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# On a symmetric space attached to polyzeta values.

Olivier Mathieu

## ABSTRACT

Quickly converging series are given to compute polyzeta numbers  $\zeta(r_1, \dots, r_k)$ . The formulas involve an intricate combination of (generalized) polylogarithms at  $1/2$ . However, the combinatoric has a very simple geometric interpretation: it corresponds with the map  $p \mapsto p^2$  on a certain symmetric space  $P$ .

### Introduction:

Let  $k \geq 1$ . For a  $k$ -uple  $(r_1, r_2, \dots, r_k)$  of positive integers, set  $\zeta(r_1, \dots, r_k) = \sum_{0 < n_1 \dots < n_k} 1/n_1^{r_1} \dots n_k^{r_k}$ . We have  $\zeta(r_1, \dots, r_k) < \infty$  if and only if  $r_k \geq 2$ . By definition, the *polyzeta values* are the  $\mathbf{Q}$ -linear combinations of the finite numbers  $\zeta(r_1, \dots, r_k)$ .

Using the definition of  $\zeta(r_1, \dots, r_k)$ , the evaluation of a polyzeta value up to the  $N^{\text{th}}$  digit requires to take into account something like  $O(10^N)$  terms. Therefore it is a very slow computation. A similar computational problem arises with the classical series  $\log 2 = -\sum_{n>0} (-1)^n/n$  and  $\pi/4 = \sum_{n \geq 0} (-1)^n 1/(2n+1)$ , which converge very slowly.

However, we easily notice that:

$$\log 2 = -\log(1 - 1/2) = \sum_{n>0} 2^{-n}/n.$$

A remarkable series for  $\pi$  has been discovered by Bailey, Borwein and Plouffe [BBP]:

$$\pi = \sum_{n \geq 0} 1/2^{4n} [4/(8n+1) - 2/(8n+4) - 1/(8n+5) - 1/(8n+6)]$$

Now to evaluate  $\log 2$  or  $\pi$  up to the  $N^{\text{th}}$  digit, one only needs the first  $O(N)$ -terms of the series and therefore  $\log 2$  and  $\pi$  can be computed very quickly. The goal of the paper is to provide similar identities for all polyzeta values.

To do so, one needs to use the functions  $L_{r_1, \dots, r_k}(z) = \sum_{0 < n_1 \dots < n_k} 1/n_1^{r_1} \dots n_k^{r_k} z^{n_k}$ , where  $r_1, \dots, r_k$  are positive integers. By definition, a  $\mathbf{Q}$ -linear combinations of the functions  $L_{r_1, \dots, r_k}(z)$  is called a *polylogarithmic function*. The obvious identity  $\zeta(r_1, \dots, r_k) = L_{r_1, \dots, r_k}(1)$  does not help to quickly evaluate polyzeta values. However, the series defining polylogarithms at  $1/2$  converges very quickly: to evaluate  $L_{r_1, \dots, r_k}(1/2)$  up to the  $N^{\text{th}}$  digit, one only needs to sum  $O(N^k)$ -terms, and this can be done in polynomial time. This remark suggests the following result:

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MAIN STATEMENT: *Any polyzeta value is the value of a certain polylogarithmic function at  $1/2$ .*

In order to get a useful statement, the corresponding polylogarithmic function is described explicitly: see Theorem 7 for a precise statement. At first glance, the combinatorics involved in Theorem 7 looks intricate and therefore no details are given in the introduction. However, we can precisely formulate the main statement in terms of very simple geometric notions.

Let  $F(2)$  be the free group on two generators  $\alpha$  and  $\beta$  and let  $s$  be the involution exchanging the generators. Let  $\Gamma = \mathbf{Q} \otimes F_2$  be the Malcev completion of  $\Gamma$  (see Section 4 for an alternative definition of  $\Gamma$ ). The involution  $s$  extends to  $\Gamma$  and there is a decomposition  $\Gamma = P.K$  where  $K$  is the subgroup of fixed points of  $s$  and where  $P = \{g \in \Gamma | s(g) = g^{-1}\}$ . The group  $\Gamma$  is proalgebraic over  $\mathbf{Q}$  and the symmetric space  $P$  is a pro-algebraic variety over  $\mathbf{Q}$ .

In section (4.8), all polyzeta values are naturally indexed by rational functions on  $P$ . Similarly, some polylogarithmic functions are naturally indexed by rational functions on  $P$ . So for  $\phi \in \mathbf{Q}[P]$ , denote by  $\zeta(\phi)$  and  $L_\phi(z)$  the corresponding polyzeta value and polylogarithmic function.

Now the square map  $\square : P \rightarrow P, p \mapsto p^2$  induces an algebra morphism  $\square : \mathbf{Q}[P] \rightarrow \mathbf{Q}[P]$ . The geometric formulation of the main result is as follows:

MAIN THEOREM: *For any  $\phi \in \mathbf{Q}[P]$ ,  $\zeta(\phi) = L_{\square\phi}(1/2)$ .*

We also express polyzeta values as values of polylogarithmic functions at  $\rho^{\pm 1} = \exp \pm i\pi/3$ . The geometric interpretation of this case is a bit more complicate because it involves an order 3 automorphism of  $\Gamma$ , see section 4, Theorem 18.

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*Summary:*

1. Polylogarithms and polyzeta values.
2. Polylogarithmic function at  $1/2$  and at  $\rho^{\pm 1}$ .
3. Explicit expressions for  $\zeta(r)$ .
4. Geometric interpretation of Theorem 7.
5. Other expressions for zeta values.
6. Conclusion.

## 1. Polylogarithms and polyzeta values.

This section is devoted to main definitions and conventions. The definitions of *polyzeta values* and *polylogarithmic functions* are not standard: see the subsections (1.14) for more comments. Moreover in this section we adopt some conventions to renormalize infinite quantities like  $\zeta(1)$  or  $\int_0^z dt/t$ .

(1.1) Shuffles: For  $N \geq 0$ , denote by  $S_N$  the symmetric group, i.e. the set of all bijections  $\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$ . Given  $n$  and  $m$  two non-negative integers, let

$S_{n,m}$  be the set of all  $\phi \in S_{n+m}$  such that  $\phi$  is increasing on the subset  $\{1, \dots, n\}$  and on the subset  $\{n+1, \dots, n+m\}$ . The elements of  $S_{n,m}$  are called *shuffles*.

(1.2) Shuffle product: Let  $\mathcal{W}$  be the set of words into the letters  $a$  and  $b$ . By convention,  $\mathcal{W}$  contains the empty word  $\emptyset$ . Set  $\mathcal{H} = \mathbf{Q}\mathcal{W}$ , i.e.  $\mathcal{H}$  is the  $\mathbf{Q}$ -vector space with basis  $\mathcal{W}$ . For any two words  $w = x_1 \dots x_n$  and  $w' = x_{n+1} \dots x_{n+m}$ , where each  $x_i \in \{a, b\}$  is a letter, define the product  $w * w' \in \mathcal{H}$  by the formula:

$$w * w' = \sum_{\sigma \in S_{n,m}} x_{\sigma(1)} \dots x_{\sigma(n+m)}$$

By convention, we have  $\emptyset * w = w * \emptyset = w$  for all word  $w$ . The product  $*$  is called the shuffle product. With respect to this product,  $\mathcal{H}$  is a commutative associative algebra, and  $\emptyset$  is its unit.

(1.3) Subalgebras of  $\mathcal{H}$ : Let  $\mathcal{W}^+$  be the set of words whose first letter is not  $b$ . Equivalently, a word  $w$  belongs to  $\mathcal{W}^+$  if  $w = \emptyset$  or if  $w$  starts with  $a$ . Similarly, let  $\mathcal{W}^{++}$  be the set of words whose first letter is not  $b$  and the last letter is not  $a$ . Set  $\mathcal{H}^+ = \mathbf{Q}\mathcal{W}^+$  and  $\mathcal{H}^{++} = \mathbf{Q}\mathcal{W}^{++}$ . It is easy to prove that  $\mathcal{H}^+$  and  $\mathcal{H}^{++}$  are subalgebras of  $\mathcal{H}$ .

LEMMA 1: *There are isomorphisms of algebras:  $\mathcal{H} = \mathcal{H}^+[b]$  and  $\mathcal{H} = \mathcal{H}^{++}[a, b]$ .*

*Proof:* For each  $n \geq 0$ , let  $\mathcal{W}_n$  be the set of words of the form  $b^n w$  with  $w \in \mathcal{W}^+$ , and set  $\mathcal{H}_n = \bigoplus_{0 \leq k \leq n} \mathbf{Q}\mathcal{W}_k$ . We have  $b * \mathcal{H}_n \subset \mathcal{H}_{n+1}$ . Moreover we have  $b * w = (n+1)bw$  modulo  $\mathcal{H}_n$  for any  $w \in \mathcal{W}_n$ . It follows easily by induction that  $\mathcal{H}_n = \bigoplus_{0 \leq k \leq n} \mathcal{H}^+ * b^k$ , i.e.  $\mathcal{H}_n$  is the space of all polynomials in  $b$  with coefficients in  $\mathcal{H}^+$  and degree  $\leq n$ . Therefore the first assertion follows.

