3. *Wiener criterion for blow up of  $U_F$ .*

**Theorem 4.3.** *For every point  $y \in F$ ,*

$$(4.9) \quad \lim_{F^c \ni x \rightarrow y} U_F(x) = \infty \iff W_F(y) = \infty.$$

*Proof.* Without loss of generality we may assume that  $y = 0$  and that  $F \subset B_1(0)$ . In order to justify the second part of this remark we observe that, for every  $m \in \mathbb{N}$ ,

$$(4.10) \quad \begin{aligned} 2^{-2m/(q-1)} U_F(2^{-m}x) &= U_{2^m F}(x) \quad \forall x \in (2^m F)^c, \\ W_F(0) &= 2^{2m/(q-1)} W_{2^m F}(0). \end{aligned}$$

Denote

$$(4.11) \quad a_m(x) = C_{2,q'}(2^m F_m(x)), \quad a_m^*(x) = C_{2,q'}(2^m F_m^*(x))$$

There exists a constant  $c = c(N, q)$  such that for every Borel set  $A \subset B_1(0)$ ,

$$(4.12) \quad C_{2,q'}(2A) \leq c C_{2,q'}(A).$$

If  $x, \xi \in \mathbb{R}^N$ ,  $|x - \xi| \leq r_m = 2^{-m}$  and  $0 \leq k \leq m$  then

$$2^k F_k^*(\xi) = 2^k (F \cap B_{r_k}(\xi)) \subset 2^k (F \cap B_{2r_k}(x)) = 2(2^{k-1} (F \cap B_{r_{k-1}}(x))).$$

Hence

$$(4.13) \quad a_k^*(\xi) \leq c a_{k-1}^*(x) \text{ for } 0 \leq k \leq m, \quad \sum_{k=0}^m 2^{2k/(q-1)} a_k^*(\xi) \leq c W_F^*(x).$$

As  $x \rightarrow \xi$ ,  $m \rightarrow \infty$  and we obtain

$$(4.14) \quad W_F^*(\xi) \leq c(N, q) \liminf_{x \rightarrow \xi} W_F^*(x).$$

By (4.1), (4.14) implies that (4.9) holds in the direction  $\Leftarrow$ .

In order to prove (4.9) in the opposite direction we derive the inequality,

$$(4.15) \quad \liminf_{F^c \ni x \rightarrow 0} W_F^*(x) \leq c(N, q) W_F^*(0).$$

If  $W_F^*(0) = \infty$  there is nothing to prove. Therefore we assume that

$$W_F^*(0) = \sum_{-\infty}^{M(0)} 2^{\frac{2m}{q-1}} C_{2,q'}(2^m F_m^*(0)) < \infty.$$

By Lemma 4.2, with  $K_m = 2^m F_m^*(0)$ , there exists an open neighborhood  $G_m$  of  $K_m$  such that

$$C_{2,q'}(G_m) \leq 4C_{2,q'}(K_m) \quad \text{and} \quad \int_{B_1(0) \setminus G_m} U_{K_m} dx \leq c(N, q)C_{2,q'}(K_m).$$

Put  $T' := [5/8 \leq |x| \leq 7/8]$  and let  $E_m$  be a compact subset of  $T' \setminus G_m$  such that

$$\begin{aligned} C_{2,q'}(E_m) &> \frac{1}{2}C_{2,q'}(T' \setminus G_m) \geq \frac{1}{2}C_{2,q'}(T') - 2C_{2,q'}(K_m) \\ &\geq \frac{1}{2}C_{2,q'}(T') - 2^{1-\frac{2m}{q-1}}W_F^*(0). \end{aligned}$$

Therefore, there exists an integer  $m_0$  such that, for  $m \geq m_0$ ,

$$\begin{aligned} (4.16) \quad \inf_{E_m} U_{K_m} &\leq |E_m|^{-1} \int_{E_m} U_{K_m} dx \\ &\leq |E_m|^{-1} c(N, q)C_{2,q'}(K_m) \leq c(N, q)C_{2,q'}(K_m)/C_{2,q'}(E_m) \\ &< 4c(N, q)C_{2,q'}(B_1(0))^{-1}C_{2,q'}(K_m) = c(N, q)C_{2,q'}(K_m). \end{aligned}$$

Hence, by (4.10),

$$\inf_{2^{-m}E_m} U_{F_m^*(0)} = 2^{2m/(q-1)} \inf_{E_m} U_{K_m} < c(N, q)2^{2m/(q-1)}C_{2,q'}(K_m),$$

which implies, for  $m \geq m_0$ ,

$$(4.17) \quad \inf_{(2^{-m}T') \setminus F} U_{F_m^*(0)} \leq c(N, q)2^{2m/(q-1)}C_{2,q'}(K_m) \leq c(N, q)W_F^*(0).$$

Fix  $j > m_0$  and let  $\xi \in (2^{-j}T') \setminus F$ . Denote

$$F^j := F \setminus B_{2^{-j+1}}(0), \quad E_{k,j} := F^j \cap \{x \in B_1(0) : |x - \xi| \leq 2^{-k}\}, \quad \bar{j} := \left\lceil \frac{j}{8} \right\rceil.$$

Since  $\text{dist}(\xi, F^j) \geq 2^{-j/8}$ ,

$$W_{F^j}(\xi) = \sum_{-\infty}^{\bar{j}} 2^{2k/(q-1)}C_{2,q'}(2^k E_{k,j}).$$

For  $k \leq \bar{j}$

$$x \in E_{k,j} \implies |x| \leq 2^{-k} + 2^{-j} \leq 2^{-k} + 2^{-8(k-1)} \leq 2^9 2^{-k}.$$

Thus  $E_{k,j} \subset F_{k-9}^*$  and

$$(4.18) \quad W_{F^j}(\xi) = \sum_{-\infty}^{\bar{j}} 2^{2k/(q-1)}C_{2,q'}(2^k F_{k-9}^*(0)) \leq c(N, q)W_F^*(0)$$

for every  $\xi \in (2^{-j}T') \setminus F$ . By (4.17), we can choose  $\xi^j \in (2^{-j}T') \setminus F$  such that

$$U_{F_j^*(0)}(\xi^j) \leq c(N, q)W_F^*(0).$$

Hence, by (4.18), bearing in mind that  $U_K \sim W_K \sim W_K^*$  for every compact  $K$  we obtain

$$(4.19) \quad U_F(\xi^j) \leq U_{F_j^*(0)}(\xi^j) + U_{F_j}(\xi^j) \leq c(N, q)W_F^*(0) \quad \forall j \geq m_0.$$

This implies (4.15) and completes the proof.  $\square$

**Corollary 4.4.** *Define*

$$\widetilde{W}_F(x) = \liminf_{y \rightarrow x} W_F(y) \quad \forall x \in \mathbb{R}^N.$$

*Then,  $\widetilde{W}_F$  is l.s.c. in  $\mathbb{R}^N$  and  $\widetilde{W}_F \sim W_F$ . In addition,  $\widetilde{W}_F$  satisfies Harnack's inequality in compact subsets of  $\mathbb{R}^N \setminus F$ .*

*Proof.* The lower semi-continuity of  $\widetilde{W}_F$  follows from its definition. The equivalence  $\widetilde{W}_F \sim W_F$  follows from (4.14) and (4.15). The last statement follows from the fact that  $U_F$  satisfies Harnack's inequality and  $\widetilde{W}_F \sim U_F$ .  $\square$

4.4.  $U_F$  is an almost large solution.

**Theorem 4.5.** *For every compact set  $F \subset \mathbb{R}^N$ ,  $U_F$  is an almost large solution.*

*Proof.* In view of Theorem 4.3 it is enough to show that there exists a set  $A \subset F$  such that

$$(4.20) \quad C_{2,q'}(A) = 0 \quad \text{and} \quad W_F(y) = \infty \quad \forall y \in F \setminus A.$$

It is known (see [1, Ch. 6]) that every point in  $F$ , with the possible exception of a set  $A_1$  of  $C_{2,q'}$  capacity zero, is a  $C_{2,q'}$ -thick point of  $F$ , i.e.,

$$(4.21) \quad \Lambda_F^{2,q'}(y) := \sum_0^\infty \left( 2^{2m/(q-1)} C_{2,q'}(F_m^*(y)) \right)^{q-1} = \infty \quad \forall y \in F \setminus A_1.$$

We show that,

$$(4.22) \quad \Lambda_F^{2,q'}(y) \leq c(N, q)(W_F(y))^{\tilde{q}}, \quad \tilde{q} = \min(1, q-1) \quad \forall q > 1.$$

Recall that,

$$C_{2,q'}(F_m^*(y)) \leq c(N, q)2^{-2m/(q-1)} C_{2,q'}(2^m F_m^*(y))$$

so that

$$(4.23) \quad \Lambda_F^{2,q'}(y) \leq c(N, q) \sum_0^\infty (C_{2,q'}(2^m F_m^*(y)))^{q-1}.$$

In view of the fact that  $C_{2,q'}(2^m F_m^*(y)) \leq C_{2,q'}(B_1)$ , if  $q \geq 2$ , (4.23) implies (4.22). If  $1 < q < 2$ ,

$$\begin{aligned} & \sum_0^\infty (C_{2,q'}(2^m F_m^*(y)))^{q-1} \leq \\ & \left( \sum_0^\infty 2^{-\frac{2m(2-q)}{q-1}} \right)^{2-q} \left( \sum_0^\infty 2^{2m/(q-1)} C_{2,q'}(2^m F_m^*(y)) \right)^{q-1}, \end{aligned}$$

which again implies (4.22). Clearly (4.21) and (4.22) imply (4.20).  $\square$

### 5. 'MAXIMAL SOLUTIONS' ON ARBITRARY SETS AND UNIQUENESS

For any Borel set  $E$  put

$$(5.1) \quad \mathcal{T}(E) := \{\mu \in W_+^{-2,q}(\mathbb{R}^N) : \mu(E^c) = 0\},$$

$$(5.2) \quad V_E := \sup\{u_\mu : \mu \in \mathcal{T}(E)\},$$

where  $u_\mu$  denotes the solution of (3.3). If  $C_{2,q'}(E) = 0$  the only measure  $\mu \in W_+^{-2,q}(\mathbb{R}^N)$  that is concentrated on  $E$  is the measure zero. Therefore in this case  $V_E = 0$ .

By Theorem 4.1,

$$(5.3) \quad E \text{ compact} \implies V_E = U_E.$$

Therefore the definition of  $V_E$  can be seen as an extension, to general sets, of the notion of 'maximal solution', previously defined for compact sets. However, by its definition,  $V_E$  *dominates only  $\sigma$ -moderate solutions in  $E^c$* , i.e., solutions of the form  $\lim u_{\mu_n}$  where  $\{\mu_n\}$  is an increasing sequence of measures in  $W_+^{-2,q}(\mathbb{R}^N)$  concentrated in  $E$ .

At this stage, it is not clear in which sense  $V_E$  is a solution of (1.1) in  $E^c$ , which, in general, is not an open set. This question will be discussed in the following sections.

The  $C_{2,q'}$  fine topology (see [1] for definition and details) plays a central role in the remaining part of the paper. If  $A$  is a set in  $\mathbb{R}^N$  we denote by  $\tilde{A}$  the closure of  $A$  in the  $C_{2,q'}$  fine topology and by  $\text{int}_q A$  the interior of  $A$  relative to this topology.

Recall that a set  $A \subset \mathbb{R}^N$  is  $C_{2,q'}$ -quasi open if, for every  $\epsilon > 0$ , there exists an open set  $G_\epsilon$  such that

$$A \subset G_\epsilon, \quad C_{2,q'}(G_\epsilon \setminus A) < \epsilon.$$

A set is  $C_{2,q'}$ -quasi closed if its complement is quasi open.

Every  $C_{2,q'}$ -finely open set is  $C_{2,q'}$ -quasi-open. On the other hand, if  $E$  is  $C_{2,q'}$ -quasi open then (see [1, Section 6.4])

$$C_{2,q'}(E \setminus \text{int}_q E) = 0.$$

This implies that every  $C_{2,q'}$ -quasi closed set  $F$  can be written in the form

$$F = \bigcup_n^\infty K_n \cup Z,$$

where  $\{K_n\}$  is an increasing sequence of compact sets and

$$C_{2,q'}(F \setminus K_n) \rightarrow 0, \quad C_{2,q'}(Z) = 0.$$

Furthermore, if  $E$  is  $C_{2,q'}$ -quasi closed then

$$C_{2,q'}(\tilde{E} \setminus E) = 0.$$

In the first two theorems below we describe some basic properties of  $V_E$ . These results are then used in order to establish a rather general uniqueness result for almost large solutions.

