Generic and Typical Ranks of Three-Way Arrays

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ABSTRACT

The concept of tensor rank, introduced in the twenties, has been popularized at the beginning of the seventies. This has allowed to carry out Factor Analysis on arrays with more than two indices. The generic rank may be seen as an upper bound to the number of factors that can be extracted from a given tensor, with certain uniqueness conditions. We explain how to obtain numerically the generic rank of tensors of arbitrary dimensions, and compare it with the rare algebraic results already known at order three. In particular, we examine the cases of symmetric tensors, tensors with symmetric matrix slices, or tensors with free entries. Related applications include antenna array processing.

Index Terms— Tensor, Generic rank, Canonical Decomposition, Factor Analysis, Parafac, Antenna arrays

1. INTRODUCTION

Generic ranks, defined in the complex field $\mathbb{C}$, have been studied for several decades [8] [13]. However, the value of the generic rank for arbitrary dimensions is not yet known in the unsymmetric case, and has been known in the symmetric case only recently [4] [3]. The typical rank of three-way arrays over the real field has been relevant for psychological data analysis since Carroll and Chang [1] and Harshman [7] independently proposed a method which they christened CANDECOMP and PARAFAC, respectively. The rank of a three-way array is the maximum number of components that CANDECOMP can extract uniquely up to scale and permutation indeterminacies. Thus, the study of typical rank of three-way arrays is of great theoretical importance for CANDECOMP. Although CANDECOMP was developed in a psychometric environment, its main area of applications has been Chemometrics, e.g. [12]. Besides, CANDECOMP has found important applications in signal processing, especially in Independent Component Analysis [6] [2] and in multi-user access in wireless communications [10] [11].

2. GENERIC AND TYPICAL RANKS

Let $\boldsymbol{T}$ be a L-way array of dimensions N_ℓ, $1 \leq \ell \leq L$, with values in a ring $\mathcal{R}$. This array always admits a decomposition into a sum of outer products as:

$$\boldsymbol{T} = \sum_{r=1}^{R} \boldsymbol{u}_r^{(1)} \circ \boldsymbol{u}_r^{(2)} \circ \ldots \circ \boldsymbol{u}_r^{(L)} \qquad (1)$$

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where $\boldsymbol{u}_r^{(\ell)}$ is a vector of dimension N_ℓ, $\forall r$, and $\circ$ denotes the tensor product.

Now consider an array $\boldsymbol{T}$ with values in a field $\mathbb{K}$. Arrays $\boldsymbol{u}_r^{(\ell)}$ may be considered as vectors of the linear space $\mathbb{K}^{N_\ell}$. Thus, as a combination of tensor products of vectors, $\boldsymbol{T}$ may be considered as a tensor. Under a linear change of coordinate system in each space $\mathbb{K}^{N_\ell}$, defined by a matrix $\boldsymbol{A}^{(\ell)}$, the tensor is represented by another array, obtained by the multi-linear transform $\{\boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}, \ldots, \boldsymbol{A}^{(L)}\}$. Since it is legitimate once a basis has been defined in the space, no distinction will be made in the remainder between the tensor and its array representation.

The *rank* of a given tensor $\boldsymbol{T}$ (and by extension, of the array defining its coordinates in a given basis) is the minimal integer R such that the decomposition (1) is exactly satisfied. Here this decomposition is referred to as the tensor Canonical Decomposition (CANDECOMP).

A property is called *typical* if it holds true on a set of nonzero volume [3] [4] [9] [15]. This supposes that some topology has been defined on $\mathbb{K}^{N_1 \times N_2 \times \ldots N_L}$; this can be the Zariski topology for instance, or an Euclidian topology. A property is said to be *generic* if it is true almost everywhere. In other words, a generic property is typical, but the converse is not true.

Let $N_1, \ldots, N_L$ be given positive integers. Then the rank of tensors of size $N_1 \times N_2 \times \cdots \times N_L$ is bounded, and one can make a partition of the tensor space, according to the rank values. One can define *typical ranks* as the ranks that are associated with subsets of nonzero volume in the latter partition. If there is a single typical rank, then it may be called the *generic rank*. For instance, there is a single generic rank if the underlying field $\mathbb{K}$ is algebraically closed (as the field of complex numbers, $\mathbb{C}$) [13] [3]. But there may be several typical ranks if $\mathbb{K}$ is the real field, $\mathbb{R}$.

