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► **To cite this version:**

| Olivier Mathieu. On a symmetric space attached to polyzeta values.. 2008. hal-00327162

HAL Id: hal-00327162

<https://hal.science/hal-00327162v1>

Preprint submitted on 7 Oct 2008

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On a symmetric space attached to polyzeta values.

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ABSTRACT

Quickly converging series are given to compute polyzeta numbers $\zeta(r_1, \dots, r_k)$. The formulas involve an intricate combination of (generalized) polylogarithms at $1/2$. However, the combinatoric has a very simple geometric interpretation: it corresponds with the map $p \mapsto p^2$ on a certain symmetric space P .

Introduction:

Let $k \geq 1$. For a k -uple (r_1, r_2, \dots, r_k) of positive integers, set $\zeta(r_1, \dots, r_k) = \sum_{0 < n_1 \dots < n_k} 1/n_1^{r_1} \dots n_k^{r_k}$. We have $\zeta(r_1, \dots, r_k) < \infty$ if and only if $r_k \geq 2$. By definition, the *polyzeta values* are the \mathbf{Q} -linear combinations of the finite numbers $\zeta(r_1, \dots, r_k)$.

Using the definition of $\zeta(r_1, \dots, r_k)$, the evaluation of a polyzeta value up to the N^{th} digit requires to take into account something like $O(10^N)$ terms. Therefore it is a very slow computation. A similar computational problem arises with the classical series $\log 2 = -\sum_{n>0} (-1)^n/n$ and $\pi/4 = \sum_{n \geq 0} (-1)^n 1/(2n+1)$, which converge very slowly.

However, we easily notice that:

$$\log 2 = -\log(1 - 1/2) = \sum_{n>0} 2^{-n}/n.$$

A remarkable series for π has been discovered by Bailey, Borwein and Plouffe [BBP]:

$$\pi = \sum_{n \geq 0} 1/2^{4n} [4/(8n+1) - 2/(8n+4) - 1/(8n+5) - 1/(8n+6)]$$

Now to evaluate $\log 2$ or π up to the N^{th} digit, one only needs the first $O(N)$ -terms of the series and therefore $\log 2$ and π can be computed very quickly. The goal of the paper is to provide similar identities for all polyzeta values.

To do so, one needs to use the functions $L_{r_1, \dots, r_k}(z) = \sum_{0 < n_1 \dots < n_k} 1/n_1^{r_1} \dots n_k^{r_k} z^{n_k}$, where r_1, \dots, r_k are positive integers. By definition, a \mathbf{Q} -linear combinations of the functions $L_{r_1, \dots, r_k}(z)$ is called a *polylogarithmic function*. The obvious identity $\zeta(r_1, \dots, r_k) = L_{r_1, \dots, r_k}(1)$ does not help to quickly evaluate polyzeta values. However, the series defining polylogarithms at $1/2$ converges very quickly: to evaluate $L_{r_1, \dots, r_k}(1/2)$ up to the N^{th} digit, one only needs to sum $O(N^k)$ -terms, and this can be done in polynomial time. This remark suggests the following result:

2000 Mathematics Subject Classification: 11M99, 17B01, 53C35

Keywords: Polyzyeta values, symmetric spaces, polylogarithms

MAIN STATEMENT: *Any polyzeta value is the value of a certain polylogarithmic function at $1/2$.*

In order to get a useful statement, the corresponding polylogarithmic function is described explicitly: see Theorem 7 for a precise statement. At first glance, the combinatorics involved in Theorem 7 looks intricate and therefore no details are given in the introduction. However, we can precisely formulate the main statement in terms of very simple geometric notions.

Let $F(2)$ be the free group on two generators α and β and let s be the involution exchanging the generators. Let $\Gamma = \mathbf{Q} \otimes F_2$ be the Malcev completion of Γ (see Section 4 for an alternative definition of Γ). The involution s extends to Γ and there is a decomposition $\Gamma = P.K$ where K is the subgroup of fixed points of s and where $P = \{g \in \Gamma | s(g) = g^{-1}\}$. The group Γ is proalgebraic over \mathbf{Q} and the symmetric space P is a pro-algebraic variety over \mathbf{Q} .

In section (4.8), all polyzeta values are naturally indexed by rational functions on P . Similarly, some polylogarithmic functions are naturally indexed by rational functions on P . So for $\phi \in \mathbf{Q}[P]$, denote by $\zeta(\phi)$ and $L_\phi(z)$ the corresponding polyzeta value and polylogarithmic function.

Now the square map $\square : P \rightarrow P, p \mapsto p^2$ induces an algebra morphism $\square : \mathbf{Q}[P] \rightarrow \mathbf{Q}[P]$. The geometric formulation of the main result is as follows:

MAIN THEOREM: *For any $\phi \in \mathbf{Q}[P]$, $\zeta(\phi) = L_{\square\phi}(1/2)$.*

We also express polyzeta values as values of polylogarithmic functions at $\rho^{\pm 1} = \exp \pm i\pi/3$. The geometric interpretation of this case is a bit more complicated because it involves an order 3 automorphism of Γ , see section 4, Theorem 18.

Acknowledgements: A special thank to Wadim Zudilin. Section 5 has been suggested by him.

Summary:

1. Polylogarithms and polyzeta values.
2. Polylogarithmic function at $1/2$ and at $\rho^{\pm 1}$.
3. Explicit expressions for $\zeta(r)$.
4. Geometric interpretation of Theorem 7.
5. Other expressions for zeta values.
6. Conclusion.

1. Polylogarithms and polyzeta values.

This section is devoted to main definitions and conventions. The definitions of *polyzeta values* and *polylogarithmic functions* are not standard: see the subsections (1.14) for more comments. Moreover in this section we adopt some conventions to renormalize infinite quantities like $\zeta(1)$ or $\int_0^z dt/t$.

(1.1) Shuffles: For $N \geq 0$, denote by S_N the symmetric group, i.e. the set of all bijections $\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$. Given n and m two non-negative integers, let

$S_{n,m}$ be the set of all $\phi \in S_{n+m}$ such that ϕ is increasing on the subset $\{1, \dots, n\}$ and on the subset $\{n+1, \dots, n+m\}$. The elements of $S_{n,m}$ are called *shuffles*.

(1.2) Shuffle product: Let \mathcal{W} be the set of words into the letters a and b . By convention, \mathcal{W} contains the empty word \emptyset . Set $\mathcal{H} = \mathbf{Q}\mathcal{W}$, i.e. \mathcal{H} is the \mathbf{Q} -vector space with basis \mathcal{W} . For any two words $w = x_1 \dots x_n$ and $w' = x_{n+1} \dots x_{n+m}$, where each $x_i \in \{a, b\}$ is a letter, define the product $w * w' \in \mathcal{H}$ by the formula:

$$w * w' = \sum_{\sigma \in S_{n,m}} x_{\sigma(1)} \dots x_{\sigma(n+m)}$$

By convention, we have $\emptyset * w = w * \emptyset = w$ for all word w . The product $*$ is called the shuffle product. With respect to this product, \mathcal{H} is a commutative associative algebra, and \emptyset is its unit.

(1.3) Subalgebras of \mathcal{H} : Let \mathcal{W}^+ be the set of words whose first letter is not b . Equivalently, a word w belongs to \mathcal{W}^+ if $w = \emptyset$ or if w starts with a . Similarly, let \mathcal{W}^{++} be the set of words whose first letter is not b and the last letter is not a . Set $\mathcal{H}^+ = \mathbf{Q}\mathcal{W}^+$ and $\mathcal{H}^{++} = \mathbf{Q}\mathcal{W}^{++}$. It is easy to prove that \mathcal{H}^+ and \mathcal{H}^{++} are subalgebras of \mathcal{H} .

LEMMA 1: *There are isomorphisms of algebras: $\mathcal{H} = \mathcal{H}^+[b]$ and $\mathcal{H} = \mathcal{H}^{++}[a, b]$.*

Proof: For each $n \geq 0$, let \mathcal{W}_n be the set of words of the form $b^n w$ with $w \in \mathcal{W}^+$, and set $\mathcal{H}_n = \bigoplus_{0 \leq k \leq n} \mathbf{Q}\mathcal{W}_k$. We have $b * \mathcal{H}_n \subset \mathcal{H}_{n+1}$. Moreover we have $b * w = (n+1)bw$ modulo \mathcal{H}_n for any $w \in \mathcal{W}_n$. It follows easily by induction that $\mathcal{H}_n = \bigoplus_{0 \leq k \leq n} \mathcal{H}^+ * b^k$, i.e. \mathcal{H}_n is the space of all polynomials in b with coefficients in \mathcal{H}^+ and degree $\leq n$. Therefore the first assertion follows.