**Theorem 5.1.** *Let  $F$  be a  $C_{2,q'}$ -quasi closed set. Then*

$$(5.4) \quad \lim_{F^c \ni x \rightarrow y} V_F(x) = \infty \quad \text{for } C_{2,q'}\text{-a.e. } y \in F$$

and  $V_F$  satisfies

$$(5.5) \quad \frac{1}{c}W_F \leq V_F \leq cW_F,$$

where  $c$  depends only on  $N, q$ . Finally, for every  $x \in F^c$ ,

$$(5.6) \quad W_F(x) < \infty \implies \lim_{\substack{C_{2,q'}(E) \rightarrow 0 \\ E \subset F}} V_E(x) = 0.$$

*Proof.* There exists an increasing sequence of compact sets  $\{K_n\}$  such that  $K_n \subset F$  and  $C_{2,q'}(F \setminus K_n) \rightarrow 0$ . By Theorem 4.1  $U_{K_n} = V_{K_n}$  and, obviously,  $V_{K_n} \leq V_F$ . By Theorem 4.5, (5.4) holds if  $F$  is replaced by  $K_n$ . Therefore, by taking the limit as  $n \rightarrow \infty$ , we obtain (5.4) in the general case.

If  $\mu \in \mathcal{T}_F$  then  $u_\mu = \lim u_{\mu_n}$  where  $\mu_n = \mu \chi_{K_n}$ . Therefore

$$(5.7) \quad V_F = \lim U_{K_n}.$$

Since  $U_{K_n}$  satisfies estimates (2.5) and (3.6) for every  $n$ , it follows that  $V_F$  satisfies (5.5).

We turn to the proof of the last assertion. Let  $\{E_j\}$  be a sequence of subsets of  $F$  such that  $C_{2,q'}(E_j) \rightarrow 0$ . We must show that

$$(5.8) \quad \xi \in F^c, \quad \limsup V_{E_j}(\xi) > 0 \implies W_F(\xi) = \infty.$$

By taking a subsequence we may assume that there exists  $a > 0$  such that  $V_{E_j}(\xi) > a$  for all  $j$ . Since  $V_{\tilde{E}_j}(\xi) = V_{E_j}(\xi)$  and  $C_{2,q'}(E_j) \rightarrow 0$  implies  $C_{2,q'}(\tilde{E}_j) \rightarrow 0$  we may assume that the sets  $E_j$  are  $C_{2,q'}$ -finely closed. By (5.7) it follows that, for every  $j$ , there exists a compact set  $K_j \subset E_j$  such that

$$(5.9) \quad U_{K_j}(\xi) > a.$$

By negation, suppose that  $W_F(\xi) < \infty$ . Then

$$\lim_{J \rightarrow \infty} \sum_J^\infty 2^{2j/(q-1)} C_{2,q'}(2^j F_j(\xi)) \rightarrow 0,$$

$F_j$  being defined as in (2.1). Pick a positive integer  $J$  such that

$$(5.10) \quad \sum_J^\infty 2^{2j/(q-1)} C_{2,q'}(2^j F_j(\xi)) < a/4C,$$

where  $C$  is the constant in (2.5).

Pick a subsequence of  $\{K_j\}$ , say  $\{K_{j_n}\}$ , such that  $C_{2,q'}(K_{j_n}) < \epsilon/2^n$ , with  $\epsilon$  to be determined. The set  $A := \bigcup_1^\infty K_{j_n}$  is  $C_{2,q'}$ -quasi closed and  $C_{2,q'}(A) \leq \sum_1^\infty C_{2,q'}(K_{j_n}) < \epsilon$ . Further,

$$\begin{aligned} W_A(\xi) &= \sum_{-\infty}^\infty 2^{2j/(q-1)} C_{2,q'}(2^j A_j(\xi)) \leq \sum_{-\infty}^{-1} 2^{2j/(q-1)} C_{2,q'}(2^j A) \\ &\quad + \sum_0^{J-1} 2^{2j/(q-1)} C_{2,q'}(2^j A) + \sum_J^\infty 2^{2j/(q-1)} C_{2,q'}(2^j F_j(\xi)), \end{aligned}$$

where  $A_j(\xi)$  is defined as in (2.1) with  $F$  replaced by  $A$ . (We used the fact that  $A_j(\xi) \subset A \subset F$ .) By (2.28),

$$\sum_{-\infty}^{-1} 2^{2j/(q-1)} C_{2,q'}(2^j A) \leq c_1(N, q) \sum_{-\infty}^{-1} 2^{2jN} C_{2,q'}(A) \leq c_2(N, q)\epsilon.$$

By (2.29),

$$\sum_0^{J-1} 2^{2j/(q-1)} C_{2,q'}(2^j A) \leq c_1(N, q, J) \sum_0^{J-1} 2^{2jN} C_{2,q'}(A) \leq c_2(N, q, J)\epsilon.$$

Therefore, choosing  $\epsilon = (a/4C)(c_2(N, q) + c_2(N, q, J))^{-1}$  and using (5.10) we obtain,

$$W_A(\xi) < a/2C.$$

Since  $V_A$  satisfies (2.5) we conclude that  $V_A(\xi) < a/2$ . As

$$U_{K_{j_n}} = V_{K_{j_n}} \leq V_A,$$

this contradicts (5.9).  $\square$

**Theorem 5.2.** *Let  $F$  be a  $C_{2,q'}$ -quasi closed set and let  $\{F_n\}$  be an increasing sequence of compact subsets of  $F$  such that  $C_{2,q'}(F \setminus F_n) \rightarrow 0$ . Then*

$$(5.11) \quad V_F = \lim U_{F_n}.$$

*Furthermore, there exists an increasing sequence of non-negative measures  $\{\mu_n\} \subset W^{-2,q}(\mathbb{R}^N)$  such that  $\mu_n(F_n^c) = 0$  and  $u_{\mu_n} \rightarrow V_F$  in  $F^c$ .*

Finally, for every  $y \in F$ ,

$$(5.12) \quad W_F(y) < \infty \iff \liminf_{F^c \ni x \rightarrow y} V_F(x) < \infty.$$

*Proof.* From the definition of  $V_F$  it follows that  $V_F = \lim V_{F_n}$ . By Theorem 4.1,  $V_{F_n} = U_{F_n}$ .

Let  $\xi \in D_n = F_n^c$  and let  $\{\tau_k^n\}_{k=1}^\infty$  be a sequence in  $W^{-2,q}(\mathbb{R}^N)$  such that  $\tau_k^n(D_n) = 0$  and  $u_{\tau_k^n}(\xi) \rightarrow U_{F_n}(\xi)$ . Note that  $w_m^n = \max(u_{\tau_1^n}, \dots, u_{\tau_m^n})$  is a subsolution of the equation

$$-\Delta w + w^q = \mu_m^n := \max(\tau_1^n, \dots, \tau_m^n) \text{ in } D_n.$$

Therefore  $v_m^n = u_{\mu_m^n}$  is the smallest solution in  $D_n$  dominating  $w_m^n$ . The sequence  $\{v_m^n\}_{m=1}^\infty$  is increasing, bounded by  $U_{F_n}$  and  $v^n := \lim_{m \rightarrow \infty} v_m^n$  is a solution of (1.1) in  $D_n$  such that  $v^n(\xi) = U_{F_n}(\xi)$ . The fact that  $v^n \leq U_{F_n}$  and equals it at a point  $\xi \in D_n$  implies that  $v^n = U_{F_n}$ .

Put

$$(5.13) \quad \tau^{(n)} := \sum_m a_m^n \mu_m^n, \quad a_m^n := 2^{-m} \|\mu_m^n\|_{W_+^{-2,q}(\mathbb{R}^N)}.$$

Then

$$(5.14) \quad U_{F_n} = \lim_{k \rightarrow \infty} u_{k\tau^{(n)}}.$$

Finally, if  $\mu_n := \sum_1^n \tau^{(j)}$  then  $\{\mu_n\}$  is increasing and  $u_{\mu_n} \rightarrow V_F$ .

The last statement of the theorem is proved exactly as in the case that  $F$  is compact (see Theorem 4.3).  $\square$

**Theorem 5.3.** *Let  $E$  be a Borel set such that  $C_{2,q'}(E) > 0$ . Then*

$$(5.15) \quad V_E = V_{\tilde{E}}$$

and, if  $\mu \in W_+^{-2,q}(\mathbb{R}^N)$ ,

$$(5.16) \quad u_\mu < V_E \iff \mu(\mathbb{R}^N \setminus \tilde{E}) = 0.$$

Furthermore, if  $E$  is  $C_{2,q'}$ -quasi closed, there exists  $\tau \in W_+^{-2,q}(\mathbb{R}^N)$  such that  $\tau(E^c) = 0$  and

$$(5.17) \quad V_E = \lim_{k \rightarrow \infty} u_{k\tau}.$$

*Proof.* We prove (5.15) under the assumption that  $E$  is bounded, say,  $\bar{E} \subset B_R$ . For the general case we observe that

$$\lim_{R \rightarrow \infty} V_{E \cap B_R} = V_E.$$

Assertion 1: Let  $\overline{\mathcal{T}}(E)$  denote the closure of  $\mathcal{T}(E)$  in  $W_+^{-2,q}(B_R)$ . If  $\mu \in \overline{\mathcal{T}}(E)$  then  $u_\mu \leq V_E$ .

Let  $\{\nu_n\}$  be a sequence in  $\mathcal{T}(E)$  such that  $\nu_n \rightarrow \nu$  in  $W_+^{-2,q}(B_R)$ . Let  $u_n$  be the solution of

$$-\Delta u_n + u_n^q = \nu_n \text{ in } B_R, \quad u_n = 0 \text{ on } \Sigma_R.$$

Then  $\{u_n\}$  converges in  $L^q(B_R)$  and the limit  $u$  is a weak solution of

$$-\Delta u + u^q = \nu \text{ in } B_R, \quad u = 0 \text{ on } \Sigma_R.$$

Since  $u_n \leq V_E$  it follows that  $u = u_\nu \leq V_E$ .

Assertion 2:

$$(5.18) \quad \nu \in \mathcal{T}(\tilde{E}) \implies \nu \in \overline{\mathcal{T}}(E).$$

Suppose that  $\nu \in \mathcal{T}(\tilde{E})$  but  $\nu \notin \overline{\mathcal{T}}_E$ . Then there exists  $\phi \in W^{2,q'}(\mathbb{R}^N)$  such that

$$(5.19) \quad \|\phi\|_{W^{2,q'}(\mathbb{R}^N)} = 1, \quad \langle \phi, \nu \rangle > 0, \quad \langle \phi, \mu \rangle = 0 \quad \forall \mu \in \mathcal{T}_E.$$

We choose  $\phi$  to be a  $C_{2,q'}$ -finely continuous representative of its equivalence class (see [1, Proposition 6.1.2]). Thus the inverse image (by  $\phi$ ) of every open interval is quasi-open (see [1, Proposition 6.4.10]). It follows that

$$A_0 := \{\sigma : \phi(\sigma) = 0\} \text{ is } C_{2,q'}\text{-finely closed.}$$

We show that

$$(5.20) \quad C_{2,q'}(\tilde{E} \setminus A_0) = 0.$$

Put  $A_1 := \tilde{E} \setminus A_0$  and

$$A_1^+ = \{x \in A_1 : \phi(x) > 0\}, \quad A_1^- = \{x \in A_1 : \phi(x) < 0\}.$$

If (5.20) does *not* hold then

$$\text{either } C_{2,q'}(A_1^+) > 0, \text{ or } C_{2,q'}(A_1^-) > 0.$$

Each of these sets is  $C_{2,q'}$ -finely open relative to  $\tilde{E}$ , i.e., there exist  $C_{2,q'}$ -finely open sets  $Q_1, Q_2$  such that  $Q_1 \cap \tilde{E} = A_1^+$  and  $Q_2 \cap \tilde{E} = A_1^-$ . If, say,  $C_{2,q'}(Q_1 \cap \tilde{E}) > 0$  then  $C_{2,q'}(Q_1 \cap E) > 0$  (because  $C_{2,q'}(G) \sim C_{2,q'}(\tilde{G})$  for any Borel set  $G \subset B_R$ ). Let  $\mu \in W_+^{-2,q}(\mathbb{R}^N)$  be a non-trivial measure, supported in a compact subset of  $Q_1 \cap E$ . Then

$$\langle \phi, \mu \rangle > 0.$$

This contradicts (5.19) and proves (5.20).

Further (5.20) implies that  $\phi = 0$   $C_{2,q'}$ -a.e. on  $\tilde{E}$  which implies  $\langle \phi, \nu \rangle = 0$  in contradiction to (5.19). This proves Assertion 2.

Combining these assertions we conclude:

$$(5.21) \quad \nu \in \mathcal{T}(\tilde{E}) \implies u_\nu \leq V_E \implies V_{\tilde{E}} = \sup\{u_\nu : \nu \in \mathcal{T}(\tilde{E})\} \leq V_E.$$

Since, trivially,  $V_E \leq V_{\tilde{E}}$  we obtain (5.15).