3. COMPUTATION OF GENERIC RANKS

The algorithm proposed is directly inspired by [4]. Equation (1) can be seen as a parametrization of tensor $\boldsymbol{T}$. In fact, given a set of vectors $\{\boldsymbol{u}_r^{(\ell)} \in \mathbb{K}^{N_\ell}, 1 \leq \ell \leq L, 1 \leq r\}$, consider the mapping φ defined from a known subspace $\mathcal{T}_R$ of $(\mathbb{K}^{N_1} \times \mathbb{K}^{N_2} \times \cdots \times \mathbb{K}^{N_L})^R$ onto $\mathbb{K}^{N_1 N_2 \ldots N_L}$ as:

$$\{\boldsymbol{u}_r^{(\ell)} \in \mathcal{T}_R,\, 1 \leq r \leq R\} \rightarrow \sum_{r=1}^{R} \boldsymbol{u}_r^{(1)} \circ \boldsymbol{u}_r^{(2)} \circ \ldots \circ \boldsymbol{u}_r^{(L)}$$

Denote $\mathcal{Z}_R = \varphi(\mathcal{T}_R)$ the image of this mapping. Then the dimension D of its closure $\bar{\mathcal{Z}}_R$ is given by the rank of the Jacobian of φ, expressed in any fixed basis of $\mathbb{K}^{N_1 N_2 \dots N_L}$. If the Jacobian is of maximal rank, that is, if its rank equals the dimension of the image space (e.g. $N_1 N_2 \dots N_L$ for unconstrained arrays), then it means that R is a typical rank. Actually, R will be either the smallest typical rank, or the generic rank. Note that it is always possible to reach the maximal Jacobian rank by increasing the number of terms R, so that the smallest typical rank is always found.

This result yields the following numerical algorithm:

• Express formally the parametrized rank-one tensor term in a canonical basis

• Express formally the gradient of the latter in this basis

• Draw randomly the parameters according to an absolutely continuous distribution, and initialize matrix $\boldsymbol{J}$ with the numerical value of the gradient, and set $R = 1$

• While rank($\boldsymbol{J}$) strictly increases, do:

- Draw randomly the parameters according to an absolutely continuous distribution, and append this new numerical value of the gradient as a new row block in $\boldsymbol{J}$

- Compute the new value of $D = \mathrm{rank}(\boldsymbol{J})$

- $R \leftarrow R + 1$

• Compute the dimension of the fiber of solutions as $F = M - D$, the difference between the number of parameters and the dimension of the image $\bar{\mathcal{Z}}_R$.

In order to clarify the description of this algorithm, we give now the exact expressions of the Jacobian in various cases.

3.1. Jacobian for 3rd order asymmetric tensors with free entries

The mapping takes the form below

$$\{\boldsymbol{a}(r), \boldsymbol{b}(r), \boldsymbol{c}(r)\} \xrightarrow{\varphi} \boldsymbol{T} = \sum_{r=1}^{R} \boldsymbol{a}(r) \circ \boldsymbol{b}(r) \circ \boldsymbol{c}(r)$$

Taking into account the presence of redundancies, the number of parameters in this parametrization is $M = R(N_1 + N_2 + N_3 - 2)$. In a canonical basis, $\boldsymbol{T}$ has the coordinate vector:

$$\sum_{r=1}^{R} \boldsymbol{a}(r) \otimes \boldsymbol{b}(r) \otimes \boldsymbol{c}(r)$$

where we may decide that $\boldsymbol{a}(r)$, $\boldsymbol{b}(r)$, and $\boldsymbol{c}(r)$ are row arrays of dimension N_1, N_2, and N_3, respectively, and $\otimes$ denotes the Kronecker product. Hence, after R iterations, the Jacobian of φ is the $R(N_1 + N_2 + N_3) \times N_1 N_2 N_3$ matrix $\boldsymbol{J}$, whose rth row block is:

$$\begin{bmatrix} \boldsymbol{I}_{N_1} & \otimes & \boldsymbol{b}(r) & \otimes & \boldsymbol{c}(r) \\ \boldsymbol{a}(r) & \otimes & \boldsymbol{I}_{N_2} & \otimes & \boldsymbol{c}(r) \\ \boldsymbol{a}(r) & \otimes & \boldsymbol{b}(r) & \otimes & \boldsymbol{I}_{N_3} \end{bmatrix} \tag{2}$$

The values of the generic rank obtained with this algorithm, called `rangj3(N1,N2,N3)`, or `rangj(N,L)`[1] for tensors of arbitrary order L and equal dimensions, are reported in tables 1, 2, and 3.