The proof of the second assertion is similar.

(1.4) The bijection $\lambda : \mathcal{W}^+ \rightarrow \Lambda$:

Let \mathbf{N} be the set of positive integer. For clarity, a word into the letters $1, 2, \dots \in \mathbf{N}$ will be called a *sequence of positive integers*. Let Λ the set of sequence $(r_1 \dots r_k)$ of positive integers. By convention, Λ contains the empty sequence \emptyset .

Any word $w \in \mathcal{W}^+$ can be uniquely factorized as: $w = ab^{t_1} ab^{t_2} \dots ab^{t_k}$, where k is the number of occurrence of a in w and where the t_i are non-negative integers. Then, the map $w \in \mathcal{W}^+ \mapsto (1 + t_1, 1 + t_2, \dots, 1 + t_k) \in \Lambda$ defines a natural bijection $\lambda : \mathcal{W}^+ \rightarrow \Lambda$.

(1.5) Polylogarithmic functions and polyzeta values:

Let $k \geq 1$ and let $r_1 \dots r_k$ be a sequence of k positive integers. Consider the following series in the complex variable z :

$$L_{r_1, \dots, r_k}(z) = \sum_{0 < n_1 < \dots < n_k} n_1^{-r_1} \dots n_k^{-r_k} z^{n_k}$$

In the infinite sum, the indices n_1, \dots, n_k are integers. The functions $L_{r_1, \dots, r_k}(z)$ are called *polylogarithms*. Set $D = \{z \in \mathbf{C} \mid |z| < 1\}$, $\overline{D} = \{z \in \mathbf{C} \mid |z| \leq 1\}$. The two points of interest for the paper are the following:

(i) if $r_k \geq 2$, the series is absolutely convergent on \overline{D} and therefore $L_{r_1, \dots, r_k}(z)$ extends to a continuous function on \overline{D} .

(ii) if $r_k = 1$, the series converges on D and $L_{r_1, \dots, r_k}(z)$ extends to a continuous function on $\overline{D} \setminus \{1\}$.

Indeed $L_{r_1, \dots, r_k}(z)$ extends to a multivalued function, see e.g. [C] and Proposition 2 below. For $r_k \geq 2$, set

$$\zeta(r_1, \dots, r_k) = \sum_{0 < n_1 < \dots < n_k} n_1^{-r_1} \dots n_k^{-r_k}$$

In the paper, the numbers $\zeta(r_1, \dots, r_k)$ will be called polyzeta values. Indeed the polyzeta value is both the value at $z = 1$ of the polylogarithm $L_{r_1, \dots, r_k}(z)$ and a value of the polyzeta function $\zeta(s_1, \dots, s_k) = \sum_{0 < n_1 < \dots < n_k} n_1^{-s_1} \dots n_k^{-s_k}$.

(1.6) New notations: Let $w \in \mathcal{W}^+$ and set $(r_1, \dots, r_k) = \lambda(w)$. It is convenient to denote the function $L_{(r_1, \dots, r_k)}(z)$ by $L_w(z)$. Similarly set $\zeta(w) = \zeta(r_1, \dots, r_k)$ if $w \in \mathcal{W}^{++}$.

(1.7) The one-forms ω_a and ω_b : Define the following one-forms on \mathbf{C} :

$$\omega_a(z) = \frac{dz}{1-z} \quad \text{and} \quad \omega_b(z) = \frac{dz}{z}$$

For an element $c = xa + yb \in \mathbf{Q}a \oplus \mathbf{Q}b$, set $\omega_c(z) = x\omega_a(z) + y\omega_b(z)$. Given a smooth path $\gamma : [0, 1] \rightarrow \mathbf{C}, t \mapsto \gamma(t)$, recall that $\gamma^*\omega_a(t) = \frac{\gamma'(t)}{1-\gamma(t)} dt$, $\gamma^*\omega_b(t) = \frac{\gamma'(t)}{\gamma(t)} dt$. and $\gamma^*\omega_c(t) = x\gamma^*\omega_a(t) + y\gamma^*\omega_b(t)$.

(1.8) Kontsevitch formula: For a positive integer n , set $\Delta_n = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n | 0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1\}$. Let $w = c_1 \dots c_n \in \mathcal{W}^+$ be a word, where each c_i is a letter. The following formula is due to Kontsevitch (see [Z]).

PROPOSITION 2: Let $w = c_1 \dots c_n \in \mathcal{W}^+$ be a word, let $z \in \overline{D}$, and let $\gamma : [0, 1] \rightarrow \overline{D}$ be a path with $\gamma(0) = 0$ and $\gamma(1) = z$.

Assume that $w \in \mathcal{W}^{++}$ or that γ does not meet 1. Then we have:

$$L_w(z) = \int_{\Delta_n} \gamma^*\omega_{c_1}(x_1)\gamma^*\omega_{c_2}(x_2)\dots\gamma^*\omega_{c_n}(x_n)$$

In [Z], Kontsevitch formula is stated for the straight path $t \mapsto zt$, but it is easy to see that the integral is homotopy invariant as long γ stay in \overline{D} (and γ stay $\overline{D} \setminus \{1\}$ if $w \notin \mathcal{W}^{++}$).

(1.9) Products: Let n, m be non negative integers and let $c_1, c_2, \dots, c_{n+m} \in \{a, b\}$ be letters with $c_1 = c_{n+1} = a$. Set $u = c_1 \dots c_n$ and $v = c_{n+1} \dots c_{n+m}$. For $\sigma \in S_{n,m}$, set $w_\sigma = c_{\sigma(1)} \dots c_{\sigma(n+m)}$.

COROLLARY 3: For $u, v \in \mathcal{W}^+$, we have $L_u(z)L_v(z) = \sum_{\sigma \in S_{n,m}} L_{w_\sigma}(z)$ for all $z \in D$. Moreover for $u, v \in \mathcal{W}^{++}$, we have $\zeta(u)\zeta(v) = \sum_{\sigma \in S_{n,m}} \zeta(w_\sigma)$.

Proof: Set

$$\begin{aligned} \omega' &= \omega_{c_1}(zx_1)\omega_{c_2}(zx_2)\dots\omega_{c_n}(zx_n), \\ \omega'' &= \omega_{c_{n+1}}(zx_{n+1})\omega_{c_{n+2}}(zx_{n+2})\dots\omega_{c_{n+m}}(zx_{n+m}), \\ \Delta_m &= \{(x_{n+1}, x_{n+2}, \dots, x_{n+m}) \in \mathbf{R}^n | 0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1\}, \end{aligned}$$

and for $\sigma \in S_{n,m}$, set

$$\Delta_\sigma = \{(x_1, \dots, x_{n+m}) \in \mathbf{R}^{n+m} \mid x_{\sigma(1)} \leq x_{\sigma(2)} \leq \dots \leq x_{\sigma(n+m)}\}.$$

Since $\Delta_n \times \Delta_m = \cup_{\sigma \in S_{n,m}} \Delta_\sigma$, we get $\int_{\Delta_n} \omega' \int_{\Delta_m} \omega'' = \int_{\Delta_n \times \Delta_m} \omega' \wedge \omega'' = \sum_{\sigma} \int_{\Delta_\sigma} \omega' \wedge \omega''$. By Proposition 2, this identity is equivalent to $L_u(z)L_v(z) = \sum_{\sigma \in S_{n,m}} L_{w_\sigma}(z)$. At $z = 1$, one gets the second identity $\zeta(u)\zeta(v) = \sum_{\sigma \in S_{n,m}} \zeta(w_\sigma)$. Q.E.D.

(1.10) Final definitions and notations for polylogarithmic functions: Up to now, the polylogarithms $L_w(z)$ are defined for $w \in \mathcal{W}^+$. In order to extend the definition to all $w \in \mathcal{W}$, a renormalization procedure is used.

Set $\Omega = D \setminus]-1, 0]$ and let $Hol(\Omega)$ be the algebra of holomorphic functions on Ω . Since ω is simply connected, let denote by $\log z$ the holomorphic function on Ω whose restriction to $]0, 1[$ is the usual logarithmic function.