If  $E$  is  $C_{2,q'}$ -quasi closed then, by Theorem 5.2, there exists an increasing sequence  $\{\mu_n\}$  in  $W_+^{-2,q}(\mathbb{R}^N)$  such that  $\mu_n(E^c) = 0$  and  $u_{\mu_n} \rightarrow V_E$ . Put

$$(5.22) \quad \tau := \sum a_n \mu_n, \quad a_n := 2^{-n} \|\mu_n\|_{W_+^{-2,q}(\mathbb{R}^N)}.$$

Then  $\tau \in W_+^{-2,q}(\mathbb{R}^N)$ ,  $\tau(E^c) = 0$  and (5.17) holds.

We turn to the proof of (5.16). The implication

$$\mu(\mathbb{R}^N \setminus \tilde{E}) = 0 \implies u_\mu < V_E$$

is a consequence of (5.15). To prove the implication in the opposite direction we may assume that  $E$  is compact. (This follows from Theorem 5.2.) By negation, suppose there exists  $\mu \in W^{-2,q}(\mathbb{R}^N)$  such that  $u_\mu < V_E$  but  $\mu(\mathbb{R}^N \setminus \tilde{E}) > 0$ . It follows that there exists a compact set  $K \subset \mathbb{R}^N \setminus \tilde{E}$  such that  $\mu(K) > 0$ . Let  $v_n := u_{n\mu\chi_K}$ . Then  $v_n \leq nu_\mu$  because  $nu_\mu$  is a supersolution of the equation  $-\Delta w + w^q = n\mu\chi_K$ . On the other hand,  $V_E$  is the largest solution dominated by  $nV_E$ , for every  $n$ . Therefore

$$(5.23) \quad v = \lim v_n \leq V_E.$$

If  $A$  is an open neighborhood of  $K$  such that  $\text{dist}(A, E) > 0$  then  $V_E \in L^q(A)$ . On the other hand

$$\int_{A \setminus K} v^q = \infty.$$

Therefore  $(v - V_E)_+$  is positive in an open subset of  $A \setminus K$ . This contradicts (5.23).  $\square$

**Theorem 5.4.** *Let  $\Omega$  be an open bounded set in  $\mathbb{R}^N$  such that  $\Omega = \cup \Omega_n$ , where  $\{\Omega_n\}$  is an increasing family of open sets satisfying*

$$(5.24) \quad C_{2,q'}(\Omega \setminus \Omega_n) \rightarrow 0.$$

Put

$$(5.25) \quad \begin{aligned} F_n &:= \partial\Omega_n, & D_n &= \mathbb{R}^N \setminus \bar{\Omega}_n, & \Omega^n &= \Omega \setminus \Omega_n \\ F &:= \partial_q \Omega = \tilde{\Omega} \setminus \Omega, & D &:= \mathbb{R}^N \setminus \tilde{\Omega} \end{aligned}$$

and assume that

$$(5.26) \quad C_{2,q'}(F_n \setminus \tilde{D}_n) \rightarrow 0.$$

Under these assumptions,  $V_{\tilde{D}}$  is the unique  $\partial_q$ -large solution in  $\Omega$ .

The proof is based on several lemmas.

**Lemma 5.5.** *Let  $\Omega$  be a bounded open set such that, with the notation  $F := \partial\Omega$ ,  $D := \mathbb{R}^N \setminus \bar{\Omega}$ ,*

$$(5.27) \quad C_{2,q'}(F \setminus \tilde{D}) = 0.$$

Then  $V_{\tilde{D}}$  is the unique  $\partial_q$ -large solution in  $\Omega$ .

*Proof.* Let  $v$  be a  $\partial_q$ -large solution in  $\Omega$ . First we show that

$$(5.28) \quad V_D \leq v \text{ in } \Omega.$$

If  $\mu \in \mathcal{T}_D$  then  $\mu = \sup\{\mu\chi_K : K \subset D, K \text{ compact}\}$  and  $u_\mu = \sup u_{\mu\chi_K}$  over compact sets  $K$  as above. Therefore it is sufficient to show that

$$(5.29) \quad u_\mu \leq v$$

for every  $\mu \in W_+^{-2,q}(\mathbb{R}^N)$  supported in a compact set  $K \subset D$ . Since  $K \cap \bar{\Omega} = \emptyset$ ,  $u_\mu$  is uniformly bounded in  $\bar{\Omega}$ .

Let

$$A_v := \{y \in F : \liminf_{\Omega \ni x \rightarrow y} v(x) < \infty\}.$$

Note that

$$\partial_q D \subset \partial D \subset \partial \Omega = F,$$

$$\partial_q D \subset \partial_q(\mathbb{R}^N \setminus \tilde{\Omega}) = \partial_q \Omega.$$

By (5.27)  $C_{2,q'}(F \setminus \partial_q D) = 0$ ; therefore  $C_{2,q'}(F \setminus \partial_q \Omega) = 0$ . Therefore any  $\partial_q$ -large solution in  $\Omega$  is an almost large solution in  $\Omega$ . Hence  $C_{2,q'}(A_v) = 0$ .

Let  $G_\epsilon$  be an open neighborhood of  $A_v(F)$  such that  $C_{2,q'}(G_\epsilon) < \epsilon$ . Put

$$\Omega_\delta = \{x \in \Omega : \text{dist}(x, F) < \delta\}, \quad \Omega'_\delta = \{x \in \Omega : \text{dist}(x, F) > \delta\}.$$

Let  $\Omega_\delta^*$  be a smooth domain such that  $\Omega'_\delta \subset \Omega_\delta^* \Subset \Omega'_{\delta/2}$ . Put

$$G_{\epsilon,\delta} := G_\epsilon \cap (\Omega \setminus \bar{\Omega}_\delta^*).$$

Then  $v + V_{G_{\epsilon,\delta}}$  is a supersolution of (1.1) in  $\Omega_\delta^*$  and, if  $\delta$  is sufficiently small,

$$u_\mu \leq v + V_{G_{\epsilon,\delta}} \quad \text{on } \partial \Omega_\delta^*.$$

Thus

$$u_\mu \leq v + V_{G_{\epsilon,\delta}} \quad \text{in } \Omega_\delta^*.$$

Since  $C_{2,q'}(G_\epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$ , Theorem 5.1 implies that, for fixed  $\delta > 0$ ,

$$\lim_{\epsilon \rightarrow 0} V_{G_{\epsilon,\delta}} = 0 \quad \text{in } \Omega_\delta^*.$$

Letting  $\delta \rightarrow 0$  we obtain (5.29) and hence (5.28). Further, by Theorem 5.3,

$$(5.30) \quad V_{\tilde{D}} = V_D \leq v \quad \text{in } \Omega.$$

Next we show that the opposite inequality,

$$(5.31) \quad v \leq V_{\tilde{D}},$$

is also valid. (A-priori this is not obvious because we do not assume that  $v$  is  $\sigma$ -moderate.)

By (5.27)  $C_{2,q'}(\bar{D} \setminus \tilde{D}) = 0$ ; hence  $V_{\bar{D}} = V_{\tilde{D}}$ .

Let  $R$  be sufficiently large so that  $\bar{\Omega} \subset B_R(0)$ . Then

$$(i) \quad V_{\bar{D}} \leq V_{\overline{D \cap B_R}} + V_{B_R^c}, \quad (ii) \quad U_{\bar{D}} \leq U_{\overline{D \cap B_R}} + U_{B_R^c}.$$

and  $V_{\bar{D}}$  (resp.  $U_{\bar{D}}$ ) is the largest solution in  $\Omega$ , dominated by the right hand side of inequality (i) (resp. (ii)). Since  $\overline{D \cap B_R}$  is compact,  $V_{\overline{D \cap B_R}} = U_{\overline{D \cap B_R}}$ . The uniqueness of large solutions in smooth domains implies that

$$U_{B_R^c} = U_{\partial B_R} = V_{\partial B_R} = V_{B_R^c}.$$

Combining these facts we conclude that

$$U_{\bar{D}} = V_{\bar{D}} = V_{\tilde{D}}.$$

By definition,  $v \leq U_{\bar{D}}$  in  $\Omega$ ; hence  $v \leq V_{\tilde{D}}$ . □

**Lemma 5.6.** *Let  $v$  be a solution of (1.1) in a bounded open set  $\Omega$ . Suppose that  $A$  is a  $C_{2,q'}$ -finely closed subset of  $\partial\Omega$  such that*

$$(5.32) \quad \lim_{x \rightarrow y} v(x) = \infty \quad \forall y \in \partial\Omega \setminus A.$$

If  $D := \mathbb{R}^N \setminus \tilde{\Omega}$  then,

$$(5.33) \quad V_{\tilde{D}} = V_D \leq v + V_A \quad \text{in } \Omega.$$

*Proof.* Let  $\mu$  be a measure in  $W_+^{-2,q}(\mathbb{R}^N)$  concentrated on a compact set  $K \subset D$ . Let  $\{O_n\}$  be a decreasing sequence of open sets such that

$$A \subset O_n, \quad C_{2,q'}(O_n \setminus A) \rightarrow 0, \quad \lim_{\Omega \ni x \rightarrow \partial\Omega \setminus O_n} v(x) = \infty.$$

Let  $\Omega_{n,\delta}^*$  be as in the proof of Lemma 5.5 and let  $\{\delta_n\}$  be a sequence of positive numbers decreasing to zero. Denote

$$G^n := O_n \cap (\Omega_n \setminus \overline{\Omega_{n,\delta_n}^*}).$$

As in the proof of Lemma 5.5, we obtain

$$u_\mu \leq V_{G^n} + v \quad \text{in } \Omega_{n,\delta_n}^*.$$

Since  $A \subset G^n$  and  $C_{2,q'}(G^n \setminus A) \rightarrow 0$  it follows that  $V_{G^n} \downarrow V_A$ . Letting  $n \rightarrow \infty$  we obtain

$$u_\mu \leq V_A + v$$

which in turn implies (5.33).  $\square$

**Lemma 5.7.** *Put*

$$S_{n,1} := \overline{D}_n \setminus \tilde{D}_n, \quad S_{n,2} = (\partial_q D_n) \Delta \partial_q(\mathbb{R}^N \setminus \tilde{\Omega}_n), \quad E_n := F_n \Delta F.$$

*Then, under the assumptions of the theorem,*

$$(5.34) \quad (a) \ C_{2,q'}(S_{n,1}) \rightarrow 0, \quad (b) \ C_{2,q'}(S_{n,2}) \rightarrow 0, \quad (c) \ C_{2,q'}(E_n) \rightarrow 0.$$

*Proof.* Since  $S_{n,1} \subset F_n \setminus \tilde{D}_n$ , (a) follows from (5.26).

Since  $D_n \subset \mathbb{R}^N \setminus \tilde{\Omega}_n$  it follows that

$$\partial_q D_n \subset \partial_q(\mathbb{R}^N \setminus \tilde{\Omega}_n) \cup \partial_q(\overline{D}_n \setminus \tilde{D}_n), \quad \partial_q(\mathbb{R}^N \setminus \tilde{\Omega}_n) \subset \partial_q D_n \cup \partial_q(\overline{D}_n \setminus \tilde{D}_n).$$

But (5.34) (a) implies that  $C_{2,q'}(\partial_q(\overline{D}_n \setminus \tilde{D}_n)) \rightarrow 0$ . Therefore, the previous relations imply (5.34) (b).