The proof of the second assertion is similar.

(1.4) The bijection  $\lambda : \mathcal{W}^+ \rightarrow \Lambda$ :

Let  $\mathbf{N}$  be the set of positive integer. For clarity, a word into the letters  $1, 2, \dots \in \mathbf{N}$  will be called a *sequence of positive integers*. Let  $\Lambda$  the set of sequence  $(r_1 \dots r_k)$  of positive integers. By convention,  $\Lambda$  contains the empty sequence  $\emptyset$ .

Any word  $w \in \mathcal{W}^+$  can be uniquely factorized as:  $w = ab^{t_1} ab^{t_2} \dots ab^{t_k}$ , where  $k$  is the number of occurrence of  $a$  in  $w$  and where the  $t_i$  are non-negative integers. Then, the map  $w \in \mathcal{W}^+ \mapsto (1 + t_1, 1 + t_2, \dots, 1 + t_k) \in \Lambda$  defines a natural bijection  $\lambda : \mathcal{W}^+ \rightarrow \Lambda$ .

(1.5) Polylogarithmic functions and polyzeta values:

Let  $k \geq 1$  and let  $r_1 \dots r_k$  be a sequence of  $k$  positive integers. Consider the following series in the complex variable  $z$ :

$$L_{r_1, \dots, r_k}(z) = \sum_{0 < n_1 < \dots < n_k} n_1^{-r_1} \dots n_k^{-r_k} z^{n_k}$$

In the infinite sum, the indices  $n_1, \dots, n_k$  are integers. The functions  $L_{r_1, \dots, r_k}(z)$  are called *polylogarithms*. Set  $D = \{z \in \mathbf{C} \mid |z| < 1\}$ ,  $\overline{D} = \{z \in \mathbf{C} \mid |z| \leq 1\}$ . The two points of interest for the paper are the following:

(i) if  $r_k \geq 2$ , the series is absolutely convergent on  $\overline{D}$  and therefore  $L_{r_1, \dots, r_k}(z)$  extends to a continuous function on  $\overline{D}$ .

(ii) if  $r_k = 1$ , the series converges on  $D$  and  $L_{r_1, \dots, r_k}(z)$  extends to a continuous function on  $\overline{D} \setminus \{1\}$ .

Indeed  $L_{r_1, \dots, r_k}(z)$  extends to a multivalued function, see e.g. [C] and Proposition 2 below. For  $r_k \geq 2$ , set

$$\zeta(r_1, \dots, r_k) = \sum_{0 < n_1 < \dots < n_k} n_1^{-r_1} \dots n_k^{-r_k}$$

In the paper, the numbers  $\zeta(r_1, \dots, r_k)$  will be called polyzeta values. Indeed the polyzeta value is both the value at  $z = 1$  of the polylogarithm  $L_{r_1, \dots, r_k}(z)$  and a value of the polyzeta function  $\zeta(s_1, \dots, s_k) = \sum_{0 < n_1 < \dots < n_k} n_1^{-s_1} \dots n_k^{-s_k}$ .

(1.6) New notations: Let  $w \in \mathcal{W}^+$  and set  $(r_1, \dots, r_k) = \lambda(w)$ . It is convenient to denote the function  $L_{(r_1, \dots, r_k)}(z)$  by  $L_w(z)$ . Similarly set  $\zeta(w) = \zeta(r_1, \dots, r_k)$  if  $w \in \mathcal{W}^{++}$ .

(1.7) The one-forms  $\omega_a$  and  $\omega_b$ : Define the following one-forms on  $\mathbf{C}$ :

$$\omega_a(z) = \frac{dz}{1-z} \quad \text{and} \quad \omega_b(z) = \frac{dz}{z}$$

For an element  $c = xa + yb \in \mathbf{Q}a \oplus \mathbf{Q}b$ , set  $\omega_c(z) = x\omega_a(z) + y\omega_b(z)$ . Given a smooth path  $\gamma : [0, 1] \rightarrow \mathbf{C}, t \mapsto \gamma(t)$ , recall that  $\gamma^*\omega_a(t) = \frac{\gamma'(t)}{1-\gamma(t)} dt$ ,  $\gamma^*\omega_b(t) = \frac{\gamma'(t)}{\gamma(t)} dt$ . and  $\gamma^*\omega_c(t) = x\gamma^*\omega_a(t) + y\gamma^*\omega_b(t)$ .

(1.8) Kontsevitch formula: For a positive integer  $n$ , set  $\Delta_n = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n | 0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1\}$ . Let  $w = c_1 \dots c_n \in \mathcal{W}^+$  be a word, where each  $c_i$  is a letter. The following formula is due to Kontsevitch (see [Z]).

PROPOSITION 2: Let  $w = c_1 \dots c_n \in \mathcal{W}^+$  be a word, let  $z \in \overline{D}$ , and let  $\gamma : [0, 1] \rightarrow \overline{D}$  be a path with  $\gamma(0) = 0$  and  $\gamma(1) = z$ .

Assume that  $w \in \mathcal{W}^{++}$  or that  $\gamma$  does not meet 1. Then we have:

$$L_w(z) = \int_{\Delta_n} \gamma^*\omega_{c_1}(x_1)\gamma^*\omega_{c_2}(x_2)\dots\gamma^*\omega_{c_n}(x_n)$$

In [Z], Kontsevitch formula is stated for the straight path  $t \mapsto zt$ , but it is easy to see that the integral is homotopy invariant as long  $\gamma$  stay in  $\overline{D}$  (and  $\gamma$  stay  $\overline{D} \setminus \{1\}$  if  $w \notin \mathcal{W}^{++}$ ).

(1.9) Products: Let  $n, m$  be non negative integers and let  $c_1, c_2, \dots, c_{n+m} \in \{a, b\}$  be letters with  $c_1 = c_{n+1} = a$ . Set  $u = c_1 \dots c_n$  and  $v = c_{n+1} \dots c_{n+m}$ . For  $\sigma \in S_{n,m}$ , set  $w_\sigma = c_{\sigma(1)} \dots c_{\sigma(n+m)}$ .

COROLLARY 3: For  $u, v \in \mathcal{W}^+$ , we have  $L_u(z)L_v(z) = \sum_{\sigma \in S_{n,m}} L_{w_\sigma}(z)$  for all  $z \in D$ . Moreover for  $u, v \in \mathcal{W}^{++}$ , we have  $\zeta(u)\zeta(v) = \sum_{\sigma \in S_{n,m}} \zeta(w_\sigma)$ .

Proof: Set

$$\begin{aligned} \omega' &= \omega_{c_1}(zx_1)\omega_{c_2}(zx_2)\dots\omega_{c_n}(zx_n), \\ \omega'' &= \omega_{c_{n+1}}(zx_{n+1})\omega_{c_{n+2}}(zx_{n+2})\dots\omega_{c_{n+m}}(zx_{n+m}), \\ \Delta_m &= \{(x_{n+1}, x_{n+2}, \dots, x_{n+m}) \in \mathbf{R}^n | 0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1\}, \end{aligned}$$

and for  $\sigma \in S_{n,m}$ , set

$$\Delta_\sigma = \{(x_1, \dots, x_{n+m}) \in \mathbf{R}^{n+m} \mid x_{\sigma(1)} \leq x_{\sigma(2)} \leq \dots \leq x_{\sigma(n+m)}\}.$$

Since  $\Delta_n \times \Delta_m = \cup_{\sigma \in S_{n,m}} \Delta_\sigma$ , we get  $\int_{\Delta_n} \omega' \int_{\Delta_m} \omega'' = \int_{\Delta_n \times \Delta_m} \omega' \wedge \omega'' = \sum_{\sigma} \int_{\Delta_\sigma} \omega' \wedge \omega''$ . By Proposition 2, this identity is equivalent to  $L_u(z)L_v(z) = \sum_{\sigma \in S_{n,m}} L_{w_\sigma}(z)$ . At  $z = 1$ , one gets the second identity  $\zeta(u)\zeta(v) = \sum_{\sigma \in S_{n,m}} \zeta(w_\sigma)$ . Q.E.D.

(1.10) Final definitions and notations for polylogarithmic functions: Up to now, the polylogarithms  $L_w(z)$  are defined for  $w \in \mathcal{W}^+$ . In order to extend the definition to all  $w \in \mathcal{W}$ , a renormalization procedure is used.

Set  $\Omega = D \setminus ]-1, 0]$  and let  $Hol(\Omega)$  be the algebra of holomorphic functions on  $\Omega$ . Since  $\omega$  is simply connected, let denote by  $\log z$  the holomorphic function on  $\Omega$  whose restriction to  $]0, 1[$  is the usual logarithmic function.

