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Higher-dimensional categories with finite derivation type

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Abstract – *The finite derivation type property is a homotopical condition on monoids. Craig Squier has proved that a monoid must satisfy it in order to admit a presentation by a finite and convergent rewriting system. We generalise the property to n -categories presented by polygraphs and we recover Squier’s theorem when n is 1. However, we prove that this result does not hold anymore for categories of dimension 2 and above. We study several examples of 2-categories presented by finite convergent polygraphs, with or without the property of finite derivation type, in order to illustrate sample cases.*

Keywords – *n -category; polygraph; finite derivation type; rewriting; low-dimensional topology.*

INTRODUCTION

Monoids with finite derivation type

Craig Squier has proved that, if a monoid admits a presentation by a finite and convergent (*i.e.* terminating and confluent) word rewriting system [5], then it has *finite derivation type* [19]. This is a homotopical property of the monoid, independent of the chosen finite presentation. Squier has used it to prove that finite convergent word rewriting systems did not provide a universal way to decide equality in monoids presented by generators and relations.

The property of finite derivation type is defined on the Squier’s combinatorial 2-complex associated to the word rewriting system: its vertices are the words, its arrows are generated by the rewriting steps and their inverses, its 2-cells are the equivalence of rewriting paths (by commutation of rules that are applied on disjoint parts of a word). Then the presentation (and the monoid) has finite derivation type when one can fill all the holes of the complex by filling finitely many.

The property of finite derivation type can be interpreted computationally: it means that, in a word rewriting system with this property, there are only finitely many choices that can be made during computation. These choices correspond to the critical branchings of the rewriting system, where two rules can be applied at the same place of a given word. Thus, the finite derivation type property ensures that making a finite number of choices is sufficient to build a rewriting strategy that makes the computation deterministic.

Generalisation to n -categories

There exist many types of rewriting systems, acting, for example, on terms, on elements of some sort of algebra or on topological objects. For instance, permutations admit a graphical presentation by one generator  and two (directed) relations:

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \rightarrow \begin{array}{|l} | \\ | \end{array} \quad \text{and} \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \rightarrow \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} .$$

It turns out that word rewriting systems and this presentation of permutations are instances of the same objects: polygraphs (or computads), which are presentations by “generators” and “relations” of some higher-dimensional categories [20, 6], see also [21, 22].

Indeed, monoids and word rewriting systems are special instances of 1-categories and 2-polygraphs, respectively. And many usual types of rewriting systems have interpretations as 3-polygraphs, seen as presentations of 2-categories, including the presentation of permutations, term rewriting systems [6, 13, 10], Petri nets [12] or formal proofs of propositional and linear logics [11].

The purpose of this document is to formalise the property of finite derivation type for n -categories and to answer the following question:

For an n -category, does having a finite convergent presentation by an $(n + 1)$ -polygraph implies that it has finite derivation type?

As we have seen, Squier has proved that this was true when n is 1. Until now, several examples of finite and convergent 3-polygraphs were studied and it was observed that all of them had finite derivation type. However, it turns out that the answer is negative.

Overview of the document

In this document, we assume basic knowledge of theories of categories, of n -categories and of rewriting. Good references to these fields include the books written by Saunders Mac Lane [15], by Eugenia Cheng and Aaron Lauda [7] and by Franz Baader and Tobias Nipkow [1].

In Section 1 we briefly recall notations and representations of n -categories. Then, in 1.4, we define polygraphs and presentations of n -categories by polygraphs, as in [6] and [16].

Section 2 contains several notions and tools we need, not directly to define the property of finite derivation type or to prove the main results, but to study the examples. In particular, we recall the definition of context of an n -category in 2.1 and introduce the notions of module over an n -category and of derivation of an n -category into a module, respectively in 2.3 and in 2.4.

In Section 3, we introduce the notion of track n -category, which is an n -category whose n -cells are invertible, or an $(n - 1)$ -category enriched in groupoid. This generalises the notion of track category, used as an algebraic model of the homotopy 2-type [2]. The fundamental groupoid of the Squier’s combinatorial 2-complex of a word rewriting system is a monoid enriched in groupoid [8]. For n -polygraphs, we use track n -categories to define homotopy relations, yielding the notion of homotopy basis of an n -category: a set of $(n + 1)$ -cells that identifies all the parallel n -cells. Then an n -category has finite derivation type when it admits a finite homotopy basis. In practice, we define this property on polygraphs and prove Proposition 3.3.4: if two n -polygraphs present the same $(n - 1)$ -category, both have finite derivation type or neither has.

In Section 4, we study the links between the property of finite derivation type and the rewriting properties of polygraphs. In particular, we prove Proposition 4.3.5: if a finite convergent polygraph has a finite set of critical branchings, then it has finite derivation type. We get Squier’s result [19] as a consequence since a finite convergent word rewriting system has a finite number of critical branchings. Finally, Theorem 4.3.9 asserts that this result does not generalise to higher dimensions, as proved by a counter-example that is studied in the final part:

Theorem. *For every natural number n with $n \geq 2$, there exists a n -category which does not have finite derivation type and admits a presentation by a finite convergent $(n + 1)$ -polygraph.*

In Section 5, we examine the case of 2-categories presented by 3-polygraphs. We give a classification of their critical branchings and we use it to get additional sufficient conditions for a finite convergent 3-polygraph to have finite derivation type. Finally, we study three examples of finite and convergent 3-polygraph, in order to illustrate sample cases:

- (5.2) The 3-polygraph of monoids has a finite number of critical branchings and, thus, has finite derivation type. We use this result to rediscover Mac Lane’s coherence theorem [15]: in a monoidal category, all the diagrams built using the monoidal structure commute.
- (5.4) The 3-polygraph of permutations has an infinite number of critical branchings. However, it was observed by Yves Lafont [13] that, for confluence study, only a finite number of them were useful. Following the same intuition, we prove that, despite the infinite number of critical branchings, this 3-polygraph has finite derivation type.
- (5.5) The final 3-polygraph we study is the proof of Theorem 4.3.9: it is finite and convergent but does not have finite derivation type. This 3-polygraph is an artificial counter-example, yet it has a topological flavour: it presents a 2-category whose morphisms could be interpreted as (open-closed, planar) necklaces with pearls, considered up to homotopy.

1. HIGHER-DIMENSIONAL CATEGORIES PRESENTED BY POLYGRAPHS

1.1. Generalities on n -categories and n -functors

Throughout this document, we only consider small, strict n -categories and strict n -functors between them. We denote by \mathbf{Cat}_n the (large) category they form.

1.1.1. Vocabulary and notations. If \mathcal{C} is an n -category, we denote by \mathcal{C}_k the set of k -cells of \mathcal{C} . If $F : \mathcal{C} \rightarrow \mathcal{D}$ is an n -functor, we denote by F_k the corresponding map from \mathcal{C}_k to \mathcal{D}_k .

In an n -category, s_k and t_k denote the k -source and k -target maps. If f is a k -cell, $s_{k-1}(f)$ and $t_{k-1}(f)$ are respectively called its *source* and *target* and respectively denoted by $s(f)$ and $t(f)$. The source and target maps satisfy the *globular relations*:

$$s_i \circ s_{i+1} = s_i \circ t_{i+1} \quad \text{and} \quad t_i \circ s_{i+1} = t_i \circ t_{i+1}.$$

Two cells f and g are *parallel* when they have same source and same target. A pair (f, g) of parallel k -cells is called a *k-sphere*. The *boundary* of a k -cell is the $(k - 1)$ -sphere $\partial f = (s(f), t(f))$. The source and target maps are extended to a k -sphere $\gamma = (f, g)$ by $s(\gamma) = f$ and $t(\gamma) = g$.

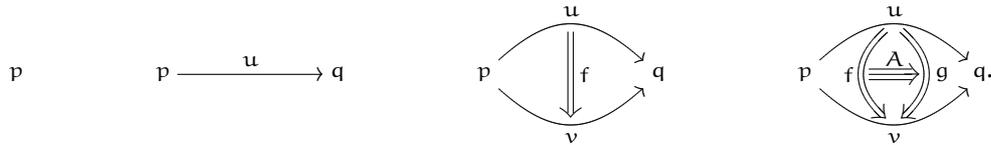
A pair (f, g) of k -cells of \mathcal{C} is *i-composable* when $t_i(f) = s_i(g)$ holds; when $i = k - 1$, one simply says *composable*. The set of i -composable pairs of k -cells is denoted by $\mathcal{C}_k \star_i \mathcal{C}_k$. The i -composite of (f, g) is denoted by $f \star_i g$, *i.e.* in the diagrammatic direction. The compositions satisfy the *exchange relation* given, for every $j < k$ and every possible cells f, f', g and g' by:

$$(f \star_j f') \star_k (g \star_j g') = (f \star_k g) \star_j (f' \star_k g').$$

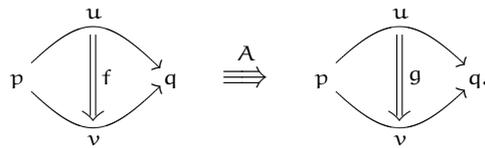
1. Higher-dimensional categories presented by polygraphs

If f is a k -cell, we denote by Id_f or 1_f its identity $(k + 1)$ -cell and, by abuse, all the higher-dimensional identity cells it generates. When 1_f is composed with cells of dimension $k + 1$ or higher, we abusively denote it by f to make expressions easier to read. A cell is *degenerate* when it is an identity cell. For $k \leq n$, a k -category \mathcal{C} can be seen as an n -category, with only degenerate cells above dimension k .

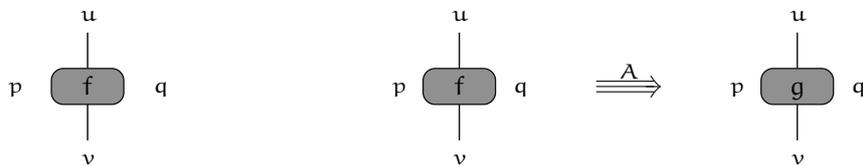
1.1.2. Graphical representations. Low-dimensional cells are written $u : p \rightarrow q$, $f : u \Rightarrow v$, $A : f \Rrightarrow g$ and pictured as usual (and so are n -categories, omitting the degenerate cells):



For readability, we also depict 3-cells as "rewriting rules" on 2-cells:



For 2-cells, circuit-like diagrams are an alternative representations, where 0-cells are parts of the plane, 1-cells are lines and 2-cells are points, inflated for emphasis:



1.2. Standard cells and spheres

1.2.1. The suspension functor. For every natural number n , the *suspension functor*

$$S_n : \mathbf{Cat}_n \rightarrow \mathbf{Cat}_{n+1}$$

informally lifts all the cells by one dimension, adding a formal 0-source and a formal 0-target for all of them; thus, in the $(n + 1)$ -category one gets, one has exactly the same compositions as in the original one. More formally, given an n -category \mathcal{C} , the $(n + 1)$ -category $S_n\mathcal{C}$ has the following cells:

$$(S_n\mathcal{C})_0 = \{-, +\} \quad \text{and} \quad (S_n\mathcal{C})_{k+1} = \mathcal{C}_k \amalg \{-, +\}.$$

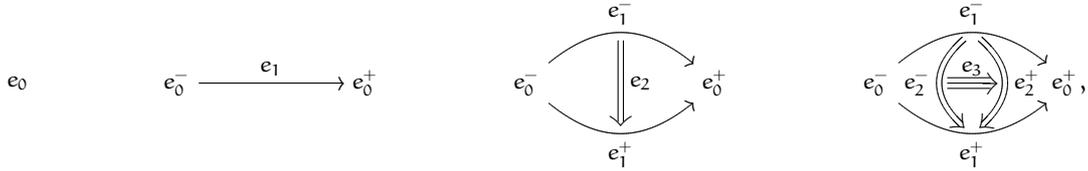
Every cell has 0-source $-$ and 0-target $+$. The $(k + 1)$ -source and $(k + 1)$ -target of a non-degenerate cell are its k -source and k -target in \mathcal{C} . The $(k + 1)$ -composable pairs are the k -composable ones of \mathcal{C} , plus pairs where at least one of the cells is an identity of $-$ or $+$.

1.2.2. The standard n -cell and n -sphere. By induction on n , we define the n -categories \mathcal{E}_n and \mathcal{S}_n , respectively called the *standard n -cell* and the *standard n -sphere*. Informally, we consider them as the n -categorical equivalents of the standard topological n -ball and n -sphere, used to build the n -categorical equivalents of (relative) CW-complexes.

The standard 0-cell \mathcal{E}_0 is defined as any chosen single-element set and the standard 0-sphere as any chosen set with two elements. Then, if $n \geq 1$, the n -categories \mathcal{E}_n and \mathcal{S}_n are defined as the suspensions of \mathcal{E}_{n-1} and \mathcal{S}_{n-1} :

$$\mathcal{E}_n = \mathcal{S}_{n-1}(\mathcal{E}_{n-1}) \quad \text{and} \quad \mathcal{S}_n = \mathcal{S}_{n-1}(\mathcal{S}_{n-1}).$$

For coherence, we define \mathcal{S}_{-1} as the empty set. Thus, the standard n -cell \mathcal{E}_n and n -sphere \mathcal{S}_n have two non-degenerate k -cells e_k^- and e_k^+ for every k in $\{0, \dots, n-1\}$, plus a non-degenerate n -cell e_n in \mathcal{E}_n . Using the cellular representations, the standard cells $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$ and \mathcal{E}_3 are respectively pictured as follows (for $\mathcal{S}_{-1}, \mathcal{S}_0, \mathcal{S}_1$ and \mathcal{S}_2 , one removes the top-dimensional cell):



If \mathcal{C} is an n -category then, for every k in $\{0, \dots, n\}$, the k -cells and k -spheres of \mathcal{C} are in bijective correspondence with the n -functors from \mathcal{E}_k to \mathcal{C} and from \mathcal{S}_k to \mathcal{C} , respectively. When the context is clear, we use the same notation for a k -cell or k -sphere and its corresponding n -functor.

As a consequence, if I is a set, the I -indexed families $(f_i)_{i \in I}$ of k -cells and k -spheres of \mathcal{C} are in bijective correspondence with the n -functors from $I \cdot \mathcal{E}_k$ to \mathcal{C} and from $I \cdot \mathcal{S}_k$ to \mathcal{C} , respectively. We recall that, for a set X and an n -category \mathcal{D} , the copower $X \cdot \mathcal{D}$ is the coproduct n -category $\coprod_{x \in X} \mathcal{D}$, whose set of k -cells is the product $X \times \mathcal{D}_k$.

1.2.3. The inclusion and collapsing n -functors. For every n , the *inclusion n -functor* J_n and the *collapsing n -functor* P_n

$$J_n : \mathcal{S}_n \rightarrow \mathcal{E}_{n+1} \quad \text{and} \quad P_n : \mathcal{S}_n \rightarrow \mathcal{E}_n$$

are respectively defined as the canonical inclusion of \mathcal{S}_n into \mathcal{E}_{n+1} and as the n -functor sending both e_n^- and e_n^+ to e_n , leaving the other cells unchanged.

1.3. Adjoining and collapsing cells

1.3.1. Definition. Let \mathcal{C} be an n -category, let k be in $\{0, \dots, n-1\}$, let I be a set and let $\Gamma : I \cdot \mathcal{S}_k \rightarrow \mathcal{C}$ be an n -functor. The *adjoining of Γ to \mathcal{C}* and the *collapsing of Γ in \mathcal{C}* are the n -categories respectively denoted by $\mathcal{C}[\Gamma]$ and \mathcal{C}/Γ and defined by the following pushouts in \mathbf{Cat}_n :

$$\begin{array}{ccc} I \cdot \mathcal{S}_k & \xrightarrow{\Gamma} & \mathcal{C} \\ I \cdot J_k \downarrow & \text{\textcircled{C}} & \downarrow \\ I \cdot \mathcal{E}_{k+1} & \longrightarrow & \mathcal{C}[\Gamma] \end{array} \quad \begin{array}{ccc} I \cdot \mathcal{S}_k & \xrightarrow{\Gamma} & \mathcal{C} \\ I \cdot P_k \downarrow & \text{\textcircled{C}} & \downarrow \\ \mathcal{E}_k & \longrightarrow & \mathcal{C}/\Gamma . \end{array}$$

1. Higher-dimensional categories presented by polygraphs

When $k = n$, one defines $\mathcal{C}[\Gamma]$ by seeing \mathcal{C} as an $(n + 1)$ -category with degenerate $(n + 1)$ -cells only. By abuse, one denotes a sphere of Γ and the induced cell of $\mathcal{C}[\Gamma]$ the same way.

To be more explicit, the n -category $\mathcal{C}[\Gamma]$ has the same cells as \mathcal{C} up to dimension k ; its $(k + 1)$ -cells are all the formal composites made of the $(k + 1)$ -cells of \mathcal{C} , plus one extra $(k + 1)$ -cell from $\Gamma(i, e_k^-)$ to $\Gamma(i, e_k^+)$; above dimension $k + 1$, its cells are the ones of \mathcal{C} , plus the identities of each extra cell.

The n -category \mathcal{C}/Γ has the same cells as \mathcal{C} up to dimension $k - 1$; its k -cells are the equivalence classes of k -cells of \mathcal{C} , for the congruence relation generated by $\Gamma(i, e_k^-) \sim \Gamma(i, e_k^+)$, for every i ; above dimension k , its cells are the formal composites of the ones of \mathcal{C} , but with sources and targets considered modulo the previous congruence.

1.3.2. Extension of an n -functor. Let \mathcal{C} and \mathcal{D} be n -categories and let $\Gamma : I \cdot \mathcal{S}_k \rightarrow \mathcal{C}$ be an n -functor. Then, by universal property of $\mathcal{C}[\Gamma]$, one extends an n -functor $F : \mathcal{C} \rightarrow \mathcal{D}$ to a unique n -functor $F : \mathcal{C}[\Gamma] \rightarrow \mathcal{D}$ by fixing, for every γ in Γ , a $(k + 1)$ -cell $F(\gamma)$ in \mathcal{D} , provided the following two equalities hold:

$$s(F(\gamma)) = F(s(\gamma)) \quad \text{and} \quad t(F(\gamma)) = F(t(\gamma)).$$

1.3.3. Occurrences. Here we see the group \mathbb{Z} of integers as an n -category: it has one cell in each dimension up to $n - 1$ and \mathbb{Z} as set of n -cells; all the compositions of n -cells are given by the addition.

Let \mathcal{C} be an n -category and let $\Gamma : I \cdot \mathcal{S}_k \rightarrow \mathcal{C}$ be an n -functor. We denote by $\|f\|_\Gamma$ the n -functor from $\mathcal{C}[\Gamma]$ to \mathbb{Z} defined by:

$$\|f\|_\Gamma = \begin{cases} 1 & \text{if } f \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

For every cell f , one calls $\|f\|_\Gamma$ the *number of occurrences of cells of Γ in f* . Indeed, intuitively, $\|f\|_\Gamma$ is the number of $(k + 1)$ -cells of Γ that f contains.

