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Anne Souquière

► **To cite this version:**

Anne Souquière. Approximation and representation of the value for some differential games with imperfect information. 2008. hal-00327540

HAL Id: hal-00327540

<https://hal.science/hal-00327540>

Preprint submitted on 13 Oct 2008

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Approximation and representation of the value for some differential games with imperfect information

A. SOUQUIERE

1-Institut TELECOM ; TELECOM Bretagne ;
UMR CNRS 3192 Lab-STICC,
Technople Brest Iroise, CS 83818, 29238 Brest Cedex 3
2- Université Européenne de Bretagne, France
Anne.Souquiere@telecom-bretagne.eu

1st October 2008

Abstract

We consider differential games with imperfect information. For special games with dynamics independent of the state of the system and linear payoffs, we give a representation formula for the value similar to the value of repeated games with lack of information on both sides. For general games, this representation formula does not hold and we introduce an approximation of the value : we build a sequence of functions converging to the value function.

Keywords

Differential games - Asymmetric information - Representation formulas - HamiltonJacobi equations - Viscosity solutions

Introduction

We consider a zero-sum differential game $G_T(t_0, x_0, p, q)$ with finite horizon T and dynamics :

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)) & t \in [t_0, T] & u(t) \in U & v(t) \in V \\ x(t_0) = x_0 & & x_0 \in \mathbb{R}^n & \end{cases} \quad (1)$$

where U and V are compact subsets of some finite dimensional spaces and $f : \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^n$ is Lipschitz continuous. We introduce asymmetric information on the payoff the following way cf. [4] :

- there exists a finite number of possible types for player I indexed by $i \in I$ and for player II indexed by $j \in J$;

- to each pair of types (i, j) we associate a final payoff function $g_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}$ and a running payoff function $l_{ij} : \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}$;
- each pair of types (i, j) is selected at random with the probability distribution $p \otimes q \in \Delta(I) \times \Delta(J)$ and the distribution on types is supposed common knowledge.

At the beginning of the game, Nature chooses at random a pair (i, j) with probability $p \otimes q$. Each player is informed only of his true type and has therefore partial information on the true payoff function of the game he's playing.

During the game, player I chooses $u_i(s)$, $s \in [t_0, T]$, in order to control the state of the system (1) and minimize the total payoff

$$g_{ij}(x_{ij}(T)) + \int_{t_0}^T l_{ij}[x_{ij}(s), u_i(s), v_j(s)] ds,$$

whereas player II chooses control v_j to maximize the payoff. We will always assume that the players observe the controls played so far.

The final payoff of this differential game with asymmetric information is the expected value of the payoffs associated to (i, j) under the product probability $p \otimes q$:

$$\mathbf{E}_{p \otimes q}[g_{ij}(x_{ij}(T)) + \int_{t_0}^T l_{ij}[x_{ij}(s), u_i(s), v_j(s)] ds]$$

These kinds of games, inspired by [1], have been studied by Cardaliaguet [4] for games without running payoff and Cardaliaguet and Rainer [3] for stochastic games with final and running payoff. Using these results, we recall in the first section that under usual regularity assumptions on the dynamics and the payoff functions, and assuming Isaacs' condition on the Hamiltonian, namely for all $(x, \xi, p, q) \in \mathbb{R}^n \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$:

$$\begin{aligned} H(x, \xi, p, q) &= \inf_{u \in U} \sup_{v \in V} [\langle f(x, u, v), \xi \rangle + \sum_{i,j} p_i q_j l_{ij}(x, u, v)] \\ &= \sup_{v \in V} \inf_{u \in U} [\langle f(x, u, v), \xi \rangle + \sum_{i,j} p_i q_j l_{ij}(x, u, v)] \end{aligned} \quad (2)$$

the game has a value characterized as the unique dual viscosity solution of the following Hamilton-Jacobi equation :

$$\frac{\partial \phi}{\partial t} + H(x, D\phi, p, q) = 0 \quad (t, x, p, q) \in [t_0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$$

In section 2, we investigate a special class of differential games with asymmetric information with dynamics and running payoff functions independent of the state of the system and linear final payoff functions $g_{ij}(x) = \langle a_{ij}, x \rangle$.

This game is interesting because it is closely related to repeated games with lack of information on both sides studied by Mertens and Zamir [8]. Indeed, the operator Φ which associates to any Lipschitz continuous function $W : \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ the unique solution of the following system of functional equations :

$$\begin{cases} \Phi W = Cav_q \min(\Phi W, W) \\ \Phi W = Vex_p \max(\Phi W, W) \end{cases}$$

allows us to link the value V of the game with imperfect information to the value W of another game with perfect information leading to the representation formula for the value :

$$V(t, x, p, q) = (T - t)\Phi[H(\sum_{i,j} p_i q_j a_{ij}, p, q)] + \sum_{i,j} p_i q_j \langle a_{ij}, x \rangle$$

This representation formula cannot be generalized to generic differential games with imperfect information, as shown in [5]. This leads to the approximation introduced in section 3, where we build a sequence of functions V_τ converging as $\tau \rightarrow 0$ to the value of the game for the topology of the uniform convergence on compacts as in [6].