In order to establish (c) we observe that,

$$F \subset \partial_q \Omega_n \cup \partial_q \Omega^n, \quad \partial_q \Omega_n \subset \partial_q \Omega^n \cup F.$$

It is known that (see [1]) there exists a constant  $c(N, q)$  such that, for every Borel set  $A$ ,

$$(5.35) \quad C_{2,q'}(\tilde{A}) \leq c C_{2,q'}(A).$$

Therefore (5.24) implies that  $C_{2,q'}(\tilde{\Omega}^n) \rightarrow 0$ , which in turn implies that  $C_{2,q'}(\partial_q \Omega^n) \rightarrow 0$ . We conclude that

$$(5.36) \quad C_{2,q'}(F \Delta \partial_q \Omega_n) \rightarrow 0.$$

Hence, as  $\partial_q \Omega_n \subset F_n$ ,

$$(5.37) \quad C_{2,q'}(F \setminus F_n) \leq C_{2,q'}(F \setminus \partial_q \Omega_n) \rightarrow 0.$$

On the other hand,

$$(5.38) \quad F_n \setminus F \subset (F_n \setminus \partial_q \Omega_n) \cup (\partial_q \Omega_n \setminus F).$$

Since

$$\partial_q \Omega_n \supseteq \partial_q \tilde{\Omega}_n = \partial_q(\mathbb{R}^N \setminus \tilde{\Omega}_n).$$

(5.34) (b) implies

$$C_{2,q'}(\partial_q D_n \setminus \partial_q \Omega_n) \rightarrow 0.$$

This fact and assumption (5.26) imply

$$(5.39) \quad C_{2,q'}(F_n \setminus \partial_q \Omega_n) \rightarrow 0.$$

Finally, (5.36), (5.38) and (5.39) imply

$$(5.40) \quad C_{2,q'}(F_n \setminus F) \rightarrow 0.$$

This together with (5.37) yields (5.34) (c).  $\square$

*Proof of Theorem 5.4.* Let  $A_n = F_n \setminus F$ . By (5.34) (b),  $C_{2,q'}(A_n) \rightarrow 0$ . If  $v$  is a  $\partial_q$ -large solution in  $\Omega$  then  $v$  blows up  $C_{2,q'}$  a.e. on  $F$  and consequently it blows up  $C_{2,q'}$  a.e. on  $F_n \setminus A_n$ . Applying Lemma 5.6 to  $v$  in  $\Omega_n$  we obtain

$$V_{\tilde{D}_n} = V_{D_n} \leq v + V_{A_n} \quad \text{in } \Omega_n.$$

Note that

$$D_n \setminus D = \tilde{\Omega} \setminus \bar{\Omega}_n = (\Omega \setminus \bar{\Omega}_n) \cup (F \setminus \bar{\Omega}_n) \subset (\Omega \setminus \bar{\Omega}_n) \cup (F \setminus F_n)$$

and

$$D \setminus D_n = \bar{\Omega}_n \setminus \tilde{\Omega} = (\Omega_n \setminus \tilde{\Omega}) \cup (F_n \setminus \tilde{\Omega}) \subset (F_n \setminus F).$$

Therefore, (5.24) and (5.34) (c) imply that

$$(5.41) \quad C_{2,q'}(D_n \Delta D) \rightarrow 0.$$

The definition of  $V_E$  (see (5.2)) implies

$$V_{D_n} \leq V_D + V_{D_n \setminus D}$$

and, by (5.41) and Theorem 5.1,  $V_{D_n \setminus D} \rightarrow 0$ . Hence

$$(5.42) \quad V_{\tilde{D}_n} \rightarrow V_{\tilde{D}}.$$

By Theorem 5.1,  $V_{A_n} \rightarrow 0$  in  $\Omega$ . Therefore, letting  $n \rightarrow \infty$ , we obtain

$$V_{\tilde{D}} \leq v \quad \text{in } \Omega.$$

It remains to show that  $v \leq V_{\tilde{D}}$ . As  $U_{\bar{D}_n}$  is the maximal solution in  $\Omega_n$ ,

$$v \leq V_{\bar{D}_n} = U_{\bar{D}_n}.$$

Lemma 5.7, implies that

$$V_{\bar{D}_n} - V_{\tilde{D}_n} \rightarrow 0 \quad \text{in } \Omega.$$

Indeed, as an immediate consequence of the definition of  $V_E$  (see (5.2)),

$$V_{\overline{D}_n} \leq V_{\tilde{D}_n} + V_{\overline{D}_n \setminus \tilde{D}_n}.$$

By (5.34) (a),  $C_{2,q'}(\overline{D}_n \setminus \tilde{D}_n) \rightarrow 0$ . Hence, by Theorem 5.1,  $V_{\overline{D}_n \setminus \tilde{D}_n} \rightarrow 0$  in  $\Omega_n$ . It follows that

$$\lim V_{\overline{D}_n} \leq \lim V_{\tilde{D}_n}.$$

The limits exist because of monotonicity. Since  $V_{\tilde{D}_n} \leq V_{\overline{D}_n}$  we obtain,

$$\lim V_{\overline{D}_n} = \lim V_{\tilde{D}_n}.$$

Therefore

$$v \leq \lim V_{\tilde{D}_n} = V_{\tilde{D}}.$$

□

**Corollary 5.8.** *Suppose that  $\Omega = \cup_1^\infty Q_n$  where  $\{Q_n\}$  is a sequence of open sets such that*

$$(5.43) \quad \sum_1^\infty C_{2,q'}(Q_n) < \infty.$$

For every  $n \in \mathbb{N}$ , put

$$S_n = \cup_1^n \overline{Q}_k, \quad D_n = \mathbb{R}^N \setminus \overline{S}_n$$

and assume that

$$(5.44) \quad C_{2,q'}(\partial S_n \setminus \tilde{D}_n) \rightarrow 0.$$

Then there exists a unique almost large solution in  $\Omega$ .

*Remark.* If  $y \in \partial S_n$  and there exists an open cone  $C_y$ , with vertex  $y$ , such that  $C_y \subset \mathbb{R}^N \setminus S_n$  then  $y \in \partial_q S_n$ . Hence if, for every  $n \in \mathbb{N}$ , this condition is satisfied  $C_{2,q'}$  a.e. on  $\partial S_n$  then (5.44) holds. In particular, if  $\{Q_n\}$  is a sequence of balls, (5.44) is satisfied.

*Proof.* Let  $\Omega_n = S_n^0 := S_n \setminus \partial S_n$ . Then  $\{\Omega_n\}$  is an increasing sequence of open sets, (5.44) implies (5.26) and (5.43) implies (5.24). Therefore the corollary is an immediate consequence of Theorem 5.4. □

*Example.* Let  $\{x^m\}$  be a sequence of distinct points in  $B_1(0)$ . Let  $\{r_n\}$  be a decreasing sequence of positive numbers such that  $\{B_{r_n}(x^n)\}$  is a sequence of balls contained in  $B_1(0)$  and

$$(5.45) \quad \begin{aligned} \sum r_n^{N-2q'} &< \infty && \text{if } N > 2q', \\ \sum (1 - \log r_n)^{1-q'} &< \infty && \text{if } N = 2q'. \end{aligned}$$

Then there exists a unique large solution in  $\Omega := \cup_1^\infty B_{r_n}(x^n)$ .

Indeed  $C_{2,q'}(B_r) \sim r^{N-2q'}$  if  $N > 2q'$  and  $C_{2,q'}(B_r) \sim \log(1 - \log r)$  if  $N = 2q'$  and  $0 < r < 1$ . Therefore the conditions of Corollary 5.8 are satisfied.

Note that  $\tilde{\Omega} = \cup \overline{B_{r_n}}(x^n)$ , but, in general  $\overline{\Omega}$  is much larger. For instance, if  $\{x^n\}$  is a dense sequence in  $B_1(0)$  then  $\overline{\Omega} = \overline{B_1(0)}$ . Therefore it is important that our conditions in Corollary 5.8 require  $C_{2,q'}(\partial_q \Omega \setminus \tilde{D}) = 0$  and not  $C_{2,q'}(\partial \Omega \setminus \tilde{D}) = 0$ .

## 6. VERY WEAK SUBSOLUTIONS

In this section  $F$  is a  $C_{2,q'}$ -finely closed set contained in  $B_1(0)$  and  $D = B_2(0) \setminus F$ . Note that  $D$  is a  $C_{2,q'}$ -finely open set, but not necessarily open in the Euclidean topology.

We denote by  $W^{2,q'}(D)$  the set  $\{h|_D : h \in W^{2,q'}(\mathbb{R}^N)\}$ . If  $f \in W^{2,q'}(\mathbb{R}^N)$  we denote by  $\text{supp}_{(2,q')} f$  (= the  $C_{2,q'}$ -fine support of  $f$ ) the intersection of all  $C_{2,q'}$ -finely closed sets  $E$  such that  $f = 0$  a.e. in  $\mathbb{R}^N \setminus E$ .

The following subspace of  $W^{2,q'}(D)$  serves as a space of test functions in our study:

$$(6.1) \quad W_{0,\infty}^{2,q'}(D) := \{h|_D : h \in W^{2,q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \text{supp}_{(2,q')} h \Subset D\}.$$

The notation  $E \Subset D$  means:  $E$  is ‘strongly contained’ in  $D$ , i.e.,  $\overline{E}$  is a compact subset of  $D$ . Some features of this space are discussed in Appendix A.

The following statement was established in [29] (see Lemma 2.6). (The framework in [29] is somewhat different, but the proof, with obvious modifications, applies to the present case as well.)

**Lemma 6.1.** *Let  $D$  be a bounded  $C_{2,q'}$ -finely open set. Then there exists an increasing sequence of compact sets  $\{E_n\}$  such that*

$$(6.2) \quad \begin{aligned} E_n &\subset \text{int}_q E_{n+1}, & \cup_1^\infty E_n &\subset D, \\ C_{2,q'}(D \setminus \cup_1^\infty E_n) &= 0, & C_{2,q'}(E_n) &\rightarrow C_{2,q'}(D). \end{aligned}$$

A sequence of sets  $\{E_n\}$  as above is called a  $q$ -exhaustion of  $D$ .

We denote by  $L_{\ell(2,q')}^q(D)$  the space of measurable functions  $f$  in  $D$  such that, for every positive  $\phi \in W_{0,\infty}^{2,q'}(D)$ ,  $f \in L^q(D; \phi)$ , i.e.,  $|f|^q \phi \in L^1(D)$ . We endow this space with the topology determined by the family of semi-norms

$$(6.3) \quad \{\|\cdot\|_{L^q(D;\phi)} : \phi \in W_{0,\infty}^{2,q'}(D), \phi \geq 0\}.$$

This topology will be denoted by  $\tau_{\ell(2,q')}(D)$ .

Further we denote by  $\mathfrak{M}_{\ell(2,q')}(D)$  the space of positive Borel measures  $\mu$  in  $D$  such that

$$(6.4) \quad \begin{aligned} (a) \quad K \subset D, K \text{ compact} &\implies \mu(K) < \infty, \\ (b) \quad E \subset D, E \text{ Borel}, C_{2,q'}(E) = 0 &\implies \mu(E) = 0. \end{aligned}$$

We observe that,

**Lemma 6.2.** *If  $\mu \in \mathfrak{M}_{\ell(2,q')}(D)$  then:*

(i) *There exists an increasing sequence  $\{\mu_n\}$  of positive, bounded measures*

in  $W^{-2,q}(\mathbb{R}^N)$  such that  $\mu_n(D^c) = 0$  and  $\mu_n \uparrow \mu$ .

(ii)  $W_{0,\infty}^{2,q'}(D) \subset L^1(\mu)$ .

*Proof.* (i) This is well known in the case that  $\mu$  is a positive, bounded measure [6] and it follows from Lemma 6.1 in the case that  $\mu$  is a positive measure in  $\mathfrak{M}_{\ell(2,q')}(D)$ .

(ii) If  $\varphi \in W_{0,\infty}^{2,q'}(D)$ , it vanishes outside a compact set  $K_\varphi \subset D$ . By definition,  $\mu(K_\varphi) < \infty$ . Furthermore  $\varphi$  is the limit  $C_{2,q'}$  a.e. of smooth functions; consequently it is  $\mu$ -measurable. Since  $\varphi$  is bounded, it is integrable relative to  $\mu$ .  $\square$

*Notation.* A sequence  $\{\mu_n\}$  as in Lemma 6.2 (i) will be called a *determining sequence* for  $\mu$ .

We introduce below a very weak type of subsolution of (1.2) defined as follows.

**Definition 6.3.** Assume that the measure  $\mu$  in (1.2) belongs to  $\mathfrak{M}_{\ell(2,q')}(D)$ . A non-negative measurable function  $u$  is a *very weak subsolution* of (1.2) in  $D$  if, for every non-negative  $\phi \in W_{0,\infty}^{2,q'}(D)$ ,

$$(6.5) \quad u \in L^q(D; \zeta) \quad \text{where} \quad \zeta := \phi^{2q'},$$

$$(6.6) \quad - \int_D u \Delta \zeta dx + \int_D u^q \zeta dx \leq \int_D \zeta d\mu.$$

*Remarks.* (a) If (6.5) holds for every non-negative  $\phi \in W_{0,\infty}^{2,q'}(D)$  then

$$(6.7) \quad u \Delta \zeta \in L^1(D).$$

This is proved in the next lemma.

(b) Let  $\phi \in W_{0,\infty}^{2,\gamma}(D)$ ,  $\gamma \geq 1$ . By interpolation,  $|\nabla \phi|^2 \in L^\gamma(D)$  and

$$(6.8) \quad \| |\nabla \phi|^2 \|_{L^\gamma(D)} \leq c(q, N) L \| D^2 \phi \|_{L^\gamma(D)}, \quad L := \|\phi\|_{L^\infty(D)}.$$

where  $|D^2 \phi| := \sum_{|\alpha|=2} |D^\alpha \phi|$ .