The *set of cells of Γ in f* is the subset $\{f\}_\Gamma$ of Γ that consists of all the cells γ of Γ such that $\|f\|_\gamma \geq 1$ holds. It is, intuitively, the set of cells of Γ required to write f . This construction can also be seen as an n -functor. One considers the set $\mathcal{P}(\Gamma)$ of parts of Γ as an n -category this way: it has one cell in each dimension up to $n - 1$ and parts of Γ as n -cells; all the compositions of n -cells are given by the union. Then $\{\cdot\}_\Gamma$ is the n -functor from $\mathcal{C}[\Gamma]$ to $\mathcal{P}(\Gamma)$ defined by:

$$\{f\}_\Gamma = \begin{cases} \{f\} & \text{if } f \in \Gamma \\ \emptyset & \text{otherwise.} \end{cases}$$

1.3.4. The n -category presented by an $(n + 1)$ -category. Let \mathcal{C} be an $(n + 1)$ -category. If f is an $(n + 1)$ -cell of \mathcal{C} , then ∂f is an n -sphere of \mathcal{C} . Thus, the set \mathcal{C}_{n+1} of $(n + 1)$ -cells of \mathcal{C} yields an $(n + 1)$ -functor from $\mathcal{C}_{n+1} \cdot \mathcal{S}_n$ to the underlying n -category of \mathcal{C} : the *n -category presented by \mathcal{C}* is the n -category denoted by $\bar{\mathcal{C}}$ one gets by collapsing the $(n + 1)$ -cells of \mathcal{C} in its underlying n -category.

1.4. Polygraphs and presentations of n -categories

Polygraphs (or *computads*) are presentations by “generators” and “relations” of some higher-dimensional categories [20, 6], see also [21, 22]. We define n -polygraphs by induction on the natural number n .

1.4. Polygraphs and presentations of n -categories

The category \mathbf{Pol}_0 of 0-polygraphs and morphisms between them is the one of sets and maps. A 0-polygraph is *finite* when it is finite as a set. A 0-cell of a 0-polygraph is one of its elements. The *free 0-category functor* is the identity functor $\mathbf{Pol}_0 \rightarrow \mathbf{Cat}_0$. Now, let us fix a non-zero natural number n and let us assume that we have defined the category \mathbf{Pol}_{n-1} of $(n-1)$ -polygraphs and morphisms between them, finite polygraphs, cells of a polygraph and the free $(n-1)$ -category functor $\mathbf{Pol}_{n-1} \rightarrow \mathbf{Cat}_{n-1}$, sending an $(n-1)$ -polygraph Σ to the $(n-1)$ -category Σ^* .

1.4.1. n -polygraphs. An n -polygraph is a pair $\Sigma = (\Sigma_{n-1}, \Sigma_n)$ made of an $(n-1)$ -polygraph Σ_{n-1} and a family Σ_n of $(n-1)$ -spheres of the $(n-1)$ -category Σ_{n-1}^* .

An n -cell of Σ is an element of Σ_n and, if $k < n$, a k -cell of Σ is a k -cell of the $(n-1)$ -polygraph Σ_{n-1} . The set of k -cells of Σ is abusively denoted by Σ_k , thus identifying it to the k -polygraph underlying Σ . An n -polygraph is *finite* when it has a finite number of cells in every dimension. The *size* of a k -cell f in Σ^* , denoted by $\|f\|$, is the natural number $\|f\|_{\Sigma_k}$, giving the number of k -cells of Σ that f is made of. For 1-cells and 3-cells, we also use $|\cdot|$ and $\|\cdot\|$ instead of $\|\cdot\|$.

The original paper [6] contains an equivalent description of n -polygraphs, where they are defined as diagrams

$$\begin{array}{ccccccc}
 \Sigma_0 & & \Sigma_1 & & (\dots) & & \Sigma_{n-1} & & \Sigma_n \\
 \parallel & \swarrow_{s_0, t_0} & \downarrow u_1 & \swarrow_{s_1, t_1} & & \swarrow_{s_{n-1}, t_{n-1}} & \downarrow u_{n-1} & \swarrow_{s_n, t_n} & \\
 \Sigma_0 & \xleftarrow{\bar{s}_0, \bar{t}_0} & \Sigma_1^* & \xleftarrow{\bar{s}_1, \bar{t}_1} & (\dots) & \xleftarrow{\bar{s}_{n-1}, \bar{t}_{n-1}} & \Sigma_{n-1}^* & &
 \end{array}$$

of sets and maps such that, for any k in $\{0, \dots, n-1\}$, the following two conditions hold:

- The diagram $\Sigma_0^* \xleftarrow{\bar{s}_0, \bar{t}_0} \Sigma_1^* \xleftarrow{\bar{s}_1, \bar{t}_1} (\dots) \xleftarrow{\bar{s}_{k-1}, \bar{t}_{k-1}} \Sigma_k^*$ is a k -category.
- The diagram $\Sigma_0^* \xleftarrow{\bar{s}_0, \bar{t}_0} \Sigma_1^* \xleftarrow{\bar{s}_1, \bar{t}_1} (\dots) \xleftarrow{\bar{s}_{k-1}, \bar{t}_{k-1}} \Sigma_k^* \xleftarrow{s_k, t_k} \Sigma_{k+1}$ is a $(k+1)$ -graph.

1.4.2. Morphisms of n -polygraphs. Let Σ and Ξ be two n -polygraphs. A *morphism of n -polygraphs from Σ to Ξ* is a pair $F = (F_{n-1}, F_n)$ where F_{n-1} is a morphism of $(n-1)$ -polygraphs from Σ_{n-1} to Ξ_{n-1} and where F_n is a map from Σ_n to Ξ_n such that the following two diagrams commute:

$$\begin{array}{ccc}
 \Sigma_n & \xrightarrow{F_n} & \Xi_n \\
 s_{n-1} \downarrow & \text{\textcircled{C}} & \downarrow s_{n-1} \\
 \Sigma_{n-1}^* & \xrightarrow{F_{n-1}^*} & \Xi_{n-1}^*
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Sigma_n & \xrightarrow{F_n} & \Xi_n \\
 t_{n-1} \downarrow & \text{\textcircled{C}} & \downarrow t_{n-1} \\
 \Sigma_{n-1}^* & \xrightarrow{F_{n-1}^*} & \Xi_{n-1}^*
 \end{array}$$

Alternatively, if $\Sigma_n : I \cdot \mathcal{S}_{n-1} \rightarrow \Sigma_{n-1}^*$ and $\Xi_n : J \cdot \mathcal{S}_{n-1} \rightarrow \Xi_{n-1}^*$ are seen as $(n-1)$ -functors, then F_n

1. Higher-dimensional categories presented by polygraphs

is a map from I to J such that the following diagram commutes in \mathbf{Cat}_{n-1} :

$$\begin{array}{ccc} I \cdot \mathcal{S}_{n-1} & \xrightarrow{\Sigma_n} & \Sigma_{n-1}^* \\ \downarrow F_n \cdot 1_{\mathcal{S}_{n-1}} & \text{\textcircled{C}} & \downarrow F_{n-1}^* \\ J \cdot \mathcal{S}_{n-1} & \xrightarrow{\Xi_n} & \Xi_{n-1}^* \end{array}$$

We denote by \mathbf{Pol}_n the category of polygraphs and morphisms between them.

1.4.3. The free n -category functor. Let Σ be an n -polygraph. The n -category freely generated by Σ is the n -category Σ^* defined as follows:

$$\Sigma^* = \Sigma_{n-1}^*[\Sigma_n].$$

This construction extends to an n -functor $(\cdot)^* : \mathbf{Pol}_n \rightarrow \mathbf{Cat}_n$ called the *free n -category functor*.

1.4.4. The n -category presented by an $(n+1)$ -polygraph. Let Σ be a $(n+1)$ -polygraph. The n -category presented by Σ is the n -category denoted by $\bar{\Sigma}$ and defined as follows:

$$\bar{\Sigma} = \Sigma_n^* / \Sigma_{n+1}.$$

Two n -polygraphs are *Tietze-equivalent* when the $(n-1)$ -categories they present are isomorphic. If \mathcal{C} is an n -category, a *presentation of \mathcal{C}* is an $(n+1)$ -polygraph Σ such that \mathcal{C} is isomorphic to the n -category $\bar{\Sigma}$ presented by Σ . One says that an n -category \mathcal{C} is *finitely generated* when it admits a presentation by an $(n+1)$ -polygraph Σ whose underlying n -polygraph Σ_n is finite. One says that \mathcal{C} is *finitely presented* when it admits a finite presentation.

1.4.5. Example : a presentation of the 2-category of permutations. The 2-category $\mathcal{P}erm$ of permutations has one 0-cell, one 1-cell for each natural number and, for each pair (m, n) of natural number, its set of 2-cells from m to n is the group S_n of permutations if $m = n$ and the empty set otherwise. The 0-composition of 1-cells is the addition of natural numbers. The 0-composition of two 2-cells $\sigma \in S_m$ and $\tau \in S_n$ is the permutation $\sigma \star_0 \tau$ defined by:

$$\sigma \star_0 \tau(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq n, \\ \tau(i - n) & \text{otherwise.} \end{cases}$$

Finally the 1-composition of 2-cells is the composition of permutations. The 2-category $\mathcal{P}erm$ is presented by the 3-polygraph with one 0-cell, one 1-cell, one 2-cell, pictured by \bowtie , and the following two 3-cells:

$$\begin{array}{c} \text{Diagram 1: } \bowtie \Rightarrow \parallel \parallel \\ \text{Diagram 2: } \text{Two } \bowtie \text{ cells} \Rightarrow \text{Two } \bowtie \text{ cells} \end{array}$$

2. CONTEXTS, MODULES AND DERIVATIONS OF n -CATEGORIES

2.1. The category of contexts of an n -category

Throughout this section, n is a fixed natural number and \mathcal{C} is a fixed n -category.

2.1.1. Contexts of an n -category. A *context of \mathcal{C}* is a pair (x, C) made of an n -sphere x of \mathcal{C} and an n -cell C in $\mathcal{C}[x]$ such that $\|C\|_x = 1$. We often denote by $C[x]$, or simply by C , such a context.

Let x and y be n -spheres of \mathcal{C} and let f be an n -cell in $\mathcal{C}[x]$ such that $\partial f = y$ holds. We denote by $C[f]$ the image of a context $C[y]$ of \mathcal{C} by the functor

$$\mathcal{C}[y] \rightarrow \mathcal{C}[x]$$

defined by extending the identity n -functor of \mathcal{C} with $y \mapsto f$.

2.1.2. The category of contexts. The *category of contexts of \mathcal{C}* is denoted by $\mathbf{C}\mathcal{C}$ and defined as follows:

- Its objects are the n -cells of \mathcal{C} .
- If f and g are n -cells of \mathcal{C} , then the morphisms of $\mathbf{C}\mathcal{C}$ from f to g are the contexts $C[\partial f]$ of \mathcal{C} such that $C[f] = g$ holds.
- If $C : f \rightarrow g$ and $D : g \rightarrow h$ are morphisms of $\mathbf{C}\mathcal{C}$ then $D \circ C : f \rightarrow h$ is $D[C]$.
- The identity context I_f on an n -cell f of \mathcal{C} is the context $\partial f[\partial f]$.

When Σ is an n -polygraph, one uses $\mathbf{C}\Sigma$ instead of $\mathbf{C}\Sigma^*$.

2.1.3. Proposition. *Every context C of $\mathcal{C}[x]$ decomposes as follows:*

$$C[x] = f_n \star_{n-1} (f_{n-1} \star_{n-2} \cdots (f_1 \star_0 x \star_0 g_1) \cdots \star_{n-2} g_{n-1}) \star_{n-1} g_n,$$

where, for every k in $\{1, \dots, n\}$, f_k and g_k are n -cells in \mathcal{C} . Moreover, one can chose these cells so that f_k and g_k are (the identities of) k -cells.

Proof. The set of n -cells f of $\mathcal{C}[x]$ such that $\|f\|_x = 1$ is a quotient of the following inductively defined set X :

- The n -cell x is in X .
- If C is in X and f is an n -cell of \mathcal{C} such that $t_i(f) = s_i(C)$ holds for some i , then $f \star_i C$ is in X .
- If C is in X and f is a n -cell of \mathcal{C} such that $t_i(C) = s_i(f)$ holds for some i , then $C \star_i f$ is in X .

Using the associativity and exchange relations satisfied by the compositions of \mathcal{C} , one can order these successive compositions to reach the required shape, or to reach the same shape with f_k and g_k being identities of k -cells. □

2. Contexts, modules and derivations of n -categories

2.1.4. Whiskers. A whisker of \mathcal{C} is a context $C[x]$ with a decomposition

$$C[x] = f_{n-1} \star_{n-2} \cdots (f_1 \star_0 x \star_0 g_1) \cdots \star_{n-2} g_{n-1}$$

such that, for every k in $\{1, \dots, n-1\}$, f_k and g_k are k -cells. We denote by \mathbf{WC} the subcategory of \mathbf{CC} with the same objects and with whiskers as morphisms. When Σ is an n -polygraph, we write $\mathbf{W}\Sigma$ instead of $\mathbf{W}\Sigma^*$.

2.1.5. Proposition. Let Σ be an n -polygraph and let f be an n -cell of Σ^* with size $k \geq 1$. Then there exist n -cells $\gamma_1, \dots, \gamma_k$ in Σ and whiskers C_1, \dots, C_k of Σ^* such that f decomposes as follows:

$$f = C_1[\gamma_1] \star_{n-1} \cdots \star_{n-1} C_k[\gamma_k].$$

Proof. We proceed by induction on the size of the n -cell f .

If it has size 1, then it contains exactly one n -cell γ of Σ , possibly composed with other ones of lower dimension. Using the relations satisfied by compositions in an n -category, one can write f as $C[\gamma]$, with C a context of Σ^* . Moreover, this context must be a whisker, since f has size 1.

Now, let us assume that we have proved that every n -cell with size at most k , for a fixed non-zero natural number k , admits a decomposition as in Proposition 2.1.5. Then let us consider an n -cell f with size $k+1$. Since $\|f\| \geq 2$ and by construction of $\Sigma^* = \Sigma_{n-1}^*[\Sigma_n]$, one gets that f can be written $g \star_i h$, where (g, h) is a pair of i -composable n -cells of Σ^* , for some i in $\{0, \dots, n-1\}$, with $\|g\|$ and $\|h\|$ at least 1. One can assume that $i = n-1$ since, otherwise, one considers the following alternative decomposition of f , thanks to the exchange relation between \star_i and \star_{n-1} :

$$f = (g \star_i s(h)) \star_{n-1} (t(g) \star_i h).$$

Since $\|f\| = \|g\| + \|h\|$, one must have $\|g\| \leq k$ and $\|h\| \leq k$. We use the induction hypothesis to decompose g and h as in 2.1.5, where j denotes $\|g\|$:

$$g = C_1[\gamma_1] \star_{n-1} \cdots \star_{n-1} C_j[\gamma_j] \quad \text{and} \quad h = C_{j+1}[\gamma_{j+1}] \star_{n-1} \cdots \star_{n-1} C_k[\gamma_k].$$

We compose the right members and use the associativity of \star_{n-1} to conclude. \square

2.2. Contexts in low dimensions

2.2.1. Contexts of a 1-category as factorizations. From Proposition 2.1.3, the contexts of a 1-category \mathcal{C} have the following shape:

$$f \star_0 x \star_0 g,$$

where f and g are 1-cells of \mathcal{C} . The morphisms in \mathbf{CC} from $h : u \rightarrow v$ to $h' : u' \rightarrow v'$ are the pairs $(f : u' \rightarrow u, g : v \rightarrow v')$ of 1-cells of \mathcal{C} such that the following diagram commutes in \mathcal{C} :

$$\begin{array}{ccc} u & \xleftarrow{f} & u' \\ h \downarrow & \text{\textcircled{C}} & \downarrow h' \\ v & \xrightarrow{g} & v' \end{array}$$

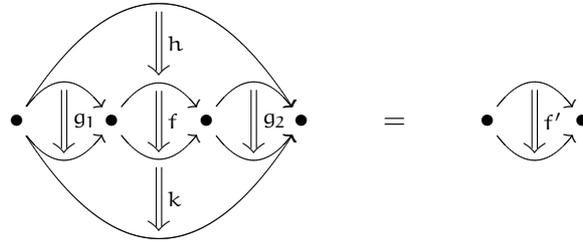
When \mathcal{C} is freely generated by a 1-polygraph, the 1-cells f and g are uniquely defined by the context. Moreover, the contexts from h to h' are in bijective correspondence with the occurrences of the word h in the word h' .

The category \mathbf{CC} has been introduced by Quillen under the name *category of factorizations of \mathcal{C}* [18]. It has been used by Leech to introduce cohomological properties of congruences on monoids [14] and by Baues and Wirsching for the cohomology of small categories [3].

2.2.2. Contexts of 2-categories. Let \mathcal{C} be a 2-category. From Proposition 2.1.3, a context of \mathcal{C} has the following shape:

$$h \star_1 (g_1 \star_0 x \star_0 g_2) \star_1 k$$

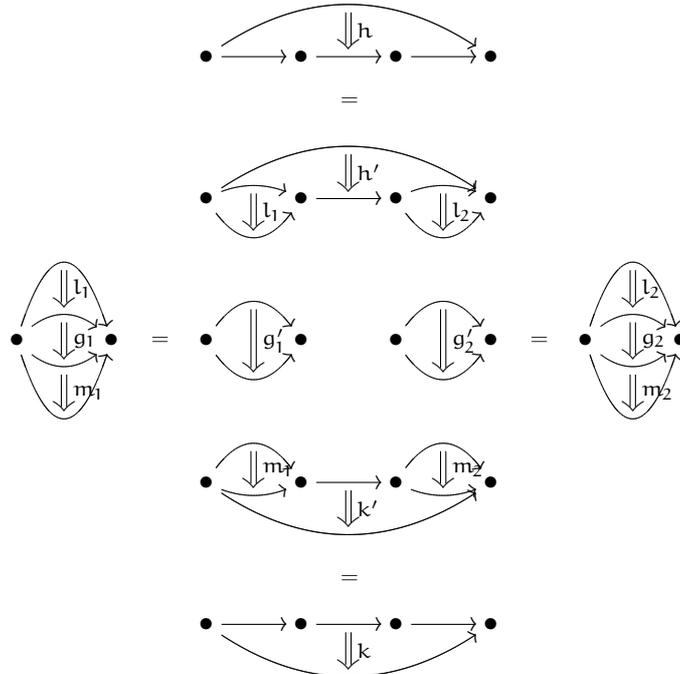
where g_1, g_2, h and k are 2-cells. Morphisms in \mathbf{CC} from a 2-cell f to a 2-cell f' are the contexts $C = h \star_1 (g_1 \star_0 x \star_0 g_2) \star_1 k$ such that $h \star_1 (g_1 \star_0 f \star_0 g_2) \star_1 k = f'$ holds:



Note that this decomposition is not unique. One proves that two decompositions

$$h \star_1 (g_1 \star_0 x \star_0 g_2) \star_1 k \quad \text{and} \quad h' \star_1 (g'_1 \star_0 x' \star_0 g'_2) \star_1 k'$$

represent the same context if and only if $x = x'$ and there exist 2-cells l_1, l_2, m_1, m_2 such that the following four relations are defined and hold in \mathcal{C} :



2. Contexts, modules and derivations of n -categories

Thus, informally, two decompositions represent the same context when one can pass from one to the other one by moving up and down 2-cells that one finds at the left side or the right side of the sphere x .