We define the function V_τ only at discrete times of the form $t_k := k\tau$ where τ is a small time step. For the final time T , we set

$$V_\tau(T, x, p, q) := \sum_{i,j} p_i q_j g_{ij}(x).$$

Then using backward induction and assuming $V_\tau(t_{k+1}, \cdot, \cdot, \cdot)$ is already built, we set

$$V_\tau(t_k, x, p, q) := \Phi \left\{ \min_{u \in U} \max_{v \in V} [V_\tau(t_{k+1}, x + \tau f(x, u, v), p, q) + \tau \sum_{i,j} p_i q_j l_{ij}(x, u, v)] \right\}$$

We prove that $V_\tau \rightarrow V$ locally uniformly as $\tau \rightarrow 0^+$. This is a generalization of the result stated in [6] that dealt with differential game with imperfect information on one side.

1 Definitions and characterization of the value

We denote by $\mathcal{U}(t_0)$ (resp. $\mathcal{V}(t_0)$) the set of measurable controls of player I (resp. player II) :

$$\begin{aligned} \mathcal{U}(t_0) &:= \{u(\cdot) : [t_0, T] \rightarrow U, u \text{ measurable}\} \\ \mathcal{V}(t_0) &:= \{v(\cdot) : [t_0, T] \rightarrow V, v \text{ measurable}\} \end{aligned}$$

Definition 1. Strategies

1. A pure strategy for player I at time t_0 is a map $\alpha : \mathcal{V}(t_0) \rightarrow \mathcal{U}(t_0)$ which satisfies the following conditions :

- α is a measurable map from $\mathcal{V}(t_0)$ to $\mathcal{U}(t_0)$ where $\mathcal{V}(t_0)$ and $\mathcal{U}(t_0)$ are endowed with the Borel σ -field associated with the L^1 distance,
- α is non anticipative with delay, i.e. there is some delay $\tau > 0$ such that for any $v_1, v_2 \in \mathcal{V}(t_0)$, if $v_1 \equiv v_2$ a.e. on $[t_0, t]$ for some $t \in [t_0, T]$, then $\alpha(v_1) \equiv \alpha(v_2)$ a.e. on $[t_0, (t + \tau) \wedge T]$

We denote by $\mathcal{A}(t_0)$ (resp. $\mathcal{B}(t_0)$) the set of pure strategies for player I (resp. player II).

2. A mixed strategy for player I at time t_0 is a pair $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha), \alpha)$ where $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha)$ is a probability space and $\alpha : \Omega_\alpha \times \mathcal{V}(t_0) \rightarrow \mathcal{U}(t_0)$ satisfies :

- α is a measurable map from $\Omega_\alpha \times \mathcal{V}(t_0)$ to $\mathcal{U}(t_0)$ where Ω_α is endowed with the σ -field \mathcal{F}_α and $\mathcal{V}(t_0)$ and $\mathcal{U}(t_0)$ with the Borel σ -field associated with the L^1 distance,
- α is non anticipative with delay, i.e. there is some delay $\tau > 0$ such that for any $\omega_\alpha \in \Omega_\alpha$, the pure strategy $\alpha(\omega_\alpha, \cdot)$ is non anticipative with delay τ

From now on, mixed strategies $((\Omega_\alpha, \mathcal{F}_\alpha, \mathbf{P}_\alpha), \alpha)$ will be simply denoted by α . We denote by $\mathcal{A}_r(t_0)$ (resp. $\mathcal{B}_r(t_0)$) the set of mixed strategies for player I (resp. player II).

Note that any pure strategy can be considered as a mixed strategy whose underlying probability space is trivial, namely $\mathcal{A}(t_0) \subset \mathcal{A}_r(t_0)$ and $\mathcal{B}(t_0) \subset \mathcal{B}_r(t_0)$.

Remark that these definitions allow us, due to the delay, to associate to any pair of pure strategies $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$ a unique pair of controls $(u_{\alpha\beta}, v_{\alpha\beta}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that $\alpha(v_{\alpha\beta}) = u_{\alpha\beta}$ and $\beta(u_{\alpha\beta}) = v_{\alpha\beta}$ as shown in [2].

Given any pair of pure strategies $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$, we denote by $(X_t^{t_0, x_0, \alpha, \beta})$ the map $t \mapsto X_t^{t_0, x_0, u_{\alpha\beta}, v_{\alpha\beta}}$ defined on $[t_0, T]$ where $X^{t_0, x_0, u_{\alpha\beta}, v_{\alpha\beta}}$ is the unique solution of the dynamics (1).

Using the results of [4], if $(\alpha, \beta) \in \mathcal{A}_r(t_0) \times \mathcal{B}_r(t_0)$, for all $\omega = (\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$, there is a unique pair of controls $(u_\omega, v_\omega) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that $\alpha(\omega_\alpha, v_\omega) = u_\omega$ and $\beta(\omega_\beta, u_\omega) = v_\omega$. The map $\omega \mapsto (u_\omega, v_\omega)$ is measurable from $\Omega_\alpha \times \Omega_\beta$ endowed with $\mathcal{F}_\alpha \otimes \mathcal{F}_\beta$ into $\mathcal{U}(t_0) \times \mathcal{V}(t_0)$ endowed with the Borel σ -field associated with the L^1 distance.