LEMMA 4: *There is a unique algebra morphism  $\Phi : \mathcal{H} \rightarrow Hol(\Omega)$  such that  $\Phi(w) = L_w(z)$  for  $w \in \mathcal{H}^+$  and  $\Phi(b) = \log z$ .*

*Proof:* This follows from Lemmma 1 and corollary 3. Q.E.D.

For any  $h \in \mathcal{H}$ , set  $L_h(z) = \Phi(h)$ . When  $h$  is a word  $w$  in  $\mathcal{W}^+$ , this new notation agrees with the previous one. Corollary 3 can be restated as:  $L_u(z)L_v(z) = L_{u*v}(z)$ . By definition the *polylogarithmic functions* are the functions  $L_h(z)$  with  $h \in \mathcal{H}$ .

This definition is a slightly different from the introduction. However, we will only use polylogarithmic functions  $L_h(z)$  with  $h \in \mathcal{H}^+$ , which are the polylogarithmic functions defined in introduction.

(1.11) Final definitions and notations for polyzeta values: Up to now, the polyzeta values  $\zeta(w)$  are defined for  $w \in \mathcal{W}^{++}$ . In order to extend the definition to all  $w \in \mathcal{W}$ , we will use a renormalization procedure as follows.

LEMMA 5: *There are three algebra morphisms  $\psi, \psi^+, \psi^- : \mathcal{H} \rightarrow \mathbf{C}$  uniquely defined by the following requirements:*

$$\begin{aligned} \psi(w) &= \psi^+(w) = \psi^-(w) = \zeta(w) \text{ if } w \in \mathcal{H}^{++} \\ \psi(a) &= 0, \psi^+(a) = i\pi, \psi^-(a) = -i\pi \\ \psi(b) &= \psi^+(b) = \psi^-(b) = 0. \end{aligned}$$

*Proof:* This follows from Lemmma 1 and corollary 3.

Similarly, this allows to define  $\zeta(h) = \psi(h)$ ,  $\zeta^\pm(h) = \psi^\pm(h)$  for any  $h \in \mathcal{H}$ . By definition the *polyzeta values* are the numbers  $\zeta(h)$  with  $h \in \mathcal{H}$ . Corollary 3 can be restated as:  $\zeta(u)\zeta(v) = \zeta(u*v)$  and  $\zeta^\pm(u)\zeta^\pm(v) = \zeta^\pm(u*v)$  for any  $u, v \in \mathcal{H}$ .

Set  $\mathcal{Z} = \zeta(\mathcal{H})$  and  $\mathcal{Z}^\pm = \zeta(\mathcal{H}^\pm)$ . By definition,  $\mathcal{Z}$  and  $\mathcal{Z}^\pm$  are subrings of  $\mathbf{C}$ , and  $\mathcal{Z}$  is the space of all polyzeta values. It is easy to compare the three algebras  $\mathcal{Z}$  and  $\mathcal{Z}^+$  and  $\mathcal{Z}^-$ .

LEMMA 6:

- (i) *As a  $\mathbf{Q}$  vector space,  $\mathcal{Z}$  is generated by all  $\zeta(w)$  with  $w \in \mathcal{W}^{++}$ .*
- (ii) *We have  $\mathcal{Z} \subset \mathbf{R}$ .*
- (iii)  *$\mathcal{Z}^\pm = \mathcal{Z} \oplus i\pi\mathcal{Z}$ .*

*Proof:* The assertions (i) and (ii) follow from Lemma 1. Moreover it follows that  $\mathcal{Z}^\pm$  is the  $\mathbf{Q}$ -algebra generated by  $\mathcal{Z}$  and  $\zeta^\pm(b) = \pm i\pi$ . However  $(\zeta^\pm(b))^2 = -\pi^2 = -6\zeta(2)$ , therefore  $(\zeta^\pm(b))^2$  belongs to  $\mathcal{Z}$  and assertion (6.3) follows.

(1.12) Hopf algebra structure: Define the linear maps  $\eta : \mathcal{H} \rightarrow \mathbf{Q}$ ,  $\iota : \mathcal{H} \rightarrow \mathcal{H}$  and  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  as follows. For any word  $w = c_1 \dots c_n \in \mathcal{W}$ , set

$$\eta(w) = 1 \text{ if } w = \emptyset \text{ and } \eta(w) = 0 \text{ otherwise}$$

$$\iota(w) = (-1)^n c_n c_{n-1} \dots c_1$$

$$\Delta(w) = \sum_{0 \leq i \leq n} c_1 \dots c_i \otimes c_{i+1} \dots c_n$$

The map  $\eta$ ,  $\iota$  and  $\Delta$  are algebra morphisms. Indeed  $\mathcal{H}$  is a Hopf algebra with co-unit  $\eta$ , inverse map  $\iota$  and coproduct  $\Delta$ .

(1.13) Concatenation product: For two words  $w = c_1 \dots c_n$  and  $w' = c_{n+1} \dots c_{n+m}$ , their concatenation is the word  $ww' = c_1 \dots c_n c_{n+1} \dots c_{n+m}$ . This induces another structure of algebra on  $\mathcal{H}$ , for which the product of two elements  $h, h'$  is simply denoted by  $hh'$ .

(1.14) Remarks on references and on the terminology:

In the classical literature, only the functions  $L_k(z) = \sum_{n>0} z^n/n^k$  are called polylogarithms, see [L] [Oe]. We did not find a standard name for the  $L_{(r_1, \dots, r_k)}(z)$ . They are defined in the Bourbaki's talk [C], where the title suggests to call them again polylogarithms.

It seems that some polyzeta values, like  $\zeta(1, 3)$ , were already known by Euler, see [C]. The general definition of  $\zeta(r_1, \dots, r_k)$  appears explicitly around 1990 in [H] and [Z]. These numbers are also called multiple zeta values in [Z], multiple harmonic sums in [H], multizeta numbers in [E], Euler-Zagier numbers in [BB] and polyzetas numbers in [C].

The fact that polyzeta values are naturally indexed by words has been observed by many authors, see [H], [H-P] and [C]. Lemma 1 and corollary 3 are well-known. Proofs are given for the convenience of the reader.

## 2. Polylogarithmic functions at $1/2$ and at $\rho^{\pm 1}$ :

Define two linear maps  $\sigma, \tau : \mathcal{H} \rightarrow \mathcal{H}$  as follows. First set  $\sigma(a) = b$ ,  $\sigma(b) = a$ ,  $\tau(a) = a + b$  and  $\tau(b) = -a$ . For any word  $w = c_1 \dots c_n \in \mathcal{W}$ , set  $\sigma(w) = \sigma(c_n) \dots \sigma(c_1)$  and  $\tau(w) = \tau(c_n) \dots \tau(c_1)$ . It is easy to see that  $\sigma$  and  $\tau$  are algebra morphisms relative to the shuffle product  $*$  (they are anti-morphism relative to the concatenation product).

Define now the two operators  $\square, \nabla : \mathcal{H} \rightarrow \mathcal{H}$  as the following composite maps:

$$\square : \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{id \otimes \sigma} \mathcal{H} \otimes \mathcal{H} \xrightarrow{*} \mathcal{H}$$

$$\nabla : \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{id \otimes \tau} \mathcal{H} \otimes \mathcal{H} \xrightarrow{*} \mathcal{H}$$

Set  $\rho = e^{i\pi/3}$ .

THEOREM 7: For any  $h \in \mathcal{H}$ , we have:

$$\zeta(h) = L_{\square(h)}(1/2)$$

$$\zeta^\pm(h) = L_{\nabla(h)}(\rho^{\pm 1})$$

*Proof:* Note that  $\square(a) = \square(b) = a+b$ ,  $L_a(z) = \log 1/(1-z)$  and  $L_b(z) = \log z$ , therefore  $L_a(1/2) + L_b(1/2) = 0 = \zeta(a) = \zeta(b)$ . Similarly,  $\nabla(a) = 2a + b$  and  $\nabla(b) = 0$ , and we have  $L_{\nabla(a)}(\rho^{\pm 1}) = -2 \log(1 - \rho^{\pm 1}) + \log(\rho^{\pm 1}) = \pm i\pi = \zeta^\pm(a)$  and  $L_{\nabla(b)}(\rho^{\pm 1}) = 0 = \zeta^\pm(b)$ .

Since  $\square$  and  $\nabla$  are algebra morphisms, and since the algebra  $\mathcal{H}$  is generated by  $a$ ,  $b$  and  $\mathcal{W}^{++}$ , it is enough to show the formulas when  $h$  is a non empty word  $w$  in  $\mathcal{W}^{++}$ . So let  $w \in \mathcal{W}^{++}$  be a word of length  $n \geq 2$ . Set  $w = c_1 \dots c_n$  where  $c_i \in \{a, b\}$  are letters with  $c_1 = a$  and  $c_n = b$ .