2.3. Modules over n -categories

2.3.1. Definition. Let \mathcal{C} be an n -category. A *module over \mathcal{C}* or \mathcal{C} -*module* is a functor from the category of contexts \mathbf{CC} to the category \mathbf{Ab} of abelian groups. Hence, a \mathcal{C} -module M is given by:

- An abelian group $M(f)$, for every n -cell f in \mathcal{C} .
- A morphism $M(C) : M(f) \rightarrow M(g)$ of groups, for every context $C(x) : f \rightarrow g$ of \mathcal{C} . When the context $C(x)$ is of the shape $h \star_i x$ (resp. $x \star_i h$) and when no confusion may occur, then one writes $h \star_i m$ (resp. $m \star_i h$) instead of $M(C)(m)$.

2.3.2. Proposition. Let Σ be an n -polygraph. A Σ^* -module M is entirely and uniquely defined by its values on the following contexts of \mathcal{C}

$$C[\varphi] \star_i x \quad \text{and} \quad x \star_i C[\varphi]$$

for every i in $\{0, \dots, n-1\}$, every generating $(i+1)$ -cell φ and every i -context $C[\partial\varphi]$.

Proof. Let f, g be two n -cells of Σ^* and let $C[x] : f \rightarrow g$ be a morphism of $\mathbf{C}\Sigma$. We decompose $C[x]$ as follows:

$$C[x] = f_n \star_{n-1} \cdots \star_1 (f_1 \star_0 x \star_0 g_1) \star_1 \cdots \star_{n-1} g_n,$$

in such a way that, for every k in $\{1, \dots, n\}$, f_k and g_k are k -cells. Thus, in the category $\mathbf{C}\Sigma$, the context $C[x]$ decomposes as

$$C[x] = C_n[x_n] \circ \cdots \circ C_1[x_1],$$

where $x_1 = x$ and, for every i in $\{1, \dots, n\}$, one has $C_i[x_i] = f_i \star_{i-1} x_i \star_{i-1} g_i$ and $x_{i+1} = \partial C_i[x_i]$. Moreover, each $C_i[x_i]$ splits into:

$$C_i[x_i] = (y_i \star_{i-1} g_i) \circ (f_i \star_{i-1} x_i),$$

where $y_i = \partial(f_i \star_{i-1} x_i)$. Thus, since M is a functor, it is entirely defined by its values on the contexts with shape $f \star_i x$ or $x \star_i f$, with i in $\{0, \dots, n-1\}$ and f a non-degenerate $(i+1)$ -cell (indeed, when f is degenerate as a i -cell, one has $x \star_i f = x$ and $M(x)$ is always an identity).

Now, let us consider the n -context $f \star_i x$, where f is an $(i+1)$ -cell of size $k \geq 1$. We decompose it as in 2.1.5:

$$f = C_1[\varphi_1] \star_i \cdots \star_i C_k[\varphi_k],$$

where $\varphi_1, \dots, \varphi_k$ are generating $(i+1)$ -cells and C_1, \dots, C_k are i -contexts. Thus, a context $f \star_i x$ decomposes into $\mathbf{C}\Sigma$ as follows:

$$f \star_i x = (C_1[\varphi_1] \star_i x_1) \circ \cdots \circ (C_k[\varphi_k] \star_i x_k),$$

where $x_k = x$ and $x_j = \partial(C_{j+1}[\varphi_{j+1}] \star_i x_{j+1})$.

Proceeding similarly with contexts of the shape $x \star_i f$, one gets the result. \square

2.3.3. Example: the trivial module. Let \mathcal{C} be an n -category. The *trivial \mathcal{C} -module* sends each n -cell of \mathcal{C} to \mathbb{Z} and each context of \mathcal{C} to the identity of \mathbb{Z} .

2.3.4. Example of modules over 2-categories. Let \mathcal{V} be a concrete category, *i.e.* a category equipped with a faithful functor into **Set**. We view it as a 2-category with:

- One 0-cell.
- Objects as 1-cells. Their 0-composition is given by the cartesian product.
- Morphisms as 2-cells. Their 0-composition is given by the cartesian product of morphisms. Their 1-composition is the composition of morphisms in \mathcal{V} , written in reverse order.

Let us fix a 2-category \mathcal{C} and let us consider 2-functors $X : \mathcal{C} \rightarrow \mathcal{V}$ and $Y : \mathcal{C}^{\text{co}} \rightarrow \mathcal{V}$, where \mathcal{C}^{co} is \mathcal{C} where one has exchanged the source and target of every 2-cell. Then, for every internal abelian group G in \mathcal{V} , the following assignments yield a \mathcal{C} -module $M(X, Y, G)$:

- Every 2-cell $f : u \Rightarrow v$ is sent to the abelian group of morphisms:

$$M(X, Y, G)(f) = \mathcal{V}(X(u) \times Y(v), G).$$

- Let C be a context from $f : u \Rightarrow v$ to $f' : u' \Rightarrow v'$ and let $C = g \star_1 (w \star_0 x \star_0 w') \star_1 g'$ be a decomposition of C , where g, g' are 2-cells and w, w' are 1-cells. By hypothesis, $X(g)$ and $Y(g')$ are morphisms in \mathcal{V} :

$$X(g) : X(u') \rightarrow X(w) \times X(u) \times X(w') \quad Y(g') : Y(v') \rightarrow Y(w) \times Y(v) \times Y(w').$$

We denote by $\tilde{X}(g) : X(u') \rightarrow X(u)$ and by $\tilde{Y}(g') : Y(v') \rightarrow Y(v)$ the morphisms they induce, thanks to the projections given by the cartesian structure. Then $M(X, Y, G)(C)$ is defined as the morphism of groups from $M(X, Y, G)(f)$ to $M(X, Y, G)(f')$ that sends a morphism $\varphi : X(u) \times Y(v) \rightarrow G$ of \mathcal{V} to the following one:

$$\varphi \circ (\tilde{X}(g), \tilde{Y}(g')) : X(u') \times Y(v') \rightarrow G.$$

When X or Y is trivial, *i.e.* sends all the cells of \mathcal{C} to the terminal object of \mathcal{V} , one denotes the corresponding \mathcal{C} -module by $M(*, Y, G)$ or $M(X, *, G)$. In particular, $M(*, *, \mathbb{Z})$ is the trivial \mathcal{C} -module.

By construction, a \mathcal{C} -module $M(X, Y, G)$ is uniquely and entirely defined by its values on the contexts $h \star_1 x$ and $x \star_1 h$, for every 2-cell h in \mathcal{C} , *i.e.* by the objects $X(u)$ and $Y(u)$, for every 1-cell u , and by the morphisms $X(f)$ and $Y(f)$ for every 2-cell f . As a consequence, when \mathcal{C} is freely generated by a 2-polygraph Σ , the \mathcal{C} -module $M(X, Y, G)$ is uniquely and entirely determined by:

- The objects $X(a)$ and $Y(a)$ of \mathcal{V} for every 1-cell a in Σ_1 .
- The morphisms $X(\gamma) : X(u) \rightarrow X(v)$ and $Y(\gamma) : Y(v) \rightarrow Y(u)$ of \mathcal{V} for every 2-cell $\gamma : u \Rightarrow v$ in Σ_2 .

In the sequel, we consider this kind of \mathcal{C} -module with \mathcal{V} being the category **Set** of sets and maps or the category **Ord** of partially ordered sets and monotone maps. In this last situation, an internal abelian group in **Ord** is a partially ordered set equipped with a structure of abelian group, such that group law is monotone in both arguments.

3. Higher-dimensional categories with finite derivation type

2.4. Derivations of n -categories

2.4.1. Definition. Let \mathcal{C} be an n -category and let M be a \mathcal{C} -module. A *derivation of \mathcal{C} into M* is a map sending every n -cell f of \mathcal{C} to an element $d(f)$ of $M(f)$ such that the following relation holds, for every i -composable pair (f, g) of n -cells of \mathcal{C} :

$$d(f \star_i g) = f \star_i d(g) + d(f) \star_i g.$$

Given a derivation d on \mathcal{C} , we define its values on contexts by

$$d(C) = \sum_{i=-n}^n f_n \star_{n-1} (f_{n-1} \star_{n-2} \cdots (d(f_i) \star_{i-1} \cdots (f_1 \star_0 x \star_0 f_{-1}) \cdots \star_{n-1} f_{-n}),$$

for any context $C = f_n \star_{n-1} \cdots (f_1 \star_0 x \star_0 f_{-1}) \cdots \star_{n-1} f_{-n}$ in $\mathcal{C}[x]$. This gives a mapping $d(C)$ from the set of 2-cells of \mathcal{C} with boundary ∂x to the abelian group $M(\partial C)$. In this way a derivation from \mathcal{C} into M satisfies:

$$d(C[f]) = d(C)[f] + C[d(f)].$$

Given a derivation d of an n -category \mathcal{C} and an n -functor $\Gamma : I \cdot \mathcal{S}_n \rightarrow \mathcal{C}$, we define d on $(n+1)$ -cells of $\mathcal{C}[\Gamma]$ by $d(\gamma) = d(s\gamma) - d(t\gamma)$. Let us note that this has no reason to be a derivation of $\mathcal{C}[\Gamma]$.

If f is a degenerate n -cell, then $d(f) = 0$ holds. Indeed, one has:

$$d(f) = d(f \star_{n-1} f) = f \star_{n-1} d(f) + d(f) \star_{n-1} f = 2 \cdot d(f).$$

2.4.2. Example: occurrences. If \mathcal{C} is an n -category and $\Gamma : I \cdot \mathcal{S}_n \rightarrow \mathcal{C}$ is an n -functor, we have defined the n -functor $\|\cdot\|_\Gamma$ counting the number of occurrences of cells of Γ in a cell of $\mathcal{C}[\Gamma]$. This construction is a derivation of \mathcal{C} into the trivial \mathcal{C} -module, sending each n -cell of \mathcal{C} to 0 and each n -cell of Γ to 1.

2.4.3. Example of derivations of free 2-categories. Let us consider a 2-polygraph Σ , a concrete category \mathcal{V} and a module of the shape $M(X, Y, G)$, as defined in 2.3.4. Then, by construction of Σ^* , a derivation d of Σ^* into $M(X, Y, G)$ is entirely and uniquely determined by a family $(d\gamma)_{\gamma \in \Sigma_2}$ made of a morphism

$$d\gamma : X(u) \times Y(v) \rightarrow G$$

of \mathcal{V} for each 2-cell $\gamma : u \Rightarrow v$ of Σ . Indeed, every 2-cell of Σ^* can be written as a formal composite of 2-cells of Σ and 1-cells of Σ^* . Moreover, the relations satisfied by a derivation imply that such a map is compatible with the axioms of n -category.

3. HIGHER-DIMENSIONAL CATEGORIES WITH FINITE DERIVATION TYPE

3.1. Track n -categories

3.1.1. Definitions. In an n -category \mathcal{C} , a k -cell f is *invertible* when there exists a k -cell g from $t(f)$ to $s(f)$ in \mathcal{C} such that both $f \star_{k-1} g = s(f)$ and $g \star_{k-1} f = t(f)$ hold. In that case, g is unique and denoted by f^{-1} . The following relations are satisfied:

$$(1_x)^{-1} = 1_x \quad \text{and} \quad (f \star_i g)^{-1} = \begin{cases} f^{-1} \star_i g^{-1} & \text{if } i < k-1 \\ g^{-1} \star_{k-1} f^{-1} & \text{otherwise.} \end{cases}$$

Moreover, if $F : \mathcal{C} \rightarrow \mathcal{D}$ is an n -functor, one has:

$$F(f^{-1}) = F(f)^{-1}.$$

A *track n -category* is an n -category whose n -cells are invertible, *i.e.* an $(n - 1)$ -category enriched in groupoid. We denote by \mathbf{Tck}_n the category of track n -categories and n -functors between them.

3.1.2. Example. Let \mathcal{C} be an n -category. Given two n -cells f from u to v and g from v to u in \mathcal{C} , we denote by $I_{f,g}$ the following n -sphere of \mathcal{C} :

$$I_{f,g} = (f \star_{n-1} g, 1_u).$$

If $\gamma = (f, g)$ is an n -sphere of \mathcal{C} , we denote by γ^{-1} the n -sphere (g, f) of \mathcal{C} . Then we define the track $(n + 1)$ -category $\mathcal{C}(\gamma)$ by:

$$\mathcal{C}(\gamma) = \mathcal{C}[\gamma, \gamma^{-1}] / \{I_{\gamma, \gamma^{-1}}, I_{\gamma^{-1}, \gamma}\}.$$

This construction is extended to a set Γ of n -spheres, yielding a track $(n + 1)$ -category $\mathcal{C}(\Gamma)$.

3.1.3. The free track category functor. Given an n -polygraph Σ , the *track n -category freely generated by Σ* is the n -category denoted by Σ^\top and defined by:

$$\Sigma^\top = \Sigma_{n-1}^*(\Sigma_n),$$

This construction extends into a functor $(\cdot)^\top : \mathbf{Pol}_n \rightarrow \mathbf{Tck}_n$ called the *free track n -category functor*.

3.2. Homotopy bases

3.2.1. Homotopy relation. Let \mathcal{C} be a n -category. A *homotopy relation* on \mathcal{C} is a track $(n + 1)$ -category \mathcal{T} with \mathcal{C} as underlying n -category. Given an n -sphere (f, g) in \mathcal{C} , one denotes by $f \approx_{\mathcal{T}} g$ the fact that there exists an $(n + 1)$ -cell from f to g in \mathcal{T} . If Γ is a set of n -spheres of \mathcal{C} , one simply uses \approx_Γ instead of $\approx_{\mathcal{C}(\Gamma)}$ and calls it the *homotopy relation on \mathcal{C} generated by Γ* .

One has $f \approx_{\mathcal{T}} g$ if and only if $\pi(f) = \pi(g)$ holds, where π is the canonical projection from \mathcal{T} to the n -category $\overline{\mathcal{T}}$ presented by \mathcal{T} , *i.e.* $\mathcal{C}/\mathcal{T}_{n+1}$. As a consequence, the relation $\approx_{\mathcal{T}}$ is a congruence relation on the parallel n -cells of \mathcal{C} , *i.e.* it is an equivalence relation compatible with every composition of \mathcal{C} .

3.2.2. Homotopy basis. A set Γ of n -spheres of \mathcal{C} is a *homotopy basis of \mathcal{C}* when, for every n -sphere (f, g) of \mathcal{C} , one has $f \approx_\Gamma g$. In other words, Γ is a homotopy basis if and only if, for every n -sphere γ of \mathcal{C} , there exists an $(n + 1)$ -cell $\overline{\gamma}$ such that $\partial\overline{\gamma} = \gamma$ holds, *i.e.* such that the following diagram commutes in \mathbf{Cat}_{n+1} :

$$\begin{array}{ccc} \mathcal{S}_n & \xrightarrow{\gamma} & \mathcal{C} \\ \downarrow J_n & \text{\textcircled{C}} & \downarrow \\ \mathcal{E}_{n+1} & \xrightarrow{\overline{\gamma}} & \mathcal{C}(\Gamma). \end{array}$$

3. Higher-dimensional categories with finite derivation type

3.2.3. Proposition. *Let \mathcal{C} be an n -category and let Γ be a homotopy basis of \mathcal{C} . If \mathcal{C} admits a finite homotopy basis, then there exists a finite subset of Γ that is a homotopy basis of \mathcal{C} .*

Proof. Let Γ' be a finite homotopy basis of \mathcal{C} . Let γ be an n -sphere of \mathcal{C} in Γ' . Since Γ is a homotopy basis of \mathcal{C} , there exists an $(n+1)$ -cell φ_γ in $\mathcal{C}(\Gamma)$ with boundary γ . This defines an $(n+1)$ -functor F from $\mathcal{C}(\Gamma')$ to $\mathcal{C}(\Gamma)$ which is the identity on cells of \mathcal{C} and which sends each γ in Γ' to φ_γ . For each φ_γ , we fix a representative in $\mathcal{C}[\Gamma, \Gamma^{-1}]$ and denote by $\{\varphi_\gamma\}_\Gamma$ the set of cells of Γ occurring in this representative. Let us denote by Γ_0 the following subset of Γ

$$\Gamma_0 = \bigcup_{\gamma \in \Gamma'} \{\varphi_\gamma\}_\Gamma,$$

consisting of all the cells of Γ contained in the cells φ_γ . The subset Γ_0 is finite since Γ' and every $\{\varphi_\gamma\}$ are. Now let us see that it is an homotopy basis of \mathcal{C} . Let us fix an n -sphere (f, g) of \mathcal{C} . By hypothesis, there exists an $(n+1)$ -cell A in $\mathcal{C}(\Gamma')$ with boundary (f, g) . By application of F , one gets an $(n+1)$ -cell $F(A)$ in $\mathcal{C}(\Gamma)$ with boundary (f, g) . Moreover, the $(n+1)$ -cell $F(A)$ is a composite of cells of the shape φ_γ : hence, it lives in $\mathcal{C}(\Gamma_0)$. As a consequence, one gets $f \approx_{\Gamma_0} g$, which concludes the proof. \square

3.3. Polygraphs with finite derivation type

3.3.1. Definition. One says that an n -polygraph Σ has *finite derivation type* when it is finite and when the track n -category Σ^\top it generates admits a finite homotopy basis.

3.3.2. Lemma. *Let Σ and Σ' be n -polygraphs. We denote by $\pi : \Sigma_{n-1}^* \rightarrow \bar{\Sigma}$ and by $\pi' : \Sigma'_{n-1} \rightarrow \bar{\Sigma}'$ the canonical $(n-1)$ -functors. Then every $(n-1)$ -functor F from $\bar{\Sigma}$ to $\bar{\Sigma}'$ can be lifted to an n -functor $\tilde{F} : \Sigma^\top \rightarrow \Sigma'^\top$ such that the following diagram commutes in \mathbf{Cat}_{n-1} :*

$$\begin{array}{ccc} \Sigma_{n-1}^* & \xrightarrow{\pi} & \bar{\Sigma} \\ \tilde{F} \downarrow & \circlearrowleft & \downarrow F \\ \Sigma'_{n-1} & \xrightarrow{\pi'} & \bar{\Sigma}' \end{array}$$

Proof. For every k -cell u in Σ^* , with k in $\{0, \dots, n-2\}$, we take $\tilde{F}(u) = F(u)$. Since π and π' are identities on cells up to dimension $n-2$, we have the relation $F \circ \pi(u) = \pi' \circ \tilde{F}(u)$.