(c) If  $\phi \in W_{0,\infty}^{2,\gamma}(D)$ ,  $\gamma \geq 1$  then

$$(6.9) \quad \begin{aligned} |\phi|^{2\gamma} &= (\phi^2)^\gamma \in W_{0,\infty}^{2,q'}(D), \\ \nabla(|\phi|^{2\gamma}) &= 2\gamma(\phi^2)^{\gamma-1/2} \nabla \phi, \\ \Delta(\phi^{2\gamma}) &= 2\gamma(2\gamma-1)|\phi|^{2\gamma-2} |\nabla \phi|^2 + 2\gamma|\phi|^{2\gamma-1} \Delta \phi. \end{aligned}$$

The last two formulas are easily verified for  $\phi \in C_c^\infty(\mathbb{R}^N)$ ; in the general case they are obtained by the usual density argument. The fact that  $|\phi|^{2\gamma} \in W_{0,\infty}^{2,q'}(D)$  is a consequence of these formulas and (6.8). (6.9) imply,

$$(6.10) \quad \| |\phi|^{2\gamma} \|_{W^{2,\gamma}(D)} \leq AL^{2\gamma-1} \max(1, L) \|\phi\|_{W^{2,\gamma}(D)}.$$

**Theorem 6.4.** (i) If  $u$  is a non-negative measurable function satisfying (6.5) then  $u\Delta\zeta \in L^1(D)$ .

(ii) If  $u$  is a very weak subsolution of (1.1) in  $D$  (i.e.  $\mu = 0$ ) then, for every non-negative  $\phi \in W_{0,\infty}^{2,q'}(D)$ ,

$$(6.11) \quad \int_D u|\Delta\zeta|dx + \int_D u^q\zeta \leq c \left( L \|D^2\phi\|_{L^\gamma(D)} \right)^\gamma,$$

where  $\zeta := \phi^{2q'}$ ,  $c = c(N, q)$  and  $L := \|\phi\|_{L^\infty(D)}$ .

(iii) Let  $\mu \in W^{-2,q}(\mathbb{R}^N)$  be a positive bounded measure vanishing outside  $D$ . If  $u$  is a non-negative very weak subsolution of (1.2) then

$$(6.12) \quad \|u\|_{L^q(D,\zeta)} \leq cL^{1/(q-1)} \left( (\|D^2\phi\|_{L^{q'}(D)})^{1/(q-1)} + (L^{1/(q-1)} \|\mu\|_{W^{-2,q'}} \|D^2\phi\|_{L^{q'}(D)})^{1/q} \right)$$

Finally, if  $L \leq \bar{L}$ ,  $u$  satisfies

$$(6.13) \quad \|u\|_{L^q(D,\phi)} \leq c(N, q, \bar{L})L^{\frac{1}{q-1}} \left( \|D^2\phi\|_{L^{q'}(D)}^{\frac{1}{q-1}} + \|\mu\|_{W^{-2,q'}}^{\frac{1}{q}} \|D^2\phi\|_{L^{q'}(D)}^{\frac{1}{q}} \right).$$

*Proof.* Let  $\phi$  be a non-negative function in  $W_{0,\infty}^{2,q'}(D)$ . By (6.9), with  $\gamma = q'$ , we obtain

$$|\Delta\zeta| \leq c(q)\zeta^{1/q}M(\phi), \quad M(\phi) := (|\nabla\phi|^2 + \phi|\Delta\phi|)$$

and hence, using (6.8),

$$(6.14) \quad \int_D u|\Delta\zeta|dx \leq c(q) \left( \int_D u^q\zeta dx \right)^{1/q} \left( \int_D M(\phi)^{q'} dx \right)^{1/q'},$$

$$(6.15) \quad \int_D M(\phi)^{q'} dx \leq c(q, N)L \|D^2\phi\|_{L^{q'}(D)}^{q'}.$$

Assuming that  $u \in L^q(D, \zeta)$  we obtain  $u\Delta\zeta \in L^1(D)$ .

We turn to the proof of (ii) and (iii). We assume that  $u$  is a non-negative very weak subsolution as in Definition 6.3. Put

$$A := \left( \int_D u^q\zeta dx \right)^{1/q}, \quad B := \left( \int_D M(\phi)^{q'} dx \right)^{1/q'}, \quad C := \|\mu\|_{W^{-2,q'}} \|\zeta\|_{W^{2,q'}(D)}$$

By (6.6) and (6.14)

$$(6.16) \quad A^q = \int_D u^q\zeta dx \leq \int_D u\Delta\zeta dx + \int_D \zeta d\mu \leq c(q, N)AB + C.$$

This implies

$$A^q \leq \frac{1}{q}A^q + \frac{1}{q'}(cB)^{q'} + C \implies A^q \leq (cB)^{q'} + q'C \leq c'(N, q) \max(B^{q'}, C).$$

Thus

$$(6.17) \quad A \leq c(q, N) \left( B^{1/(q-1)} + C^{1/q} \right).$$

By Poincaré's inequality

$$\|\zeta\|_{W^{2,q'}(D)} \leq c(q, N) \|D^2\zeta\|_{L^{q'}}$$

and therefore, by the same computation as in (6.9),

$$\|\zeta\|_{W^{2,q'}(D)} \leq c(q, N) \left( \left\| \phi^{2/(q-1)} (\nabla\phi)^2 \right\|_{L^{q'}} + \left\| \phi^{(1+q)/(q-1)} D^2\phi \right\|_{L^{q'}} \right).$$

Therefore by (6.8) and (6.15),

$$(6.18) \quad \|\zeta\|_{W^{2,q'}(D)} \leq L^{\frac{1}{q}} (L + L^{\frac{1}{q}}) \|D^2\phi\|_{L^{q'}}.$$

This estimate and (6.17) imply (6.12). Further, if  $\mu = 0$ , (6.12), (6.14) and (6.15) imply (6.11).

Now let  $\psi$  be a non-negative function in  $W_{0,\infty}^{2,q'}(D)$  and put

$$\phi := (1 + \psi)^{\frac{1}{2q'}} - 1, \quad \zeta := \phi^{2q'}.$$

Then  $\phi \in W_{0,\infty}^{2,q'}(D)$ ,  $\zeta \sim \psi$  and

$$\|D^2\phi\|_{L^{q'}(D)} \leq c(N, q) \|D^2\psi\|_{L^{q'}(D)} (1 + \|\psi\|_{L^\infty(D)}).$$

This inequality and (6.12) imply (6.13). □

**Lemma 6.5.** *If  $F$  is a Borel set such that  $C_{2,q'}(F) = 0$  then the only non-negative very weak subsolution of (1.1) in  $D = F^c$  is the trivial solution.*

*Remark.* A set of capacity zero is  $C_{2,q'}$ -finely closed by definition. Therefore the notion of very weak subsolution in  $F^c$  is well defined in the present case.

*Proof.* Since  $C_{2,q'}(F) = 0$ , there exists a sequence  $\{\eta_n\}$  in  $W^{2,q'}(\mathbb{R}^N)$  such that  $0 \leq \eta_n \leq 1$ ,  $\|\eta_n\|_{W^{2,q'}} \rightarrow 0$  and  $\eta_n = 1$  on a neighborhood of  $F$  (depending on  $n$ ). Applying (6.11) to  $u$  and  $\phi_n = 1 - \eta_n$  yields:

$$\int_D u^q \phi_n^{2q'} dx \leq c \|D^2\eta_n\|_{L^{q'}(D)}^{q'} \rightarrow 0.$$

Since  $\|\eta_n\|_{L^{q'}} \rightarrow 0$  it follows that there exists a subsequence converging to zero a.e.. Therefore

$$\liminf \int_D u^q \phi_n^{2q'} dx \geq \int_D u^q dx.$$

This implies that  $u = 0$  a.e. □

## 7. THE EQUATION (1.2) IN FINELY OPEN SETS

We start with the definition of 'strong' solutions of (1.2) in a  $C_{2,q'}$ -finely open set or more generally in a  $C_{2,q'}$ -quasi open set. We recall that a set  $E$  is  $C_{2,q'}$ -quasi open if, for every  $\epsilon > 0$  there exists an open set  $O$  such that  $E \subset O$  and  $C_{2,q'}(O \setminus E) < \epsilon$ . Every  $C_{2,q'}$ -finely open set is  $C_{2,q'}$ -quasi open; if  $E$  is  $C_{2,q'}$ -quasi open then  $E \stackrel{q}{\approx} \text{int}_q E$ , (see [1, Chapter 6]).

**Definition 7.1.** Let  $D$  be a  $C_{2,q'}$ -quasi open set, let  $\mu \in \mathfrak{M}_{\ell(2,q')}(D)$  be a non-negative measure and let  $\{\mu_n\}$  be a determining sequence for  $\mu$  (see Lemma 6.2).

(i) A positive function  $u \in L^q_{\ell(2,q')}(D)$  is a *strong solution* of (1.2) in  $D$  if there exists a decreasing sequence of open sets  $\{\Omega_n\}$ , such that  $D \subset \Omega_n$  and, for each  $n$ , there exists a positive solution  $u_n \in L^q_{\text{loc}}(\Omega_n)$  of the equation

$$(7.1) \quad -\Delta u_n + u_n^q = \mu_n$$

such that

$$(7.2) \quad u_n \rightarrow u \text{ in } L^q_{\ell(2,q')}(D).$$

We say that  $\{(u_n, \Omega_n)\}$  is a *determining sequence for  $u$  in  $D$* .

(ii) A strong subsolution is defined in the same way as above except that  $u_n$  is only required to be a subsolution of (7.1) in  $\Omega_n$ .

(iii) A positive strong solution of (1.2) in  $D$  is  $\sigma$ -*moderate* if, in addition, the sequence  $\{u_n\}$  is non-decreasing and there exists a sequence  $\{v_n\}$  such that  $v_n \in L^1(\Omega_n)$  and

$$(7.3) \quad -\Delta v_n = \mu_n, \quad u_n \leq v_n \text{ in } \Omega_n, \quad n = 1, 2, \dots$$

(iv) If  $\mu$  is bounded and  $\{\|v_n\|_{L^1(\Omega_n)}\}$  is bounded we say that  $u$  is a *moderate solution*.

*Remark.* If  $D$  is an open set we may choose  $\Omega_n = D$  for every  $n$ . Therefore any non-negative solution of (1.2) in  $D$  is a strong solution in  $D$ . Furthermore, if  $u$  is a  $\sigma$ -moderate solution of (1.1) in  $D$  in the standard sense (i.e. the limit of an increasing sequence of moderate solutions) then it is a  $\sigma$ -moderate strong solution in the sense of part (iii) of the above definition.

**Definition 7.2.** (a) A strong solution  $v$  of (1.1) in  $D$  is called a  $\partial_q$ -*large solution* if

$$(7.4) \quad \lim_{x \rightarrow \partial_q D}^q v(x) = \infty \quad C_{2,q'} \text{ a.e. at } \partial_q D.$$

This condition is understood as follows. There exists a determining sequence  $\{(v_n, \Omega_n)\}$  for  $v$  in  $D$  such that

$$(7.5) \quad \sum_1^{\infty} C_{2,q'}(\Omega_n \setminus D) < \infty,$$

and, for every  $M > 0$ ,  $k \in \mathbb{N}$ , there exists an open set  $Q_{k,M}$  such that

$$(7.6) \quad \begin{aligned} \bigcup_{n=k}^{\infty} \widetilde{\Omega_n \setminus D} \subset Q_{k,M}, \quad \lim_{k \rightarrow \infty} C_{2,q'}(Q_{k,M}) = 0, \\ \liminf_{\substack{x \rightarrow \partial_q D \setminus Q_{k,M} \\ x \in \Omega_n}} v_n(x) \geq M \quad \forall n \geq k. \end{aligned}$$

Note that  $\partial_q D \setminus Q_{k,M} \subset \partial \Omega_n$  for all  $n \geq k$ .