Let  $\gamma : [0, 1] \rightarrow \overline{D}$ ,  $t \mapsto t$  be the straight path from 0 to 1. Choose two smooth paths  $\gamma_\pm : [0, 1] \rightarrow \overline{D}$  with the following properties:  $\gamma_\pm(0) = 0$ ,  $\gamma_\pm(1) = 1$ ,  $\gamma_\pm(1/2) = \rho^{\pm 1} = 1/2 \pm i\sqrt{3}/2$  and  $\text{Re } \gamma_\pm(t) \geq 1/2$  for all  $t \in [1/2, 1]$ . Set  $\Delta_n = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid 0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1\}$ . By Proposition 2, we have:

$$\zeta(w) = \int_{\Delta_n} \eta^* \omega_{c_1}(x_1) \eta^* \omega_{c_2}(x_2) \dots \eta^* \omega_{c_n}(x_n)$$

where  $\eta$  is the path  $\gamma$ , or  $\gamma^+$  or  $\gamma^-$ .

For  $0 \leq i \leq n$ , set  $\Delta'_i = \{(x_1, x_2, \dots, x_i) \in \mathbf{R}^n \mid 0 \leq x_1 \leq x_2 \leq \dots \leq x_i \leq 1/2\}$  and  $\Delta''_i = \{(x_{i+1}, x_{i+2}, \dots, x_n) \in \mathbf{R}^n \mid 1/2 \leq x_{i+1} \leq x_{i+2} \leq \dots \leq x_n \leq 1\}$ . From the decomposition:  $\Delta = \cup_{0 \leq i \leq n} \Delta'_i \times \Delta''_i$ , it follows that  $\zeta(w) = \sum_{0 \leq i \leq n} L'_i L''_i = \sum_{0 \leq i \leq n} L_i^{\pm} L_i^{\pm}$ , where the numbers  $L'_i$ ,  $L''_i$ ,  $L_i^{\pm}$ ,  $L_i^{\pm}$  are the following integrals:

$$L'_i = \int_{\Delta'_i} \gamma^* \omega_{c_1}(x_1) \gamma^* \omega_{c_2}(x_2) \dots \gamma^* \omega_{c_i}(x_i)$$

$$L''_i = \int_{\Delta''_i} \gamma^* \omega_{c_{i+1}}(x_{i+1}) \gamma^* \omega_{c_{i+2}}(x_{i+2}) \dots \gamma^* \omega_{c_n}(x_n)$$

$$L_i^{\pm} = \int_{\Delta'_i} \gamma_\pm^* \omega_{c_1}(x_1) \gamma_\pm^* \omega_{c_2}(x_2) \dots \gamma_\pm^* \omega_{c_i}(x_i)$$

$$L_i^{\pm} = \int_{\Delta''_i} \gamma_\pm^* \omega_{c_{i+1}}(x_{i+1}) \gamma_\pm^* \omega_{c_{i+2}}(x_{i+2}) \dots \gamma_\pm^* \omega_{c_n}(x_n)$$

Using Kontsevitch formula, we get  $L'_i = L_{w'_i}(1/2)$  and  $L_i^{\pm} = L_{w'_i}(\rho^{\pm 1})$ , where  $w'_i = c_1 \dots c_i$ . To evaluate  $L''_i$ , one needs to introduce some new notations. Define by  $S, T : \mathbf{C} \rightarrow \mathbf{C}$  the rational maps:  $S(z) = 1 - z$  and  $T(z) = 1 - 1/z$ . Define the new paths  $\delta, \delta_\pm : [0, 1/2] \rightarrow \mathbf{C}$  by  $\delta(t) = 1 - \gamma(1-t) = S \circ \gamma(1-t)$  and  $\delta_\pm(t) = 1 - 1/\gamma_\pm(1-t) = T \circ \gamma(1-t)$ . Clearly,  $\delta$  is the straight path from 0 to 1/2. Since  $T(\rho^{\pm 1}) = \rho^{\pm 1}$  and  $T(1) = 0$ ,  $\delta_\pm$  is



a path from 0 to  $\rho^{\pm 1}$ . Since  $\operatorname{Re} \gamma_{\pm}(t) \geq 1/2$  for all  $t \in [1/2, 1]$ , it follows that  $\delta_{\pm}$  lies in  $\overline{D} \setminus \{1\}$ .

With the convention of (1.7), we get:

$$\gamma^* \omega_c(t) = \delta^* \omega_{\sigma(c)}(1-t) \text{ and } \gamma_{\pm}^* \omega_c(t) = \delta_{\pm}^* \omega_{\tau(c)}(1-t)$$

for any  $c \in \mathbf{Q}a \oplus \mathbf{Q}b$ . Using the new variables  $y_j = 1 - x_j$ , we thus get:

$$L''_i = \int_{\overline{\Delta''_i}} \delta^* \omega_{\sigma(c_n)}(y_n) \delta^* \omega_{\sigma(c_{n-1})}(y_{n-1}) \dots \delta^* \omega_{\sigma(c_{i+1})}(y_{i+1})$$

$$L''_{\pm i} = \int_{\overline{\Delta''_{\pm i}}} \delta_{\pm}^* \omega_{\tau(c_n)}(y_n) \delta_{\pm}^* \omega_{\tau(c_{n-1})}(y_{n-1}) \dots \delta_{\pm}^* \omega_{\tau(c_{i+1})}(y_{i+1})$$

where  $\overline{\Delta''_i} = \{(y_n, y_{n-1}, \dots, y_{i+1}) \in \mathbf{R}^n \mid 0 \leq y_n \leq y_{n-1} \leq \dots \leq y_{i+1} \leq 1/2\}$ . It follows from Proposition 2 that  $L''_i = L_{\sigma(w''_i)}(1/2)$  and  $L''_{\pm i} = L_{\tau(w''_i)}(\rho^{\pm 1})$  where  $w''_i$  is the word  $c_{i+1} \dots c_n$ . Therefore we get

$$\zeta(w) = \sum_{0 \leq i \leq n} L_{w'_i}(1/2) L_{\sigma(w''_i)}(1/2), \text{ and}$$

$$\zeta(w) = \sum_{0 \leq i \leq n} L_{w'_i}(\rho^{\pm 1}) L_{\tau(w''_i)}(\rho^{\pm 1}).$$

Since  $\Delta(w) = \sum_{0 \leq i \leq n} w'_i \otimes w''_i$ , it is clear that  $\square(w) = \sum_{0 \leq i \leq n} w'_i * \sigma(w''_i)$  and  $\nabla(w) = \sum_{0 \leq i \leq n} w'_i * \tau(w''_i)$ , and therefore the formula follows from Corollary 3. Q.E.D.

### 3. Explicit expressions for $\zeta(r)$ .

Theorem 7 provides a combinatorial way to express any polyzeta value as a polylogarithmic functions at  $1/2$  or at  $\rho$  or at  $\bar{\rho}$ . In this section, Theorem 10 and Corollary 12 provided closed formulas for zeta values  $\zeta(r)$ , where  $r \geq 2$  is a given integer. The formulas are derived from Theorem 7. However, for general polyzeta values  $\zeta(r_1, \dots, r_k)$  the combinatorics seem too intricate to find a simple combinatorial formula.

The concatenation product  $hh'$ , which is not commutative, should not be confused with the commutative shuffle product  $h * h'$ . The following conventions will be used. First, for  $h \in \mathcal{H}$  and  $n \geq 1$ , the notation  $h^n$  will be the  $n^{\text{th}}$  power of  $h$  with respect to the concatenation product. Moreover, the concatenation product takes precedence of the shuffle product. For example, the expression  $hh' * h^n$  should be understood as  $(hh') * h^n$ .

*LEMMA 8: We have:*

$$\square(ab^{r-1}) = 2a^{r-1}(a+b) + \sum_{1 \leq j \leq r-2} a^j (a+b)^{r-j}$$

$$\nabla(ab^{r-1}) = (-1)^{r+1} 3a^r + \sum_{1 \leq j \leq r-2} (-1)^{j+1} a^j (b-a)^{r-j}$$

*Proof:* We have:

$$\begin{aligned}\square(ab^{r-1}) &= \sigma(ab^{r-1}) + \sum_{0 \leq j \leq r-1} ab^j * \sigma(b^{r-1-j}) \\ &= a^{r-1}b + \sum_{u+v=r-1} ab^u * a^v.\end{aligned}$$

Similarly, we have:

$$\begin{aligned}\nabla(ab^{r-1}) &= \tau(ab^{r-1}) + \sum_{0 \leq j \leq r-1} ab^j * \tau(b^{r-1-j}) \\ &= (-1)^{r+1}a^{r-1}(a+b) + \sum_{u+v=r-1} (-1)^v ab^u * a^v.\end{aligned}$$