LEMMA 4: *There is a unique algebra morphism $\Phi : \mathcal{H} \rightarrow Hol(\Omega)$ such that $\Phi(w) = L_w(z)$ for $w \in \mathcal{H}^+$ and $\Phi(b) = \log z$.*

Proof: This follows from Lemmma 1 and corollary 3. Q.E.D.

For any $h \in \mathcal{H}$, set $L_h(z) = \Phi(h)$. When h is a word w in \mathcal{W}^+ , this new notation agrees with the previous one. Corollary 3 can be restated as: $L_u(z)L_v(z) = L_{u*v}(z)$. By definition the *polylogarithmic functions* are the functions $L_h(z)$ with $h \in \mathcal{H}$.

This definition is a slightly different from the introduction. However, we will only use polylogarithmic functions $L_h(z)$ with $h \in \mathcal{H}^+$, which are the polylogarithmic functions defined in introduction.

(1.11) Final definitions and notations for polyzeta values: Up to now, the polyzeta values $\zeta(w)$ are defined for $w \in \mathcal{W}^{++}$. In order to extend the definition to all $w \in \mathcal{W}$, we will use a renormalization procedure as follows.

LEMMA 5: *There are three algebra morphisms $\psi, \psi^+, \psi^- : \mathcal{H} \rightarrow \mathbf{C}$ uniquely defined by the following requirements:*

$$\begin{aligned} \psi(w) &= \psi^+(w) = \psi^-(w) = \zeta(w) \text{ if } w \in \mathcal{H}^{++} \\ \psi(a) &= 0, \psi^+(a) = i\pi, \psi^-(a) = -i\pi \\ \psi(b) &= \psi^+(b) = \psi^-(b) = 0. \end{aligned}$$

Proof: This follows from Lemmma 1 and corollary 3.

Similarly, this allows to define $\zeta(h) = \psi(h)$, $\zeta^\pm(h) = \psi^\pm(h)$ for any $h \in \mathcal{H}$. By definition the *polyzeta values* are the numbers $\zeta(h)$ with $h \in \mathcal{H}$. Corollary 3 can be restated as: $\zeta(u)\zeta(v) = \zeta(u*v)$ and $\zeta^\pm(u)\zeta^\pm(v) = \zeta^\pm(u*v)$ for any $u, v \in \mathcal{H}$.

Set $\mathcal{Z} = \zeta(\mathcal{H})$ and $\mathcal{Z}^\pm = \zeta(\mathcal{H}^\pm)$. By definition, \mathcal{Z} and \mathcal{Z}^\pm are subrings of \mathbf{C} , and \mathcal{Z} is the space of all polyzeta values. It is easy to compare the three algebras \mathcal{Z} and \mathcal{Z}^+ and \mathcal{Z}^- .

LEMMA 6:

- (i) *As a \mathbf{Q} vector space, \mathcal{Z} is generated by all $\zeta(w)$ with $w \in \mathcal{W}^{++}$.*
- (ii) *We have $\mathcal{Z} \subset \mathbf{R}$.*
- (iii) *$\mathcal{Z}^\pm = \mathcal{Z} \oplus i\pi\mathcal{Z}$.*

Proof: The assertions (i) and (ii) follow from Lemma 1. Moreover it follows that \mathcal{Z}^\pm is the \mathbf{Q} -algebra generated by \mathcal{Z} and $\zeta^\pm(b) = \pm i\pi$. However $(\zeta^\pm(b))^2 = -\pi^2 = -6\zeta(2)$, therefore $(\zeta^\pm(b))^2$ belongs to \mathcal{Z} and assertion (6.3) follows.

(1.12) Hopf algebra structure: Define the linear maps $\eta : \mathcal{H} \rightarrow \mathbf{Q}$, $\iota : \mathcal{H} \rightarrow \mathcal{H}$ and $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ as follows. For any word $w = c_1 \dots c_n \in \mathcal{W}$, set

$$\eta(w) = 1 \text{ if } w = \emptyset \text{ and } \eta(w) = 0 \text{ otherwise}$$

$$\iota(w) = (-1)^n c_n c_{n-1} \dots c_1$$

$$\Delta(w) = \sum_{0 \leq i \leq n} c_1 \dots c_i \otimes c_{i+1} \dots c_n$$

The map η , ι and Δ are algebra morphisms. Indeed \mathcal{H} is a Hopf algebra with co-unit η , inverse map ι and coproduct Δ .

(1.13) Concatenation product: For two words $w = c_1 \dots c_n$ and $w' = c_{n+1} \dots c_{n+m}$, their concatenation is the word $ww' = c_1 \dots c_n c_{n+1} \dots c_{n+m}$. This induces another structure of algebra on \mathcal{H} , for which the product of two elements h, h' is simply denoted by hh' .

(1.14) Remarks on references and on the terminology:

In the classical literature, only the functions $L_k(z) = \sum_{n>0} z^n/n^k$ are called polylogarithms, see [L] [Oe]. We did not find a standard name for the $L_{(r_1, \dots, r_k)}(z)$. They are defined in the Bourbaki's talk [C], where the title suggests to call them again polylogarithms.

It seems that some polyzeta values, like $\zeta(1, 3)$, were already known by Euler, see [C]. The general definition of $\zeta(r_1, \dots, r_k)$ appears explicitly around 1990 in [H] and [Z]. These numbers are also called multiple zeta values in [Z], multiple harmonic sums in [H], multizeta numbers in [E], Euler-Zagier numbers in [BB] and polyzetas numbers in [C].

The fact that polyzeta values are naturally indexed by words has been observed by many authors, see [H], [H-P] and [C]. Lemma 1 and corollary 3 are well-known. Proofs are given for the convenience of the reader.

2. Polylogarithmic functions at $1/2$ and at $\rho^{\pm 1}$:

Define two linear maps $\sigma, \tau : \mathcal{H} \rightarrow \mathcal{H}$ as follows. First set $\sigma(a) = b$, $\sigma(b) = a$, $\tau(a) = a + b$ and $\tau(b) = -a$. For any word $w = c_1 \dots c_n \in \mathcal{W}$, set $\sigma(w) = \sigma(c_n) \dots \sigma(c_1)$ and $\tau(w) = \tau(c_n) \dots \tau(c_1)$. It is easy to see that σ and τ are algebra morphisms relative to the shuffle product $*$ (they are anti-morphism relative to the concatenation product).

Define now the two operators $\square, \nabla : \mathcal{H} \rightarrow \mathcal{H}$ as the following composite maps:

$$\square : \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{id \otimes \sigma} \mathcal{H} \otimes \mathcal{H} \xrightarrow{*} \mathcal{H}$$

$$\nabla : \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{id \otimes \tau} \mathcal{H} \otimes \mathcal{H} \xrightarrow{*} \mathcal{H}$$

Set $\rho = e^{i\pi/3}$.

THEOREM 7: For any $h \in \mathcal{H}$, we have:

$$\zeta(h) = L_{\square(h)}(1/2)$$

$$\zeta^\pm(h) = L_{\nabla(h)}(\rho^{\pm 1})$$

Proof: Note that $\square(a) = \square(b) = a+b$, $L_a(z) = \log 1/(1-z)$ and $L_b(z) = \log z$, therefore $L_a(1/2) + L_b(1/2) = 0 = \zeta(a) = \zeta(b)$. Similarly, $\nabla(a) = 2a + b$ and $\nabla(b) = 0$, and we have $L_{\nabla(a)}(\rho^{\pm 1}) = -2 \log(1 - \rho^{\pm 1}) + \log(\rho^{\pm 1}) = \pm i\pi = \zeta^\pm(a)$ and $L_{\nabla(b)}(\rho^{\pm 1}) = 0 = \zeta^\pm(b)$.

Since \square and ∇ are algebra morphisms, and since the algebra \mathcal{H} is generated by a , b and \mathcal{W}^{++} , it is enough to show the formulas when h is a non empty word w in \mathcal{W}^{++} . So let $w \in \mathcal{W}^{++}$ be a word of length $n \geq 2$. Set $w = c_1 \dots c_n$ where $c_i \in \{a, b\}$ are letters with $c_1 = a$ and $c_n = b$.