This property allows to compute the payoff of the game with types (i, j) if player I plays $\alpha \in \mathcal{A}_r(t_0)$ and player II plays $\beta \in \mathcal{B}_r(t_0)$:

$$\begin{aligned} \mathfrak{J}_{ij}(t_0, x_0, \alpha, \beta) &:= \mathbf{E}_{\alpha\beta}[g_{ij}(X_T^{t_0, x_0, \alpha, \beta}) + \int_{t_0}^T l_{ij}(X_s^{t_0, x_0, \alpha, \beta}, u_{\alpha\beta}(s), v_{\alpha\beta}(s)) ds] \\ &= \int_{\Omega_\alpha \times \Omega_\beta} [g_{ij}(X_T^{t_0, x_0, u_\omega, v_\omega}) \\ &\quad + \int_{t_0}^T l_{ij}(X_s^{t_0, x_0, u_\omega, v_\omega}, u_\omega(s), v_\omega(s)) ds] d\mathbf{P}_\alpha \otimes d\mathbf{P}_\beta(\omega) \end{aligned}$$

Note that in order to take into account his information, in $G_T(t_0, x_0, p, q)$, player I has to choose a strategy vector $\hat{\alpha} = (\alpha_i) \in \mathcal{A}_r(t_0)^I$ and player II picks a strategy $\hat{\beta} = (\beta_j) \in \mathcal{B}_r(t_0)^J$. The final payoff in $G_T(t_0, x_0, p, q)$ in this case becomes :

$$\mathfrak{J}(t_0, x_0, p, q, \hat{\alpha}, \hat{\beta}) = \mathbf{E}_{p \otimes q}[\mathfrak{J}_{ij}(t_0, x_0, \alpha_i, \beta_j)] = \sum_{ij} p_i q_j \mathfrak{J}_{ij}(t_0, x_0, \alpha_i, \beta_j)$$

Definition 2. Dual Hamilton-Jacobi equation

We define the dual Hamilton-Jacobi equation by

$$\frac{\partial \phi}{\partial t} + H^*(x, D\phi, p, q) = 0 \quad (t, x, p, q) \in (t_0, T) \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$$

and $H^*(x, \xi, p, q) := -H(x, -\xi, p, q)$, or

$$\begin{aligned} H^*(x, \xi, p, q) &= \sup_{u \in U} \inf_{v \in V} [\langle f(x, u, v), \xi \rangle - \sum_{i,j} p_i q_j l_{ij}(x, u, v)] \\ &= \inf_{v \in V} \sup_{u \in U} [\langle f(x, u, v), \xi \rangle - \sum_{i,j} p_i q_j l_{ij}(x, u, v)] \end{aligned}$$

Definition 3. Dual viscosity solution

The dual viscosity solution of the Hamilton-Jacobi equation :

$$\frac{\partial \phi}{\partial t} + H(x, D\phi, p, q) = 0 \quad (t, x, p, q) \in [t_0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$$

is a Lipschitz continuous function $w(t, x, p, q)$, convex in p , concave in q and such that

- if we design by w^* the convex Fenchel conjugate of w , namely $w^*(t, x, \hat{p}, q) = \sup_{p \in \Delta(I)} \{\langle \hat{p}, p \rangle - w(t, x, p, q)\}$, for any test function $\phi \in C^1((t_0, T) \times \mathbb{R}^n)$ such that $(t, x) \mapsto w^*(t, x, \hat{p}, q) - \phi(t, x)$ has a maximum at some point (\bar{t}, \bar{x}) for some $(\hat{p}, q) \in \mathbb{R}^I \times \Delta(J)$ such that $\frac{\partial w^*}{\partial \hat{p}}(\bar{t}, \bar{x}, \hat{p}, q)$ exists, we have:

$$\phi_t(\bar{t}, \bar{x}) + H^*(\bar{x}, D\phi(\bar{t}, \bar{x}), p, q) \geq 0 \text{ where } p = \frac{\partial w^*}{\partial \hat{p}}(\bar{t}, \bar{x}, \hat{p}, q)$$

- if we design by $w^\#$ the concave Fenchel conjugate of w , namely $w^\#(t, x, p, \hat{q}) = \inf_{q \in \Delta(J)} \{ \langle \hat{q}, q \rangle - w(t, x, p, q) \}$, for any test function $\phi \in C^1((t_0, T) \times \mathbb{R}^n)$ such that $(t, x) \mapsto w^\#(t, x, p, \hat{q}) - \phi(t, x)$ has a minimum at some point (\bar{t}, \bar{x}) for some $(p, \hat{q}) \in \mathbb{R}^I \times \Delta(J)$ such that $\frac{\partial w^\#}{\partial \hat{q}}(\bar{t}, \bar{x}, p, \hat{q})$ exists, we have:

$$\phi_t(\bar{t}, \bar{x}) + H^*(\bar{x}, D\phi(\bar{t}, \bar{x}), p, q) \leq 0 \text{ where } q = \frac{\partial w^\#}{\partial \hat{q}}(\bar{t}, \bar{x}, p, \hat{q})$$

A Lipschitz continuous function $w(t, x, p, q)$, convex in p , concave in q , satisfying only the first item of the definition will be called a dual super-solution of the Hamilton-Jacobi equation, and a dual sub-solution if it satisfies only the second item of the definition.

We assume usual regularity assumptions [3] :

$$\left\{ \begin{array}{l} U \text{ and } V \text{ are compact subsets of some finite dimensional spaces,} \\ f : \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^n \text{ is bounded, continuous and uniformly} \\ \quad \text{Lipschitz continuous w.r.t. } x \\ g_{ij} : \mathbb{R}^n \rightarrow \mathbb{R} \text{ are Lipschitz continuous and bounded} \\ l_{ij} : \mathbb{R}^n \times U \times V \rightarrow \mathbb{R} \text{ are Lipschitz continuous and bounded} \end{array} \right. \quad (3)$$

We recall a result stated in [3] :