Use now the formula,  $cw * a^v = \sum_{i+j=v} a^j c(w * a^i)$ , which holds for any word  $w$  and any letter  $c$ . Thus we have  $ab^u * a^v = \sum_{i+j=v} a^{j+1}(b^u * a^i)$  and we get:

$$\begin{aligned}\square(ab^{r-1}) &= a^{r-1}b + \sum_{j+u+v=r-1} a^{j+1}(b^u * a^v), \text{ and} \\ \nabla(ab^{r-1}) &= (-1)^{r+1}a^{r-1}(a+b) + \sum_{j+u+v=r-1} (-1)^{j+v} a^{j+1}(b^u * a^v).\end{aligned}$$

Using now the formulas:

$$(a+b)^N = \sum_{u+v=N} a^u * b^v \text{ and } (b-a)^N = \sum_{u+v=N} (-1)^u a^u * b^v$$

we get:

$$\begin{aligned}\square(ab^{r-1}) &= a^{r-1}b + \sum_{0 \leq j \leq r-1} a^{j+1}(a+b)^{r-1-j} \\ &= a^{r-1}b + \sum_{1 \leq j \leq r} a^j(a+b)^{r-j} \\ &= a^{r-1}b + a^r + \sum_{1 \leq j \leq r-1} a^j(a+b)^{r-j} \\ &= a^{r-1}(a+b) + \sum_{1 \leq j \leq r-1} a^j(a+b)^{r-j} \\ &= 2a^{r-1}(a+b) + \sum_{1 \leq j \leq r-2} a^j(a+b)^{r-j}.\end{aligned}$$

We also get:

$$\begin{aligned}\nabla(ab^{r-1}) &= (-1)^{r+1}a^{r-1}(a+b) + \sum_{0 \leq j \leq r-1} (-1)^j a^{j+1}(b-a)^{r-1-j} \\ &= (-1)^{r+1}a^{r-1}(a+b) + \sum_{1 \leq j \leq r} (-1)^{j+1} a^j(b-a)^{r-j} \\ &= (-1)^{r+1}a^{r-1}(a+b) + (-1)^{r+1}a^r + (-1)^r a^{r-1}(b-a) \\ &\quad + \sum_{1 \leq j \leq r-2} (-1)^{j+1} a^j(b-a)^{r-j} \\ &= (-1)^{r+1}3a^r + \sum_{1 \leq j \leq r-2} (-1)^{j+1} a^j(b-a)^{r-j}. \text{ Q.E.D.}\end{aligned}$$

Let  $\mathcal{W}_r$  be the set of words of length  $r$ . Any  $w \in \mathcal{W}_r$  can be written as  $w = a^j u$ , where  $u$  does not start with  $a$ . Let  $k$  be the number of occurrence of  $a$  in  $w$ . Set  $j(w) = j$ ,  $k(w) = k$  and define the numbers  $c(w)$  and  $c^\pm(w)$  as follows.

- (i) If  $w = a^r$ , set  $c(w) = r$  and  $c^\pm(w) = (-1)^{r+1}(r+1)$ .
- (ii) If  $w = a^{r-1}b$ , set  $c(w) = r$  and  $c^\pm(w) = (-1)^r(r-2)$ .
- (iii) Otherwise, set  $c(w) = j$  and  $c^\pm(w) = (-1)^{k+1}j$ .

*LEMMA 9:* We have:

$$\square(ab^{r-1}) = \sum_{w \in \mathcal{W}_r} c(w)w$$

$$\nabla(ab^{r-1}) = \sum_{w \in \mathcal{W}_r} c^\pm(w)w$$

*Proof:* The first identity of Lemma 8 can be written as:

$$\square(ab^{r-1}) = a^{r-1}b + \sum_{1 \leq i \leq r} a^i(a+b)^{r-i}$$

Since  $(a+b)^{r-i} = \sum_{u \in \mathcal{W}_{r-i}} u$ , we thus get

$$\square(ab^{r-1}) = a^{r-1}b + \sum_{1 \leq i \leq r} \sum_{u \in \mathcal{W}_{r-i}} a^i u$$

The word  $w = a^{j(w)}v$  belongs to  $a^i\mathcal{W}_{r-i}$  for all  $i \leq j(w)$ . Therefore

$\sum_{1 \leq i \leq r} a^i(a+b)^{r-i} = \sum_{w \in \mathcal{W}_r} j(w)w$ . Since  $c(a^{r-1}b) = j(a^{r-1}b) + 1$  and  $c(w) = j(w)$  otherwise, the formula  $\square(ab^{r-1}) = \sum_{w \in \mathcal{W}_r} c(w)w$  is now proved.

Set  $\bar{a} = -a$  and  $\bar{b} = b$ . For a word  $w = c_1 \dots c_n$ , set  $\bar{w} = \bar{c}_1 \dots \bar{c}_n$  and for a general element  $h = \sum c_w w$  in  $\mathcal{H}$  set  $\bar{h} = \sum c_w \bar{w}$ . Since the involution  $h \mapsto \bar{h}$  is a morphism relative to the concatenation product, it follows from Lemma 8 that:

$$\begin{aligned} -\overline{\nabla(ab^{r-1})} &= 3a^r + \sum_{1 \leq i \leq r-2} a^i(b+a)^{r-i} \\ &= a^r - a^{r-1}b + \sum_{1 \leq i \leq r} a^i(b+a)^{r-i} \end{aligned}$$

It follows from the previous proof that  $\sum_{1 \leq i \leq r} a^i(a+b)^{r-i} = \sum_{w \in \mathcal{W}_r} j(w)w$ . Therefore, one gets:

$$-\overline{\nabla(ab^{r-1})} = a^r - a^{r-1}b + \sum_{w \in \mathcal{W}_r} j(w)w.$$

Note that  $\bar{w} = (-1)^{k(w)} w$  for all words  $w$ . Thus:

$$\nabla(ab^{r-1}) = (-1)^{r+1}a^r - (-1)^r a^{r-1}b + \sum_{w \in \mathcal{W}_r} (-1)^{(1+k(w))} j(w)w$$

Since  $c^\pm(a^r) = (-1)^{r+1}(j(a^r) + 1)$ ,  $c^\pm(a^{r-1}b) = (-1)^r(j(a^{r-1}b) - 1)$  and  $c^\pm(w) = (-1)^{1+k(w)} j(w)$  otherwise, the formula  $\nabla(ab^{r-1}) = \sum_{w \in \mathcal{W}_r} c^\pm(w)w$  is now proved. Q.E.D.

Let  $r \geq 2$  be an integer. Let  $\Lambda_r$  be the set of all  $\mathbf{m} = (m_1, \dots, m_k) \in \Lambda$  with  $m_1 + \dots + m_k = r$ . For  $\mathbf{m} = (m_1, \dots, m_k) \in \Lambda_r$ , set  $k(\mathbf{m}) = k$  and define the integers  $j(\mathbf{m})$ ,  $b(\mathbf{m})$  and  $b^\pm(\mathbf{m})$  as follows.

- (i) If  $m_1 = m_2 = \dots = m_r = 1$ , set  $j(\mathbf{m}) = r$ ,  $b(\mathbf{m}) = r$  and  $b^\pm(\mathbf{m}) = (-1)^{r+1}(r+1)$ . Otherwise, let  $j(\mathbf{m}) = j$  be the index such that  $m_1 = m_2 = \dots = m_{j-1} = 1$  and  $m_j \geq 2$ .
- (ii) If  $m_1 = m_2 = \dots = m_{r-2} = 1$  and  $m_{r+1} = 2$ , set  $b(\mathbf{m}) = r$  and  $b^\pm(\mathbf{m}) = (-1)^r(r-2)$ .
- (iii) Otherwise, set  $b(\mathbf{m}) = j$  and  $b^\pm(\mathbf{m}) = (-1)^{k+1}j$ , where  $j = j(\mathbf{m})$  and  $k = k(\mathbf{m})$ .

**THEOREM 10:** For  $r \geq 2$ , we have:

$$\zeta(r) = \sum_{\mathbf{m} \in \Lambda_r} b(\mathbf{m})L_{\mathbf{m}}(1/2)$$

$$\zeta(r) = \sum_{\mathbf{m} \in \Lambda_r} b^\pm(\mathbf{m}) L_{\mathbf{m}}(\rho^{\pm 1})$$

*Proof:* It is clear that  $c(w)$  and  $c^\pm(w)$  vanish if  $w \notin \mathcal{W}^+$ . Therefore it follows from Theorem 7 and Lemma 9 that

$$\zeta(r) = \sum_{w \in \mathcal{W}_r^+} c(w) L_w(1/2)$$

$$\zeta(r) = \sum_{w \in \mathcal{W}_r^+} c^\pm(w) L_w(\rho^{\pm 1})$$

where  $\mathcal{W}_r^+ = \mathcal{W}^+ \cap \mathcal{W}_r$ . Note that the map  $\lambda$  of section 1.4 provides a bijection  $\lambda : \mathcal{W}_r^+ \rightarrow \Lambda_r$ . It is easy to check that  $j(w) = j(\lambda(w))$  and  $k(w) = k(\lambda(w))$  for all  $w \in \mathcal{W}_r^+$ , and therefore

$$c(w) = b(\lambda(w)) \quad \text{and} \quad c^\pm(w) = b^\pm(\lambda(w))$$

for all  $w \in \mathcal{W}_r^+$ . Therefore Theorem 10 is proved. Q.E.D.