Let $\gamma : [0, 1] \rightarrow \overline{D}$, $t \mapsto t$ be the straight path from 0 to 1. Choose two smooth paths $\gamma_\pm : [0, 1] \rightarrow \overline{D}$ with the following properties: $\gamma_\pm(0) = 0$, $\gamma_\pm(1) = 1$, $\gamma_\pm(1/2) = \rho^{\pm 1} = 1/2 \pm i\sqrt{3}/2$ and $\text{Re } \gamma_\pm(t) \geq 1/2$ for all $t \in [1/2, 1]$. Set $\Delta_n = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid 0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1\}$. By Proposition 2, we have:

$$\zeta(w) = \int_{\Delta_n} \eta^* \omega_{c_1}(x_1) \eta^* \omega_{c_2}(x_2) \dots \eta^* \omega_{c_n}(x_n)$$

where η is the path γ , or γ^+ or γ^- .

For $0 \leq i \leq n$, set $\Delta'_i = \{(x_1, x_2, \dots, x_i) \in \mathbf{R}^n \mid 0 \leq x_1 \leq x_2 \leq \dots \leq x_i \leq 1/2\}$ and $\Delta''_i = \{(x_{i+1}, x_{i+2}, \dots, x_n) \in \mathbf{R}^n \mid 1/2 \leq x_{i+1} \leq x_{i+2} \leq \dots \leq x_n \leq 1\}$. From the decomposition: $\Delta = \cup_{0 \leq i \leq n} \Delta'_i \times \Delta''_i$, it follows that $\zeta(w) = \sum_{0 \leq i \leq n} L'_i L''_i = \sum_{0 \leq i \leq n} L_i^{\pm} L_i^{\mp}$, where the numbers L'_i , L''_i , L_i^{\pm} , L_i^{\mp} are the following integrals:

$$L'_i = \int_{\Delta'_i} \gamma^* \omega_{c_1}(x_1) \gamma^* \omega_{c_2}(x_2) \dots \gamma^* \omega_{c_i}(x_i)$$

$$L''_i = \int_{\Delta''_i} \gamma^* \omega_{c_{i+1}}(x_{i+1}) \gamma^* \omega_{c_{i+2}}(x_{i+2}) \dots \gamma^* \omega_{c_n}(x_n)$$

$$L_i^{\pm} = \int_{\Delta'_i} \gamma_\pm^* \omega_{c_1}(x_1) \gamma_\pm^* \omega_{c_2}(x_2) \dots \gamma_\pm^* \omega_{c_i}(x_i)$$

$$L_i^{\mp} = \int_{\Delta''_i} \gamma_\pm^* \omega_{c_{i+1}}(x_{i+1}) \gamma_\pm^* \omega_{c_{i+2}}(x_{i+2}) \dots \gamma_\pm^* \omega_{c_n}(x_n)$$

Using Kontsevitch formula, we get $L'_i = L_{w'_i}(1/2)$ and $L_i^{\pm} = L_{w'_i}(\rho^{\pm 1})$, where $w'_i = c_1 \dots c_i$. To evaluate L''_i , one needs to introduce some new notations. Define by $S, T : \mathbf{C} \rightarrow \mathbf{C}$ the rational maps: $S(z) = 1 - z$ and $T(z) = 1 - 1/z$. Define the new paths $\delta, \delta_\pm : [0, 1/2] \rightarrow \mathbf{C}$ by $\delta(t) = 1 - \gamma(1-t) = S \circ \gamma(1-t)$ and $\delta_\pm(t) = 1 - 1/\gamma_\pm(1-t) = T \circ \gamma(1-t)$. Clearly, δ is the straight path from 0 to 1/2. Since $T(\rho^{\pm 1}) = \rho^{\pm 1}$ and $T(1) = 0$, δ_\pm is

a path from 0 to $\rho^{\pm 1}$. Since $\operatorname{Re} \gamma_{\pm}(t) \geq 1/2$ for all $t \in [1/2, 1]$, it follows that δ_{\pm} lies in $\overline{D} \setminus \{1\}$.

With the convention of (1.7), we get:

$$\gamma^* \omega_c(t) = \delta^* \omega_{\sigma(c)}(1-t) \text{ and } \gamma_{\pm}^* \omega_c(t) = \delta_{\pm}^* \omega_{\tau(c)}(1-t)$$

for any $c \in \mathbf{Q}a \oplus \mathbf{Q}b$. Using the new variables $y_j = 1 - x_j$, we thus get:

$$L''_i = \int_{\overline{\Delta''_i}} \delta^* \omega_{\sigma(c_n)}(y_n) \delta^* \omega_{\sigma(c_{n-1})}(y_{n-1}) \dots \delta^* \omega_{\sigma(c_{i+1})}(y_{i+1})$$

$$L''_{\pm i} = \int_{\overline{\Delta''_{\pm i}}} \delta_{\pm}^* \omega_{\tau(c_n)}(y_n) \delta_{\pm}^* \omega_{\tau(c_{n-1})}(y_{n-1}) \dots \delta_{\pm}^* \omega_{\tau(c_{i+1})}(y_{i+1})$$

where $\overline{\Delta''_i} = \{(y_n, y_{n-1}, \dots, y_{i+1}) \in \mathbf{R}^n \mid 0 \leq y_n \leq y_{n-1} \leq \dots \leq y_{i+1} \leq 1/2\}$. It follows from Proposition 2 that $L''_i = L_{\sigma(w''_i)}(1/2)$ and $L''_{\pm i} = L_{\tau(w''_i)}(\rho^{\pm 1})$ where w''_i is the word $c_{i+1} \dots c_n$. Therefore we get

$$\zeta(w) = \sum_{0 \leq i \leq n} L_{w'_i}(1/2) L_{\sigma(w''_i)}(1/2), \text{ and}$$

$$\zeta(w) = \sum_{0 \leq i \leq n} L_{w'_i}(\rho^{\pm 1}) L_{\tau(w''_i)}(\rho^{\pm 1}).$$

Since $\Delta(w) = \sum_{0 \leq i \leq n} w'_i \otimes w''_i$, it is clear that $\square(w) = \sum_{0 \leq i \leq n} w'_i * \sigma(w''_i)$ and $\nabla(w) = \sum_{0 \leq i \leq n} w'_i * \tau(w''_i)$, and therefore the formula follows from Corollary 3. Q.E.D.

3. Explicit expressions for $\zeta(r)$.

Theorem 7 provides a combinatorial way to express any polyzeta value as a polylogarithmic functions at $1/2$ or at ρ or at $\bar{\rho}$. In this section, Theorem 10 and Corollary 12 provided closed formulas for zeta values $\zeta(r)$, where $r \geq 2$ is a given integer. The formulas are derived from Theorem 7. However, for general polyzeta values $\zeta(r_1, \dots, r_k)$ the combinatorics seem too intricate to find a simple combinatorial formula.

The concatenation product hh' , which is not commutative, should not be confused with the commutative shuffle product $h * h'$. The following conventions will be used. First, for $h \in \mathcal{H}$ and $n \geq 1$, the notation h^n will be the n^{th} power of h with respect to the concatenation product. Moreover, the concatenation product takes precedence of the shuffle product. For example, the expression $hh' * h^n$ should be understood as $(hh') * h^n$.