Proposition 1.1. Characterization of the value

Under regularity assumptions (3) and Isaacs'condition (2), the game with imperfect information and running and final payoff $G_T(t_0, x_0, p, q)$ has a value V , namely :

$$\begin{aligned} V(t_0, x_0, p, q) &= \inf_{\hat{\alpha} \in \mathcal{A}_r(t_0)^I} \sup_{\hat{\beta} \in \mathcal{B}_r(t_0)^J} \mathfrak{J}(t_0, x_0, p, q, \hat{\alpha}, \hat{\beta}) \\ &= \sup_{\hat{\beta} \in \mathcal{B}_r(t_0)^J} \inf_{\hat{\alpha} \in \mathcal{A}_r(t_0)^I} \mathfrak{J}(t_0, x_0, p, q, \hat{\alpha}, \hat{\beta}) \end{aligned}$$

Furthermore, the value V is characterized as the only dual viscosity solution of the Hamilton-Jacobi equation

$$\phi_t + H(x, D\phi, p, q) = 0 \quad (4)$$

with terminal condition

$$V(T, x, p, q) = \sum_{i,j} p_i q_j g_{ij}(x) \quad (5)$$

The following Lemma gives a more convenient characterization of dual solutions.

Lemma 1.2. For all function $w : [t_0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ Lipschitz continuous, convex in p and concave in q ,

1. there is equivalence between :

- w is a dual super-solution of $\phi_t + H(x, D\phi, p, q) = 0$
- for all fixed $q \in \Delta(J)$, for all test function $\phi = \phi(t, x, p)$ C^1 convex in p and such that $(t, x, p) \mapsto w(t, x, p, q) - \phi(t, x, p)$ has a strict global minimum at $(\bar{t}, \bar{x}, \bar{p}) \in (t_0, T) \times \mathbb{R}^n \times \Delta(I)$:

$$\phi_t(\bar{t}, \bar{x}, \bar{p}) + H(\bar{x}, D\phi(\bar{t}, \bar{x}, \bar{p}), \bar{p}, q) \leq 0$$

2. there is equivalence between :

- w is a dual sub-solutions of $\phi_t + H(x, D\phi, p, q) = 0$
- for all fixed $p \in \Delta(I)$, for all test function $\phi = \phi(t, x, q)$ C^1 concave in q and such that $(t, x, q) \mapsto w(t, x, p, q) - \phi(t, x, q)$ has a strict global maximum at $(\bar{t}, \bar{x}, \bar{q}) \in [t_0, T) \times \mathbb{R}^n \times \Delta(J)$:

$$\phi_t(\bar{t}, \bar{x}, \bar{q}) + H(\bar{x}, D\phi(\bar{t}, \bar{x}, \bar{q})) \geq 0$$

2 Representation formula for a specific class of differential games

In this section, we consider a very special class of games for which :

- the dynamics f and the running payoffs l_{ij} do not depend on the state of the system x
- the final payoffs $g_{ij}(x)$ are linear of the form $g_{ij}(x) := \langle a_{ij}, x \rangle$, for fixed $a_{ij} \in \mathbb{R}^n$

We will assume that regularity assumptions (3) and Isaacs'condition (2) are fulfilled.

The dynamics is now :

$$\begin{cases} x'(t) = f(u(t), v(t)) & t \in [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

Note that in this game the Hamiltonian $H(x, \xi, p, q)$ does not depend on x any more and will be denoted by :

$$\begin{aligned} H(\xi, p, q) &= \inf_u \sup_v [\langle f(u, v), \xi \rangle + \sum_{i,j} p_i q_j l_{ij}(u, v)] \\ &= \sup_v \inf_u [\langle f(u, v), \xi \rangle + \sum_{i,j} p_i q_j l_{ij}(u, v)] \end{aligned}$$

These games have been introduced by Cardaliaguet [5] and have the same kind of properties as repeated games studied by Aumann and Maschler [1]. Indeed, in case player I only is informed of the true state of nature (namely $I > 1$ and $J = 1$), the value V of the game is linked to the value W of a game with perfect information that can be derived from the original game (the non-revealing game) through the formula

$$V(t, x, p) = \text{Vex}_p(W)(t, x, p)$$

We extend this formula to games with lack of information on both sides.

2.1 The non-revealing game

We introduce now the so-called "non-revealing game", which is the game where none of the player is using his private information, or equivalently no one is informed of the true state of nature. The non-revealing game may be defined as the game with final payoff function $\sum_{i,j} p_i q_j g_{ij}$ and running payoff $\sum_{i,j} p_i q_j l_{ij}$. It is well known that this differential game with perfect information has a value which will be designed by $W(t, x, p, q)$. The value of the non-revealing game is the only viscosity solution of :

$$\begin{cases} \frac{\partial \phi}{\partial t} + H(D\phi, p, q) = 0 & (t, x, p, q) \in [t_0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \\ \phi(T, x) = \sum_{i,j} p_i q_j \langle a_{ij}, x \rangle & x \in \mathbb{R}^n \end{cases}$$

Following Hopf formula, we can check that the following function

$$W(t, x, p, q) = (T - t)H\left(\sum_{i,j} p_i q_j a_{ij}, p, q\right) + \sum_{i,j} p_i q_j \langle a_{ij}, x \rangle \quad (6)$$

which is C^1 w.r.t. (t, x) and Lipschitz continuous w.r.t. (p, q) is a classical solution of the previous Hamilton-Jacobi equation.

2.2 Representation formula for the value of the game

In case of games with imperfect information on both sides, the operator Vex_p has to be replaced with the operator Φ introduced in [8].

We now recall the exact definition and some properties of this operator Φ .