Now, let us consider an $(n-1)$ -cell u in Σ_{n-1} . One arbitrarily chooses an $(n-1)$ -cell of Σ'^* , hence of Σ'^\top , that is sent on $F \circ \pi(u)$ by π' , and one fixes $\tilde{F}(u)$ to that $(n-1)$ -cell. One extends \tilde{F} to every $(n-1)$ -cell of Σ^* thanks to the universal property of Σ^* .

Then, let f be an n -cell from u to v in Σ_n . Then $\pi(u) = \pi(v)$ holds by definition of π . Applying F on both members and using the property satisfied by \tilde{F} , one gets $\pi' \circ \tilde{F}(u) = \pi' \circ \tilde{F}(v)$. By definition of π' and of Σ'^\top , this means that there exists an n -cell from $\tilde{F}(u)$ to $\tilde{F}(v)$ in Σ'^\top . One takes one such n -cell for $\tilde{F}(f)$. Finally, one extends \tilde{F} to every n -cell of Σ^\top . \square

3.3. Polygraphs with finite derivation type

3.3.3. Lemma. Let Σ and Σ' be n -polygraphs and let $F : \Sigma^\top \rightarrow \Sigma'^\top$ be an n -functor. Given a set Γ of n -spheres of Σ^\top , we define $F(\Gamma)$ as the following set of n -spheres of Σ'^\top :

$$F(\Gamma) = \{ (F(g), F(g')) \mid (g, g') \in \Gamma \} .$$

Then, for every n -sphere (f, f') of Σ^\top such that $f \approx_\Gamma f'$ holds, we have $F(f) \approx_{F(\Gamma)} F(f')$.

Proof. It is an immediate consequence of functoriality of F . \square

3.3.4. Proposition. Let Σ and Σ' be Tietze-equivalent finite n -polygraphs. Then Σ has finite derivation type if and only if Σ' has.

Proof. Let us assume that Σ and Σ' are n -polygraphs which present the same $(n-1)$ -category, say \mathcal{C} . Let us assume that Σ has finite derivation type, so that we can fix a finite homotopy basis Γ of Σ^\top . Using Lemma 3.3.2 twice on the $(n-1)$ -functor $\text{Id}(\mathcal{C})$, we get two n -functors $F : \Sigma^\top \rightarrow \Sigma'^\top$ and $G : \Sigma'^\top \rightarrow \Sigma^\top$ such that the following diagrams commute in \mathbf{Cat}_{n-1} :

$$\begin{array}{ccc} \Sigma_{n-1}^* & \xrightarrow{\pi} & \mathcal{C} \\ F \downarrow & \circlearrowleft & \downarrow \text{Id}(\mathcal{C}) \\ \Sigma_{n-1}'^* & \xrightarrow{\pi'} & \mathcal{C} \end{array} \quad \begin{array}{ccc} \Sigma_{n-1}^* & \xrightarrow{\pi} & \mathcal{C} \\ G \uparrow & \circlearrowleft & \uparrow \text{Id}(\mathcal{C}) \\ \Sigma_{n-1}'^* & \xrightarrow{\pi'} & \mathcal{C} \end{array}$$

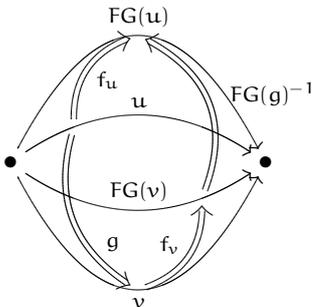
In particular, both π and π' are the identity on k -cells, for every $k < n-1$, hence so are F and G .

Let us consider an $(n-1)$ -cell a in Σ_{n-1}' . Then $\pi' \circ FG(a) = \pi \circ G(a) = \pi'(a)$. Thus, there exists an n -cell denoted by f_a from a to $FG(a)$ in Σ'^\top . From these cells, we define f_u for every $(n-1)$ -cell u in Σ'^* , hence of Σ'^\top , using the following relations:

- For every degenerate $(n-1)$ -cell u , f_u is defined as u .
- For every i -composable pair (u, v) of $(n-1)$ -cells, $f_{u \star_i v}$ is defined as $f_u \star_i f_v$.

We have that, for every $(n-1)$ -cell u , the n -cell f_u goes from u to $FG(u)$: to check this, we argue that FG is an n -functor which is the identity on degenerate $(n-1)$ -cells.

Now, let us consider an n -cell g from u to v in Σ'^\top . We denote by f_g the following n -cell from u to u in Σ'^\top , with a cellular representation giving the intuition for the case $n = 2$:

$$f(g) = g \star_{n-1} f_v \star_{n-1} FG(g)^{-1} \star_{n-1} f_u^{-1}$$


3. Higher-dimensional categories with finite derivation type

Let us prove that, for any composable pair (g, h) of n -cells in Σ'^{\top} , we have:

$$f_{g \star_{n-1} h} = g \star_{n-1} f_h \star_{n-1} g^{-1} \star_{n-1} f_g.$$

For that, we assume that g has source u and target v , while h has source v and target w . Then we compute:

$$\begin{aligned} & g \star_{n-1} f_h \star_{n-1} g^{-1} \star_{n-1} f_g \\ &= g \star_{n-1} \left(h \star_{n-1} f_w \star_{n-1} FG(h)^{-1} \star_{n-1} f_v^{-1} \right) \\ & \quad \star_{n-1} g^{-1} \star_{n-1} \left(g \star_{n-1} f_v \star_{n-1} FG(g)^{-1} \star_{n-1} f_u^{-1} \right) \\ &= g \star_{n-1} h \star_{n-1} f_w \star_{n-1} FG(h)^{-1} \star_{n-1} FG(g)^{-1} \star_{n-1} f_u^{-1} \\ &= (g \star_{n-1} h) \star_{n-1} f_w \star_{n-1} FG(g \star_{n-1} h)^{-1} \star_{n-1} f_u^{-1} \\ &= f_{g \star_{n-1} h}. \end{aligned}$$

Now, let us consider an n -cell g and a whisker $C[x]$ in Σ^{\top} such that $x = \partial(g_{n-1}^-)$. We note that, by definition of f_g , it has the same $(n-1)$ -source and $(n-1)$ -target as g , so that $C[f_g]$ is defined. Let us prove that the following relation holds:

$$f_{C[g]} = C[f_g].$$

From the decomposition of contexts, it is sufficient to prove that the following relation holds

$$f_{u \star_i g \star_i v} = u \star_i f_g \star_i v$$

for every n -cell g , every possible k -cells u and v , with $k < n-1$, and every $i < k$ such that $u \star_i g \star_i v$ is defined. Let us assume that g has source w and target w' and compute, from the left-hand side of this relation:

$$\begin{aligned} f_{u \star_i g \star_i v} &= (u \star_i g \star_i v) \star_{n-1} f_{u \star_i w' \star_i v} \star_{n-1} FG(u \star_i g \star_i v)^{-1} \star_{n-1} f_{u \star_i w \star_i v}^{-1} \\ &= (u \star_i g \star_i v) \star_{n-1} (u \star_i f_{w'} \star_i v) \star_{n-1} (u \star_i FG(g)^{-1} \star_i v) \star_{n-1} (u \star_i f_w^{-1} \star_i v) \\ &= u \star_i f_g \star_i v. \end{aligned}$$

Now, we denote by Γ' the set of n -spheres $(f_g, \text{Id}_{s(g)})$, for every n -cell g in Σ'_n . Then, it follows from the previous relations that, for every n -cell g in Σ'^{\top} , one has:

$$f_g \approx_{\Gamma} \text{Id}_{s(g)}.$$

Let us consider an n -sphere (g, g') in Σ'^{\top} . Then $(G(g), G(g'))$ is an n -sphere in Σ^{\top} . Since Γ is a homotopy basis for Σ^{\top} , we have $G(g) \approx_{\Gamma} G(g')$, so that, by Lemma 3.3.3, one gets $FG(g) \approx_{F(\Gamma)} FG(g')$.

Finally, let us denote Γ'' the set of n -spheres of Σ'^{\top} defined by $\Gamma'' = \Gamma' \cup F(\Gamma)$ and let us prove that Γ'' is a finite homotopy basis of Σ'^{\top} . Since both Σ'_n and Γ are finite, so is Γ'' . Let us consider an n -sphere (g, g') in Σ'^{\top} , with source w and target w' , and let us prove that $g \approx_{\Gamma''} g'$ holds. We start by using the definition of f_g to get:

$$g = f_g \star_{n-1} f_{w'}^{-1} \star_{n-1} FG(g) \star_{n-1} f_w^{-1}.$$

Using the definition of $f_{g'}$, one gets a similar formula for g' . We have seen that $f_g \approx_{\Gamma'} w$, $f_{g'} \approx_{\Gamma'} w'$ and $FG(g) \approx_{\Gamma''} FG(g')$ hold. Thus one gets $g \approx_{\Gamma''} g'$. \square

Proposition 3.3.4 shows that the property is invariant by Tietze-equivalence for finite polygraphs. We will illustrate in Example 4.3.10 that this is not the case for infinite ones. Thus we define

3.3.5. Definition. An n -category has *finite derivation type* when it admits a presentation by an $(n + 1)$ -polygraph with finite derivation type.

4. CRITICAL BRANCHINGS AND FINITE DERIVATION TYPE

4.1. Rewriting properties of polygraphs

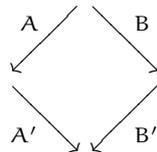
In this section we review abstract rewriting properties on polygraphs. Proof of claims in this section are intuitively the same as the ones for abstract rewriting systems, which can be found in [1]. We fix an $(n + 1)$ -polygraph Σ and an n -cell f in Σ^* .

4.1.1. Reductions and normal forms. One says that f *reduces* into some n -cell g when there exists a non-degenerate $(n + 1)$ -cell A from f to g in Σ^* . A *reduction sequence* is a family $(f_k)_k$ of n -cells such that each f_k reduces into f_{k+1} . One says that f is a *normal form* (for Σ_{n+1}) when every $(n + 1)$ -cell with source f is degenerate, *i.e.* it reduces only in itself. A *normal form for f* is a normal form g such that f reduces into g . The polygraph Σ is *normalizing at f* when f admits a normal form. It is *normalizing* when it is at every n -cell of Σ^* .

4.1.2. Termination. One says that Σ *terminates at f* when there exists no infinite reduction sequence starting at f is stationary. One says that Σ *terminates* when it does at every n -cell of Σ^* .

If Σ terminates at f , then it is normalizing at f , *i.e.* every n -cell has at least one normal form. Moreover, in case of termination, one can prove properties using a form of induction, usually called *Noetherian induction*: for that, one proves the property on normal forms; then one fixes an n -cell f , one assumes that the result holds for every g such that f reduces into g and one proves that, under those hypotheses, the n -cell f satisfies the property.

4.1.3. Confluence. A *branching of Σ* is a pair (A, B) of $(n + 1)$ -cells of Σ^* with same source; this n -cell is called the *source* of the branching (A, B) . A branching (A, B) is *local* when $\|A\| = \|B\| = 1$. A *confluence of Σ* is a pair (A, B) of $(n + 1)$ -cells of Σ^* with same target. A branching (A, B) is *confluent* when there exists a confluence (A', B') such that both $t_n(A) = s_n(A')$ and $t_n(B) = s_n(B')$ hold, as in the following diagram:



Such a pair (A', B') is called a *confluence for (A, B)* . Branchings and confluences are only considered up to symmetry, so that (A, B) and (B, A) are considered equal. The polygraph Σ is *(locally) confluent at f* when every (local) branching with source f is confluent. It is *(locally) confluent* when it is at every n -cell.

4. Critical branchings and finite derivation type

If Σ is confluent then every n -cell of Σ^* has at most one normal form. Thus, normalization and confluence imply that the n -cell f has exactly one normal form, written \widehat{f} . In a terminating polygraph, local confluence and confluence are equivalent: this was proved in the case of word rewriting systems (a subclass of 2-polygraphs) by Newman [17] and, since, the result is called Newman's lemma.

4.1.4. Convergence. The polygraph Σ is *convergent at* f when it terminates and it is confluent at f . It is *convergent* when it is at every n -cell.

If Σ is convergent at f , then f has exactly one normal form. Thanks to Newman's lemma, one gets convergence from termination and local confluence. If Σ is convergent, we have $f \approx_{\Sigma_{n+1}} g$ if and only if $\widehat{f} = \widehat{g}$: thus, a convergent $(n+1)$ -polygraph provides a decision procedure to the equivalence of n -cells in the n -category it presents.

4.1.5. Critical branchings in polygraphs. Given a branching $b = (A, B)$ of Σ with source f and a whisker $C[\partial f]$ of Σ^* , the pair $C[b] = (C[A], C[B])$ is a branching of Σ , with source $C[f]$. Furthermore, if b is local and C is a whisker, then $C[b]$ is also local. We define by \preceq the order relation on branchings of Σ given by $b \preceq b'$ when there exists a whisker C such that $C[b] = b'$ holds.

A branching is *minimal* when it is minimal for the order relation \preceq . A branching is *trivial* when it can be written either as (A, A) , for a $(n+1)$ -cell A , or as $(A \star_i s_n(B), s_n(A) \star_i B)$, for $(n+1)$ -cells A and B and a i in $\{0, \dots, n-1\}$. A branching is *critical* when it is minimal and not trivial.

A polygraph is locally confluent if and only if all of its critical branchings are confluent. To prove that, one checks that trivial branchings are always confluent and that a non-minimal branching is confluent if and only if the corresponding minimal branching is (there is a minimal one by a size argument, allowed by the fact that we work in a polygraph).

4.2. Using derivations for proving termination of a 3-polygraph

A method to prove termination of a 3-polygraph has been introduced in [9], see also [10, 11]; in special cases, it can also provide complexity bounds [4]. It turns out that the method uses interpretations that are a special case of derivations, as described here. Here we only give an outline of the proof.

4.2.1. Theorem. *Let Σ be a 3-polygraph. Let $X : \Sigma_2^* \rightarrow \mathbf{Ord}$ and $Y : (\Sigma^*)_2^{\text{co}} \rightarrow \mathbf{Ord}$ be 2-functors, let G be an abelian group in \mathbf{Ord} and let d be a derivation of Σ_2^* into $M(X, Y, G)$ such that the following conditions hold:*

- *For every 1-cell a in Σ_1 , the sets $X(a)$ and $Y(a)$ are not empty.*
- *For every 3-cell from f to g in Σ_3 , the inequalities $X(f) \geq X(g)$, $Y(f) \geq Y(g)$ and $d(f) > d(g)$ are satisfied.*
- *The addition of G is strictly monotone in both arguments.*
- *Every bounded decreasing sequence of elements of G is stationary.*
- *There exists an element z in G such that, for every 2-cell f in Σ_2^* , one has $d(f) \geq z$.*

Then the 3-polygraph Σ terminates.

Proof. Let us assume that $A : f \Rightarrow g$ is a 3-cell of Σ^* with size 1. Then there exists a 3-cell $\alpha : \varphi \Rightarrow \psi$ of Σ and a context C of Σ_2^* such that $A = C[\alpha]$ holds, *i.e.* such that $f = C[\varphi]$ and $g = C[\psi]$ hold. Thus, one gets:

$$d(f) = d(C)[\varphi] + C[d(\varphi)] \quad \text{and} \quad d(g) = d(C)[\psi] + C[d(\psi)].$$

We use the fact $d(\varphi) > d(\psi)$ holds by hypothesis to get $C[d(\varphi)] > C[d(\psi)]$. Moreover, since X and Y are 2-functors into **Ord** and since d sends every 2-cell to a monotone map, one gets $d(C)[\varphi] \geq d(C)[\psi]$. Finally, one uses the hypothesis on the strict monotony of addition in G to get $d(f) > d(g)$. Then one deduces that, for every non-degenerate 3-cell $A : f \Rightarrow g$, one has $d(f) > d(g)$. Thus, every infinite reduction sequence $(f_k)_k$ would produce an infinite, strictly decreasing sequence $(d(f_k))_k$ in G , the existence of which is prohibited by the last two hypotheses. \square

4.2.2. Special cases. The sequel contains several examples where derivations are used to prove termination. Other examples can be found in [10] or [4]. In some of those, we take at the trivial 2-functor for at least one of the 2-functors X and Y . Also, we often take G to be \mathbb{Z} , with the derivation satisfying $d(f) > 0$ for every f . One can check that those situations match the hypotheses of Theorem 4.2.1.

4.3. Branchings and homotopy bases

In the case of convergent word rewriting systems, *i.e.* convergent 2-polygraphs with exactly one 0-cell, the critical branchings generate a homotopy basis [19]. In this section, we generalise this result to any polygraph.

In particular, we recover Squier's theorem as Corollary 4.3.7, stating that a finite convergent 2-polygraph has finite derivation type. However, this result fails to generalise to higher-dimensional polygraphs, as stated in Theorem 4.3.9. Indeed, starting with $n = 3$, there exist finite convergent n -polygraphs with an infinite number of critical branchings. The detailed proof can be found in 5.5.

4.3.1. Notation. When Σ is a locally confluent $(n + 1)$ -polygraph, we assume that, for every critical branching $b = (A, B)$, a confluence (A', B') has been chosen. We denote by Γ_Σ the set of all the $(n + 1)$ -spheres $(A \star_n A', B \star_n B')$ of Σ , for each critical branching $b = (A, B)$.

4.3.2. Lemma. *Let Σ be a locally confluent $(n + 1)$ -polygraph. Then every local branching $b = (A, B)$ admits a confluence (A', B') such that $A \star_n A' \approx_{\Gamma_\Sigma} B \star_n B'$ holds.*

Proof. First, let us examine the case where b is a trivial branching. If $A = B$, then $(t_n(A), t_n(B))$ is a confluence that satisfies the required property. Otherwise, let us assume that there exist $(n + 1)$ -cells A_1 and B_1 in Σ^* and a i in $\{0, \dots, n - 2\}$ such that $A = A_1 \star_i s_n(B_1)$ and $B = s_n(A_1) \star_i B_1$ hold: then $(t_n(A_1) \star_i B_1, A_1 \star_i t_n(B_1))$ is a confluence that satisfies the required property.