For  $1 \leq i \leq r-1$ , set

$$C_i = \{\mathbf{n} = (n_1 \dots n_k) \in \mathbf{Z}^r \mid 0 < n_1 < \dots < n_i \leq n_{i+1} \leq \dots \leq n_k\}.$$

Also, set  $C_r = C_{r-1}$ .

LEMMA 11: *We have:*

$$\zeta(r) = \sum_{1 \leq i \leq r} \sum_{\mathbf{n} \in C_i} \frac{2^{-n_r}}{n_1 n_2 \dots n_r}$$

*Proof:* Set  $c = a + b$ . For any word  $w = d_1 \dots d_r$  into the letters  $a, b$  and  $c$ , let  $C_w$  be the set of all  $\mathbf{n} = (n_1 \dots n_k) \in \mathbf{Z}^r$  satisfying the following property:

$$0 \mathcal{R}_1 n_1 \mathcal{R}_2 n_2 \dots \mathcal{R}_r n_r$$

where  $\mathcal{R}_i$  stands for the symbol  $<$  if  $d_i = a$ ,  $\mathcal{R}_i$  stands for the symbol  $=$  if  $d_i = b$  and  $\mathcal{R}_i$  stands for the symbol  $\leq$  if  $d_i = c$ . So if  $w$  is a word into the letters  $a, b$  and  $c$ , we get  $L_w(z) = \sum_{\mathbf{n} \in C_w} \frac{z^{n_r}}{n_1 \dots n_r}$ .

By Lemma 8, we have:

$$\square(ab^{r-1}) = a^{r-1}(a+b) + \sum_{1 \leq j \leq r-1} a^j (a+b)^{r-j}$$

Since  $C_i = C_{a^i c^{r-i}}$  for all  $i \leq r-1$ , and  $C_r = C_{a^{r-1} c}$ , we get

$$L_{\square(ab^{r-1})}(z) = \sum_{1 \leq i \leq r} \sum_{\mathbf{n} \in C_i} \frac{z^{-n_r}}{n_1 n_2 \dots n_r}$$

Therefore, Lemma 11 follows from Theorem 7. Q.E.D.

Set

$$C = \{\mathbf{n} = (n_1 \dots n_r) \in \mathbf{Z}^r \mid 0 < n_1 \leq n_2 \leq \dots \leq n_r\}.$$

For  $\mathbf{n} = (n_1 \dots n_r) \in C$ , define the number  $a(\mathbf{n})$  as follows. If we have  $0 < n_1 < \dots < n_{r-1}$  set  $a(\mathbf{n}) = k$ . Otherwise, there exist an index  $i \leq r-2$  such that  $0 < n_1 < n_2 \leq \dots \leq n_i = n_{i+1}$ . In such a case, set  $a(\mathbf{n}) = i$ . Note that  $a(\mathbf{n})$  does not depend on the last component  $n_r$  of  $\mathbf{n}$ , and the function  $\mathbf{n} \mapsto a(\mathbf{n})$  takes value in the set  $\{1, 2, \dots, r-2, r\}$

COROLLARY 12: *We have:*

$$\zeta(r) = \sum_{\mathbf{n} \in C} a(\mathbf{n}) \frac{2^{-n_r}}{n_1 n_2 \dots n_r}$$

*Proof:* It is easy to check that  $a(\mathbf{n})$  is precisely the number of indices  $i$ ,  $1 \leq i \leq r$  such that  $\mathbf{n}$  belongs to  $C_i$ . Therefore the formula of Corollary 7 follows from Lemma 11. Q.E.D.

Examples: For  $r = 2$ , then  $a((m, n)) = 2$  for all  $(m, n) \in C$ . Therefore, we get

$$\zeta(2) = 2 \sum_{0 < m \leq n} \frac{2^{-n}}{nm} = 2L_2(1/2) + \log^2 2$$

Accordingly to [C], this formula is due to Euler.

For  $r = 5$ , we have  $a((k, l, m, n, p)) = 1$  if  $k = l$ ,  $a((k, l, m, n, p)) = 2$  if  $k < l = m$ ,  $a((k, l, m, n, p)) = 3$  if  $k < l < m = n$  and  $a((k, l, m, n, p)) = 5$  if  $k < l < m < n$ . Therefore, we get the following expansion for  $\zeta(5)$

$$\sum_{0 < l \leq m \leq n \leq p} \frac{2^{-p}}{l^2 m n p} + 2 \sum_{0 < l < m \leq n \leq p} \frac{2^{-p}}{l m^2 n p} + 3 \sum_{0 < l < m < n \leq p} \frac{2^{-p}}{l m n^2 p} + 5 \sum_{0 < k < l < m < n \leq p} \frac{2^{-p}}{k l m n p}$$

#### 4. Geometric interpretation of Theorem 7.

Theorem 7 provides a combinatorial way to express any polyzeta value as the value of a polylogarithmic functions at  $1/2$  or at  $\rho^{\pm 1}$ . The combinatorics seem very intricate: e.g. the explicit formulas for zeta values  $\zeta(r)$  of Section 3 are difficult to extend for general polyzeta values  $\zeta(r_1, \dots, r_k)$ .

In this section, Theorem 7 is reformulated in terms of simple geometric notions.

(4.1) First, the free pro-algebraic group on two generators  $\Gamma$  and its Lie algebra  $\mathfrak{g}$  are defined.

Let  $F$  be the free Lie  $\mathbf{Q}$ -algebra with two generators  $\alpha$  and  $\beta$ , let  $C^n F$  be its central descending series and set  $\mathfrak{g} = \varprojlim F/C^n F$ . Since  $F/C^n F$  is a nilpotent Lie algebra, the Campbell-Hausdorff series defines a structure of algebraic group on  $F/C^n F$ , denoted by  $\Gamma_n$ . Then  $\Gamma = \varprojlim \Gamma_n$  is a proalgebraic group (an alternative definition of  $\Gamma$  is given in the introduction). As pro-algebraic varieties,  $\mathfrak{g}$  and  $\Gamma$  are identical, and the corresponding isomorphism is denoted by  $\exp : \mathfrak{g} \rightarrow \Gamma$ .

Let  $F = \bigoplus_{n \geq 1} F_n$  be the grading of  $F$  such that  $F_1 = \mathbf{Q}\alpha \oplus \mathbf{Q}\beta$ . Then we have  $\mathfrak{g} = \prod_{n \geq 1} F_n$ , so any  $x \in \mathfrak{g}$  can be written as the series  $x = \sum_{i > 0} x_i$  where  $x_i \in F_i$ . The multiplicative group  $\mathbf{Q}^*$  acts linearly on  $\mathfrak{g}$  as follows:  $t.x = \sum_{i > 0} t^i x_i$ , for any  $t \in \mathbf{Q}^*$ .

*LEMMA 13:* Let  $\Phi : \mathfrak{g} \rightarrow \mathfrak{g}$  be a morphism of pro-algebraic varieties. Assume that  $\Phi$  is  $\mathbf{Q}^*$ -invariant and that  $d\Phi_0$  is invertible, then  $\Phi$  is an isomorphism.

*Proof:* One can assume that  $d\Phi_0$  is the identity. Then choose a basis of  $F$  consisting of homogenous elements  $(e_n)_{n \geq 1}$  with  $d_n \leq d_m$  if  $n < m$ , where  $d_n$  is the degree of  $e_n$ . Accordingly, we have  $\Phi(\sum_{n \geq 1} x_n e_n) = \sum_{n \geq 1} \Phi_n(x) e_n$ , where each  $\Phi_n$  is a polynomial in  $x = (x_1, x_2, \dots)$ . By hypothesis, the linear part of  $\Phi_n(x)$  is  $x_n$  and for any monomial  $x_{i_1} \dots x_{i_k}$  occuring in  $\Phi_n(x)$  we have  $d_{i_1} + \dots + d_{i_k} = d_n$ . It follows that  $\Phi_n(x) - x_n$  depends only on  $x_1, \dots, x_{n-1}$ , so we can write:  $\Phi_n(x) = x_n + H_n(x_1, \dots, x_{n-1})$ . Since  $\Phi$  is triangular, it is an isomorphism.

(4.2) There is an isomorphism of Hopf algebras  $\mathbf{Q}[\Gamma] \simeq \mathcal{H}$ , see [P]. A natural group isomorphism  $\psi : \Gamma \rightarrow \text{Spec } \mathcal{H}$  is now described.