[1] Corresponding Matlab and Scilab codes can be downloaded from `www.i3s.unice.fr/~pcomon`.

3.2. Jacobian for 3rd order asymmetric tensors with symmetric matrix slices

In this section, we consider tensors of size $N_2 \times N_2 \times N_3$, having symmetric $N_2 \times N_2$ matrix slices, hence the mapping:

$$\{\boldsymbol{b}(r), \boldsymbol{c}(r)\} \longrightarrow \boldsymbol{T} = \sum_{r=1}^{R} \boldsymbol{b}(r) \circ \boldsymbol{b}(r) \circ \boldsymbol{c}(r).$$

Our code `rgindscal3(N2,N3)` implements the computation of the rank of the Jacobian $\boldsymbol{J}$, whose rth row block is given below, when its size increases according to the algorithm described in section 3:

$$\begin{bmatrix} \boldsymbol{I}_{N_2} \otimes \boldsymbol{b}(r) \otimes \boldsymbol{c}(r) + \boldsymbol{b}(r) \otimes \boldsymbol{I}_{N_2} \otimes \boldsymbol{c}(r) \\ \boldsymbol{b}(r) \otimes \boldsymbol{b}(r) \otimes \boldsymbol{I}_{N_3} \end{bmatrix} \tag{3}$$

After R iterations, this matrix is of size $R(N_2+N_3) \times N_2^2 N_3$. The number of parameters in this parametrization is $M = R(N_2 + N_3 - 1)$. Values of the generic rank are reported in table 4.

3.3. Jacobian for 3rd order double centered tensors with symmetric matrix slices

Now, take again $N_2 \times N_2 \times N_3$ tensors with symmetric $N_2 \times N_2$ matrix slices, but assume in addition that every row and column in the latter matrix slices are zero-mean. In order to achieve this, it is sufficient to generate vectors $\boldsymbol{b}(r)$ with zero mean [5]; in other words, only $N_2 - 1$ random numbers need to be drawn, the last entry of each vector $\boldsymbol{b}(r)$ being obtained via $b_{N_2} = -\sum_{n_2=1}^{N_2-1} b_{n_2}$. The Jacobian is then built from R row blocks of the form

$$\begin{bmatrix} [\boldsymbol{I}_{N_2-1},\ -\mathbf{1}] \otimes \boldsymbol{b}(r) \otimes \boldsymbol{c}(r) + \boldsymbol{b}(r) \otimes [\boldsymbol{I}_{N_2-1},\ -\mathbf{1}] \otimes \boldsymbol{c}(r) \\ \boldsymbol{b}(r) \otimes \boldsymbol{b}(r) \otimes \boldsymbol{I}_{N_3} \end{bmatrix} \tag{4}$$

where $\mathbf{1}$ denotes a column of ones of size $N_2 - 1$. At the Rth iteration, this matrix is of size $R(N_2 + N_3 - 1) \times N_2^2 N_3$. The number of parameters in this parametrization is $M = R(N_2 + N_3 - 2)$. Table 5 reports some numerical values obtained with the code `rgindscal2z`.

3.4. Jacobian for 3rd order tensors with double centered matrix slices

The previous reasoning can be applied to $N_1 \times N_2 \times N_3$ tensors with no symmetry constraint and whose $N_1 \times N_2$ matrix slices have zero-mean rows and column. As before, it is sufficient to generate vectors $\boldsymbol{a}(r)$ and $\boldsymbol{b}(r)$ with zero mean. The Jacobian is then composed of row blocks of the form::

$$\begin{bmatrix} [\boldsymbol{I}_{N_1-1},\ -\mathbf{1}] \otimes \boldsymbol{b}(r) \otimes \boldsymbol{c}(r) \\ \boldsymbol{a}(r) \otimes [\boldsymbol{I}_{N_2-1},\ -\mathbf{1}] \otimes \boldsymbol{c}(r) \\ \boldsymbol{a}(r) \otimes \boldsymbol{b}(r) \otimes \boldsymbol{I}_{N_3} \end{bmatrix} \tag{5}$$

At the Rth iteration, this matrix is of size $R(N_1 + N_2 + N_3 - 2) \times N_1 N_2 N_3$. The number of parameters in this parametrization is $M = R(N_1 + N_2 + N_3 - 3)$. The numerical values obtained with the code `rangj3z` are not reported, since we always have, for any triplet (N_1, N_2, N_3):

`rangj3z(N1,N2,N3)`=`rangj3(N1-1,N2-1,N3)`. In other words, as far as the generic rank is concerned, centering in a given mode of dimension N_i yields the same effect as reducing the dimension to $N_i - 1$, which makes sense.