Now, let us assume that b is not trivial. Let $b_1 = (A_1, B_1)$ be a minimal branching such that $b_1 \preceq b$, with a whisker C such that $b = C[b_1]$ holds. Since (A, B) is not trivial, then b_1 cannot be trivial, so that it is critical. Then we consider its fixed confluence (A', B') . Then $(C[A'], C[B'])$ is a confluence for (A, B) . Furthermore, one has

$$A \star_n C[A'] = C[A_1] \star_n C[A'] = C[A_1 \star_n A']$$

4. Critical branchings and finite derivation type

and, similarly, $B \star_n C[B'] = C[B_1 \star_n B']$. Since C is a whisker and since, by definition of C , one has $A_1 \star_n A' \approx_{\Gamma_\Sigma} B_1 \star_n B'$, one gets that $(C[A'], C[B'])$ satisfies the required property. \square

4.3.3. Lemma. *Let Σ be a convergent $(n + 1)$ -polygraph and let (A, B) be a branching of Σ such that both $t_n(A)$ and $t_n(B)$ are in normal form. Then one has $t_n(A) = t_n(B)$ and $A \approx_{\Gamma_\Sigma} B$.*

Proof. Since Σ is terminating, we can prove the result by induction on the source of the branching.

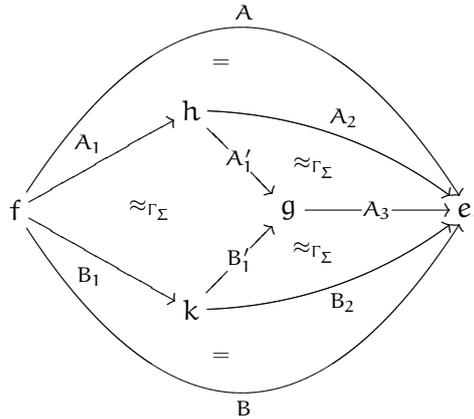
First, if this source f is a normal form, then by definition of normal form, both A and B must be identities. Hence $t_n(A)$ and $t_n(B)$ are equal, and so do A and B . Thus $A \approx_{\Gamma_\Sigma} B$ holds.

Now, we fix a n -cell f , which is not a normal form. We assume that the result holds for every branching (A, B) such that the targets of A and B are normal forms and such that there exists a non trivial $(n + 1)$ -cell from f to their source. Let (A, B) be a branching with source f and such that the targets of A and B are normal forms. Since f is not a normal form, A and B cannot be identities, hence one can decompose them into $A = A_1 \star_n A_2$ and $B = B_1 \star_n B_2$ with A_1 and B_1 being $(n + 1)$ -cells of size 1.

The pair (A_1, B_1) is a local branching. Thus, using Lemma 4.3.2, one gets a confluence (A'_1, B'_1) for (A_1, B_1) such that $A_1 \star_n A'_1 \approx_{\Gamma_\Sigma} B_1 \star_n B'_1$ holds. Let us denote by g the common target of A'_1 and B'_1 , by e its normal form and by A_3 a n -cell from g to e .

Then we consider the branching $(A_2, A'_1 \star_n A_3)$, whose source is denoted by h . The targets of A_2 and $A'_1 \star_n A_3$ are normal forms and A_1 is a non trivial $(n + 1)$ -cell from f to h : thus, the induction hypothesis can be applied to this branching, yielding that A_2 has target e and that $A_2 \approx_{\Gamma_\Sigma} A'_1 \star_n A_3$ holds.

We proceed similarly to prove that B_2 satisfies the same properties, so that one gets that A and B have the same target and that $A \approx_{\Gamma_\Sigma} B$ holds. The constructions we have done are summarized in the following diagram:

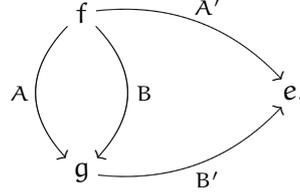


\square

4.3.4. Proposition. *Let Σ be a convergent $(n + 1)$ -polygraph. Then Γ_Σ is a homotopy basis for Σ^\top .*

Proof. Let (A, B) be an $(n + 1)$ -sphere in Σ^\top , with source and target denoted by f and g , respectively. Since Σ is convergent, these n -cells share the same normal form, which we denote by e . Let A' and B'

be $(n + 1)$ -cells in Σ^* , from f to e and from g to e respectively:



We apply the previous result to the branching $(A \star_n B', A')$, yielding $A \star_n B' \approx_{\Gamma_\Sigma} A'$ and, hence, $A \approx_{\Gamma_\Sigma} A' \star_n B'^{-1}$. We proceed similarly with the branching $(B \star_n B', A')$ to get $B \approx_{\Gamma_\Sigma} A' \star_n B'^{-1}$ and, thus, $A \approx_{\Gamma_\Sigma} B$. \square

4.3.5. Proposition. *A finite convergent polygraph with a finite set of critical branchings has finite derivation type.*

Proof. If Σ has a finite set of critical branchings, then the set $\Gamma(\Sigma)$ is finite. \square

4.3.6. Corollary. *A terminating polygraph with no critical branching has finite derivation type.*

4.3.7. Corollary ([19]). *A finite convergent 2-polygraph has finite derivation type.*

Proof. If Σ is a finite convergent 2-polygraph with one 0-cell, *i.e.* a word rewriting system, then its set of critical branchings is finite. Indeed, it is equal to the number of possible overlaps between the words corresponding to the sources of 2-cells: there are finitely many 2-cells and finitely many letters in each word. If Σ has more than one 0-cell, then the number of possible overlaps is bounded by the number of overlaps in Σ' , built from Σ by identification of all its 0-cells. \square

From this result Squier has proved that, if a finitely presented monoid admits a presentation by a finite convergent word rewriting system, then it has finite derivation type, [19]. Now we prove that this result is false for n -categories when $n \geq 2$.

4.3.8. Proposition. *For every natural number n with $n \geq 3$, there exists a finite convergent n -polygraph without finite derivation type.*

Proof. We consider the 3-polygraph Σ with one 0-cell, one 1-cell, three 2-cells \bullet , \cap , \cup and the following four 3-cells:

$$\bullet \cap \bullet \equiv \cap \bullet, \quad \bullet \cup \bullet \equiv \cup \bullet, \quad \cap \cup \equiv |, \quad \cup \cap \equiv |.$$

The 3-polygraph Σ is finite and convergent. However, the first and second 3-cells create an infinite number of critical branchings whose confluence diagrams cannot be presented by a finite homotopy basis. These facts are proved in 5.5.

Then we apply suspension functors on Σ to get an n -polygraph, for some $n \geq 3$. It has exactly the same cells and compositions in dimensions $n - 3$, $n - 2$, $n - 1$ and n as Σ in dimensions 0, 1, 2 and 3; on top of that, it has two cells in each dimension up to $n - 4$ and no other possible compositions, except with degenerate cells. Thus, we conclude that the n -polygraph we have built is finite and convergent, yet it still fails to have finite derivation type. \square

4. Critical branchings and finite derivation type

4.3.9. Theorem. *For every natural number n with $n \geq 2$, there exists a n -category which does not have finite derivation type and admits a presentation by a finite convergent $(n + 1)$ -polygraph.*

Proof. Let $n \geq 2$, by Proposition 4.3.8, there exists a finite convergent $(n + 1)$ -polygraph Σ without finite derivation type. Let denote by \mathcal{C} the n -category presented by the polygraph Σ . By proposition 3.3.4 any finite $(n + 1)$ -polygraph presenting \mathcal{C} does not have finite derivation type. This proves that \mathcal{C} does not have finite derivation type. \square

We end this section by an example which illustrates that the property of finite derivation type is not Tietze-invariant for infinite polygraphs.

4.3.10. Example. Let \mathcal{C} be the 2-category presented by the 3-polygraph Σ with one 0-cell, one 1-cell, three 2-cells $\square, \blacklozenge, \circ$ and the following two 3-cells:

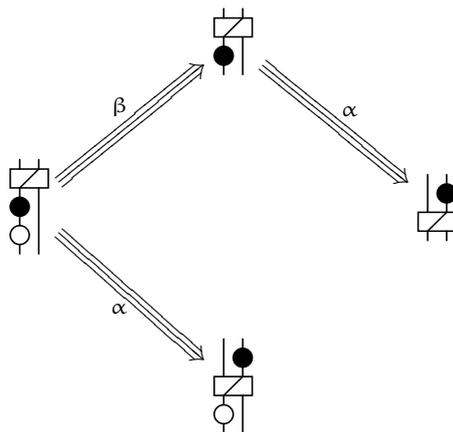
$$\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \xRightarrow{\alpha} \begin{array}{c} \square \\ | \\ \bullet \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ | \\ \circ \end{array} \xRightarrow{\beta} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}.$$

The polygraph Σ terminates and does not have critical branching. By Corollary 4.3.6 it follows that Σ has finite derivation type and, thus, so does \mathcal{C} .

Now let us consider another presentation of the 2-category \mathcal{C} , namely the 3-polygraph Ξ defined the same way as Σ except for the 3-cells:

$$\begin{array}{c} \square \\ | \\ \bullet \end{array} \xRightarrow{\alpha} \begin{array}{c} \bullet \\ | \\ \square \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ | \\ \circ \end{array} \xRightarrow{\beta} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}.$$

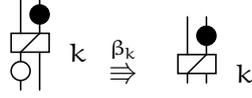
The polygraph Ξ still terminates, but it has the following non-confluent critical branching:



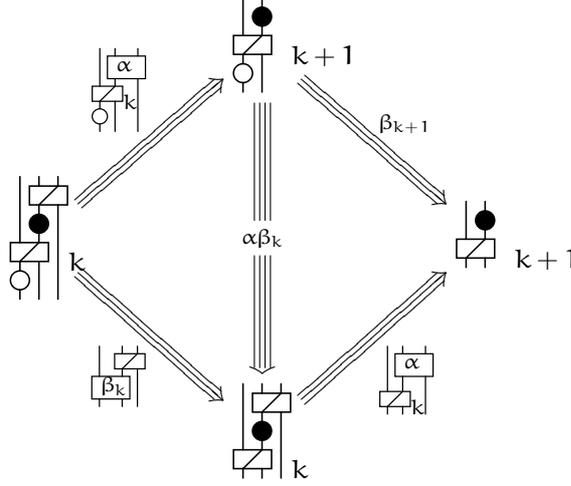
We define, by induction on the natural number $k \geq 1$, the 2-cell \square_k as follows:

$$\square_1 = \square \quad \text{and} \quad \square_{k+1} = \begin{array}{c} \square \\ | \\ \square_k \end{array}.$$

Then, we complete the polygraph Ξ into an infinite convergent polygraph $\Xi_\infty = \Xi \cup \{\beta_k, k \geq 1\}$, where β_k is the 3-cell:



We define β_0 as β . For any natural number k , we have the confluence diagrams:



By Proposition 4.3.4, the set $\Gamma = \{\alpha\beta_k \mid k \in \mathbb{N}\}$ form a homotopy basis of Ξ_∞^\top .

Let us prove that the polygraph Ξ_∞ does not have finite derivation type. On the contrary, suppose that Ξ_∞ does have finite derivation type. Then, following Proposition 3.2.3, there is a finite subset Γ_0 of Γ which is a homotopy base of Ξ_∞^\top . Subset Γ_0 being finite, there is a natural number $l > \max\{k \mid \alpha\beta_k \in \Gamma_0\}$ such that the 4-cell $\alpha\beta_l$ is not in Γ_0 . However, since Γ_0 is a homotopy base we have

$$s(\alpha\beta_l) \approx_{\Gamma_0} t(\alpha\beta_l).$$

That is there is a 4-cell Φ in $\Xi_\infty^\top(\Gamma_0)$ such that $s\Phi = s(\alpha\beta_l)$ and $t\Phi = t(\alpha\beta_l)$ hold. Thus there exist 4-cells $\Phi_1, \Phi_2 \in \Xi_\infty^\top(\Gamma_0)$ and a context C of Ξ^\top such that

$$\Phi = \Phi_1 \star_3 C[\Psi^\varepsilon] \star_3 \Phi_2,$$

where $\Psi \in \Gamma_0$ and $\varepsilon \in \{-1, 1\}$. We have

$$s_2\Phi = s_2(C[\Psi^\varepsilon]) = s_2(C)[s_2(\Psi)].$$

Since $\Psi \in \Gamma_0$, there exist $k < l$ such that $\Psi = \alpha\beta_k$. Hence $s_2\Phi = s_2(C)[\alpha\beta_k]$. In an other hand, we have $s_2\Phi = s_2(\alpha\beta_l)$. This means that there are 2-cells f_1, f_2, g_1, g_2 such that

$$f_2 \star_1 (f_1 \star_0 \left[\begin{array}{c} \square \\ \bullet \\ \square \\ \circ \end{array} \right]_k \star_0 g_1) \star_1 g_2 = \left[\begin{array}{c} \square \\ \bullet \\ \square \\ \circ \end{array} \right]_l,$$

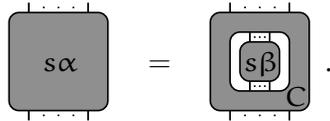
holds in Ξ_∞^* , which is impossible with $k < l$. This proves that Ξ_∞ does not have finite derivation type.

5. THE CASE OF 3-POLYGRAPHS

5.1. Classification of critical branchings

5.1.1. Definitions. Let Σ be a 3-polygraph and let (A, B) be a critical branching of Σ . Let us denote by α and β the 3-cells of Σ that generate A and B . Then (A, B) falls in one of three cases.

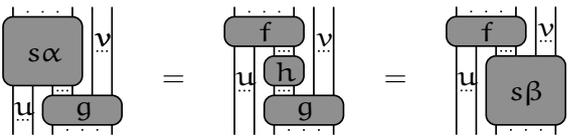
The first possibility is that there exists a context C of Σ_2^* such that $s\alpha = C[s\beta]$ holds. Then, the source of the branching (A, B) is:



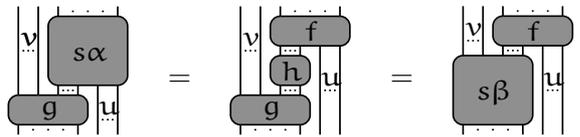
In that case, (A, B) is an *inclusion* critical branching.

If the branching (A, B) is not an inclusion one, the second possibility is that there exist 1-cells u, v and 2-cells f, g, h such that $s\alpha$ and $s\beta$ decompose in one of the following ways.

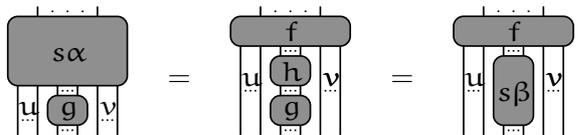
- One has $s\alpha = f \star_1 (u \star_0 h)$ and $s\beta = (h \star_0 v) \star_1 g$, so that the source of (A, B) is:



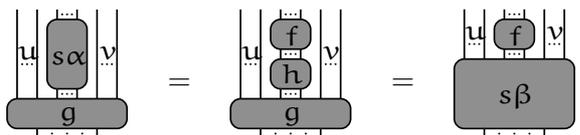
- One has $s\alpha = f \star_1 (h \star_0 u)$ and $s\beta = (v \star_0 h) \star_1 g$:



- One has $s\alpha = f \star_1 (u \star_0 h \star_0 v)$ and $s\beta = h \star_1 g$:



- One has $s\alpha = f \star_1 h$ and $s\beta = (u \star_0 h \star_0 v) \star_1 g$:

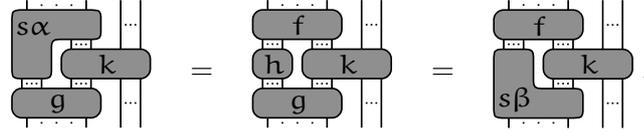


If (A, B) matches one of these cases, then it is called a *regular* critical branching.

Finally, when the branching (A, B) is not an inclusion or regular one, there exist 1-cells u, v and 2-cells f, g, h such $s\alpha$ and $s\beta$ decompose in one of the following ways.

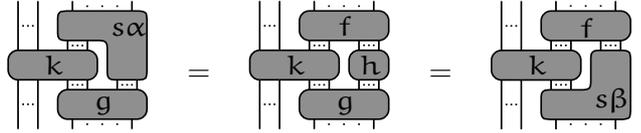
5.1. Classification of critical branchings

- One has $s\alpha = f \star_1 (h \star_0 u)$ and $s\beta = (h \star_0 v) \star_1 g$, so that there exists a 2-cell k such that the source of (A, B) is:



In that case, one can write $(A, B) = (C[k], D[k])$ for appropriate contexts C and D of Σ^* . The family $(C[k], D[k])_k$, where k ranges over the 2-cells with appropriate boundary and such that $(C[k], D[k])$ is a minimal branching, is called a *right-indexed* critical branching.

- One has $s\alpha = f \star_1 (u \star_0 h)$ and $s\beta = (v \star_0 h) \star_1 g$, so that there exists a 2-cell k such that the source of (A, B) is:



In that case, one can write $(A, B) = (C[k], D[k])$ for appropriate contexts C and D of Σ^* . The family $(C[k], D[k])_k$, where k ranges over the 2-cells with appropriate boundary and such that $(C[k], D[k])$ is a minimal branching, is called a *left-indexed* critical branching.

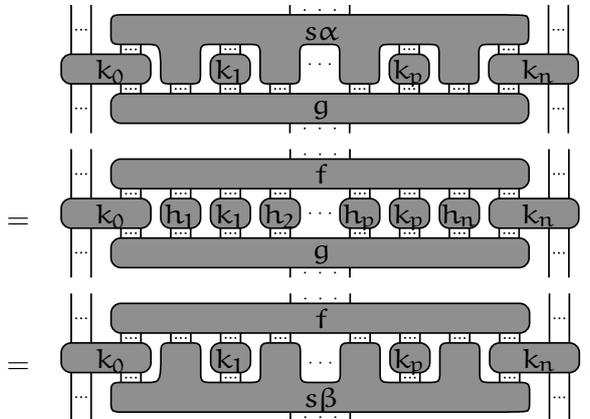
- One is not in the right-indexed or left-indexed cases and one has

$$s\alpha = f \star_1 (u_0 \star_0 h_1 \star_0 u_1 \star_0 h_2 \star_0 \cdots \star_0 u_{n-1} \star_0 h_n \star_0 u_n)$$

and

$$s\beta = (v_0 \star_0 h_1 \star_0 v_1 \star_0 h_2 \star_0 \cdots \star_0 v_{n-1} \star_0 h_n \star_0 v_n) \star_1 g,$$

so that there exist 2-cells k_0, \dots, k_n such that the source of (A, B) is as follows, where we write p instead of $n - 1$ for size reasons:



5. The case of 3-polygraphs

In that case, one can write $(A, B) = (C[k_0, \dots, k_n], D[k_0, \dots, k_n])$ for appropriate 3-cells C and D in some $\Sigma^*[x_0, \dots, x_n]$. The family $(C[k_0, \dots, k_n], D[k_0, \dots, k_n])_{k_0, \dots, k_n}$, where the k_i 's range over the 2-cells with appropriate boundary and such that $(C[k_0, \dots, k_n], D[k_0, \dots, k_n])$ is a minimal branching, is called a *multi-indexed critical branching*.