LEMMA 8: We have:

$$\square(ab^{r-1}) = 2a^{r-1}(a+b) + \sum_{1 \leq j \leq r-2} a^j (a+b)^{r-j}$$

$$\nabla(ab^{r-1}) = (-1)^{r+1} 3a^r + \sum_{1 \leq j \leq r-2} (-1)^{j+1} a^j (b-a)^{r-j}$$

Proof: We have:

$$\begin{aligned}\square(ab^{r-1}) &= \sigma(ab^{r-1}) + \sum_{0 \leq j \leq r-1} ab^j * \sigma(b^{r-1-j}) \\ &= a^{r-1}b + \sum_{u+v=r-1} ab^u * a^v.\end{aligned}$$

Similarly, we have:

$$\begin{aligned}\nabla(ab^{r-1}) &= \tau(ab^{r-1}) + \sum_{0 \leq j \leq r-1} ab^j * \tau(b^{r-1-j}) \\ &= (-1)^{r+1}a^{r-1}(a+b) + \sum_{u+v=r-1} (-1)^v ab^u * a^v.\end{aligned}$$

Use now the formula, $cw * a^v = \sum_{i+j=v} a^j c(w * a^i)$, which holds for any word w and any letter c . Thus we have $ab^u * a^v = \sum_{i+j=v} a^{j+1}(b^u * a^i)$ and we get:

$$\begin{aligned}\square(ab^{r-1}) &= a^{r-1}b + \sum_{j+u+v=r-1} a^{j+1}(b^u * a^v), \text{ and} \\ \nabla(ab^{r-1}) &= (-1)^{r+1}a^{r-1}(a+b) + \sum_{j+u+v=r-1} (-1)^{j+v} a^{j+1}(b^u * a^v).\end{aligned}$$

Using now the formulas:

$$(a+b)^N = \sum_{u+v=N} a^u * b^v \text{ and } (b-a)^N = \sum_{u+v=N} (-1)^u a^u * b^v$$

we get:

$$\begin{aligned}\square(ab^{r-1}) &= a^{r-1}b + \sum_{0 \leq j \leq r-1} a^{j+1}(a+b)^{r-1-j} \\ &= a^{r-1}b + \sum_{1 \leq j \leq r} a^j (a+b)^{r-j} \\ &= a^{r-1}b + a^r + \sum_{1 \leq j \leq r-1} a^j (a+b)^{r-j} \\ &= a^{r-1}(a+b) + \sum_{1 \leq j \leq r-1} a^j (a+b)^{r-j} \\ &= 2a^{r-1}(a+b) + \sum_{1 \leq j \leq r-2} a^j (a+b)^{r-j}.\end{aligned}$$

We also get:

$$\begin{aligned}\nabla(ab^{r-1}) &= (-1)^{r+1}a^{r-1}(a+b) + \sum_{0 \leq j \leq r-1} (-1)^j a^{j+1}(b-a)^{r-1-j} \\ &= (-1)^{r+1}a^{r-1}(a+b) + \sum_{1 \leq j \leq r} (-1)^{j+1} a^j (b-a)^{r-j} \\ &= (-1)^{r+1}a^{r-1}(a+b) + (-1)^{r+1}a^r + (-1)^r a^{r-1}(b-a) \\ &\quad + \sum_{1 \leq j \leq r-2} (-1)^{j+1} a^j (b-a)^{r-j} \\ &= (-1)^{r+1}3a^r + \sum_{1 \leq j \leq r-2} (-1)^{j+1} a^j (b-a)^{r-j}. \text{ Q.E.D.}\end{aligned}$$

Let \mathcal{W}_r be the set of words of length r . Any $w \in \mathcal{W}_r$ can be written as $w = a^j u$, where u does not start with a . Let k be the number of occurrence of a in w . Set $j(w) = j$, $k(w) = k$ and define the numbers $c(w)$ and $c^\pm(w)$ as follows.

- (i) If $w = a^r$, set $c(w) = r$ and $c^\pm(w) = (-1)^{r+1}(r+1)$.
- (ii) If $w = a^{r-1}b$, set $c(w) = r$ and $c^\pm(w) = (-1)^r(r-2)$.
- (iii) Otherwise, set $c(w) = j$ and $c^\pm(w) = (-1)^{k+1}j$.

LEMMA 9: We have:

$$\square(ab^{r-1}) = \sum_{w \in \mathcal{W}_r} c(w)w$$

$$\nabla(ab^{r-1}) = \sum_{w \in \mathcal{W}_r} c^\pm(w)w$$

Proof: The first identity of Lemma 8 can be written as:

$$\square(ab^{r-1}) = a^{r-1}b + \sum_{1 \leq i \leq r} a^i(a+b)^{r-i}$$

Since $(a+b)^{r-i} = \sum_{u \in \mathcal{W}_{r-i}} u$, we thus get

$$\square(ab^{r-1}) = a^{r-1}b + \sum_{1 \leq i \leq r} \sum_{u \in \mathcal{W}_{r-i}} a^i u$$

The word $w = a^{j(w)}v$ belongs to $a^i\mathcal{W}_{r-i}$ for all $i \leq j(w)$. Therefore

$\sum_{1 \leq i \leq r} a^i(a+b)^{r-i} = \sum_{w \in \mathcal{W}_r} j(w)w$. Since $c(a^{r-1}b) = j(a^{r-1}b) + 1$ and $c(w) = j(w)$ otherwise, the formula $\square(ab^{r-1}) = \sum_{w \in \mathcal{W}_r} c(w)w$ is now proved.

Set $\bar{a} = -a$ and $\bar{b} = b$. For a word $w = c_1 \dots c_n$, set $\bar{w} = \bar{c}_1 \dots \bar{c}_n$ and for a general element $h = \sum c_w w$ in \mathcal{H} set $\bar{h} = \sum c_w \bar{w}$. Since the involution $h \mapsto \bar{h}$ is a morphism relative to the concatenation product, it follows from Lemma 8 that:

$$\begin{aligned} -\overline{\nabla(ab^{r-1})} &= 3a^r + \sum_{1 \leq i \leq r-2} a^i(b+a)^{r-i} \\ &= a^r - a^{r-1}b + \sum_{1 \leq i \leq r} a^i(b+a)^{r-i} \end{aligned}$$

It follows from the previous proof that $\sum_{1 \leq i \leq r} a^i(a+b)^{r-i} = \sum_{w \in \mathcal{W}_r} j(w)w$. Therefore, one gets:

$$-\overline{\nabla(ab^{r-1})} = a^r - a^{r-1}b + \sum_{w \in \mathcal{W}_r} j(w)w.$$

Note that $\bar{w} = (-1)^{k(w)} w$ for all words w . Thus:

$$\nabla(ab^{r-1}) = (-1)^{r+1}a^r - (-1)^r a^{r-1}b + \sum_{w \in \mathcal{W}_r} (-1)^{(1+k(w))} j(w)w$$

Since $c^\pm(a^r) = (-1)^{r+1}(j(a^r) + 1)$, $c^\pm(a^{r-1}b) = (-1)^r(j(a^{r-1}b) - 1)$ and $c^\pm(w) = (-1)^{1+k(w)} j(w)$ otherwise, the formula $\nabla(ab^{r-1}) = \sum_{w \in \mathcal{W}_r} c^\pm(w)w$ is now proved. Q.E.D.

Let $r \geq 2$ be an integer. Let Λ_r be the set of all $\mathbf{m} = (m_1, \dots, m_k) \in \Lambda$ with $m_1 + \dots + m_k = r$. For $\mathbf{m} = (m_1, \dots, m_k) \in \Lambda_r$, set $k(\mathbf{m}) = k$ and define the integers $j(\mathbf{m})$, $b(\mathbf{m})$ and $b^\pm(\mathbf{m})$ as follows.

- (i) If $m_1 = m_2 = \dots = m_r = 1$, set $j(\mathbf{m}) = r$, $b(\mathbf{m}) = r$ and $b^\pm(\mathbf{m}) = (-1)^{r+1}(r+1)$. Otherwise, let $j(\mathbf{m}) = j$ be the index such that $m_1 = m_2 = \dots = m_{j-1} = 1$ and $m_j \geq 2$.
- (ii) If $m_1 = m_2 = \dots = m_{r-2} = 1$ and $m_{r+1} = 2$, set $b(\mathbf{m}) = r$ and $b^\pm(\mathbf{m}) = (-1)^r(r-2)$.
- (iii) Otherwise, set $b(\mathbf{m}) = j$ and $b^\pm(\mathbf{m}) = (-1)^{k+1}j$, where $j = j(\mathbf{m})$ and $k = k(\mathbf{m})$.

THEOREM 10: For $r \geq 2$, we have:

$$\zeta(r) = \sum_{\mathbf{m} \in \Lambda_r} b(\mathbf{m})L_{\mathbf{m}}(1/2)$$

$$\zeta(r) = \sum_{\mathbf{m} \in \Lambda_r} b^\pm(\mathbf{m}) L_{\mathbf{m}}(\rho^{\pm 1})$$

Proof: It is clear that $c(w)$ and $c^\pm(w)$ vanish if $w \notin \mathcal{W}^+$. Therefore it follows from Theorem 7 and Lemma 9 that

$$\zeta(r) = \sum_{w \in \mathcal{W}_r^+} c(w) L_w(1/2)$$

$$\zeta(r) = \sum_{w \in \mathcal{W}_r^+} c^\pm(w) L_w(\rho^{\pm 1})$$

where $\mathcal{W}_r^+ = \mathcal{W}^+ \cap \mathcal{W}_r$. Note that the map λ of section 1.4 provides a bijection $\lambda : \mathcal{W}_r^+ \rightarrow \Lambda_r$. It is easy to check that $j(w) = j(\lambda(w))$ and $k(w) = k(\lambda(w))$ for all $w \in \mathcal{W}_r^+$, and therefore

$$c(w) = b(\lambda(w)) \quad \text{and} \quad c^\pm(w) = b^\pm(\lambda(w))$$

for all $w \in \mathcal{W}_r^+$. Therefore Theorem 10 is proved. Q.E.D.