If  $F$  is a quasi closed subset of  $\partial D$ , the condition

$$(7.7) \quad \lim_{x \rightarrow F}^q v(x) = \infty \quad C_{2,q'} \text{ a.e. at } F$$

is defined in the same way except that the second line in (7.6) reads

$$\liminf_{x \rightarrow F \setminus Q_{k,M}} v_n(x) \geq M \quad \forall n \geq k.$$

(b) Let  $v$  be a non-negative strong subsolution of (1.1) in  $D$ . The condition

$$(7.8) \quad \lim_{x \rightarrow \partial D}^q v(x) = 0 \quad C_{2,q'} \text{ a.e. at } \partial_q D$$

is understood as follows. There exists a determining sequence  $\{(v_n, \Omega_n)\}$  for  $v$  in  $D$  satisfying (7.5) and a family of open sets

$$\{Q_{k,\epsilon} : \epsilon > 0, k \in \mathbb{N}\}$$

such that,

$$(7.9) \quad \bigcup_{n=k}^{\infty} \widetilde{\Omega_n \setminus D} \subset Q_{k,\epsilon}, \quad \lim_{k \rightarrow \infty} C_{2,q'}(Q_{k,\epsilon}) = 0,$$

$$(7.10) \quad \limsup_{\substack{x \rightarrow \partial_q D \setminus Q_{k,\epsilon} \\ x \in \Omega_n}} v_n(x) \leq \epsilon \quad \forall n \geq k.$$

If  $F$  is a quasi closed subset of  $\partial D$ , the condition

$$(7.11) \quad \lim_{x \rightarrow F}^q v(x) = 0 \quad C_{2,q'} \text{ a.e. at } F$$

is defined in the same way except that (7.10) is replaced by

$$(7.12) \quad \limsup_{x \rightarrow F \setminus Q_{k,\epsilon}} v_n(x) \leq \epsilon \quad \forall n \geq k.$$

(c) Let  $v$  be a non-negative strong solution of (1.1) in  $D$  and let  $u$  be a non-negative classical solution in a domain  $G \supseteq D$ . We say that  $u \stackrel{q}{\leq} v$  at  $\partial_q D$  if there exists a determining sequence  $\{(v_n, \Omega_n)\}$  for  $v$  in  $D$  satisfying (7.5) and a family of open sets  $\{Q_{k,\epsilon} : \epsilon > 0, k \in \mathbb{N}\}$  satisfying (7.9) such that

$$(7.13) \quad \limsup_{\substack{x \rightarrow \partial_q D \setminus Q_{k,\epsilon} \\ x \in \Omega_n}} (u - v_n)(x) \leq \epsilon \quad \forall n \geq k.$$

If  $F$  is a quasi closed subset of  $\partial D$ , the condition  $u \stackrel{q}{\leq} w$  at  $F$  is defined in the same way except that (7.13) is replaced by

$$(7.14) \quad \limsup_{x \rightarrow F \setminus Q_{k,\epsilon}} (u - v_n)(x) \leq \epsilon \quad \forall n \geq k.$$

We present several results concerning strong solutions. The main ingredients in these proofs are: Theorem 6.4, the results of Section 5 concerning  $V_F$  and the results of Appendix A.

**Theorem 7.3.** *For every  $\bar{L} > 0$ , there exist constants  $c = c(N, q)$  and  $c = c(\bar{L})$  such that, for every non-negative measure  $\mu \in \mathfrak{M}_{\ell(2, q')}(D)$  and every non-negative strong solution  $u$  of (1.2) in  $D$  the following holds:*

$$(7.15) \quad \|u\|_{L^q(D, \phi)}^q \leq c(N, q) \left( (\|\phi\|_{L^\infty(D)} \|D^2 \phi\|_{L^{q'}(D)})^{q'} + c(\bar{L}) \|\mu\|_{W^{-2, q'}} \|D^2 \phi\|_{L^{q'}(D)} \right)$$

for every  $\phi \in W_{0, \infty}^{2, q'}(D)$  such that  $0 \leq \phi \leq \bar{L}$ .

Every strong solution  $u$  as above satisfies,

$$(7.16) \quad u^q \phi \in L^1(D), \quad u \Delta(\phi \psi) \in L^1(D)$$

$$(7.17) \quad - \int_D u \Delta(\phi \psi) dx + \int_D u^q(\phi \psi) dx = \int_D \phi \psi d\mu$$

for any non-negative  $\phi, \psi \in W_{0, \infty}^{2, q'}(D)$ . Finally  $u$  satisfies the estimate

$$(7.18) \quad u \leq c(N, q) W_F \quad \text{a.e. in } D.$$

*Proof.* We use the notation of Definition 7.1. If  $u_n$  is a solution of (1.2) in  $\Omega_n$  and  $\phi \in W_{0, \infty}^{2, q'}(D)$  then

$$(7.19) \quad \int_D u_n \Delta \phi dx + \int_D u_n^q \phi dx = \int_D \phi d\mu_n$$

Evidently,  $u_n$  is, in particular, a very weak subsolution in  $D$ ; consequently it satisfies inequality (6.13). By assumption,  $u_n \rightarrow u$  in  $L_{\ell(2, q')}^q$ ; hence  $u$  satisfies (6.13).

Assume that  $\phi, \psi \in W_{0, \infty}^{2, q'}(D)$  and  $0 \leq \phi \leq \bar{L}$  and the same for  $\psi$ . Clearly, (7.17) holds for  $u_n$ . In addition,

$$\begin{aligned} \int_D u_n^q(\phi \psi) dx &\rightarrow \int_D u^q(\phi \psi) dx, \\ \int_D (\phi \psi) d\mu_n &\rightarrow \int_D (\phi \psi) d\mu. \end{aligned}$$

Further,

$$\Delta(\phi \psi) = \phi \Delta \psi + \psi \Delta \phi + 2 \nabla \phi \cdot \nabla \psi,$$

so that

$$\int_D u_n |\Delta(\phi \psi)| dx \leq \int_D u_n (\phi |\Delta \psi| + \psi |\Delta \phi|) dx + 2 \int_D u_n |\nabla \phi \cdot \nabla \psi| dx.$$

Using again the fact that  $u_n \rightarrow u$  in  $L^q_{\ell(2,q')}$

$$\int_D u_n(\phi\Delta\psi + \psi\Delta\phi) dx \rightarrow \int_D u(\phi\Delta\psi + \psi\Delta\phi) dx.$$

In addition,

$$\int_D u_n |\nabla\phi \cdot \nabla\psi| dx \leq \left( \int_D u_n (\nabla\phi)^2 dx \right)^{1/2} \left( \int_D u_n (\nabla\psi)^2 dx \right)^{1/2}$$

By (7.19)

$$\int_D u_n \Delta\phi^2 dx + \int_D u_n^q \phi^2 dx = \int_D \phi^2 d\mu_n$$

so that

$$\int_D u_n (\nabla\phi)^2 dx \leq \frac{1}{2} \int_D \phi^2 d\mu_n + \int_D u_n \phi |\Delta\phi| dx.$$

By Fatou, this implies,

$$\int_D u (\nabla\phi)^2 dx \leq \frac{1}{2} \int_D \phi^2 d\mu + \int_D u \phi |\Delta\phi| dx.$$

Now assume temporarily that  $\{u_n\}$  is non-decreasing so that  $u_n \uparrow u$ . Then, by the dominated convergence theorem,

$$(7.20) \quad \int_D u_n (\nabla\phi \cdot \nabla\psi) dx \rightarrow \int_D u (\nabla\phi \cdot \nabla\psi) dx.$$

The convergence results obtained above and (7.19) imply (7.16) and (7.17). This in turn implies, by Theorem 6.4, the estimate (7.15).

Discarding the assumption of monotonicity, put  $v_n := \max(u_1, \dots, u_n)$ . Then  $v_n$  is a subsolution of the equation

$$-\Delta v + v^q = \mu_n \quad \text{in } \Omega_n$$

and there exists a solution  $\bar{v}_n$  of this equation which is the smallest among those dominating  $v_n$ . Then  $\{\bar{v}_n\}$  is non-decreasing and, by Theorem 6.4,

$$\sup_n \int_D \bar{v}_n^q \phi dx < \infty$$

for any non-negative  $\phi \in W_{0,\infty}^{2,q'}(D)$ . Therefore  $w = \lim \bar{v}_n \in L^q_{\ell(2,q')}(D)$  and by the previous part of the proof  $w$  is a strong solution in  $D$ . In particular

$$w |\nabla\phi| |\nabla\psi| \in L^1(D) \quad \forall \phi, \psi \in W_{0,\infty}^{2,q'}(D).$$

Clearly,

$$u_n |\nabla\phi \cdot \nabla\psi| \leq w |\nabla\phi| |\nabla\psi|.$$

Therefore, once again by the dominated convergence theorem, we obtain (7.20) which together with the previous convergence results imply (7.15), (7.16) and (7.17).

Put  $F_n = B_R \setminus \tilde{\Omega}_n$ ,  $F = B_R \setminus \tilde{D}$ . In order to prove the last assertion, we observe that, in  $\Omega_n$ ,  $u_n \leq c(N, q)W_{F_n}$  with constant independent of  $n$ . As  $\{F_n\}$  increases,  $W_{F_n} \uparrow W_F$  everywhere in  $D$ . By (7.2) and Lemma A.4, we

can extract a subsequence of  $\{u_n\}$  which converges to  $u$  a.e. in  $D$ . Hence  $u \leq cW_F$ .  $\square$

**Theorem 7.4.** (i) *If  $F$  is a Borel set such that  $C_{2,q'}(F) = 0$  then the only non-negative strong subsolution of (1.1) in  $F^c$  is the trivial solution.*

(ii) *If  $F$  is a  $C_{2,q'}$ -finely closed set then  $V_F$  is a  $\sigma$ -moderate strong solution in  $F^c$ .*

(iii) *Let  $F$  be a  $C_{2,q'}$ -finely closed set and put  $D = \mathbb{R}^N \setminus F$ . If  $v$  is a strong solution in  $D$  then  $v \leq V_F$ .*

*Proof.* (i) By definition, a strong solution  $u$  in  $D = \mathbb{R}^N \setminus F$  is the limit of classical solutions in open sets containing  $D$ . In the case that  $C_{2,q'}(F) = 0$ , any such classical solution is the zero solution. Hence  $u = 0$ .

(ii) This is a consequence of Theorem 5.2.

(iii) By Theorem 7.3 :

$$v \leq c(N, q)W_F \leq c'(N, q)V_F \quad \text{a.e. in } \mathbb{R}^N \setminus F.$$

In addition, for every  $\alpha \geq 1$ ,

$$\sup\{u : u \text{ strong solution in } F^c, u \leq \alpha V_F\} = V_F.$$

Hence  $v \leq V_F$ .  $\square$

**Theorem 7.5.** *Let  $D$  be a  $C_{2,q'}$ -finely open set and let  $\{v_k\}$  be a sequence of non-negative strong solutions of (1.1) in  $D$  converging a.e. in  $D$ . Then  $v := \lim v_k$  is a strong solution in  $D$ .*

*Proof.* By Lemma A.4 there exists an increasing sequence of compact sets  $\{E'_n\}$  such that  $\cup E'_n \subset D$  and  $C_{2,q'}(D \setminus \cup E'_n) = 0$  and  $\{v_k\}$  is uniformly bounded in  $L^q(E'_n)$  for every  $n$ . Since  $\{v_k\}$  converges a.e. it follows that it converges in  $L^1(E'_n)$  for every  $n$ . By Theorem 7.4 (iii)  $V_{D^c}$  dominates  $\{v_k\}$ . By the dominated convergence theorem,  $v_k \rightarrow v$  in the topology  $\tau_{\ell(2,q')}^q(D)$ . By assumption, for each  $k$ ,  $v_k$  is a strong solution. This means that there exists a decreasing sequence of open sets  $\{\Omega_{m,k}\}_{m=1}^\infty$ , such that

$$(7.21) \quad D \subset \Omega_{m,k}, \quad \lim_{m \rightarrow \infty} C_{2,q'}(\Omega_{m,k} \setminus D) = 0$$

and, for each  $m$ , there exists a positive solution  $u_{m,k} \in L_{\text{loc}}^q(\Omega_{m,k})$  of the equation  $-\Delta u + u^q = 0$  in  $\Omega_{m,k}$  such that

$$u_{m,k} \rightarrow v_k \quad \text{in } L_{\ell(2,q')}^q(D).$$

By Lemma A.3 the space  $L_{\ell(2,q')}^q(D)$  with the topology  $\tau_{\ell(2,q')}^q(D)$  is a metric space. We denote a metric for this topology by  $d_{\ell(2,q')}$ . For each  $k$  let  $m_k$  be sufficiently large so that

$$d_{\ell(2,q')}(v_k, u_{m_k,k}) < 2^{-k} \quad \text{and} \quad C_{2,q'}(\Omega_{m_k,k} \setminus D) < 2^{-k}.$$

Denote  $v'_k = u_{m_k,k}$  and  $\Omega'_k = \cap_{j=1}^{m_k} \Omega_{m_j,j}$ . Then  $\{(v'_k, \Omega'_k)\}$  is a determining sequence for  $v$  in  $D$ .  $\square$

**Theorem 7.6.** *Let  $F$  be a  $C_{2,q'}$ -finely closed set and let  $\{A_n\}$  be a sequence of  $C_{2,q'}$ -finely closed subsets of  $F$ . For each  $n$ , let  $v_n$  be a strong solution in  $D_n := \mathbb{R}^N \setminus A_n$ .*

*If  $C_{2,q'}(A_n) \rightarrow 0$  then  $v_n \rightarrow 0$  a.e. in  $\mathbb{R}^N \setminus F$ .*

*In particular, if  $\sum C_{2,q'}(A_n) < \infty$  and  $v_n^*$  denotes the extension of  $v_n$  to  $\mathbb{R}^N$  such that  $v_n^* = \infty$  in  $A_n$  then,*

$$(7.22) \quad v_n^* \rightarrow 0 \quad \text{a.e. in } \mathbb{R}^N.$$

*Proof.* By Theorem 7.4(iii)

$$v_n \leq c(N, q)W_{A_n} \leq c'(N, q)V_{A_n} \quad \text{a.e. in } \mathbb{R}^N \setminus A_n.$$