We denote by \mathcal{F} the set of all Lipschitz continuous functions : $\Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$. We denote by $\Phi : \mathcal{F} \rightarrow \mathcal{F}$ the operator associating to any φ the unique Lipschitz continuous function $\Phi\varphi$ solution of the system :

$$\begin{cases} \Phi\varphi(p, q) = \text{Vex}_p\{\max(\varphi, \Phi\varphi)\}(p, q) \\ \Phi\varphi(p, q) = \text{Cav}_q\{\min(\varphi, \Phi\varphi)\}(p, q) \end{cases}$$

Existence and uniqueness of $\Phi\varphi$ is established in this case in [7].

Lemma 2.1. *The operator Φ is monotonic :*

$$\forall \varphi, \psi \in \mathcal{F}, \varphi \leq \psi \Rightarrow \Phi\varphi \leq \Phi\psi ;$$

homogeneous :

$$\forall \lambda \in \mathbb{R}, \forall \varphi \in \mathcal{F}, \Phi(\lambda\varphi) = \lambda\Phi\varphi$$

and for all bilinear function $L : \mathbb{R}^I \times \mathbb{R}^J \rightarrow \mathbb{R}$:

$$\forall \varphi \in \mathcal{F}, \Phi(\varphi + L) = \Phi(\varphi) + L$$

Proof of Lemma 2.1 :

Monotonicity and homogeneity are well known [9].

To prove the third property, we fix $\varphi \in \mathcal{F}$ and $L \in \mathcal{L}(\mathbb{R}^I \times \mathbb{R}^J, \mathbb{R})$. By definition of $\Phi\varphi = \text{Vex}_p \max(\varphi, \Phi\varphi)$:

$$\begin{aligned} \text{Vex}_p \max(\varphi + L, \Phi\varphi + L) &= \text{Vex}_p(\max(\varphi, \Phi\varphi) + L) \\ &= \text{Vex}_p \max(\varphi, \Phi\varphi) + L \\ &= \Phi\varphi + L \end{aligned}$$

and $\Phi\varphi + L$ is Lipschitz continuous.

We prove the same way that $\text{Cav}_q \min(\varphi + L, \Phi\varphi + L) = \Phi\varphi + L$, implying $\Phi(\varphi + L) = \Phi\varphi + L$. \square

Theorem 2.2. *The value $V(t, x, p, q)$ of the game with imperfect information where the dynamics and running payoff do not depend on the state of the system and with linear final payoffs is the unique Lipschitz continuous function solving the system :*

$$\begin{cases} V = \text{Vex}_p \max(V, W) \\ V = \text{Cav}_q \min(V, W) \end{cases}$$

denoting by W the value of the non revealing game. The representation formula for V is given by :

$$V(t, x, p, q) = (T - t)\Phi(H(\sum_{i,j} p_i q_j a_{ij}, p, q)) + \sum_{i,j} p_i q_j \langle a_{ij}, x \rangle$$

and V satisfies for all $(t, x, p, q) \in [t_0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$:

$$\frac{\partial \phi}{\partial t} + \Phi(H(D\phi, p, q)) = 0$$

Proof of Theorem 2.2 :

The idea is to prove that V and ΦW are dual viscosity solutions of the same Hamilton-Jacobi equation. Using uniqueness of dual solutions satisfying the same terminal condition, we will have at the same time $V = \Phi W$ and the representation formula.

If we apply the properties of Φ to the value function of the non-revealing game $W(t, x, p, q)$ defined by (6) for any fixed (t, x) , we get a representation formula for ΦW :

$$\begin{aligned}\Phi W(t, x, p, q) &= (T - t)\Phi(H(\sum_{i,j} p_i q_j a_{ij}, p, q)) + \sum_{i,j} p_i q_j \langle a_{ij}, x \rangle \\ &= (T - t)\tilde{h}(p, q) + \sum_{i,j} p_i q_j \langle a_{ij}, x \rangle\end{aligned}$$

denoting by

$$\tilde{h}(p, q) := \Phi(H(\sum_{i,j} p_i q_j a_{ij}, p, q)).$$

In order to prove that $V = \Phi W$, we use the characterization of the value as the unique dual solution of the Hamilton-Jacobi equation :

$$\phi_t + H(D\phi, p, q) = 0 \quad (7)$$

We will prove that ΦW is a dual super-solution of (7). Indeed, ΦW is Lipschitz continuous, convex in p and concave in q . We use the characterization of dual super-solution with test functions. For a fixed $q \in \Delta(J)$, we choose a test function $\phi = \phi(t, x, p)$ C^1 convex in p and such that $(t, x, p) \mapsto \Phi W(t, x, p, q) - \phi(t, x, p)$ has a strict global minimum at $(t^0, x^0, p^0) \in (t_0, T) \times \mathbb{R}^n \times \Delta(I)$, and we seek to prove that :

$$\phi_t(t^0, x^0, p^0) + H(D\phi(t^0, x^0, p^0), p^0, q) \leq 0$$

The test function ϕ is convex, and the global minimum is strict, so that p^0 is an extreme point of the epigraph of $p \mapsto \Phi W(t^0, x^0, p, q)$. By definition of ΦW , p^0 is an extreme point of the epigraph of $p \mapsto \tilde{h}(p, q)$.