For  $t \in \mathbf{Q}$ , define two points  $\phi_a(t)$  and  $\phi_b(t)$  in  $\text{Spec } \mathcal{H}$  as follows. Since words  $w \in \mathcal{W}$  are functions on  $\text{Spec } \mathcal{H}$ , one needs to evaluate  $w$  at the points  $\phi_a(t)$  and  $\phi_b(t)$ . The rule is as follows:

$$\begin{aligned} w(\phi_a(t)) &= t^n/n! \text{ if } w = a^n \text{ and } w(\phi_a(t)) = 0 \text{ if } b \text{ occurs in } w. \\ w(\phi_b(t)) &= t^n/n! \text{ if } w = b^n \text{ and } w(\phi_b(t)) = 0 \text{ if } a \text{ occurs in } w. \end{aligned}$$

Then it is clear that  $\phi_a(t)$  and  $\phi_b(t)$  are two one-parameter groups in  $\text{Spec } \mathcal{H}$ . Since  $\Gamma$  is freely generated (as a proalgebraic group) by the two one-parameter groups  $\exp \mathbf{Q}\alpha$  and  $\exp \mathbf{Q}\beta$ , the isomorphism  $\psi$  is prescribed by the requirements  $\psi(\exp t\alpha) = \phi_a(t)$  and  $\psi(\exp t\beta) = \phi_b(t)$  for all  $t \in \mathbf{Q}$ .

(4.3) From now on, we identify  $\mathcal{H}$  and  $\mathbf{Q}[\Gamma]$ . Since  $\mathcal{H} = \mathbf{Q}[\Gamma]$ , any function  $\phi \in \mathbf{Q}[\Gamma]$  defines a polylogarithmic function  $L_\phi(z)$  and the polyzeta value  $\zeta(\phi)$  and the numbers  $\zeta^\pm(\phi)$ .

(4.4) The maps  $\sigma, \tau : \mathcal{H} \rightarrow \mathcal{H}$  are anti-isomorphisms of Hopf algebras, and therefore they induce two anti-isomorphisms of  $\Gamma$  and of its Lie algebra  $\mathfrak{g}$ . These are again denoted by  $\sigma$  and  $\tau$ . They are uniquely characterized by the requirements:

$$\begin{aligned} \sigma \exp t\alpha &= \exp t\beta \text{ and } \sigma \exp t\beta = \exp t\alpha \\ \tau \exp t\alpha &= \exp t(\alpha + \beta) \text{ and } \tau \exp t\beta = \exp -t\alpha, \end{aligned}$$

for all  $t \in \mathbf{Q}$ . We have  $\sigma^2(g) = g$  and  $\tau^3(g) = g^{-1}$  for any  $g \in G$ .

(4.5) Since  $\mathcal{H} = \mathbf{Q}[\Gamma]$ , the maps  $\square, \nabla$  occuring in Theorem 7 are now identified with some algebra morphisms  $\square, \nabla : \mathbf{Q}[\Gamma] \rightarrow \mathbf{Q}[\Gamma]$ .

*LEMMA 14:* Let  $\phi \in \mathbf{Q}[\Gamma]$ . Then for any  $g \in \Gamma$ , we have

$$\square \phi(g) = \phi(g\sigma(g)) \quad \text{and} \quad \nabla \phi(g) = \phi(g\tau(g))$$

*Proof:* Using their definitions,  $\square$  and  $\nabla$  are the composition of the following maps:

$$\Gamma \xrightarrow{\text{diag}} \Gamma \times \Gamma \xrightarrow{id \times \sigma} \Gamma \times \Gamma \xrightarrow{\mu} \Gamma$$

$$\Gamma \xrightarrow{\text{diag}} \Gamma \times \Gamma \xrightarrow{id \times \tau} \Gamma \times \Gamma \xrightarrow{\mu} \Gamma$$

where  $\text{diag}(g) = (g, g)$  and  $\mu(g_1, g_2) = g_1 \cdot g_2$ . Therefore we have

$$\square \phi(g) = \phi(g \cdot \sigma(g)) \quad \text{and} \quad \nabla \phi(g) = \phi(g \cdot \tau(g)).$$

(4.6) In this subsection, the symmetric space associated with  $\sigma$  is defined.

Set  $\mathfrak{k} = \{x \in \mathfrak{g} \mid \sigma(x) = -x\}$ ,  $K = \{g \in \Gamma \mid \sigma(g) = g^{-1}\}$ ,  $\mathfrak{p} = \{x \in \mathfrak{g} \mid \sigma(x) = x\}$   
 $P = \{p \in \Gamma \mid \sigma(p) = p\}$ .

Since  $\sigma$  is an anti-involution,  $K$  is a subgroup in  $\Gamma$  and  $\mathfrak{k}$  is its a Lie algebra. Obviously we have  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Since  $\Gamma$  is a pro-unipotent group, we have  $K = \exp \mathfrak{k}$ ,  $P = \exp \mathfrak{p}$  and  $\Gamma = P.K$ . So any element  $g \in \Gamma$  can be written as  $g = p.k$ , where  $k \in K$  and  $p \in P$ . Moreover  $P \simeq \Gamma/K$  is a symmetric space.

LEMMA 15: Let  $\phi \in \mathbf{Q}[\Gamma]$ . Then for any  $g = p.k \in \Gamma$ , we have

$$\square \phi(p.k) = \phi(p^2)$$

In particular,  $\zeta(\phi) = 0$  if  $\phi|_P \equiv 0$ .

*Proof:* This follows from Lemma 14 and Theorem 7.

(4.7) Note that  $\tau$  is not an involution, but an "ordrer three" anti-isomorphism, i.e.  $\tau^3(g) = g^{-1}$ . In this sub-section, we introduce a space  $Q$  which is analogous to a symmetric space.

Set  $\mathfrak{l} = \{x \in \mathfrak{g} \mid \tau(x) = -x\}$ ,  $L = \{g \in \Gamma \mid \tau(g) = g^{-1}\}$ ,  $\mathfrak{q} = \{x \in \mathfrak{g} \mid \tau^2(x) - \tau(x) + x = 0\}$ . Also define  $Q$  as the image of the map  $g \in \Gamma \mapsto g\tau(g)$ . Note that  $L$  is a subgroup with Lie algebra  $\mathfrak{l}$ .

LEMMA 16: The subset  $Q$  is a closed subvariety of  $\Gamma$  and the natural map:  $Q \times L \rightarrow \Gamma$ ,  $(q, l) \mapsto ql$  is a isomorphism of pro-algebraic varieties.

*Proof* It is easy to prove that  $\Gamma = \exp \mathfrak{q}.L$ . For  $g = \exp q.l$ , with  $q \in \mathfrak{q}$  and  $l \in L$ , we have  $g\tau(g) = \exp q \exp \tau(q)$ , therefore  $Q$  is the set all  $\exp q \exp \tau(q)$  for  $q \in \mathfrak{q}$ .

Let  $\Phi : \mathfrak{q} \oplus \mathfrak{l} \rightarrow \Gamma$  be define by  $\Phi(q, l) = \exp q \exp \tau(q) \exp l$ . Note that  $d\Phi_0$  is the linear map from  $\mathfrak{g}$  to  $\mathfrak{g}$  which is the identity on  $\mathfrak{l}$  and whose restriction to  $\mathfrak{q}$  is  $1 + \tau$ . Therefore,  $d\Phi_0$  is invertible. By Lemma 13 that  $\Phi$  is an isomorphism and Lemma 16 follows easily. Q.E.D.

The definition of  $Q$  is slighly more complicated than the definition of  $P$  because  $Q \neq \exp \mathfrak{q}$ . However, the map  $\mathfrak{q} \rightarrow Q$ ,  $q \mapsto \exp q \exp \tau(q)$  is an isomorphism from  $\mathfrak{q}$  to  $Q$ . Any element  $g \in \Gamma$  can be written as  $g = q.l$ , where  $l \in L$  and  $q \in Q$

LEMMA 17: Let  $\psi \in \mathbf{Q}[\Gamma]$ . Then for any  $g = q.l \in \Gamma$ , we have

$$\nabla \psi(q.l) = \psi(q\tau(q))$$

In particular,  $\zeta(\psi) = 0$  if  $\psi|_Q \equiv 0$ .

*Proof:* This follows from Lemma 14 and Theorem 7.

(4.8) Let  $\phi \in \mathbf{Q}[P]$  be a rational function on  $P$ . The notations  $\zeta(\phi)$  and  $L_\phi(z)$  are now defined. Set  $\zeta(\phi) = \zeta(\hat{\phi})$  where  $\hat{\phi}$  is any function on  $\Gamma$  extending  $\phi$ . By Lemma 15,  $\zeta(\phi)$  is well defined. Since  $P \simeq \Gamma/K$ ,  $\phi$  can be uniquely extended to a right  $K$ -invariant function  $\Phi$  on  $\Gamma$ . Then set  $L_\phi(z) = L_\Phi(z)$ .