3.5. Jacobian for symmetric tensors

In the case of symmetric tensors of dimension N and order L, the mapping φ is defined from $\mathbb{K}^{NR}$ to the space of symmetric tensors [4], or equivalently to $\mathbb{K}^p$ with $p = \binom{N+L-1}{L}$, as:

$$\{\boldsymbol{a}(r) \in \mathbb{K}^N,\ 1 \leq r \leq R\} \xrightarrow{\varphi} \sum_{r=1}^{R} \boldsymbol{a}(r)^{\circ L}$$

where $\circ$ stands for the tensor (outer) product; once a basis is chosen, the tensor product may be replaced by a Kronecker product, yielding exactly the same expression. In the case of order-3 tensors ($L = 3$) and after R iterations, the Jacobian of φ R blocks of the following form, somewhat simpler than the previous cases:

$$\boldsymbol{I}_N \otimes \boldsymbol{a}(r) \otimes \boldsymbol{a}(r) + \boldsymbol{a}(r) \otimes \boldsymbol{I}_N \otimes \boldsymbol{a}(r) + \boldsymbol{a}(r) \otimes \boldsymbol{a}(r) \otimes \boldsymbol{I}_N \quad (6)$$

This matrix is of size $RN \times N^3$, but we know that its rank cannot exceed $\binom{N+2}{3} = N(N+1)(N+2)/6$. The number of parameters in this parametrization is $M = RN$. Numerical values of the generic rank obtained with `rangjs(N,L)` are reported in table 6.

N_3	2				3			4	
N_2	2	3	4	5	3	4	5	4	5
N_1									
2	2,3	3	4	4	3,4	4	5	4,5	5
3	3	3,4	4	5	5	**5**	5,6	**6**	**6**
4	4	4	4,5	5	5	**6**	**6**	**7**	**8**
5	4	5	5	5,6	5,6	**6**	**8**	**8**	**9**
6	4	6	6	6	6	**7**	**8**	**8**	**10**
7	4	6	7	7	7	**7**	**9**	**9**	**10**
8	4	6	8	8	8	8,9	**9**	**10**	**11**
9	4	6	8	9	9	9	**9**	**10**	**12**
10	4	6	8	10	9	10	10	**10**	**12**
11	4	6	8	10	9	11	11	**11**	**13**
12	4	6	8	10	9	12	12	12,13	**13**

Table 1. Typical ranks for 2-, 3- and 4-slice unconstrained arrays.

4. NUMERICAL RESULTS

The available results on unconstrained, slicewise symmetric, and double centered arrays can be compared with the numerical values delivered by the computer codes.

Tensors with free entries. Table 1 reports typical ranks for 2-slice, 3-slice, and 4-slice arrays. The smallest of the known typical rank values [14, 17], in plain, coincides with the generic rank computed with `rangj3`. For the yet unknown entries, results from `rangj3` are inserted in bold.

We report values of the smallest typical/generic rank of 3-way arrays with equal dimensions in table 2. Kruskal [9, p. 9] refers to a "much studied $9 \times 9 \times 9$ array whose rank has been bounded between 18 and 23 but is still unknown". `rgindscal(9,9)` yields 19 as a typical rank value, which is within the range $\{18, 23\}$ given by Kruskal. From this it may be conjectured that the array in question had symmetric slices and either rank 19 or 20.

Now the algorithm can be run on tensors of order higher than 3. For simplicity, table 3 reports values of the generic rank obtained for asymmetric tensors with equal dimensions, N, and order L, with an algorithm referred to as `rangj(N,L)`. We also indicate the dimensionality of the fiber of solutions. This number is simply defined as the difference:

$$F(N, L) = \bar{R}(N, L)\,(LN - L + 1) - N^L$$

For those values of dimension and order for which $F = 0$, only a finite number of different CANDECOMP are possible.