We use now the characterization of the operator Φ due to Laraki [7]:

For all $\varphi \in \mathcal{F}$, $\Phi\varphi$ is the unique Lipschitz continuous function such that

$$\left\{ \begin{array}{l} \forall q \in \Delta(J), p^0 \text{ is an extreme point of the epigraph of } \Phi\varphi(\cdot, q) \\ \quad \Rightarrow \Phi\varphi(p^0, q) \geq \varphi(p^0, q) \\ \forall p \in \Delta(I), q^0 \text{ is an extreme point of the hypograph of } \Phi\varphi(p, \cdot) \\ \quad \Rightarrow \Phi\varphi(p, q^0) \leq \varphi(p, q^0) \end{array} \right. \quad (8)$$

The point p^0 is an extreme point of the epigraph of $\tilde{h}(\cdot, q)$ implies :

$$\tilde{h}(p^0, q) \geq H(\sum_{i,j} p_i^0 q_j a_{ij}, p^0, q)$$

Since ΦW and ϕ are C^1 w.r.t. (t, x) and (t^0, x^0, p^0) is a minimum point of $\Phi W - \phi$, the first order optimality condition is :

$$\begin{aligned}\phi_t(t^0, x^0, p^0) &= \frac{\partial \Phi W}{\partial t}(t^0, x^0, p^0, q) = -\tilde{h}(p^0, q) \\ D\phi(t^0, x^0, p^0) &= D\Phi W(t^0, x^0, p^0, q) = \sum_{i,j} p_i^0 q_j a_{ij}\end{aligned}$$

and we get

$$\phi_t(t^0, x^0, p^0) + H(D\phi(t^0, x^0, p^0), p^0, q) = -\tilde{h}(p^0, q) + H\left(\sum_{i,j} p_i^0 q_j a_{ij}, p^0, q\right) \leq 0.$$

We have just proven that ΦW is a dual super-solution of the Hamilton-Jacobi equation (7). We could prove using the same kind of arguments that ΦW is a dual sub-solution. The terminal condition for ΦW is computed from the terminal condition for W :

$$\Phi W(T, x, p, q) = \Phi\left(\sum_{i,j} p_i q_j \langle a_{ij}, x \rangle\right) = \sum_{i,j} p_i q_j \langle a_{ij}, x \rangle.$$

This implies ΦW is a dual solution of the Hamilton-Jacobi equation (7). Uniqueness of the dual solution implies the value of the game is related to the value of the non-revealing game through

$$V(t, x, p, q) = \Phi W(t, x, p, q)$$

□

3 Approximation of the value

In this section, we consider a zero-sum differential game with imperfect information on both sides, with payoff functions non necessarily linear and taking into accounts running payoffs and a dynamics that may depend on the state of the system. Cardaliaguet proved in [5] that for general differential games, there is no way to extend the previous connection between the value of the game and the value of the non-revealing game, so that we seek to build an approximation of the value function $V(t, x, p, q)$ for discrete values of time. Such an approximation was proposed in [6] for differential games with imperfect information on one side and final payoff functions. We build an approximation for games with imperfect information on both sides with running payoff.

We will always assume that the regularity assumptions (3) are fulfilled and assume Isaacs'condition, in order to ensure that the game has a value.

We set for large $L \in \mathbb{N}$, $\tau = \frac{T}{L}$ the time step and we denote by $t_k := k\tau$ for $k \in \{0, \dots, L\}$.

We define the function $V_\tau : \{0, \dots, t_L\} \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ using backward induction on k by

$$\text{for } k = L, \quad V_\tau(T, x, p, q) := \sum_{i,j} p_i q_j g_{ij}(x) \quad (9)$$

and assuming $V_\tau(t_{k+1}, \cdot, \cdot, \cdot)$ is already built

$$V_\tau(t_k, x, p, q) := \Phi\left(\min_{u \in U} \max_{v \in V} [V_\tau(t_{k+1}, x + \tau f(x, u, v), p, q) + \tau \sum_{i,j} p_i q_j l_{ij}(x, u, v)]\right)$$

Theorem 3.1. *The sequence of functions V_τ converge to the value function V for the topology of uniform convergence on compacts of $[0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$.*

In order to prove the theorem, we will need the following Lemma :

Lemma 3.2. *The functions $V_\tau(t_k, x, p, q)$ are Lipschitz continuous with a Lipschitz constant independent of τ .*

Proof of Lemma 3.2 :

The Lipschitz continuity of V_τ w.r.t. x is clear for $t_L = T$ because the g_{ij} are Lipschitz continuous. We assume first that $V_\tau(t_{k+1}, x, p, q)$ is Lipschitz continuous w.r.t. x with Lipschitz constant M_{k+1} . Denoting by M_f an upper-bound of the Lipschitz constants of f and the l_{ij} , we prove that

$$x \mapsto \min_u \max_v [V_\tau(t_{k+1}, x + \tau f(x, u, v), p, q) + \tau \sum_{i,j} p_i q_j l_{ij}(x, u, v)]$$

is Lipschitz continuous. Fix $(u, v) \in U \times V$. For all $(x, x') \in \mathbb{R}^n \times \mathbb{R}^n$, we have :

$$\begin{aligned} & V_\tau(t_{k+1}, x + \tau f(x, u, v), p, q) + \tau \sum_{i,j} p_i q_j l_{ij}(x, u, v) \\ & \quad - V_\tau(t_{k+1}, x' + \tau f(x', u, v), p, q) - \tau \sum_{i,j} p_i q_j l_{ij}(x', u, v) \\ & \leq M_{k+1} \|x - x' + \tau(f(x, u, v) - f(x', u, v))\| \\ & \quad + \tau \sum_{i,j} p_i q_j |l_{ij}(x, u, v) - l_{ij}(x', u, v)| \\ & \leq (M_{k+1} + (1 + M_{k+1})\tau M_f) \|x' - x\| \end{aligned}$$