By Theorem 5.1  $V_{A_n} \rightarrow 0$  a.e. in  $\mathbb{R}^N \setminus F$ . This proves the first assertion. To verify the second assertion we apply the first to the sequence  $\{A_n\}_{n=k}^\infty$  with  $F$  replaced by  $F^k = \cup_{n=k}^\infty A_n$ . Note that  $F^k$  is  $C_{2,q'}$ -finely closed up to a set of capacity zero.  $\square$

**Theorem 7.7.** *Suppose that  $z$  is a non-negative strong subsolution of (1.1) in a  $C_{2,q'}$ -quasi open set  $D$ . Then there exists a strong solution dominating it.*

*Proof.* Let  $\{(z_n, \Omega_n)\}$  be a determining sequence for  $z$ . Since  $(z_n)_+$  is also a subsolution we may assume that  $z_n \geq 0$ . Let  $Z_n$  be the smallest solution in  $\Omega_n$  which dominates  $\max(z_1, \dots, z_n)$ . Then  $Z_n \leq Z_{n+1}$  in  $\Omega_{n+1}$ . Furthermore, by Theorem 7.4(iii)  $Z_n \leq V_{D^c}$  in  $D$ . Therefore, by Theorem 7.5,  $Z = \lim Z_n$  is a strong solution in  $D$ .  $\square$

**Theorem 7.8.** *Let  $\Omega$  be a  $C_{2,q'}$ -quasi open set. Suppose that there exists a sequence of open sets  $\{G_n\}$  such that*

$$(7.23) \quad \begin{aligned} (a) \quad & C_{2,q'}(G_n \Delta \Omega) \rightarrow 0, \\ (b) \quad & C_{2,q'}(\partial G_n \setminus \partial_q \tilde{G}_n) \rightarrow 0. \end{aligned}$$

*Assume also that, for every  $n \in \mathbb{N}$ ,  $\partial G_n$  has zero Lebesgue measure. If  $v$  is a  $\partial_q$ -large solution of (1.1) in  $\Omega$  then  $v = V_{\tilde{D}}$  in  $\Omega$ ,  $D := \mathbb{R}^N \setminus \tilde{\Omega}$ . Thus  $V_{\tilde{D}}$  is the unique  $\partial_q$ -large solution in  $\Omega$ .*

*Remark.* Every  $C_{2,q'}$ -quasi open set  $\Omega$  is  $C_{2,q'}$ -equivalent to the intersection of a sequence of open sets  $\{O_n\}$  such that  $C_{2,q'}(O_n \setminus \Omega) \rightarrow 0$ . However, in the statement of the theorem, we do not require that  $G_n$  contain  $\Omega$ . Instead we require (7.23) (b).

The proof of the theorem is based on several lemmas. The first collects several useful formulas:

**Lemma 7.9.** *Let  $A, E_1, E_2$  be sets in  $\mathbb{R}^N$ . Then the following relations hold:*

$$(7.24) \quad \begin{aligned} (i) \quad & \partial_q A^c = \partial_q A, \\ (ii) \quad & \partial_q(E_1 \cup E_2) \subset \partial_q E_1 \bigcup \partial_q E_2, \\ (iii) \quad & \partial_q(E_1 \cap E_2) \subset \partial_q E_1 \bigcup \partial_q E_2, \\ (iv) \quad & \partial_q E_1 \subset \partial_q E_2 \bigcup \partial_q(E_2 \setminus E_1) \bigcup \partial_q(E_1 \setminus E_2), \\ (v) \quad & \partial_q E_1 \Delta \partial_q E_2 \subset \partial_q(E_2 \setminus E_1) \bigcup \partial_q(E_1 \setminus E_2), \\ (vi) \quad & \partial_q A \subset \partial A, \quad \partial_q \tilde{A} \subset \partial_q A. \end{aligned}$$

*Proof.* (i),(ii) and (vi) follow immediately from the definition of boundary. (iii) follows from (i), (ii) and the relation

$$(E_1 \cap E_2)^c = (E_1^c \cup E_2^c).$$

By (ii),

$$\partial_q E_1 \subset \partial_q(E_1 \cap E_2) \bigcup \partial_q(E_1 \setminus E_2).$$

By (i) and (iii), the relation,

$$E_1 \cap E_2 = E_2 \cap (E_2 \setminus E_1)^c,$$

implies that

$$\partial_q(E_1 \cap E_2) \subset \partial_q E_2 \bigcup \partial_q(E_2 \setminus E_1).$$

These relations imply (iv) which in turn implies (v). □

*Notation.* Let  $\{A_n\}$  and  $\{B_n\}$  be two sequences of sets.

(a) The notation  $A_n \stackrel{lim}{\subset} B_n$  means that  $C_{2,q'}(A_n \setminus B_n) \rightarrow 0$ .

(b) The notation  $A_n \stackrel{lim}{\sim} B_n$  means that  $C_{2,q'}(A_n \Delta B_n) \rightarrow 0$ .

**Lemma 7.10.** *Under the assumptions of the theorem,*

$$(7.25) \quad \partial_q G_n \stackrel{lim}{\sim} \partial_q \tilde{G}_n \stackrel{lim}{\sim} \partial G_n, \quad \tilde{G}_n \stackrel{lim}{\sim} \overline{G}_n$$

and

$$(7.26) \quad \partial_q G_n \stackrel{lim}{\sim} \partial_q \Omega.$$

Finally

$$(7.27) \quad C_{2,q'}(\overline{G}_n \setminus \tilde{\Omega}) \rightarrow 0.$$

*Proof.* By (7.23)(b) and Lemma 7.9 (vi) we have

$$\partial_q G_n \subset \partial G_n \stackrel{lim}{\subset} \partial_q \tilde{G}_n \subset \partial_q G_n.$$

This proves (7.25).

Condition (7.23) (a) implies that

$$(7.28) \quad C_{2,q'}(\partial_q(G_n \setminus \Omega)) \rightarrow 0, \quad C_{2,q'}(\partial_q(\Omega \setminus G_n)) \rightarrow 0.$$

This fact and Lemma 7.9 (v) imply (7.26).

By (7.25) and (7.26),

$$\overline{G}_n \setminus \Omega \stackrel{lim}{\sim} \tilde{G}_n \setminus \Omega = (G_n \setminus \Omega) \cup (\partial_q G_n \setminus \Omega).$$

By (7.23)(a) and (7.24) (v),

$$C_{2,q'}(G_n \setminus \Omega) \rightarrow 0, \quad C_{2,q'}(\partial_q G_n \Delta \partial_q \Omega) \rightarrow 0.$$

Since  $\Omega$  is  $C_{2,q'}$ -quasi open,

$$C_{2,q'}(\partial_q \Omega \cap \Omega) = 0.$$

These relations imply (7.27).  $\square$

**Lemma 7.11.** *Let  $G$  be an open set and  $Q$  be a  $C_{2,q'}$ -quasi open set. Assume that  $C_{2,q'}(\partial G \setminus \partial_q G) = 0$ . Let  $v$  be a strong solution in  $G' = G \setminus \tilde{Q}$  and let  $u$  be a (classical) solution of (1.1) in a domain  $G_0$  such that  $\bar{G} \subset G_0$ . Suppose that  $u, v$  are non-negative and*

$$(7.29) \quad u \stackrel{q}{\leq} v \quad \text{at } F := \partial_q G \setminus Q.$$

Then

$$(7.30) \quad u \leq v + V_{\tilde{Q}} \quad \text{in } G'.$$

*Proof.* Let  $\epsilon$  be a positive number. Condition (7.29) means that there exists a determining sequence  $\{(v_n, \Omega_n)\}$  for the strong solution  $v$  in  $G'$  and a family of open sets  $\{Q_{k,\epsilon}\}$  satisfying (7.5), (7.9) and (7.14) (with  $D$  replaced by  $G'$ ). We may and shall assume that  $\Omega_n \subset G$ , that  $\{\Omega_n\}$  is decreasing and that, for every  $\epsilon > 0$ ,  $\{Q_{k,\epsilon}\}_{k=1}^\infty$  is decreasing.

In the next part of the proof we keep  $\epsilon$  fixed. If  $K$  is a compact subset of  $F \setminus Q_{n,\epsilon}$  then (7.14) implies that there exists an open neighborhood of  $K$ , say  $O_K$ , such that

$$u - v_n \leq \epsilon \quad \text{in } O_K \cap \Omega_n.$$

Therefore there exists an increasing sequence of compact sets  $\{K_{n,\epsilon}\}$  and a sequence of open sets  $\{O_{n,\epsilon}\}$  such that

$$(7.31) \quad K_{n,\epsilon} \subset F \setminus Q_{n,\epsilon}, \quad C_{2,q'}(F \setminus K_{n,\epsilon}) \rightarrow 0,$$

$$(7.32) \quad K_{n,\epsilon} \subset O_{n,\epsilon}, \quad u - v_n \leq \epsilon \quad \text{in } O_{n,\epsilon} \cap \Omega_n.$$

Let  $\{O'_{n,\epsilon}\}$  be a decreasing family of open sets such that

$$(7.33) \quad (\partial G \setminus \partial_q G) \cup (F \setminus K_{n,\epsilon}) \cup \tilde{Q} \subset O'_{n,\epsilon}, \\ C_{2,q'}(O'_{n,\epsilon}) \rightarrow C_{2,q'}(Q), \quad \bigcap_{n=1}^\infty O'_{n,\epsilon} \stackrel{q}{\sim} \tilde{Q}.$$

Then  $E_{n,\epsilon} := O_{n,\epsilon} \cup O'_{n,\epsilon}$  is an open neighborhood of  $\partial G$  and

$$G \setminus E_{n,\epsilon} \subset G' \subset \Omega_n.$$

Consequently there exist smooth domains  $\Omega_{n,\epsilon}$  such that

$$\{x \in \Omega_n : \text{dist}(x, \partial \Omega_n) \geq 2^{-n}\} \subset \Omega_{n,\epsilon} \subset \overline{\Omega}_{n,\epsilon} \subset \Omega_n, \quad \partial \Omega_{n,\epsilon} \subset E_{n,\epsilon}.$$

The function  $w_{n,\epsilon} := (u - v_n - \epsilon)_+$  is a classical subsolution in  $\Omega_n$  and it vanishes in  $\Omega_n \cap O_{n,\epsilon}$ . Put  $S_{n,\epsilon} = \partial\Omega_{n,\epsilon} \setminus O_{n,\epsilon}$  and

$$z_{n,\epsilon} := \begin{cases} w_{n,\epsilon} & \text{in } \bar{\Omega}_{n,\epsilon} \setminus S_{n,\epsilon} \\ 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega}_{n,\epsilon}. \end{cases}$$

Then  $z_{n,\epsilon}$  is a (classical) subsolution in  $\mathbb{R}^N \setminus S_{n,\epsilon}$ . Since  $v_n \rightarrow v$  in  $L^q_{\ell(2,q')}(G')$ , it follows that there exists a subsequence (still denoted  $\{v_n\}$ ) such that  $v_n \rightarrow v$  a.e. in  $G'$ . Therefore  $\{z_{n,\epsilon}\}$  converges a.e. in  $D := \mathbb{R}^N \setminus \tilde{Q}$  to the function

$$z_\epsilon := \begin{cases} (u - v - \epsilon)_+ & \text{in } G' \\ 0 & \text{in } \mathbb{R}^N \setminus G. \end{cases}$$

In addition

$$\sup_{\mathbb{R}^N \setminus S_{n,\epsilon}} z_{n,\epsilon} \leq \sup_{\bar{G}} u < \infty.$$

Note that  $D \subset \mathbb{R}^N \setminus S_{n,\epsilon}$  for all  $n$ . Therefore, by the dominated convergence theorem,  $z_{n,\epsilon} \rightarrow z_\epsilon$  in  $L^q_{\ell(2,q')}(D)$ ; consequently  $z_\epsilon$  is a strong subsolution in  $D$ . In fact  $\{(z_{n,\epsilon}, \mathbb{R}^N \setminus S_{n,\epsilon})\}$  is a determining sequence for  $z_\epsilon$  in  $D$ .

By Theorem 7.7, there exists a strong solution  $Z_\epsilon$  in  $D$  such that  $z_\epsilon \leq Z_\epsilon$ . By Theorem 7.4 (iii),  $Z_\epsilon \leq V_{\tilde{Q}}$  in  $D$ . Thus  $z \leq V_{\tilde{Q}}$  and so  $u - v - \epsilon \leq V_{\tilde{Q}}$  in  $G'$ . Letting  $\epsilon \rightarrow 0$  we obtain (7.30).  $\square$

**Corollary 7.12.** *Let  $G, Q, G'$  and  $u, v$  be as in the statement of the lemma. If*

$$(7.34) \quad \lim_{x \rightarrow \partial_q G'}^q v(x) = \infty \quad C_{2,q'} \text{ a.e. at } \partial_q G'$$

then (7.30) holds.