N	2	3	4	5	6	7	8	9
$\bar{R}$	2	5	**7**	**10**	**14**	**19**	**24**	**30**

Table 2. Smallest typical rank $\bar{R}$ of unconstrained arrays of dimension $N \times N \times N$.

L \ N	2	3	4	5	6	7	8
3	2	5	**7**	**10**	**14**	**19**	**24**
4	**4**	**9**	**20**	**37**	**62**	**97**	
L \ N	2	3	4	5	6	7	8
3	0	8	6	5	8	18	16
4	4	0	4	4	6	24	

Table 3. Top: smallest typical rank $\bar{R}$ of unconstrained arrays of equal dimensions, N, and order L. In $\mathbb{C}$ these values are generic. Bottom: Number F of remaining degrees of freedom; when $F = 0$, there are only a finite number of CANDECOMP.

Tensors with symmetric matrix slices. We next turn to the $N_1 \times N_2 \times N_2$ arrays with N_1 symmetric slices (Table 4). Again, known values coincide with numerical ones delivered by the code `rgindscal3`. We inserted results obtained from `rgindscal3` alone in bold face. As far as can be determined, all results are again in agreement with previously known values [16].

Tensors with double centered symmetric matrix slices. When the matrix slices are symmetric and also row-wise (or column-wise, which is the same thing) zero-mean, the code `rgindscal2z` yielded the values reported in table 5. Note that the generic rank computed by `rgindscal2z(N2,N1)` is the same as that computed by `rgindscal3(N2-1,N1)`, at least according to the values explored in table 4. This illustrates the point made earlier in this paper.

Tensors with double centered matrix slices without symmetry constraint. A similar observation holds also true when the centered matrix slices are not symmetric. We do not separately report typical rank values for the case of double-centered (non symmetric) slices. Instead, we verified that the values obtained numerically with centering coincided

with the values obtained numerically for uncentered arrays: `rangj3z(N1,N2,N3)=rangj3(N1-1,N2-1,N3)`.

N_1 N_2	2	3	4	5
2	2,3	3,4	4,5	5,6
3	3	4	**6**	**7**
4	3	4,5	**6**	**8**
5	3	5,6	**7**	**9**
6	3	6	**7**	**9**
7	3	6	**7**	**10**
8	3	6	**8**	**10**
9	3	6	9,10	**11**
10	3	6	10	**11**

Table 4. Typical ranks for $N_1 \times N_2 \times N_2$ arrays, with $N_2 \times N_2$ symmetric slices. Bold: smallest typical ranks computed numerically. Plain: known typical ranks; in $\mathbb{C}$, the smallest value is generic.

N_1 N_2	2	3	4	5
2	1	**2**	**3**	**4**
3	1	3	4	**6**
4	1	3	**4**	**6**
5	1	3	**5**	**7**
6	1	3	6	**7**
7	1	3	6	**7**
8	1	3	6	**8**
9	1	3	6	**9**
10	1	3	6	10

Table 5. Smallest typical rank $\bar{R}$ for $N_1 \times N_2 \times N_2$ arrays, with $N_2 \times N_2$ symmetric slices having zero-mean columns. In the complex field, these values are generic.

Symmetric tensors. In table 6, generic ranks obtained with the code `rangjs` for 3-way or 4-way symmetric are reported. They are the same as in [4]. The dimensionality of the fiber of solutions is $F(N,L) = \bar{R}\,N - \binom{N+L-1}{L}$. It is interesting to compare the ranks with those of the unsymmetric case, obviously larger, reported in table 3. In particular, one can observe that that the case $F = 0$ is again rarely met with generic arrays, but less rarely than in the non-symmetric case.

Conclusion. The values reported in table 1 demonstrate that the bound given by Kruskal, which ensures uniqueness of the CANDECOMP, is sufficient but not necessary. This motivates the design of numerical algorithms, other than Kruskal's ALS, able to compute the CANDECOMP under assumptions less retrictive than [7, 9, 11, 12], i.e. for any sub-generic rank.

L N	2	3	4	5	6	7	8
3	2	4	5	8	10	12	15
4	3	6	10	15	21	30	42
L N	2	3	4	5	6	7	8
3	0	2	0	5	4	0	0
4	1	3	5	5	0	0	6

Table 6. (top) Smallest typical ranks $\bar{R}$ of symmetric arrays of dimension N and order L. In the complex field, these values are generic. (bottom) Number F of remaining degrees of freedom; when $F = 0$, there are only a finite number of CANDECOMP possibilities.

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