In all those cases, the branching (A, B) is said to be an *instance* of the corresponding right-indexed or left-indexed or multi-indexed one. It is a *normal instance* when the indexing 2-cell k (resp. 2-cells k_0, \dots, k_n) is a normal form (resp. are normal forms).

A 3-polygraph is *non-indexed* when each of its critical branchings is an inclusion one or a regular one. It is *right-indexed* (resp. *left-indexed*) when each of its critical branchings is either an inclusion one, a regular one or an instance of a right-indexed (resp. left-indexed) one.

5.1.2. Proposition. *A 3-polygraph with a finite set of 3-cells has a finite number of inclusion and regular critical branchings.*

Proof. Let Σ be a polygraph with $\Sigma_3 = \{\alpha_1, \dots, \alpha_p\}$ finite. Given $i \in \{1, \dots, p\}$, Σ_3 being finite, the set J of j such that there is a morphism $s(\alpha_j) \rightarrow s(\alpha_i)$ in $\mathbf{W}\Sigma$ is finite. Moreover, for a $j \in J$, the set of morphisms of $\mathbf{W}\Sigma$ from $s(\alpha_j)$ to $s(\alpha_i)$ is finite. This proves that there is a finite set of inclusion critical branchings.

Now, let us prove that Σ has a finite number of regular critical branchings. Let us fix some i and j in $\{1, \dots, p\}$ and let us assume that there exist two whiskers C and D of Σ^* such that $(C[\alpha_i], D[\alpha_j])$ is a regular branching, with source f . Then there exist a 2-cell h and whiskers C' and D' of Σ^* such that $C[s(\alpha)_i] = C'[h] = D'[h] = D[s(\alpha)_j]$ holds. But the sets $\mathbf{W}\Sigma(s(\alpha_i), f)$, $\mathbf{W}\Sigma(s(\alpha_j), f)$, $\mathbf{W}\Sigma(h, C[s(\alpha)_i])$ and $\mathbf{W}\Sigma(h, D[s(\alpha)_j])$ are finite. Hence there exist finitely regular branchings of this form, with i and j fixed. Since Σ_3 is finite, the 3-polygraph Σ has finitely many regular branchings. \square

5.1.3. Theorem. *A finite convergent non-indexed 3-polygraph has finite derivation type.*

Proof. We use the previous result and, then, we apply Proposition 4.3.5. \square

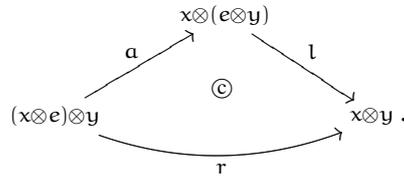
5.2. Mac Lane's coherence theorem revisited

5.2.1. Monoidal categories. A *monoidal category* is a data $(\mathcal{C}, \otimes, e, \alpha, l, r)$ made of a category \mathcal{C} , a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an object e of \mathcal{C} and three natural isomorphisms

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z), \quad l_x : e \otimes x \rightarrow x, \quad r_x : x \otimes e \rightarrow x,$$

such that $l_e = r_e$ holds and such that the following two diagrams commute in \mathcal{C} :

$$\begin{array}{ccccc}
 & & (x \otimes (y \otimes z)) \otimes t & \xrightarrow{\alpha} & x \otimes ((y \otimes z) \otimes t) \\
 & \nearrow \alpha & & & \searrow \alpha \\
 ((x \otimes y) \otimes z) \otimes t & & & \text{\textcircled{C}} & & x \otimes (y \otimes (z \otimes t)) \\
 & \searrow \alpha & & & \nearrow \alpha \\
 & & (x \otimes y) \otimes (z \otimes t) & &
 \end{array}$$

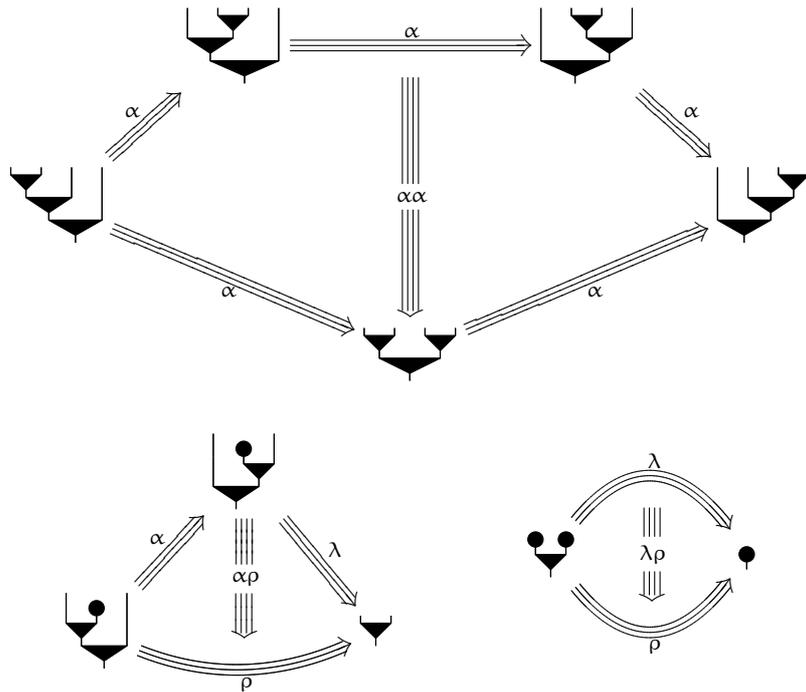


Mac Lane's coherence theorem [15] states that, in a such monoidal category, all the diagrams whose arrows are built from \otimes , e , l and r commute. Thereafter, we give a proof of this fact by building a homotopy basis of a 3-polygraph.

5.2.2. The 3-polygraph of monoids. We denote by Σ_{Mon} the 3-polygraph with one 0-cell, one 1-cell, two 2-cells ∇ and \bullet and the following three 3-cells:

$$\nabla \rightrightarrows_{\alpha} \nabla, \quad \bullet \rightrightarrows_{\lambda} \bullet, \quad \nabla \bullet \rightrightarrows_{\rho} \bullet.$$

We denote by Γ the set made of the following 4-cells $\alpha\alpha$, $\alpha\rho$ and $\lambda\rho$, where we commit the abuse of denoting a 3-cell of Σ^* with size 1 like its generating 3-cell:



5.2.3. Theorem. The set Γ of 4-cells form a homotopy basis of the track 3-category $\Sigma_{\text{Mon}}^{\top}$.

Proof. Let us prove that Σ_{Mon} terminates by using a derivation. We consider the Σ_2^* -module $M(X, *, \mathbb{Z})$ generated by the following values:

$$X(|) = \mathbb{N} \setminus \{0\}, \quad X(\nabla)(i, j) = i + j, \quad X(\bullet) = 1.$$

5. The case of 3-polygraphs

The functor X satisfies the following equalities:

$$X \left(\begin{array}{c} \blacktriangledown \\ \blacktriangledown \\ \blacktriangledown \end{array} \right) (i, j, k) = i + j + k = X \left(\begin{array}{c} \blacktriangledown \\ \blacktriangledown \end{array} \right) ,$$

$$X \left(\begin{array}{c} \bullet \\ \blacktriangledown \\ \blacktriangledown \end{array} \right) (i) = i = X \left(\begin{array}{c} | \\ | \end{array} \right) (i) \quad \text{and} \quad X \left(\begin{array}{c} \blacktriangledown \\ \bullet \\ \blacktriangledown \end{array} \right) (i) = i = X \left(\begin{array}{c} | \\ | \end{array} \right) (i) .$$

Now, let d be the derivation of Σ_2^* into $M(X, *, \mathbb{Z})$ generated by the following values:

$$d \blacktriangledown (i, j) = i \quad \text{and} \quad d \bullet = 0 .$$

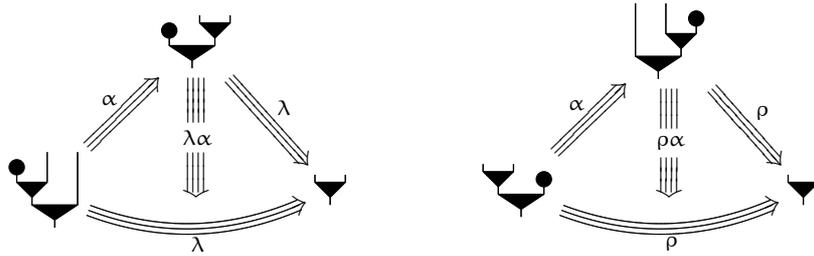
The derivation d satisfies the following strict inequalities:

$$d \left(\begin{array}{c} \blacktriangledown \\ \blacktriangledown \\ \blacktriangledown \end{array} \right) (i, j, k) = 2i + j > i + j = d \left(\begin{array}{c} \blacktriangledown \\ \blacktriangledown \end{array} \right) (i, j, k) ,$$

$$d \left(\begin{array}{c} \bullet \\ \blacktriangledown \\ \blacktriangledown \end{array} \right) (i) = 1 > 0 = d \left(\begin{array}{c} | \\ | \end{array} \right) (i) \quad \text{and} \quad d \left(\begin{array}{c} \blacktriangledown \\ \bullet \\ \blacktriangledown \end{array} \right) (i) = i > 0 = d \left(\begin{array}{c} | \\ | \end{array} \right) (i) .$$

We apply Theorem 4.2.1 to conclude.

The 3-polygraph has five critical branchings, all of the regular type, and all of them are confluent. Their confluence diagrams are given by the boundaries of the 4-cells of Γ , plus the following two ones:

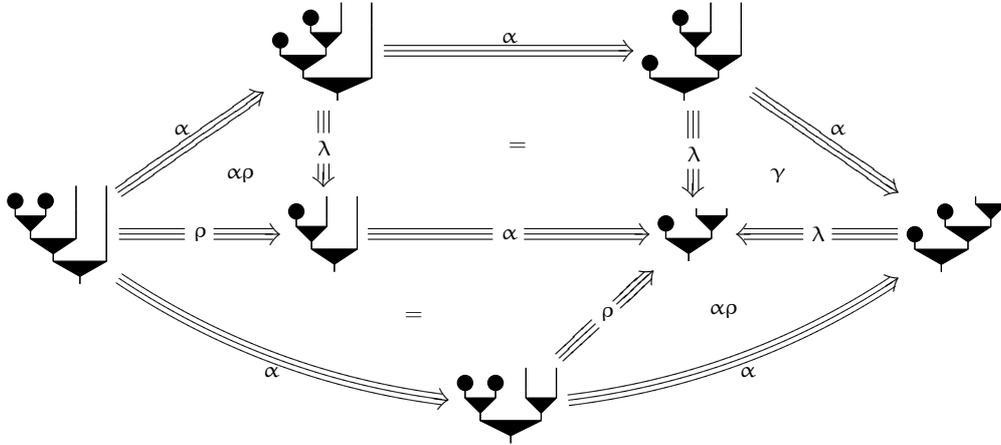


Since Σ terminates and has all its critical branchings confluent, it is convergent as a consequence of Newman's lemma. Thus we know that the set $\{\alpha\alpha, \lambda\rho, \lambda\alpha, \alpha\rho, \rho\alpha\}$ of 4-cells is a (finite) homotopy basis of Σ^\top .

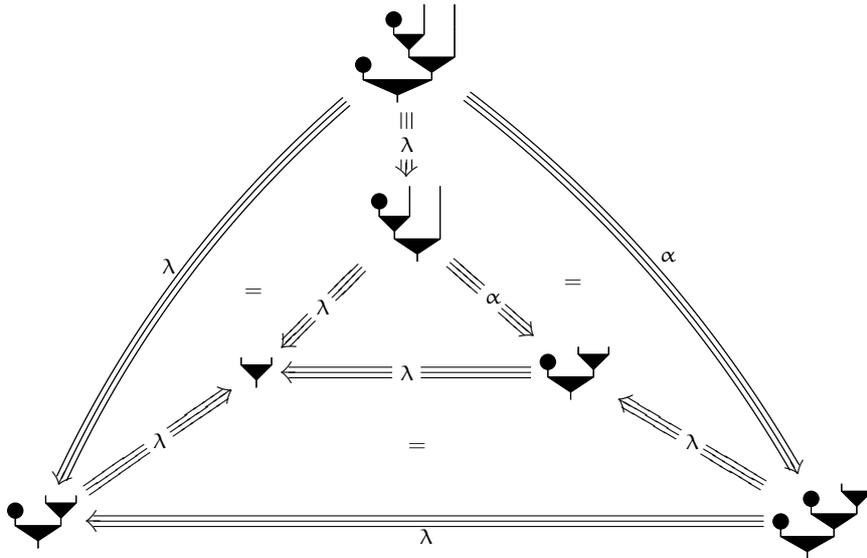
To get the result, we check that $\lambda\alpha$ and $\rho\alpha$ are superfluous in this homotopy basis, since their boundaries are also the boundaries of 4-cells of $\Sigma^\top(\Gamma)$. Let us detail the proof for $\lambda\alpha$.

We consider the 4-cell $(\bullet \bullet | |) \star_1 \alpha\alpha$ of $\Sigma^\top(\Gamma)$. We partially fill its boundary with other 4-cells of

$\Sigma^\top(\Gamma)$ and equalities, yielding a 3-sphere of Σ^\top denoted by γ :



As a consequence of this construction, we have $s\gamma \approx_\Gamma t\gamma$. Then we build the following diagram, proving that $s(\lambda\alpha) \approx_\Gamma t(\lambda\alpha)$ also holds:



For the 4-cell $\rho\alpha$, one proceeds in a similar way, starting with the 4-cell $(\text{||} \bullet \bullet) \star_1 \alpha\alpha$. □

5.2.4. Corollary (Mac Lane's coherence theorem [15]). *In a monoidal category $(\mathcal{C}, \otimes, e, \alpha, \iota, r)$, all the diagrams whose arrows are built from $\otimes, e, \alpha, \iota$ and r are commutative.*

Proof. We see \mathbf{Cat}_1 as a (large) 3-category with one 0-cell, categories as 1-cells, functors as 2-cells and natural transformations as 3-cells. The 0-composition is the cartesian product of categories, the 1-composition is the composition of functors and the 2-composition is the "vertical" composition of natural transformations.

5. The case of 3-polygraphs

Then monoidal categories are exactly the 3-functors from Σ^\top to \mathbf{Cat}_1 such that the source and target of every 4-cell of Γ are identified. The correspondence between a monoidal category $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho)$ and such a 3-functor M is given by:

$$M(\cdot) = \mathcal{C}, \quad M(\blacktriangledown) = \otimes, \quad M(\bullet) = e, \quad M(\alpha) = \alpha, \quad M(\lambda) = \lambda, \quad M(\rho) = \rho.$$

As a consequence, a diagram D in \mathcal{C} whose arrows are built from $\otimes, e, \alpha, \lambda$ and ρ is the image by M of a 3-sphere γ of Σ^\top . Since Γ is a homotopy basis of Σ^\top , we have $s\gamma \approx_\Gamma t\gamma$. Since M is a 3-functor and since it identifies the source and the target of every 3-sphere of Γ , it does the same with the ones of γ , *i.e.* we have $M(s\gamma) = M(t\gamma)$, which means that the diagram D commutes. \square

5.3. Right-indexed and left-indexed polygraphs

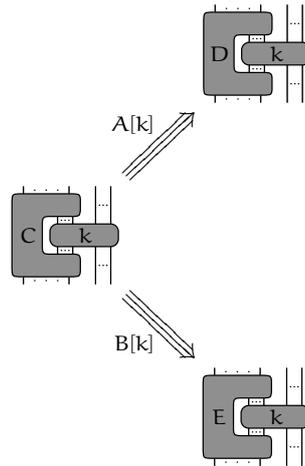
5.3.1. Proposition. *Let Σ be a terminating right-indexed (resp. left-indexed) 3-polygraph. Then Σ is confluent if and only if the following two conditions are satisfied:*

1. *Each of its inclusion and regular critical branchings is confluent.*
2. *Each normal instance of each of its right-indexed (resp. left-indexed) critical branchings is confluent.*

Proof. If Σ is confluent then, by definition, all of its branchings are confluent: in particular, its inclusion and regular critical branchings and the normal instances of its right-indexed or left-indexed ones.

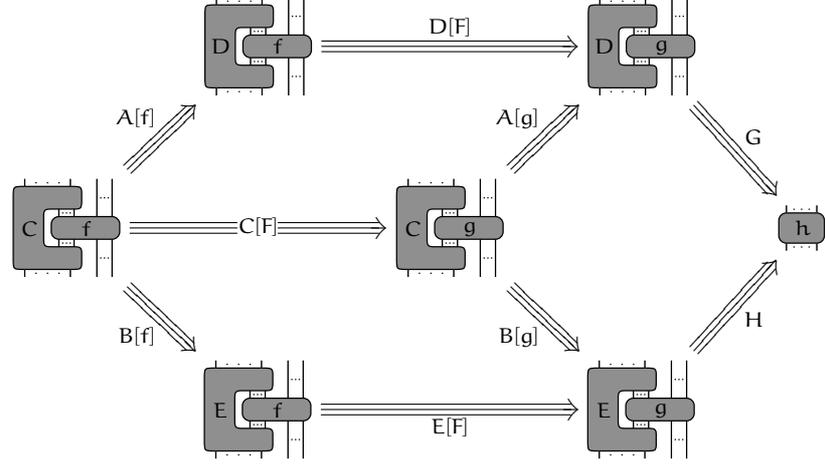
Conversely, let us assume that Σ has all of its inclusion and regular critical branchings and all of the normal instances of its right-indexed (resp. left-indexed) critical branchings that are confluent. Since Σ is terminating, we know that it is sufficient to prove that it is locally confluent. Moreover, since Σ is right-indexed (resp. left-indexed), there remain to prove that every non normal instance of the right-indexed (resp. left-indexed) critical branching is confluent.

From now on, we assume that Σ is right-indexed, the proof in the left-indexed case being similar. Let us consider a right-indexed critical branching $(A[k], B[k])_k$, which has the following shape by definition:



Now, let f be a 2-cell such that $(A[f], B[f])$ is a non normal instance of $(A[k], B[k])_k$. Since Σ is terminating, f admits a normal form, say g . We denote by F a 3-cell from f to g . Since g is in normal form,

the branching $(A[g], B[g])$ is a normal instance of $(A[k], B[k])_k$ so that, by hypothesis, it is confluent: let us denote by (G, H) a confluence for this branching, with target h . With all those ingredients, one builds the following confluence diagram for the critical branching $(A[f], B[f])$, thus concluding the proof:



□

5.3.2. Homotopy bases of indexed 3-polygraphs. Let Σ be a locally confluent and right-indexed (resp. left-indexed) 3-polygraph. We assume that a confluence has been chosen for:

- Each inclusion and regular critical branching.
- Each normal instance of each right-indexed (resp. left-indexed) critical branching.