We now take the minimum on U and the maximum on V :

$$\begin{aligned} & \min_u \max_v [V_\tau(t_{k+1}, x + \tau f(x, u, v), p, q) + \tau \sum_{i,j} p_i q_j l_{ij}(x, u, v)] \\ & \leq \min_u \max_v [V_\tau(t_{k+1}, x' + \tau f(x', u, v), p, q) + \tau \sum_{i,j} p_i q_j l_{ij}(x', u, v)] \\ & \quad + (M_{k+1} + (1 + M_{k+1})\tau M_f) \|x' - x\| \end{aligned}$$

We deduce from the monotonicity of the operator Φ that $V_\tau(t_k, x, p, q)$ is Lipschitz continuous w.r.t. x with Lipschitz constant $(M_{k+1} + (1 +$

$M_{k+1})\tau M_f)$. The function V_τ is then globally Lipschitz continuous w.r.t. x with Lipschitz constant $M_V^x := M_L e^{TM_f} + TM_f$ independent of τ recalling $\tau = \frac{T}{L}$.

The Lipschitz continuity of V_τ w.r.t. p and q can be proved the same way. At time T , it comes from the fact that the functions g_{ij} are bounded. Assuming $V_\tau(t_{k+1}, \cdot, \cdot, \cdot)$ is Lipschitz continuous w.r.t. p with Lipschitz constant M_{k+1}^p for the L^1 -norm on $\Delta(I)$, we have, using the same arguments, for all $(p, p') \in \Delta(I)^2$:

$$\begin{aligned} & \min_u \max_v [V_\tau(t_{k+1}, x + \tau f(x, u, v), p, q) + \tau \sum_{i,j} p_i q_j l_{ij}(x, u, v)] \\ & \leq \min_u \max_v [V_\tau(t_{k+1}, x + \tau f(x, u, v), p', q) + \tau \sum_{i,j} p'_i q_j l_{ij}(x, u, v)] \\ & \quad + (M_{k+1}^p + \tau \|l\|_\infty) \|p - p'\|_1 \end{aligned}$$

As before, we deduce $V_\tau(t_k, x, p, q)$ is Lipschitz continuous w.r.t. p with Lipschitz constant $(M_{k+1}^p + \tau \|l\|_\infty)$ and V_τ is globally Lipschitz continuous w.r.t. p with Lipschitz constant $(M_L + T \|l\|_\infty)$, independent of τ . The proof is the same for Lipschitz continuity w.r.t. q .

In order to prove Lipschitz continuity w.r.t. t_k , we fix (x, p, q) . We use the fact that V_τ is convex in p and concave in q for any fixed (t_k, x) and Lipschitz continuous w.r.t. x with Lipschitz constant M_V^x independent of τ :

$$\begin{aligned} & |V_\tau(t_{k+1}, x, p, q) - V_\tau(t_k, x, p, q)| \\ & = |V_\tau(t_{k+1}, x, p, q) \\ & \quad - \Phi(\min_u \max_v [V_\tau(t_{k+1}, x + \tau f(x, u, v), p, q) + \tau \sum_{i,j} p_i q_j l_{ij}(x, u, v)])| \\ & \leq |V_\tau(t_{k+1}, x, p, q) - \Phi(V_\tau(t_{k+1}, x, p, q))| \\ & \quad + \tau (M_V^x \|f\|_\infty + \|l\|_\infty) \\ & \leq (M_V^x \|f\|_\infty + \|l\|_\infty) |t_{k+1} - t_k| \end{aligned}$$

because we assumed f and l_{ij} are bounded. By induction on k , we deduce that V_τ is Lipschitz continuous w.r.t. t on $\{0, \dots, t_L = T\}$ with Lipschitz constant $(M_V^x \|f\|_\infty + \|l\|_\infty)$ independent of τ .

As a conclusion, $(t_k, x, p, q) \mapsto V_\tau(t_k, x, p, q)$ is Lipschitz continuous with Lipschitz constant independent of τ . \square

Proof of theorem 3.1 :

We now prove that the sequence of Lipschitz continuous functions V_τ converges to V for the topology of uniform convergence on the compact subsets of $[0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$.

The set of functions $\{V_\tau\}_{\tau>0}$ is equicontinuous, so that using Arzel-Ascoli theorem, $\{V_\tau, \tau > 0\}$ is relatively compact for the topology of uniform convergence. Let w be a cluster point of the V_τ as $\tau \rightarrow 0^+$. Then w is Lipschitz continuous, convex in p , concave in q . It remains to prove that w is a dual

solution of equation (4) : we already know that w satisfies the terminal condition because of (9).

We will prove w is a dual super-solution of (4). We fix $q \in \Delta(J)$. Let $\phi = \phi(t, x, p)$ a test function C^1 convex in p such that $(t, x, p) \mapsto w(t, x, p, q) - \phi(t, x, p)$ has a strict global minimum at (t^0, x^0, p^0) . There exists a sequence (t_τ, x_τ, p_τ) converging to (t^0, x^0, p^0) and such that $(t, x, p) \mapsto V_\tau(t, x, p, q) - \phi(t, x, p)$ has a global minimum at (t_τ, x_τ, p_τ) . In order to get a strict minimum, we introduce some perturbation on ϕ . We design by :

$$\begin{aligned}\phi_\gamma(t, x, p) &:= \phi - \gamma\|x - x^0\|^2 \\ \phi_{\epsilon\gamma}(t, x, p) &:= \phi_\gamma(t, x, p) + \epsilon\|p\|^2 = \phi + \epsilon\|p\|^2 - \gamma\|x - x^0\|^2\end{aligned}$$

with $\epsilon > 0$ and $\gamma > 0$. This way, $V_\tau(t, x, p, q) - \phi_{\epsilon\gamma}(t, x, p)$ admits a global minimum denoted by $(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p_{\epsilon\gamma})$ as continuous and coercive in x .