For $1 \leq i \leq r-1$, set

$$C_i = \{\mathbf{n} = (n_1 \dots n_k) \in \mathbf{Z}^r \mid 0 < n_1 < \dots < n_i \leq n_{i+1} \leq \dots \leq n_k\}.$$

Also, set $C_r = C_{r-1}$.

LEMMA 11: *We have:*

$$\zeta(r) = \sum_{1 \leq i \leq r} \sum_{\mathbf{n} \in C_i} \frac{2^{-n_r}}{n_1 n_2 \dots n_r}$$

Proof: Set $c = a + b$. For any word $w = d_1 \dots d_r$ into the letters a, b and c , let C_w be the set of all $\mathbf{n} = (n_1 \dots n_k) \in \mathbf{Z}^r$ satisfying the following property:

$$0 \mathcal{R}_1 n_1 \mathcal{R}_2 n_2 \dots \mathcal{R}_r n_r$$

where \mathcal{R}_i stands for the symbol $<$ if $d_i = a$, \mathcal{R}_i stands for the symbol $=$ if $d_i = b$ and \mathcal{R}_i stands for the symbol \leq if $d_i = c$. So if w is a word into the letters a, b and c , we get $L_w(z) = \sum_{\mathbf{n} \in C_w} \frac{z^{n_r}}{n_1 \dots n_r}$.

By Lemma 8, we have:

$$\square(ab^{r-1}) = a^{r-1}(a+b) + \sum_{1 \leq j \leq r-1} a^j (a+b)^{r-j}$$

Since $C_i = C_{a^i c^{r-i}}$ for all $i \leq r-1$, and $C_r = C_{a^{r-1} c}$, we get

$$L_{\square(ab^{r-1})}(z) = \sum_{1 \leq i \leq r} \sum_{\mathbf{n} \in C_i} \frac{z^{-n_r}}{n_1 n_2 \dots n_r}$$

Therefore, Lemma 11 follows from Theorem 7. Q.E.D.

Set

$$C = \{\mathbf{n} = (n_1 \dots n_r) \in \mathbf{Z}^r \mid 0 < n_1 \leq n_2 \leq \dots \leq n_r\}.$$

For $\mathbf{n} = (n_1 \dots n_r) \in C$, define the number $a(\mathbf{n})$ as follows. If we have $0 < n_1 < \dots < n_{r-1}$ set $a(\mathbf{n}) = k$. Otherwise, there exist an index $i \leq r-2$ such that $0 < n_1 < n_2 \leq \dots \leq n_i = n_{i+1}$. In such a case, set $a(\mathbf{n}) = i$. Note that $a(\mathbf{n})$ does not depend on the last component n_r of \mathbf{n} , and the function $\mathbf{n} \mapsto a(\mathbf{n})$ takes value in the set $\{1, 2, \dots, r-2, r\}$

COROLLARY 12: *We have:*

$$\zeta(r) = \sum_{\mathbf{n} \in C} a(\mathbf{n}) \frac{2^{-n_r}}{n_1 n_2 \dots n_r}$$

Proof: It is easy to check that $a(\mathbf{n})$ is precisely the number of indices i , $1 \leq i \leq r$ such that \mathbf{n} belongs to C_i . Therefore the formula of Corollary 7 follows from Lemma 11. Q.E.D.

Examples: For $r = 2$, then $a((m, n)) = 2$ for all $(m, n) \in C$. Therefore, we get

$$\zeta(2) = 2 \sum_{0 < m \leq n} \frac{2^{-n}}{nm} = 2L_2(1/2) + \log^2 2$$

Accordingly to [C], this formula is due to Euler.

For $r = 5$, we have $a((k, l, m, n, p)) = 1$ if $k = l$, $a((k, l, m, n, p)) = 2$ if $k < l = m$, $a((k, l, m, n, p)) = 3$ if $k < l < m = n$ and $a((k, l, m, n, p)) = 5$ if $k < l < m < n$. Therefore, we get the following expansion for $\zeta(5)$

$$\sum_{0 < l \leq m \leq n \leq p} \frac{2^{-p}}{l^2 m n p} + 2 \sum_{0 < l < m \leq n \leq p} \frac{2^{-p}}{l m^2 n p} + 3 \sum_{0 < l < m < n \leq p} \frac{2^{-p}}{l m n^2 p} + 5 \sum_{0 < k < l < m < n \leq p} \frac{2^{-p}}{k l m n p}$$

4. Geometric interpretation of Theorem 7.

Theorem 7 provides a combinatorial way to express any polyzeta value as the value of a polylogarithmic functions at $1/2$ or at $\rho^{\pm 1}$. The combinatorics seem very intricate: e.g. the explicit formulas for zeta values $\zeta(r)$ of Section 3 are difficult to extend for general polyzeta values $\zeta(r_1, \dots, r_k)$.

In this section, Theorem 7 is reformulated in terms of simple geometric notions.

(4.1) First, the free pro-algebraic group on two generators Γ and its Lie algebra \mathfrak{g} are defined.

Let F be the free Lie \mathbf{Q} -algebra with two generators α and β , let $C^n F$ be its central descending series and set $\mathfrak{g} = \varprojlim F/C^n F$. Since $F/C^n F$ is a nilpotent Lie algebra, the Campbell-Hausdorff series defines a structure of algebraic group on $F/C^n F$, denoted by Γ_n . Then $\Gamma = \varprojlim \Gamma_n$ is a proalgebraic group (an alternative definition of Γ is given in the introduction). As pro-algebraic varieties, \mathfrak{g} and Γ are identical, and the corresponding isomorphism is denoted by $\exp : \mathfrak{g} \rightarrow \Gamma$.

Let $F = \bigoplus_{n \geq 1} F_n$ be the grading of F such that $F_1 = \mathbf{Q}\alpha \oplus \mathbf{Q}\beta$. Then we have $\mathfrak{g} = \prod_{n \geq 1} F_n$, so any $x \in \mathfrak{g}$ can be written as the series $x = \sum_{i > 0} x_i$ where $x_i \in F_i$. The multiplicative group \mathbf{Q}^* acts linearly on \mathfrak{g} as follows: $t.x = \sum_{i > 0} t^i x_i$, for any $t \in \mathbf{Q}^*$.

LEMMA 13: Let $\Phi : \mathfrak{g} \rightarrow \mathfrak{g}$ be a morphism of pro-algebraic varieties. Assume that Φ is \mathbf{Q}^* -invariant and that $d\Phi_0$ is invertible, then Φ is an isomorphism.

Proof: One can assume that $d\Phi_0$ is the identity. Then choose a basis of F consisting of homogenous elements $(e_n)_{n \geq 1}$ with $d_n \leq d_m$ if $n < m$, where d_n is the degree of e_n . Accordingly, we have $\Phi(\sum_{n \geq 1} x_n e_n) = \sum_{n \geq 1} \Phi_n(x) e_n$, where each Φ_n is a polynomial in $x = (x_1, x_2, \dots)$. By hypothesis, the linear part of $\Phi_n(x)$ is x_n and for any monomial $x_{i_1} \dots x_{i_k}$ occuring in $\Phi_n(x)$ we have $d_{i_1} + \dots + d_{i_k} = d_n$. It follows that $\Phi_n(x) - x_n$ depends only on x_1, \dots, x_{n-1} , so we can write: $\Phi_n(x) = x_n + H_n(x_1, \dots, x_{n-1})$. Since Φ is triangular, it is an isomorphism.

(4.2) There is an isomorphism of Hopf algebras $\mathbf{Q}[\Gamma] \simeq \mathcal{H}$, see [P]. A natural group isomorphism $\psi : \Gamma \rightarrow \text{Spec } \mathcal{H}$ is now described.