*Proof.* Since  $u$  is bounded in  $G$ , (7.34) implies (7.29). Therefore the previous lemma implies (7.30).  $\square$

*Proof of Theorem 7.8.* Let  $K$  be a compact subset of  $\mathbb{R}^N \setminus \tilde{\Omega} = \emptyset$  and let  $\mu \in W^{-2,q}(\mathbb{R}^N)$  be a non-negative measure supported in  $K$ . We prove that

$$(7.35) \quad u_\mu \leq v \text{ in } \Omega$$

As  $\mu(\tilde{\Omega}) = 0$ , (7.23) (a) implies

$$(7.36) \quad \mu'_n := \mu \chi_{\bar{G}_n} = \mu \chi_{\bar{G}_n \setminus \tilde{\Omega}} \rightarrow 0.$$

If  $\nu$  is a bounded measure such that  $\nu(\bar{G}_n) = 0$  and  $O_{n,k}$  is a sequence of open neighborhoods of  $\bar{G}_n$  such that  $\cap_k O_{n,k} = \bar{G}_n$  then

$$\nu(O_{n,k} \setminus \bar{G}_n) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Applying this observation to  $\nu = \mu - \mu'_n$  and using (7.36) we conclude that, for every  $n \in \mathbb{N}$ , there exists a non-negative measure  $\mu_n \leq \mu$  such that

$$(7.37) \quad \mu_n \leq \mu, \quad \text{supp } \mu_n \cap \bar{G}_n = \emptyset, \quad \mu_n \uparrow \mu.$$

As  $K_n := \text{supp } \mu_n \subset K$ ,  $K_n$  is a compact set disjoint from  $\overline{G_n}$ . Therefore  $u_{\mu_n}$  is a bounded solution of (1.1) in a neighborhood of  $\overline{G_n}$ .

Let  $Q_n := G_n \setminus \widetilde{\Omega}$  so that  $\widetilde{Q}_n = \widetilde{G_n} \setminus \Omega$  and, by (7.23),

$$(7.38) \quad C_{2,q'}(\widetilde{Q}_n) \rightarrow 0.$$

Applying Corollary 7.12 to  $G_n, Q_n$  with  $u = u_{\mu_n}$  and  $w = v$  we obtain

$$(7.39) \quad u_{\mu_n} \leq v + V_{\widetilde{Q}_n} \quad \text{in } G_n \setminus \widetilde{Q}_n.$$

By (7.37)  $u_{\mu_n} \rightarrow u_\mu$  and, by (7.38),  $V_{\widetilde{Q}_n} \rightarrow 0$ . Therefore, by (7.23),

$$(7.40) \quad u_\mu \leq v \quad C_{2,q'} \text{ a.e. in } \Omega.$$

This holds for every non-negative measure  $\mu \in W^{-2,q}(\mathbb{R}^N)$  supported in a compact subset of  $\mathbb{R}^N \setminus \widetilde{\Omega}$ . Therefore  $V_{\mathbb{R}^N \setminus \widetilde{\Omega}} \leq v$  and by Theorem 5.3

$$(7.41) \quad V_{\mathbb{R}^N \setminus \Omega} \leq v \quad C_{2,q'} \text{ a.e. in } \Omega.$$

On the other hand, every strong solution in  $\Omega$  is dominated by  $V_{\mathbb{R}^N \setminus \Omega}$ . Thus  $v = V_{\mathbb{R}^N \setminus \Omega}$ . □

The following example shows that, in a sense, Theorem 7.8 complements Theorem 5.4.

*Example.* Let  $\{x^m\}$  be a sequence of distinct points in  $B_1(0)$ . Let  $\{r_n\}$  be a decreasing sequence of positive numbers such that (5.45) holds and  $\overline{B_{r_n}}(x^n) \subset B_1(0)$ . Put

$$\Omega_n = B_1(0) \setminus \cup_1^n \overline{B_{r_k}}(x^k).$$

Then there exists a unique large solution in

$$\Omega := \cap_1^\infty \Omega_n = B_1(0) \setminus \cup_1^\infty \overline{B_{r_k}}(x^k).$$

#### APPENDIX A. ON THE SPACE $W_{0,\infty}^{2,q'}$

We establish some features of the space  $W_{0,\infty}^{2,q'}$  which show that it is sufficiently rich in order to serve as a space of test functions in  $C_{2,q'}$ -finely open sets. These are used mainly in section 7.

**Lemma A.1.** *Suppose that  $D$  a  $C_{2,q'}$  finely open set and  $K$  is a bounded  $C_{2,q'}$ -finely closed subset of  $D$ . Then, for every  $a > 0$ , there exists  $\phi_a \in W^{2,q'}(\mathbb{R}^N)$  such that:*

$$(A.1) \quad \begin{aligned} (i) \quad & 0 \leq \phi_a \leq 1, \quad (ii) \quad \text{supp}_{(2,q')} \phi_a \Subset D, \\ (iii) \quad & C_{2,q'}(\{x \in K : \phi_a(x) < 1\}) < a. \end{aligned}$$

*Proof.* Let  $0 < \epsilon(1 + 2^{q'}) < a$ . Let  $K'$  be a compact set and  $D'$  be an open set such that,

$$K' \subset K, \quad D \subset D', \quad C_{2,q'}(K \setminus K') < \epsilon, \quad C_{2,q'}(\widetilde{D'} \setminus D) < \epsilon.$$

Let  $\phi$  be a smooth function with compact support in  $D'$  such that  $0 \leq \phi \leq 1$  and  $\phi = 1$  on a neighborhood of  $K'$ . Let  $\{A_n\}$  be a decreasing sequence of open neighborhoods of  $\widetilde{D' \setminus D}$  such that

$$C_{2,q'}(\tilde{A}_n) \rightarrow C_{2,q'}(\widetilde{D' \setminus D}).$$

Further, let  $\{\eta_n\}$  be a sequence of functions in  $W^{2,q'}(\mathbb{R}^N)$  such that

$$0 \leq \eta_n, \quad \eta_n \geq 1 \text{ } C_{2,q'} \text{ a.e. in } \tilde{A}_n, \quad \|\eta_n\|_{W^{2,q'}(\mathbb{R}^N)}^{q'} = C_{2,q'}(\tilde{A}_n).$$

(See [1, Thm.2.3.10] for the existence of such functions.)

Let  $\alpha \in (0, 1)$  and put  $E_n = \{x \in D : \eta_n(x) \geq 1 - \alpha\}$ . Then

$$(1 - \alpha)^{-q'} \|\eta_n\|_{W^{2,q'}(\mathbb{R}^N)} \geq C_{2,q'}(E_n)$$

so that

$$\limsup C_{2,q'}(E_n) \leq C_{2,q'}(\tilde{A}_n)/(1 - \alpha)^{q'} < \epsilon/(1 - \alpha)^{q'}.$$

Let  $h$  be a monotone, smooth cutoff function such that

$$h(t) = \begin{cases} 0 & \text{if } t < \alpha/4 \\ h(t) = t & \text{if } t > \alpha/2. \end{cases}$$

Then  $\phi_n := h \circ (\phi - \eta_n) \in W^{2,q'}(\mathbb{R}^N)$  and

$$\phi_n \geq \alpha \text{ on } K'_n := K' \setminus E_n, \quad \phi_n = 0 \text{ } C_{2,q'} \text{ a.e. in } A_n.$$

Thus, choosing  $\alpha = 1/2$ ,

$$(A.2) \quad \begin{cases} \phi_n/\alpha \geq 1 \text{ on } K'_n, & \text{supp}_{(2,q')} \phi_n \subset (\text{supp } \phi) \setminus A_n \Subset D, \\ \limsup C_{2,q'}(K \setminus K'_n) < \epsilon(1 + (1 - \alpha)^{-q'}). \end{cases}$$

By applying (to  $\phi_n/\alpha$ ) another smooth cutoff function which approximates  $\min(\cdot, 1)$ , we obtain a sequence of functions which, for  $n$  sufficiently large, satisfy the statement of the lemma.  $\square$

**Corollary A.2.** *Let  $D$  be a bounded  $C_{2,q'}$ -finely open set and let  $\{E_n\}$  be a  $q$ -exhaustion of  $D$  (see Lemma 6.1). Then there exists a sequence  $\{\varphi_n\}$  in  $W^{2,q'}(\mathbb{R}^N)$  such that:*

$$(A.3) \quad \begin{aligned} (i) & \quad 0 \leq \varphi_n \leq 1, & (ii) & \quad \text{supp}_{(2,q')} \varphi_n \Subset E_{n+1}, \\ (iii) & \quad \sum_{n=1}^{\infty} C_{2,q'}(E_n \setminus [\varphi_n = 1]) < \infty, & (iv) & \quad \{\varphi_n\} \text{ is non-decreasing.} \end{aligned}$$

In particular  $\varphi_n \uparrow 1$   $C_{2,q'}$  a.e. in  $D$ .

*Proof.* We construct  $\phi_n$  as in Lemma A.1 with  $K$  and  $D$  replaced by  $E_n$  and  $\text{int}_q E_{n+1}$ ,  $a = 2^{-n}$  and  $\alpha = 1/2$ . Then we put  $\tilde{\varphi}_n := \sum_1^n \varphi_m$  and finally apply to  $2\tilde{\varphi}_n$  a smooth cutoff function which approximates  $\min(\cdot, 1)$ .  $\square$

**Lemma A.3.** *Let  $D$  be a  $C_{2,q'}$ -finely open set and let  $\tau_{\ell(2,q')}(D)$  be the topology in  $L^q_{\ell(2,q')}(D)$  defined by the family of seminorms (6.3). Then  $\tau_{\ell(2,q')}(D)$  is a metric topology.*

*Proof.* It is sufficient to show that the space is separable. For each fixed  $\phi \in W_{0,\infty}^{2,q'}(D)$ , the space  $L^q(D; \phi)$  is separable. Let  $\{\varphi_n\}$  be as in Corollary A.2. Then, for every  $f \in L^q_{\ell(2,q')}(D)$ ,

$$\int_D f\psi(1 - \varphi_m)dx \rightarrow 0 \quad \forall \psi \in W_{0,\infty}^{2,q'}(D).$$

Therefore, if  $\{h_{k,m}\}_{k=1}^\infty$  is a dense set in  $L^q(D; \varphi_m)$  then

$$\{h_{k,m} : k, m \in \mathbb{N}\}$$

is a dense set in  $L^q_{\ell(2,q')}(D)$ .  $\square$

**Lemma A.4.** *Assume that  $F$  is a  $C_{2,q'}$ -finely closed set and  $F \subset B_{R/2}(0)$ . Put  $D = B_R(0) \setminus F$ . Let  $\{E_n\}$  be a  $q$ -exhaustion of  $D$ . Then there exists a  $q$ -exhaustion  $\{E'_n\}$  such that*

$$(A.4) \quad E'_n \subset E_n, \quad C_{2,q'}(E_n \setminus E'_n) \rightarrow 0,$$

for which the following statement holds:

*The set of non-negative very weak subsolutions of (1.1) in  $D$  is uniformly bounded in  $L^q(E'_n)$  for every  $n \in \mathbb{N}$ .*

*Proof.* Let  $\{\varphi_n\}$  be as in Corollary A.2 and let  $A_{n,k}$  be an open neighborhood of  $E_n \setminus [\varphi_n = 1]$  such that

$$C_{2,q'}(A_{n,k}) \leq (1+2^{-k})C_{2,q'}(E_n \setminus [\varphi_n = 1]), \quad \tilde{A}_{n,k+1} \subset A_{n,k} \quad \forall k \geq n, n \in \mathbb{N}.$$

Put  $E'_n = E_n \setminus \bigcup_{k=n}^\infty A_{n,k}$ . Then  $\{E'_n\}$  is a  $q$ -exhaustion of  $D$  and (A.4) holds. Furthermore,  $\varphi_n = 1$  on  $E'_n$ . Hence, by Theorem 6.4, every non-negative very weak subsolution  $u$  of (1.1) in  $D$  satisfies

$$\int_{E'_n} u^q dx \leq \int_D u^q \varphi_n^{2q'} dx \leq c(q, N) \|D^2 \varphi_n\|_{L^{q'}(D)}^{q'}.$$

$\square$

**Lemma A.5.** *Let  $K$  be a bounded  $C_{2,q'}$ -finely closed subset of  $D$ . Then, for every  $\epsilon > 0$ , there exists a compact set  $K_\epsilon \subset K$  such that*

$$C_{2,q'}(K \setminus K_\epsilon) < \epsilon, \quad f \in L^q(K_\epsilon) \quad \forall f \in L^q_{\ell(2,q')}(D)$$

*Proof.* This is an immediate consequence of Lemma A.1.  $\square$

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DEPARTMENT OF MATHEMATICS, TECHNION HAIFA, ISRAEL  
E-mail address: `marcusm@math.technion.ac.il`

LABORATOIRE DE MATHÉMATIQUES, FACULTÉ DES SCIENCES PARC DE GRANDMONT,  
37200 TOURS, FRANCE