We denote by Γ_Σ the collection of the 2-spheres of Σ^* corresponding to these confluence diagrams.

5.3.3. Proposition. *Let Σ be a convergent right-indexed (resp. left-indexed) 3-polygraph. Then Γ_Σ is a homotopy basis of Σ^\top .*

Proof. The proof follows the same scheme than the results of 4.3, where it was proved that the family of 3-spheres associated to the confluence diagrams of all the critical branchings was a homotopy basis.

First, we prove that every local branching of (A, B) of Σ admits a confluence (A', B') such that $A \star_2 A' \approx_{\Gamma_\Sigma} B \star_2 B'$ holds. The proof is the same as in 4.3 when (A, B) is a trivial or when it is generated by an inclusion or a regular critical branching.

There remains to check the cases of local branchings of the shape $C(A[f], B[f])$, where $(A[k], B[k])_k$ is a right-indexed (resp. left-indexed) critical branching and where C is a context. For that, we proceed by Noetherian induction on the indexing 2-cell f , thanks to the termination of Σ .

When f is a normal form, then $(A[f], B[f])$ is a normal instance of the branching $(A[k], B[k])_k$. To build Γ_Σ we have fixed a confluence for this branching, say (A', B') . Then we have:

$$C[A[f]] \star_2 A' \approx_{\Gamma_\Sigma} C[B[f]] \star_2 B'.$$

5. The case of 3-polygraphs

Let us assume that f is a non normal 2-cell such that $(A[f], B[f])$ is an instance of the branching $(A[k], B[k])_k$. Moreover, we assume that, for every 2-cell g such that f reduces into g and $(A[g], B[g])$ is an instance of $(A[k], B[k])_k$, there exists a confluence (A', B') for $(A[g], B[g])$ such that $A[g] \star_2 A' \approx_{\Gamma_\Sigma} B[g] \star_2 B'$ holds.

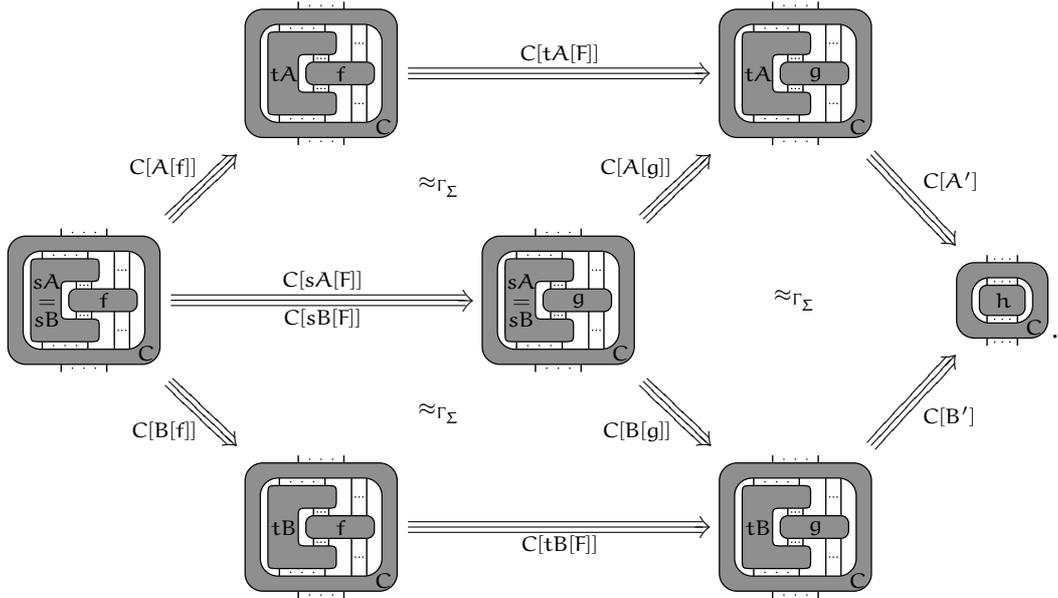
Since f is not in normal form, we can choose a 2-cell g such that f reduces into g , through a 3-cell F . Since f and g have the same boundary, we have an instance $(A[g], B[g])$ of the branching $(A[k], B[k])_k$. We apply the induction hypothesis to g to get a confluence (A', B') , with target denoted by h , such that $A[g] \star_2 A' \approx_{\Gamma_\Sigma} B[g] \star_2 B'$ holds. Moreover, the branchings $(C[A[f]], C[sA[F]])$ and $(C[B[f]], C[sB[F]])$ are trivial branchings, yielding:

$$C[A[f]] \star_2 C[tA[F]] \approx_{\Gamma_\Sigma} C[sA[F]] \star_2 C[A[g]]$$

and

$$C[B[f]] \star_2 C[tB[F]] \approx_{\Gamma_\Sigma} C[sB[F]] \star_2 C[B[g]].$$

With these constructions, we build the following diagram, where we have assumed that the considered branching was right-indexed – the case of a left-indexed critical branching is similar:



Hence, composing the three 4-cells of $\Sigma^\top(\Gamma_\Sigma)$ of that diagram, one proves that the confluence $(C[tA[F]] \star_2 C[A'], C[tB[F]] \star_2 C[B'])$ satisfies the required

$$C[A[f]] \star_2 C[tA[F]] \star_2 C[A'] \approx_{\Gamma_\Sigma} C[B[f]] \star_2 C[tB[F]] \star_2 C[B'],$$

thus concluding the first part of the proof.

The remainder of the proof is exactly the same as in 4.3, with the following two steps:

- For every branching (A, B) such that tA and tB are in normal form, one has $tA = tB$ and $A \approx_{\Gamma_\Sigma} B$. For that, we use the first part of the proof and we proceed by noetherian induction on the source of the branching (A, B) .

- We consider two parallel 3-cells A and B in Σ^T and prove that $A \approx_{\Gamma_\Sigma} B$ holds.

□

5.3.4. Theorem. *A finite convergent 3-polygraph with finitely many normal instances of indexed critical branchings has finite derivation type.*

5.4. The 3-polygraph of permutations

Here we see an example of a 3-polygraph that is finite, convergent, right-indexed and, thus, with an infinite number of critical branchings, yet with finite derivation type. Another proof for termination and the ideas for proving confluence we use here can be found in [13].

5.4.1. Definition. The 3-polygraph Σ_{Perm} has one 0-cell $*$, one 1-cell $|$, one 2-cell $\begin{array}{c} \diagup \\ \diagdown \end{array}$, and the following two 3-cells:

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \xRightarrow{\alpha} || \quad \text{and} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \xRightarrow{\beta} \begin{array}{c} \diagup \\ \diagdown \end{array}.$$

5.4.2. Termination. We consider the $(\Sigma_{\text{Perm}})_2^*$ -module $M(X, *, \mathbb{Z})$ with:

$$X(|) = \mathbb{N} \quad \text{and} \quad X\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)(i, j) = (j + 1, i).$$

Then we consider the derivation d of $(\Sigma_{\text{Perm}})_2^*$ into $M(X, *, \mathbb{Z})$ given by:

$$d\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)(i, j) = i.$$

The 2-functors X, Y and the derivation d satisfy the conditions of Theorem 4.2.1. Indeed, the derivation d sends every 2-cell of Σ_{Perm}^* to a map with values in \mathbb{N} and the following inequalities hold:

$$\begin{aligned} X\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)(i, j) &= (i + 1, j + 1) \geq (i, j) = X\left(|\right|)(i, j), \\ X\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)(i, j, k) &= (k + 2, j + 1, i) = X\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)(i, j, k), \\ d\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)(i, j) &= i + j + 1 > 0 = d\left(|\right|)(i, j), \\ d\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)(i, j, k) &= 2i + j + 1 = 2i + j = d\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)(i, j, k). \end{aligned}$$

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5.4.3. Normal forms. Let f be a 2-cell of Σ_{Perm}^* satisfying:

$$d(f)(0, \dots, 0) = 0.$$

Then f is in normal form. Otherwise, there exists a context C and a 2-cell g such that $f = C[g]$ holds and g is the source of one of the two 3-cells of Σ_{Perm} . As a consequence, there exists a family (i_1, \dots, i_n) of natural numbers, with $n = 2$ or $n = 3$, such that the following inequalities hold:

$$d(f)(0, \dots, 0) \geq d(g)(i_1, \dots, i_n) \geq 1.$$

Let us define N_0 as the set of 2-cells given by the following inductive construction:

$$\text{[Diagram: A 2-cell with two strands crossing, shaded with a diagonal line]} = \text{[Diagram: Two strands crossing]} \text{ or } \text{[Diagram: A 2-cell with two strands crossing, shaded with a diagonal line, and a vertical line to the right]}.$$

First, we check that the following holds:

$$X(\text{[Diagram: A 2-cell with two strands crossing, shaded with a diagonal line]}) (i_1, \dots, i_n, j) = (j + n, i_1, \dots, i_n).$$

We proceed by structural induction, using the definition and the functoriality of X :

$$X(\text{[Diagram: Two strands crossing]}) (i, j) = (j + 1, i)$$

and

$$\begin{aligned} X(\text{[Diagram: A 2-cell with two strands crossing, shaded with a diagonal line, and a vertical line to the right]}) (i_1, \dots, i_n, i_{n+1}, j) &= (X(\text{[Diagram: A 2-cell with two strands crossing, shaded with a diagonal line]}) \times \text{Id}_{\mathbb{N}}) (i_1, \dots, i_n, j + 1, i_{n+1}) \\ &= (j + n + 1, i_1, \dots, i_{n+1}). \end{aligned}$$

Now, let us prove that the 2-cells of N_0 are in normal form, still by structural induction. For the base case, we have, by definition of d :

$$d(\text{[Diagram: Two strands crossing]}) (0, 0) = 0.$$

Then, for the inductive case, we have, using the fact that d is a derivation:

$$d(\text{[Diagram: A 2-cell with two strands crossing, shaded with a diagonal line, and a vertical line to the right]}) (0, \dots, 0) = d(\text{[Diagram: A 2-cell with two strands crossing, shaded with a diagonal line]}) (0, \dots, 0) + d(\text{[Diagram: Two strands crossing]}) (0, 0) = 0.$$

Let us denote by N the set of 2-cells of Σ_{Perm}^* given by the following inductive graphical scheme:

$$\text{[Diagram: A rounded rectangle with dots above and below]} = * \text{ or } | \text{ [Diagram: A rounded rectangle with dots above and below]} \text{ or } \text{[Diagram: A rounded rectangle with dots above and below, and a 2-cell with two strands crossing, shaded with a diagonal line, on top]}.$$

Let us prove that the 2-cells of N are in normal form, by structural induction. First, one has $d(*) = 0$ since d is a derivation. Then, one gets:

$$d\left(\left| \text{[Diagram: A rounded rectangle with dots above and below]} \right.\right) (i_1, \dots, i_n + 1) = d(|) (i_1) + d\left(\text{[Diagram: A rounded rectangle with dots above and below]}\right) (i_2, \dots, i_n) = 0.$$

And, finally, using the values of X on N_0 :

$$\begin{aligned} & d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right) (i_1, \dots, i_m, j, k_1, \dots, k_n) \\ &= d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \end{array} \right) (i_1, \dots, i_m, j) + d \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) (i_1, \dots, i_m, k_1, \dots, k_n) = 0. \end{aligned}$$

Finally, we prove that every 2-cell of Σ_{Perm}^* in normal form is in N . We proceed by induction on the pair (m, n) of natural numbers, where m is the size of the 2-cells and n is the size of their source.

The only 2-cells of Σ_{Perm}^* with size 0 are the 1_n , where n denotes the 1-cell with size n . All of them are in normal form. Moreover, they belong to N : 1_0 is $*$ and, for every natural number n , $1_{n+1} = 1_1 \star_0 1_n$.

The only 2-cell of Σ_{Perm}^* whose source has size 0 is $1_0 = *$, which is in normal form and belongs to N .

Then, let us fix two non-zero natural numbers m and n . We assume that, every 2-cell g of Σ_{Perm}^* which is in normal form and such that $(\|g\|, |sg|) < (m, n)$ holds is in N , where compare pairs of natural numbers with the product order.

Let us consider a 2-cell f of Σ_{Perm}^* in normal form, with size m and whose source has size n . Since $\|f\| = m \geq 1$ and since $\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$ is the only 2-cell of Σ_{Perm} , there exists a 2-cell g such that f decomposes into:

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} f = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \text{---} \end{array} g.$$

Since f is in normal form, then so does g . Moreover, g has size $m - 1$ and its source has size n . We apply the induction hypothesis to g : this 2-cell is in N . Its source is $n \geq 1$, so that $g \neq *$; there remains two possibilities, by definition of N :

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} g = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} h \quad \text{or} \quad \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \text{---} \end{array} g = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \text{---} \end{array} h.$$

In the first case, the 2-cell h is in normal form, has size $m - 1$ and its source has size $n - 1$. By induction hypothesis, we know that h is in N . There are two subcases for the decomposition of f :

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} f = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \text{---} \end{array} h \quad \text{or} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} f = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} h.$$

The first decomposition is a proof that f is in N , since h is in N and $\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$ is in N_0 . The second decomposition tells us that $f = \begin{array}{c} \text{---} \\ \star_0 \\ \text{---} \end{array} f'$, where f' is in normal form (otherwise f would not), has size m and its source has size $n - 1$; we apply the induction hypothesis to get that f' is in N ; then we get that f is in N .

Now, let us examine the second case: the 2-cell h is in normal form, has size at most $m - 2$ and its source has size $n - 1$; hence, by induction hypothesis, h is in N . There are three subpossibilities:

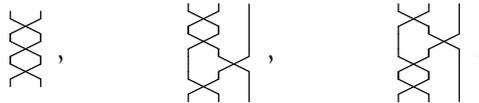
$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} f = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \text{---} \end{array} h \quad \text{or} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} f = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \text{---} \end{array} h \quad \text{or} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} f = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} h.$$

The first subcase is, in fact, impossible since f would contain the source of a 3-cell, which contradicts the assumption that f is in normal form. The second case gives that f is in N . In the third case, we have

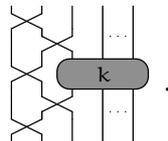
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a decomposition of f into $(f' \star_0 1_p) \star_1 (1_1 \star_0 f'')$ where f' is in N_0 and f'' is a normal form (otherwise f would not), has size at most $m - 1$ and has source $n - 1$: thus, we apply the induction hypothesis to get that f'' and, hence, f are in N .

5.4.4. Confluence. The 3-polygraph Σ_{Perm} has three regular critical branchings, with the following sources:

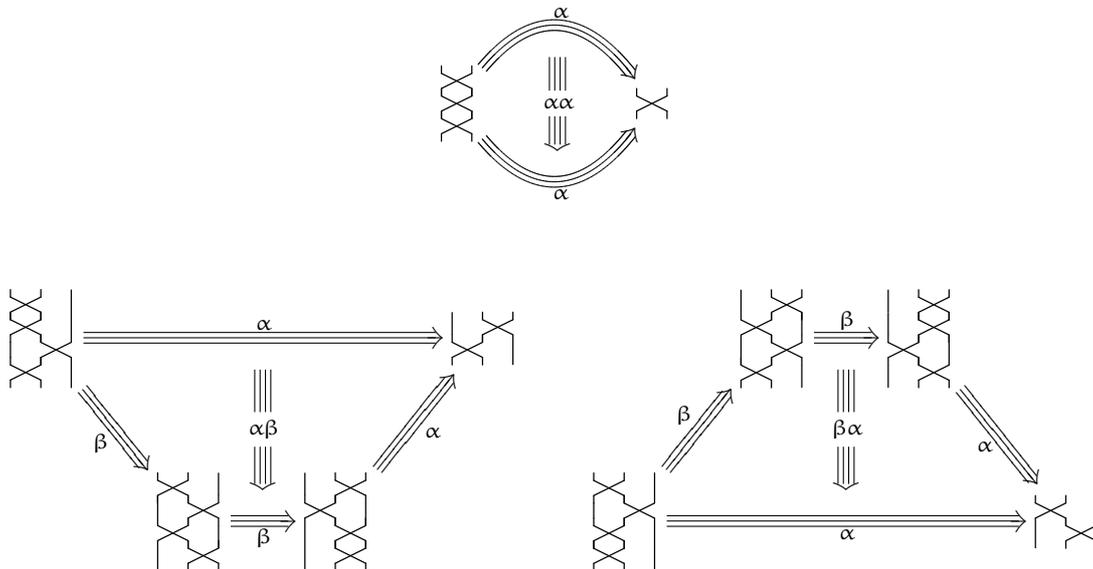


plus a right-indexed critical branching, with source:



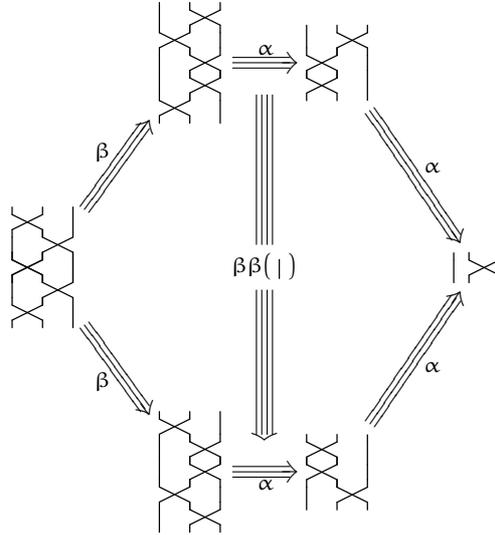
From Theorem 5.3.1, we know that, to get confluence of Σ_{Perm} , it is sufficient to prove that the three regular critical branchings are confluent and that each normal instance of the right-indexed one is.