Similarly, for  $\psi \in \mathbf{Q}[Q]$ , the notations  $\zeta^\pm(\psi)$  and  $L_\psi(z)$  are defined as follows. Set  $\zeta^\pm(\psi) = \zeta^\pm(\hat{\psi})$  where  $\hat{\psi}$  is any function on  $\Gamma$  extending  $\psi$ . By Lemma 17,  $\zeta^\pm(\psi)$  is well defined. By Lemma 16, we have  $Q \simeq \Gamma/L$ , therefore  $\psi$  can be uniquely extended to a right  $L$ -invariant function  $\Psi$  on  $\Gamma$ . Then set  $L_\psi(z) = L_\Psi(z)$ .

Define the algebra morphisms  $\square : \mathbf{Q}[P] \rightarrow \mathbf{Q}[P]$  and  $\nabla : \mathbf{Q}[Q] \rightarrow \mathbf{Q}[Q]$  by

$$\square \phi(p) = \phi(p^2), \quad \text{for } \phi \in \mathbf{Q}[P]$$

$$\nabla \psi(q) = \psi(q\tau(q)), \quad \text{for } \psi \in \mathbf{Q}[Q]$$

These operators are simply the restrictions to  $P$  and to  $Q$  of the already defined operators  $\square, \nabla : \mathbf{Q}[\Gamma] \rightarrow \mathbf{Q}[\Gamma]$ . So using the same notations should not bring confusions.

THEOREM 18: For any  $\phi \in \mathbf{Q}[P]$  and  $\psi \in \mathbf{Q}[Q]$ , we have

$$\zeta(\psi) = L_{\square(\psi)}(1/2) \quad \text{and} \quad \zeta^\pm(\psi) = L_{\nabla(\psi)}(\rho^\pm)$$

*Proof:* It follows immediately from Theorem 7, and Lemmas 15 and 17.

## 5. Other expressions for zeta values.

In this section, we follow a suggestion of W. Zudilin

(5.1) Theorem 7 shows that any polyzeta value is the value of a polylogarithmic function at  $1/2$  or at  $\rho^\pm$ . However, there is a much more simple way to express the zeta values  $\zeta(r)$  as a value of polylogarithmic functions at  $1/2$  or at  $\rho^\pm$ , see Corollary 20. It is surprizing that the two approaches give different expressions, except for  $\zeta(2)$ . Moreover, this simpler approach does not generalize to polyzeta values.

(5.2) Let  $\sigma' : \mathcal{H} \rightarrow \mathcal{H}$  be the linear map defined as follows. Set  $\sigma'(a) = -a$  and  $\sigma'(b) = a + b$ . For a word  $w = c_1 \dots c_n$ , where  $c_i \in \{a, b\}$  are letters, set  $\sigma'(w) = \sigma'(c_1) \dots \sigma'(c_n)$ . It is easy to prove that  $\sigma'$  is an algebra morphism relative to the schuffle product and that  $\sigma'(\mathcal{H}^+) = \mathcal{H}^+$ .

LEMMA 19 For any  $h \in \mathcal{H}^+$ , we have:



$$L_h(-1) = L_{\sigma'(h)}(1/2) \text{ and } L_h(\bar{\rho}) = L_{\sigma'(h)}(\rho).$$

*Proof:* One can assume that  $h$  is a word  $w = c_1 \dots c_n \in \mathcal{W}^+$ . Set  $F = \{z \in \mathbf{C} \mid |z| \leq 1 \text{ and } \text{Im}z \leq 1/2\}$  and choose a two paths  $\gamma, \gamma_- : [0, 1] \rightarrow F$  with  $\gamma(0) = \gamma_-(0) = 0$ ,  $\gamma(1) = -1$  and  $\gamma_-(1) = \bar{\rho}$ . By Proposition 2, we have:

$$L_w(-1) = \int_{\Delta_n} \gamma^* \omega_{c_1}(x_1) \gamma^* \omega_{c_2}(x_2) \dots \gamma^* \omega_{c_n}(x_n)$$

$$L_w(\bar{\rho}) = \int_{\Delta_n} \gamma_-^* \omega_{c_1}(x_1) \gamma_-^* \omega_{c_2}(x_2) \dots \gamma_-^* \omega_{c_n}(x_n)$$

For  $z \in F$ , set  $S'(z) = z/(z-1)$  and set  $\delta = S' \circ \gamma$  and  $\delta_- = S' \circ \gamma_-$ . We have  $S'^* \omega_{\sigma'(c)} = \omega_c$  for  $c = a$  or  $b$ , and therefore we get  $\gamma^* \omega_c = \delta^* \omega_{\sigma'(c)}$  and  $\gamma_-^* \omega_c = \delta_-^* \omega_{\sigma'(c)}$ . It follows that

$$L_w(-1) = \int_{\Delta_n} \delta^* \omega_{\sigma'(c_1)}(x_1) \delta^* \omega_{\sigma'(c_2)}(x_2) \dots \delta^* \omega_{\sigma'(c_n)}(x_n)$$

$$L_w(\bar{\rho}) = \int_{\Delta_n} \delta_-^* \omega_{\sigma'(c_1)}(x_1) \delta_-^* \omega_{\sigma'(c_2)}(x_2) \dots \delta_-^* \omega_{\sigma'(c_n)}(x_n)$$

We have  $S'(F) = F$ ,  $S'(0) = 0$ ,  $S'(-1) = 1/2$  and  $S'(\bar{\rho}) = \rho$ . Therefore  $\delta$  is a path from 0 to 1/2 and  $\delta_-$  is a path from  $o$  to  $\rho$ . Thus, these integrals can be identified by Proposition 2, and we get  $L_h(-1) = L_{\sigma'(h)}(1/2)$ , and  $L_h(\bar{\rho}) = L_{\sigma'(h)}(\rho)$ . Q.E.D.

*LEMMA 20:* Let  $r \geq 1$ . We have

$$\zeta(r+1) = \frac{-1}{1-2^{-r}} L_{r+1}(-1)$$

$$\zeta(r+1) = \frac{1}{(1-2^{-r})(1-3^{-r})} [L_{r+1}(\rho) + L_{r+1}(\bar{\rho})]$$

*Proof:* For each positive integer  $a$ , set  $\delta_a(n) = 1$  if  $a$  divides  $n$  and  $\delta_a(n) = 0$  otherwise.

From the formula  $(-1)^n = -\delta_1(n) + 2\delta_2(n)$ , we get

$$\begin{aligned} L_{r+1}(-1) &= \sum_{n>0} (-1)^n / n^{r+1} \\ &= - \sum_{n>0} 1/n^{r+1} + 2 \sum_{n>0} 1/(2n)^{r+1} \\ &= (-1 + 2^{-r}) \zeta(r) \end{aligned}$$

from which the first formula follows.

From the formula  $\rho^n + \bar{\rho}^n = \delta_1(n) - 2\delta_2(n) - 3\delta_3(n) + 6\delta_6(n)$  we get

$$\begin{aligned} L_{r+1}(\rho) + L_{r+1}(\bar{\rho}) &= \sum_{n>0} [\rho^n + \bar{\rho}^n] / n^{r+1} \\ &= \sum_{n>0} 1/n^{r+1} - 2 \sum_{n>0} 1/(2n)^{r+1} - 3 \sum_{n>0} 1/(3n)^{r+1} + 6 \sum_{n>0} 1/(6n)^{r+1} \\ &= (1 - 2^{-r} - 3^{-r} + 6^{-r}) \zeta(r+1) \\ &= (1 - 2^{-r})(1 - 3^{-r}) \zeta(r+1), \end{aligned}$$

from which the second formula follows.

COROLLARY 21: Let  $r \geq 1$ . We have

$$\zeta(r+1) = \frac{2^r}{(2^r-1)} L_{a(a+b)^r}(1/2)$$

$$\zeta(r+1) = \frac{6^r}{(2^r-1)(3^r-1)} [L_{ab^r}(\rho) - L_{a(a+b)^r}(\rho)]$$

*Proof:* The corollary follows from Lemmas 18 and 19.

Examples:

$$\zeta(2) = 2 \sum_{0 < m \leq n} \frac{2^{-n}}{nm}$$

$$\zeta(3) = 8/7 \sum_{0 < l \leq m \leq n} \frac{2^{-n}}{lnm}$$

The expression for  $\zeta(2)$  is the same as in section 3. However for all other zeta values  $\zeta(r)$  with  $r \geq 3$ , the expressions are different: e.g., the formula of corollary 20 uses non integral coefficients. Moreover, this simpler approach only concerns zeta values but not the polyzeta values.

**6. Conclusion:** Polyzyta values are mixed periods  $[\mathbf{G}]$ ,  $[\mathbf{T}]$ ,  $[\mathbf{Z}]$ . In the philosophy of motives, there is a proalgebraic group  $G$  and periods should be regular functions on  $G$ : more precisely, the algebra of periods should be a  $\overline{\mathbf{Q}}$ -form of  $\mathbf{C}[G]$ , modulo conjectures. Here polyzyta values are attached to some symmetric space. Does there is a motivic interpretation of this construction? Note however that the map  $\mathbf{Q}[P] \rightarrow \mathcal{Z}$  is not injective.

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