For $t \in \mathbf{Q}$, define two points $\phi_a(t)$ and $\phi_b(t)$ in $\text{Spec } \mathcal{H}$ as follows. Since words $w \in \mathcal{W}$ are functions on $\text{Spec } \mathcal{H}$, one needs to evaluate w at the points $\phi_a(t)$ and $\phi_b(t)$. The rule is as follows:

$$\begin{aligned} w(\phi_a(t)) &= t^n/n! \text{ if } w = a^n \text{ and } w(\phi_a(t)) = 0 \text{ if } b \text{ occurs in } w. \\ w(\phi_b(t)) &= t^n/n! \text{ if } w = b^n \text{ and } w(\phi_b(t)) = 0 \text{ if } a \text{ occurs in } w. \end{aligned}$$

Then it is clear that $\phi_a(t)$ and $\phi_b(t)$ are two one-parameter groups in $\text{Spec } \mathcal{H}$. Since Γ is freely generated (as a proalgebraic group) by the two one-parameter groups $\exp \mathbf{Q}\alpha$ and $\exp \mathbf{Q}\beta$, the isomorphism ψ is prescribed by the requirements $\psi(\exp t\alpha) = \phi_a(t)$ and $\psi(\exp t\beta) = \phi_b(t)$ for all $t \in \mathbf{Q}$.

(4.3) From now on, we identify \mathcal{H} and $\mathbf{Q}[\Gamma]$. Since $\mathcal{H} = \mathbf{Q}[\Gamma]$, any function $\phi \in \mathbf{Q}[\Gamma]$ defines a polylogarithmic function $L_\phi(z)$ and the polyzeta value $\zeta(\phi)$ and the numbers $\zeta^\pm(\phi)$.

(4.4) The maps $\sigma, \tau : \mathcal{H} \rightarrow \mathcal{H}$ are anti-isomorphisms of Hopf algebras, and therefore they induce two anti-isomorphisms of Γ and of its Lie algebra \mathfrak{g} . These are again denoted by σ and τ . They are uniquely characterized by the requirements:

$$\begin{aligned} \sigma \exp t\alpha &= \exp t\beta \text{ and } \sigma \exp t\beta = \exp t\alpha \\ \tau \exp t\alpha &= \exp t(\alpha + \beta) \text{ and } \tau \exp t\beta = \exp -t\alpha, \end{aligned}$$

for all $t \in \mathbf{Q}$. We have $\sigma^2(g) = g$ and $\tau^3(g) = g^{-1}$ for any $g \in G$.

(4.5) Since $\mathcal{H} = \mathbf{Q}[\Gamma]$, the maps \square, ∇ occuring in Theorem 7 are now identified with some algebra morphisms $\square, \nabla : \mathbf{Q}[\Gamma] \rightarrow \mathbf{Q}[\Gamma]$.

LEMMA 14: Let $\phi \in \mathbf{Q}[\Gamma]$. Then for any $g \in \Gamma$, we have

$$\square \phi(g) = \phi(g\sigma(g)) \quad \text{and} \quad \nabla \phi(g) = \phi(g\tau(g))$$

Proof: Using their definitions, \square and ∇ are the composition of the following maps:

$$\Gamma \xrightarrow{\text{diag}} \Gamma \times \Gamma \xrightarrow{id \times \sigma} \Gamma \times \Gamma \xrightarrow{\mu} \Gamma$$

$$\Gamma \xrightarrow{\text{diag}} \Gamma \times \Gamma \xrightarrow{id \times \tau} \Gamma \times \Gamma \xrightarrow{\mu} \Gamma$$

where $\text{diag}(g) = (g, g)$ and $\mu(g_1, g_2) = g_1 \cdot g_2$. Therefore we have

$$\square \phi(g) = \phi(g \cdot \sigma(g)) \quad \text{and} \quad \nabla \phi(g) = \phi(g \cdot \tau(g)).$$

(4.6) In this subsection, the symmetric space associated with σ is defined.

Set $\mathfrak{k} = \{x \in \mathfrak{g} \mid \sigma(x) = -x\}$, $K = \{g \in \Gamma \mid \sigma(g) = g^{-1}\}$, $\mathfrak{p} = \{x \in \mathfrak{g} \mid \sigma(x) = x\}$
 $P = \{p \in \Gamma \mid \sigma(p) = p\}$.

Since σ is an anti-involution, K is a subgroup in Γ and \mathfrak{k} is its a Lie algebra. Obviously we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since Γ is a pro-unipotent group, we have $K = \exp \mathfrak{k}$, $P = \exp \mathfrak{p}$ and $\Gamma = P.K$. So any element $g \in \Gamma$ can be written as $g = p.k$, where $k \in K$ and $p \in P$. Moreover $P \simeq \Gamma/K$ is a symmetric space.

LEMMA 15: Let $\phi \in \mathbf{Q}[\Gamma]$. Then for any $g = p.k \in \Gamma$, we have

$$\square \phi(p.k) = \phi(p^2)$$

In particular, $\zeta(\phi) = 0$ if $\phi|_P \equiv 0$.

Proof: This follows from Lemma 14 and Theorem 7.

(4.7) Note that τ is not an involution, but an "ordrer three" anti-isomorphism, i.e. $\tau^3(g) = g^{-1}$. In this sub-section, we introduce a space Q which is analogous to a symmetric space.

Set $\mathfrak{l} = \{x \in \mathfrak{g} \mid \tau(x) = -x\}$, $L = \{g \in \Gamma \mid \tau(g) = g^{-1}\}$, $\mathfrak{q} = \{x \in \mathfrak{g} \mid \tau^2(x) - \tau(x) + x = 0\}$. Also define Q as the image of the map $g \in \Gamma \mapsto g\tau(g)$. Note that L is a subgroup with Lie algebra \mathfrak{l} .

LEMMA 16: The subset Q is a closed subvariety of Γ and the natural map: $Q \times L \rightarrow \Gamma$, $(q, l) \mapsto ql$ is a isomorphism of pro-algebraic varieties.

Proof It is easy to prove that $\Gamma = \exp \mathfrak{q}.L$. For $g = \exp q.l$, with $q \in \mathfrak{q}$ and $l \in L$, we have $g\tau(g) = \exp q \exp \tau(q)$, therefore Q is the set all $\exp q \exp \tau(q)$ for $q \in \mathfrak{q}$.

Let $\Phi : \mathfrak{q} \oplus \mathfrak{l} \rightarrow \Gamma$ be define by $\Phi(q, l) = \exp q \exp \tau(q) \exp l$. Note that $d\Phi_0$ is the linear map from \mathfrak{g} to \mathfrak{g} which is the identity on \mathfrak{l} and whose restriction to \mathfrak{q} is $1 + \tau$. Therefore, $d\Phi_0$ is invertible. By Lemma 13 that Φ is an isomorphism and Lemma 16 follows easily. Q.E.D.

The definition of Q is slighly more complicated than the definition of P because $Q \neq \exp \mathfrak{q}$. However, the map $\mathfrak{q} \rightarrow Q$, $q \mapsto \exp q \exp \tau(q)$ is an isomorphism from \mathfrak{q} to Q . Any element $g \in \Gamma$ can be written as $g = q.l$, where $l \in L$ and $q \in Q$

LEMMA 17: Let $\psi \in \mathbf{Q}[\Gamma]$. Then for any $g = q.l \in \Gamma$, we have

$$\nabla \psi(q.l) = \psi(q\tau(q))$$

In particular, $\zeta(\psi) = 0$ if $\psi|_Q \equiv 0$.

Proof: This follows from Lemma 14 and Theorem 7.

(4.8) Let $\phi \in \mathbf{Q}[P]$ be a rational function on P . The notations $\zeta(\phi)$ and $L_\phi(z)$ are now defined. Set $\zeta(\phi) = \zeta(\hat{\phi})$ where $\hat{\phi}$ is any function on Γ extending ϕ . By Lemma 15, $\zeta(\phi)$ is well defined. Since $P \simeq \Gamma/K$, ϕ can be uniquely extended to a right K -invariant function Φ on Γ . Then set $L_\phi(z) = L_\Phi(z)$.

Similarly, for $\psi \in \mathbf{Q}[Q]$, the notations $\zeta^\pm(\psi)$ and $L_\psi(z)$ are defined as follows. Set $\zeta^\pm(\psi) = \zeta^\pm(\hat{\psi})$ where $\hat{\psi}$ is any function on Γ extending ψ . By Lemma 17, $\zeta^\pm(\psi)$ is well defined. By Lemma 16, we have $Q \simeq \Gamma/L$, therefore ψ can be uniquely extended to a right L -invariant function Ψ on Γ . Then set $L_\psi(z) = L_\Psi(z)$.