We introduce now

$$\psi_{\epsilon\gamma}(t, x, p) := \phi_{\epsilon\gamma}(t, x, p) - \epsilon\|p - p_{\epsilon\gamma}\|^2 - \epsilon\|(t, x) - (t_{\epsilon\gamma}, x_{\epsilon\gamma})\|^2$$

The function $\psi_{\epsilon\gamma}$ is convex in p and $V_\tau(t, x, p, q) - \psi_{\epsilon\gamma}(t, x, p)$ has a strict global minimum at $(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p_{\epsilon\gamma})$.

Notice that the sequence $(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p_{\epsilon\gamma})$ is bounded for γ strictly positive. We consider a subsequence converging to a point $(t_\gamma, x_\gamma, p_\gamma)$ as $\epsilon, \tau \rightarrow 0$ with fixed γ . This limit is a global minimum of $w(\cdot, \cdot, \cdot, q) - \phi_\gamma$. We know that $w(\cdot, \cdot, \cdot, q) - \phi$ admits (t^0, x^0, p^0) as strict global minimum so that $w(t, x, p, q) - \phi_\gamma(t, x, p) = w(t, x, p, q) - \phi(t, x, p) + \gamma\|x - x^0\|^2$ admits the same strict global minimum and finally $(t_\gamma, x_\gamma, p_\gamma) = (t^0, x^0, p^0)$.

The condition of strict global minimum is : for all $(\tau, X, p) \neq (0, x_{\epsilon\gamma}, p_{\epsilon\gamma})$

$$V_\tau(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p_{\epsilon\gamma}, q) - \psi_{\epsilon\gamma}(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p_{\epsilon\gamma}) < V_\tau(t_{\epsilon\gamma} + \tau, X, p, q) - \psi_{\epsilon\gamma}(t_{\epsilon\gamma} + \tau, X, p)$$

The function $\psi_{\epsilon\gamma}$ being convex in p , and the minimum being strict, remark that $p_{\epsilon\gamma}$ is an extreme point of $p \mapsto V_\tau(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p, q)$. Using the characterization of the operator Φ (cf. (8)) we have :

$$\begin{aligned}\min_{u \in U} \max_{v \in V} [V_\tau(t_{\epsilon\gamma} + \tau, x_{\epsilon\gamma} + \tau f(x_{\epsilon\gamma}, u, v), p_{\epsilon\gamma}, q) \\ + \tau \sum_{i,j} p_{\epsilon\gamma i} q_j l_{ij}(x_{\epsilon\gamma}, u, v)] \leq V_\tau(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p_{\epsilon\gamma}, q).\end{aligned}$$

Writing the strict global minimum condition at $(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p_{\epsilon\gamma})$ gives :

$$\begin{aligned}\min_{u \in U} \max_{v \in V} \{ \psi_{\epsilon\gamma}(t_{\epsilon\gamma} + \tau, x_{\epsilon\gamma} + \tau f(x_{\epsilon\gamma}, u, v), p_{\epsilon\gamma}) + \tau \sum_{i,j} p_{\epsilon\gamma i} q_j l_{ij}(x_{\epsilon\gamma}, u, v) \} \\ - \psi_{\epsilon\gamma}(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p_{\epsilon\gamma}) \leq 0\end{aligned}$$

so that by definition of $\psi_{\epsilon\gamma}$ and dividing by τ :

$$\min_{u \in U} \max_{v \in V} \left\{ \frac{1}{\tau} (\phi_\gamma(t_{\epsilon\gamma} + \tau, x_{\epsilon\gamma} + \tau f(x_{\epsilon\gamma}, u, v), p_{\epsilon\gamma}) - \phi_\gamma(t_{\epsilon\gamma}, x_{\epsilon\gamma}, p_{\epsilon\gamma})) \right. \\ \left. + \sum_{i,j} p_{\epsilon\gamma i} q_j l_{ij}(x_{\epsilon\gamma}, u, v) \right\} - \epsilon\tau(1 + \|f\|_\infty^2) \leq 0$$

The function ϕ_γ being C^1 , as $\epsilon, \tau \rightarrow 0$, the limit of the previous inequality is :

$$\frac{\partial}{\partial t} \phi_\gamma(t^0, x^0, p^0) \\ + \min_{u \in U} \max_{v \in V} [\langle f(x^0, u, v), D\phi_\gamma(t^0, x^0, p^0) \rangle + \sum_{i,j} p_i^0 q_j l_{ij}(x^0, u, v)] \leq 0,$$

Then, because of the definition of ϕ_γ :

$$\frac{\partial}{\partial t} \phi(t^0, x^0, p^0) \\ + \min_{u \in U} \max_{v \in V} [\langle f(x^0, u, v), D\phi(t^0, x^0, p^0) \rangle + \sum_{i,j} p_i^0 q_j l_{ij}(x^0, u, v)] \leq 0,$$

and finally w is a dual super-solution of the Hamilton-Jacobi equation(4). We would prove symmetrically, using the same kind of arguments, that w is a dual sub-solution of the Hamilton-Jacobi equation. The terminal condition being fulfilled, the uniqueness of the dual solution of (4) implies $w = V$, and the functions V_τ uniformly converge to the value of the game on compacts. \square

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