Define the algebra morphisms $\square : \mathbf{Q}[P] \rightarrow \mathbf{Q}[P]$ and $\nabla : \mathbf{Q}[Q] \rightarrow \mathbf{Q}[Q]$ by

$$\square \phi(p) = \phi(p^2), \quad \text{for } \phi \in \mathbf{Q}[P]$$

$$\nabla \psi(q) = \psi(q\tau(q)), \quad \text{for } \psi \in \mathbf{Q}[Q]$$

These operators are simply the restrictions to P and to Q of the already defined operators $\square, \nabla : \mathbf{Q}[\Gamma] \rightarrow \mathbf{Q}[\Gamma]$. So using the same notations should not bring confusions.

THEOREM 18: For any $\phi \in \mathbf{Q}[P]$ and $\psi \in \mathbf{Q}[Q]$, we have

$$\zeta(\psi) = L_{\square(\psi)}(1/2) \quad \text{and} \quad \zeta^\pm(\psi) = L_{\nabla(\psi)}(\rho^\pm)$$

Proof: It follows immediately from Theorem 7, and Lemmas 15 and 17.

5. Other expressions for zeta values.

In this section, we follow a suggestion of W. Zudilin

(5.1) Theorem 7 shows that any polyzeta value is the value of a polylogarithmic function at $1/2$ or at ρ^\pm . However, there is a much more simple way to express the zeta values $\zeta(r)$ as a value of polylogarithmic functions at $1/2$ or at ρ^\pm , see Corollary 20. It is surprizing that the two approaches give different expressions, except for $\zeta(2)$. Moreover, this simpler approach does not generalize to polyzeta values.

(5.2) Let $\sigma' : \mathcal{H} \rightarrow \mathcal{H}$ be the linear map defined as follows. Set $\sigma'(a) = -a$ and $\sigma'(b) = a + b$. For a word $w = c_1 \dots c_n$, where $c_i \in \{a, b\}$ are letters, set $\sigma'(w) = \sigma'(c_1) \dots \sigma'(c_n)$. It is easy to prove that σ' is an algebra morphism relative to the schuffle product and that $\sigma'(\mathcal{H}^+) = \mathcal{H}^+$.

LEMMA 19 For any $h \in \mathcal{H}^+$, we have:

$$L_h(-1) = L_{\sigma'(h)}(1/2) \text{ and } L_h(\bar{\rho}) = L_{\sigma'(h)}(\rho).$$

Proof: One can assume that h is a word $w = c_1 \dots c_n \in \mathcal{W}^+$. Set $F = \{z \in \mathbf{C} \mid |z| \leq 1 \text{ and } \text{Im}z \leq 1/2\}$ and choose a two paths $\gamma, \gamma_- : [0, 1] \rightarrow F$ with $\gamma(0) = \gamma_-(0) = 0$, $\gamma(1) = -1$ and $\gamma_-(1) = \bar{\rho}$. By Proposition 2, we have:

$$L_w(-1) = \int_{\Delta_n} \gamma^* \omega_{c_1}(x_1) \gamma^* \omega_{c_2}(x_2) \dots \gamma^* \omega_{c_n}(x_n)$$

$$L_w(\bar{\rho}) = \int_{\Delta_n} \gamma_-^* \omega_{c_1}(x_1) \gamma_-^* \omega_{c_2}(x_2) \dots \gamma_-^* \omega_{c_n}(x_n)$$

For $z \in F$, set $S'(z) = z/(z-1)$ and set $\delta = S' \circ \gamma$ and $\delta_- = S' \circ \gamma_-$. We have $S'^* \omega_{\sigma'(c)} = \omega_c$ for $c = a$ or b , and therefore we get $\gamma^* \omega_c = \delta^* \omega_{\sigma'(c)}$ and $\gamma_-^* \omega_c = \delta_-^* \omega_{\sigma'(c)}$. It follows that

$$L_w(-1) = \int_{\Delta_n} \delta^* \omega_{\sigma'(c_1)}(x_1) \delta^* \omega_{\sigma'(c_2)}(x_2) \dots \delta^* \omega_{\sigma'(c_n)}(x_n)$$

$$L_w(\bar{\rho}) = \int_{\Delta_n} \delta_-^* \omega_{\sigma'(c_1)}(x_1) \delta_-^* \omega_{\sigma'(c_2)}(x_2) \dots \delta_-^* \omega_{\sigma'(c_n)}(x_n)$$

We have $S'(F) = F$, $S'(0) = 0$, $S'(-1) = 1/2$ and $S'(\bar{\rho}) = \rho$. Therefore δ is a path from 0 to 1/2 and δ_- is a path from o to ρ . Thus, these integrals can be identified by Proposition 2, and we get $L_h(-1) = L_{\sigma'(h)}(1/2)$, and $L_h(\bar{\rho}) = L_{\sigma'(h)}(\rho)$. Q.E.D.

LEMMA 20: Let $r \geq 1$. We have

$$\zeta(r+1) = \frac{-1}{1-2^{-r}} L_{r+1}(-1)$$

$$\zeta(r+1) = \frac{1}{(1-2^{-r})(1-3^{-r})} [L_{r+1}(\rho) + L_{r+1}(\bar{\rho})]$$

Proof: For each positive integer a , set $\delta_a(n) = 1$ if a divides n and $\delta_a(n) = 0$ otherwise.

From the formula $(-1)^n = -\delta_1(n) + 2\delta_2(n)$, we get

$$\begin{aligned} L_{r+1}(-1) &= \sum_{n>0} (-1)^n / n^{r+1} \\ &= - \sum_{n>0} 1/n^{r+1} + 2 \sum_{n>0} 1/(2n)^{r+1} \\ &= (-1 + 2^{-r}) \zeta(r) \end{aligned}$$

from which the first formula follows.

From the formula $\rho^n + \bar{\rho}^n = \delta_1(n) - 2\delta_2(n) - 3\delta_3(n) + 6\delta_6(n)$ we get

$$\begin{aligned} L_{r+1}(\rho) + L_{r+1}(\bar{\rho}) &= \sum_{n>0} [\rho^n + \bar{\rho}^n] / n^{r+1} \\ &= \sum_{n>0} 1/n^{r+1} - 2 \sum_{n>0} 1/(2n)^{r+1} - 3 \sum_{n>0} 1/(3n)^{r+1} + 6 \sum_{n>0} 1/(6n)^{r+1} \\ &= (1 - 2^{-r} - 3^{-r} + 6^{-r}) \zeta(r+1) \\ &= (1 - 2^{-r})(1 - 3^{-r}) \zeta(r+1), \end{aligned}$$

from which the second formula follows.

COROLLARY 21: Let $r \geq 1$. We have

$$\zeta(r+1) = \frac{2^r}{(2^r-1)} L_{a(a+b)^r}(1/2)$$

$$\zeta(r+1) = \frac{6^r}{(2^r-1)(3^r-1)} [L_{ab^r}(\rho) - L_{a(a+b)^r}(\rho)]$$

Proof: The corollary follows from Lemmas 18 and 19.

Examples:

$$\zeta(2) = 2 \sum_{0 < m \leq n} \frac{2^{-n}}{nm}$$

$$\zeta(3) = 8/7 \sum_{0 < l \leq m \leq n} \frac{2^{-n}}{lnm}$$

The expression for $\zeta(2)$ is the same as in section 3. However for all other zeta values $\zeta(r)$ with $r \geq 3$, the expressions are different: e.g., the formula of corollary 20 uses non integral coefficients. Moreover, this simpler approach only concerns zeta values but not the polyzeta values.

6. Conclusion: Polyzyta values are mixed periods $[\mathbf{G}]$, $[\mathbf{T}]$, $[\mathbf{Z}]$. In the philosophy of motives, there is a proalgebraic group G and periods should be regular functions on G : more precisely, the algebra of periods should be a $\overline{\mathbf{Q}}$ -form of $\mathbf{C}[G]$, modulo conjectures. Here polyzyta values are attached to some symmetric space. Does there is a motivic interpretation of this construction? Note however that the map $\mathbf{Q}[P] \rightarrow \mathcal{Z}$ is not injective.

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