In the following confluence diagrams, we commit the abuse of naming 3-cells with their generator only. First, we check that the three regular critical branchings are confluent:

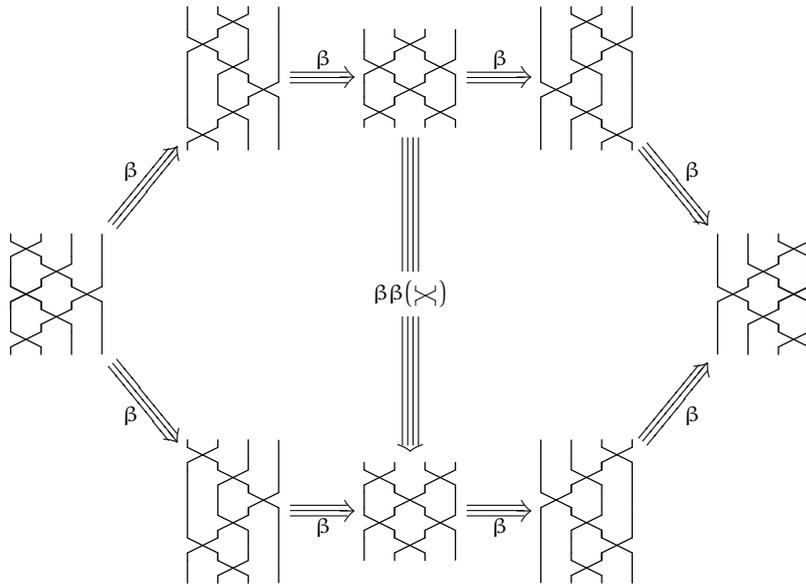


From the inductive characterization of the set N of normal forms we have given, we deduce that there are two normal instances of the right-indexed critical branching: for $k = |$ and $k = \times$. We check that

both are confluent:



and:



5.4.5. Theorem. *The 3-polygraph Σ_{Perm} has finite derivation type.*

Proof. Indeed, the 3-polygraph Σ_{Perm} is finite, convergent, right-indexed and has finitely many normal instances of its right-indexed critical branchings. Thus Theorem 5.3.3 tells us that Σ_{Perm} has finite derivation type. More precisely, the five 4-cells $\alpha\alpha$, $\alpha\beta$, $\beta\alpha$, $\beta\beta(|)$ and $\beta\beta(X)$ form a homotopy basis of the track 3-category $\Sigma_{\text{Perm}}^{\top}$. \square

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5.5. Main counterexample

Let us consider the 3-polygraph Σ with one 0-cell, one 1-cell, three 2-cells \blacklozenge , \cap and \cup and the following four 3-cells:

$$\blacklozenge \xrightarrow{\alpha} \cap, \quad \cup \xrightarrow{\beta} \blacklozenge, \quad \cup \xrightarrow{\gamma} |, \quad \cap \xrightarrow{\delta} |.$$

We define by induction on the natural number k the 2-cell \blacklozenge^k as follows:

$$\blacklozenge^0 = | \quad \text{and} \quad \blacklozenge^{k+1} = \blacklozenge^k \star_1 \blacklozenge.$$

5.5.1. Termination. To prove that the 3-polygraph Σ terminates, we use two subsequent derivations of Σ_2^* , as in Theorem 4.2.1.

We consider the derivation $\|\cdot\|_{\cap}$, into the trivial module $M(*, *, \mathbb{Z})$. One checks that the following holds:

$$\|\alpha\|_{\cap} = 0, \quad \|\beta\|_{\cap} = 0, \quad \|\gamma\|_{\cap} = 1, \quad \|\delta\|_{\cap} = 1,$$

Let us note that one would have the same results with the derivation $\|\cdot\|_{\cup}$. As a consequence, one gets that the 3-polygraph Σ terminates if and only if the 3-polygraph $\{\alpha, \beta\}$ terminates. To prove the latter, we consider the derivation d into the Σ_2^* -module $M(X, Y, \mathbb{Z})$, with:

$$\begin{aligned} X(|) &= \mathbb{N}, & X(\cap) &= (0, 0), & X(\blacklozenge) &= i + 1. \\ Y(|) &= \mathbb{N}, & Y(\cup) &= (0, 0), & Y(\blacklozenge) &= i + 1. \\ d(\cap)(i, j) &= i, & d(\cup)(i, j) &= i, & d(\blacklozenge)(i, j) &= 0. \end{aligned}$$

Since d is a derivation, one gets:

$$\begin{aligned} d\alpha &= d(\cap) - d(\blacklozenge) \\ &= d(\cap) \star_1 (\blacklozenge |) + \cap \star_1 (d(\blacklozenge) \star_0 |) - d(\cap) \star_1 (| \blacklozenge) - \cap \star_1 (| \star_0 d(\blacklozenge)). \end{aligned}$$

Thus, for every natural numbers i and j , one gets:

$$\begin{aligned} d\alpha(i, j) &= d(\cap)(i + 1, j) + d(\blacklozenge)(0, i) - d(\cap)(i, j + 1) - d(\blacklozenge)(0, j) \\ &= (i + 1) + 0 - i - 0 \\ &= 1. \end{aligned}$$

Similarly, one gets $d\beta(i, j) = 1$ for every natural numbers i and j , yielding the termination of $\{\alpha, \beta\}$ and, thus, of Σ .

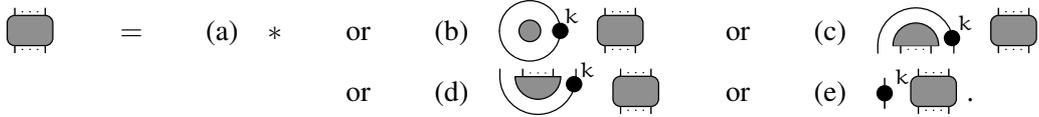
5.5.2. Normal forms. Let f be a 2-cell of Σ^* , that cannot be reduced by the 3-cells γ and δ and which satisfies:

$$d(f)(0, \dots, 0) = 0.$$

Then f is in normal form. Indeed, otherwise there exists a context C such that $f = C[g]$, with either $g = s\alpha$ or $g = s\beta$. As a consequence, there exist two natural numbers i and j such that the following inequalities hold:

$$df(0, \dots, 0) \geq dg(i, j) \geq 1.$$

Now, we define N as the set of 2-cells given by the following inductive construction scheme:



We use the special graphical representations \bullet , $\text{[semi-circle with dot]}$ and $\text{[semi-circle with dot]}$ for 2-cells of N which have, respectively, degenerate source and target, degenerate source, degenerate target.

We start by checking that the 2-cells of N are in normal form. For that, one proceeds by structural induction, using the construction scheme, in order to prove two properties.

The first one is that each 2-cell of N is irreducible by the 3-cells γ and δ : this is an observation that the given construction scheme do not allow any 2-cell of N to contain either [wavy line 1] or [wavy line 2] .

The second one is that, for a 2-cell f of N , one has $df(0, \dots, 0) = 0$. For the base case (a), one has $d(*) = 0$ since d is a derivation. Then, for the induction, there are four cases:

$$\begin{aligned}
 \text{(b)} \quad & d\left(\text{[circle with dot]}^k \text{[rectangle with 4 dots]}\right)(0, \dots, 0) \\
 &= d(\cap)(0, k) + d(\cup)(0, k) + k \cdot d(\bullet)(0, 0) + d(\bullet) + d\left(\text{[rectangle with 4 dots]}\right)(0, \dots, 0) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & d\left(\text{[semi-circle with dot]}^k \text{[rectangle with 4 dots]}\right)(0, \dots, 0) \\
 &= d(\cap)(0, k) + k \cdot d(\bullet)(0, 0) + d(\text{[semi-circle with dot]})(0, \dots, 0) + d\left(\text{[rectangle with 4 dots]}\right)(0, \dots, 0) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & d\left(\text{[semi-circle with dot]}^k \text{[rectangle with 4 dots]}\right)(0, \dots, 0) \\
 &= d(\cup)(0, k) + k \cdot d(\bullet)(0, 0) + d(\text{[semi-circle with dot]})(0, \dots, 0) + d\left(\text{[rectangle with 4 dots]}\right)(0, \dots, 0) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & d\left(\text{[dot]}^k \text{[rectangle with 4 dots]}\right)(0, \dots, 0) \\
 &= k \cdot d(\bullet)(0, 0) + d\left(\text{[rectangle with 4 dots]}\right)(0, \dots, 0) \\
 &= 0.
 \end{aligned}$$

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Now, let us prove that every 2-cell of Σ^* that is in normal form is contained in the set N . We proceed by induction on the triple (m, n, p) of natural numbers, where m is the size of the 2-cells, n the size of their source, p the size of their target.

The only 2-cells of Σ^* with size 0 are the 1_n , where n denotes the 1-cell with size n . All of them are in normal form and belong to N . Indeed, each 1_n can be formed, from $*$, by n subsequent applications of the construction rule (e) with $k = 0$.

The 2-cells of Σ^* with size 1 are the $1_p \star_0 \varphi \star_0 1_q$, where φ is one of \cap , \cup and \bullet . Such a 2-cell is always in normal form and belongs to N . Indeed, we have seen that 1_q is in N . Then $\varphi \star_0 1_q$ is in N by using the following case, with $\square = 1_q$, depending on φ :

- If $\varphi = \cap$: case (c) with $\cap = *$ and $k = 0$.
- if $\varphi = \cup$: case (d) with $\cup = *$ and $k = 0$.
- If $\varphi = \bullet$: case (e) with $k = 1$.

Finally, $1_p \star_0 \varphi \star_0 1_q$ is in N , thanks to case (e), applied p times in sequence, each time with $k = 0$.

Now, let us fix a non-zero natural number m and two natural numbers n and p . We assume that we have proved the result for each g in normal form of size at most $m - 1$ or with size m and such that the inequality $(|sg|, |tg|) < (n, p)$ holds.

Let us consider a 2-cell f , in normal form and such that $\|f\| = m$, $|sf| = n$ and $|tf| = p$ hold. Since f has size at least 1, there exists a 2-cell g such that f decomposes in one of the three following ways:

$$\boxed{f} = \boxed{g} \text{ or } \boxed{g} \text{ or } \boxed{g}$$

One denotes by φ the corresponding generating 2-cell. Since f is in normal form, so does g and g has size $m - 1$: we apply the induction hypothesis to it, so that we know that g is in N . Thus, g decomposes into one of the five following ways:

$$\boxed{g} = \text{(a) } * \text{ or (b) } \textcircled{\bullet}^k \boxed{h} \text{ or (c) } \overbrace{\cap}^k \boxed{h} \text{ or (d) } \underbrace{\cup}_k \boxed{h} \text{ or (e) } \bullet^k \boxed{h}.$$

We study all the possible decompositions of f , depending on the one of g and on φ . In case (a), *i.e.* when $g = *$, we have $\varphi = \cup$, since this is the only possibility to have $t\varphi$ degenerate. We have already seen that \cup is in N . In case (b), one has the following possibilities, depending on φ :

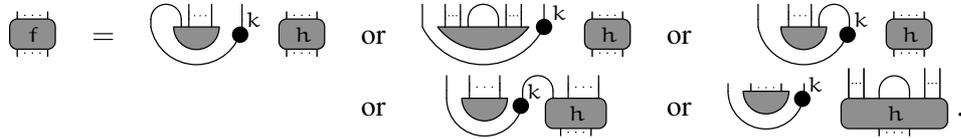
$$\boxed{f} = \textcircled{\bullet}^k \boxed{h} \text{ or } \textcircled{\cap}^k \boxed{h} \text{ or } \textcircled{\cup}^k \boxed{h}$$

The following 2-cells must be in normal form, since f is, and they have size at most $m - 2$:

$$\boxed{h}, \quad \boxed{h}, \quad \bullet \boxed{h}.$$

We apply the induction hypothesis to each one: they belong to N . Thus f is in N by case (b). The proof for case (c) is obtained from case (b) by replacing the 2-cell $\textcircled{\cap}^k$ by $\overbrace{\cap}^k$. In case (d), the reasoning depends on φ :

- When $\varphi = \cap$, one has the following possibilities, depending where φ connects to g :



The first and third case cannot occur. Indeed, one proves, by structural induction that a normal form with source of size at least 1 and with degenerate target has the following shape:



As a consequence, such a decomposition of f would contain either \cap or \cup , preventing it from being in normal form.

For the second case, one applies the induction hypothesis to the 2-cell \cup : indeed, it is a 2-cell with size at most $m - 1$ that must be in normal form, otherwise f would not. Thus, f is built from 2-cells of N following case (d) and, as such, is in N .

The fourth decomposition contains either \cap or \cup , respectively when $k \geq 1$ and $k = 0$. Thus it is not possible that f decomposes this way, since it is a normal form.

For the fifth decomposition, one applies the induction hypothesis to \cup , which is a 2-cell that must be in normal form, with size at most $m - 1$.

- When $\varphi = \cup$, one has the following possible decompositions of f :

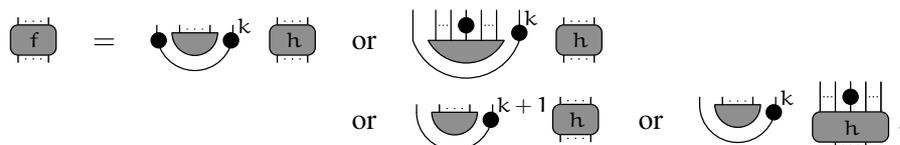


The first case shows that f is in N : it is built with case (d), applied with $\cup = \cup$ and $h = h$, which is g that we already know to belong to N , with $\cup = *$ and $k = 0$.

In the second case, we apply the induction hypothesis to \cup : it is a normal form of size at most $m - 1$. Thus f is built from case (d).

In the third case, one applies the induction hypothesis to \cup : it is a normal form of size at most $m - 1$. Thus f is built from case (d).

- When $\varphi = \bullet$, the 2-cell can decompose as follows:



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The first case cannot occur: otherwise, f would contain  and, thus, it would not be in normal form.

In the second case, we apply the induction hypothesis to : this is a normal form with size at most $m - 1$. This proves that f is in \mathbb{N} , built following case (d).

In the third case, f is in \mathbb{N} , built following case (d).

In the fourth case, we apply the induction hypothesis to : this is a normal form with size at most $m - 1$. Thus f is in \mathbb{N} , built from case (d).

Case (e) depends on the values of φ :

- When $\varphi = \cap$, we have the following possible decompositions of f :

$$\text{f} = \text{dot} \overset{k}{\cap} \text{h} \quad \text{or} \quad \text{dot} \overset{k}{\cup} \text{h}.$$

In the first case, one must have $k = 0$: otherwise, f would contain  which is not a normal form. Thus the 2-cell h is a normal form of size $m - 1$: we apply the induction hypothesis to get that h is in \mathbb{N} . Then, by structural induction on h , one shows that it decomposes in one of the following two ways:

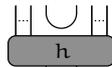
$$\text{h} = \text{dot} \overset{k}{\cap} \text{h} \quad \text{or} \quad \text{dot} \overset{k}{\cup} \text{h}.$$

The first decomposition is impossible since, otherwise, f would contain  and, thus, it would not be a normal form. The second decomposition gives that f is in \mathbb{N} , built from case (c).

- When $\varphi = \cup$, we have the following possible decompositions of f :

$$\text{f} = \text{dot} \overset{k}{\cup} \text{h} \quad \text{or} \quad \text{dot} \overset{k}{\cap} \text{h}.$$

In the first case, f is in \mathbb{N} , built from h in two subsequent steps: with case (e), then with case (d).

In the second case, one applies the induction hypothesis to  which is a 2-cell in normal form, with either size at most $m - 1$, when $k > 0$, or with size m and source of size $n - 1$. Thus this 2-cell is in \mathbb{N} , and so does f , which is built following case (e).

- When $\varphi = \bullet$, we have the following possible decompositions of f :

$$\text{f} = \text{dot} \overset{k+1}{\bullet} \text{h} \quad \text{or} \quad \text{dot} \overset{k}{\bullet} \text{h}.$$

In the first case, f is built from h by application of case (e) and, as such, is in \mathbb{N} .

In the second case, one applies the induction hypothesis to  which is a 2-cell in normal form, with either size at most $m - 1$, when $k > 0$, or with size m and source of size $n - 1$. As a consequence, this 2-cell is in \mathbb{N} , proving that f is built following case (e) and, thus, it is in \mathbb{N} .

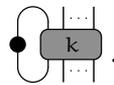
We have proved that the 2-cells of Σ^* that are in normal form are exactly the 2-cells of \mathbb{N} . In particular, we denote by \mathbb{N}_0 the set of normal forms with degenerate source and target, which are defined by the following inductive scheme:

$$\bullet = * \text{ or } \bigcirc^k \bullet .$$

5.5.3. Confluence. Let us examine the critical branchings of Σ . The 3-polygraph Σ has four regular critical branchings whose sources are:

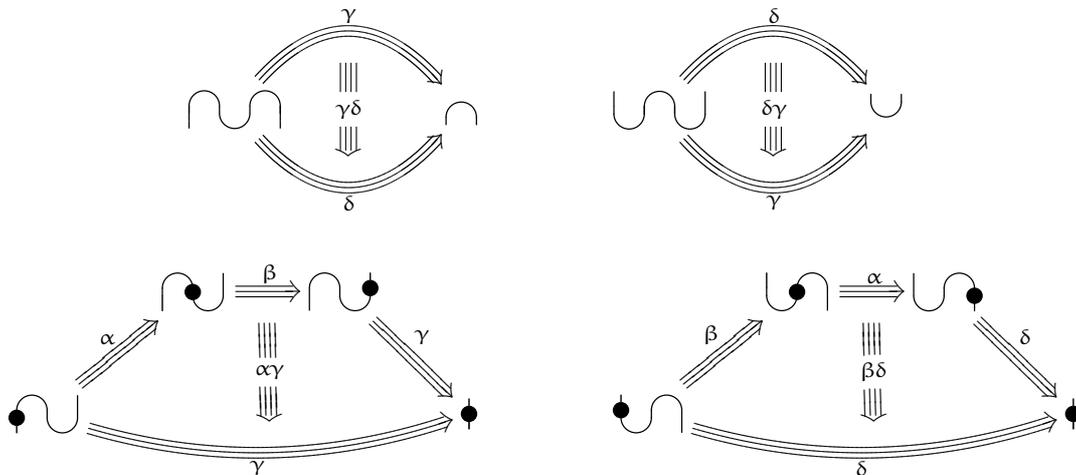


plus one right-indexed critical branching, generated by α and β , with source:



Thus Σ is a terminating and right-indexed 3-polygraph. By application of Theorem 5.3.1, we get confluence of Σ by proving that its four regular critical branchings and all normal instances of its right-indexed critical branchings are confluent.

For the regular ones, we have the following confluence diagrams:



From the characterization of normal forms of Σ , the normal instances of the right-indexed critical branching $\alpha\beta_{\boxplus}$ are the instances corresponding to the following 2-cells:

$$\boxplus = \cup, \quad \boxplus = \cap, \quad \boxplus = \cup |,$$

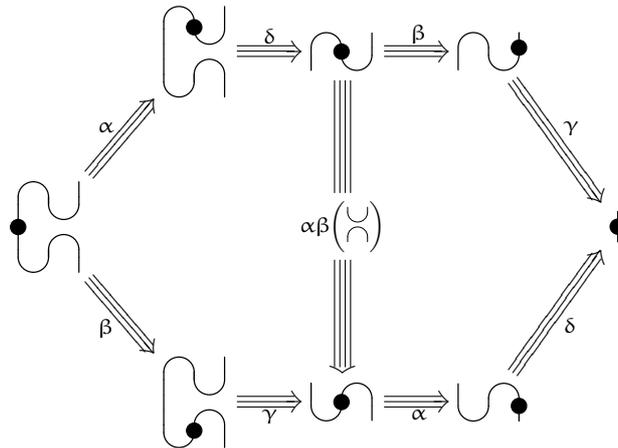
plus:

$$\boxplus = \bullet \bullet^n,$$

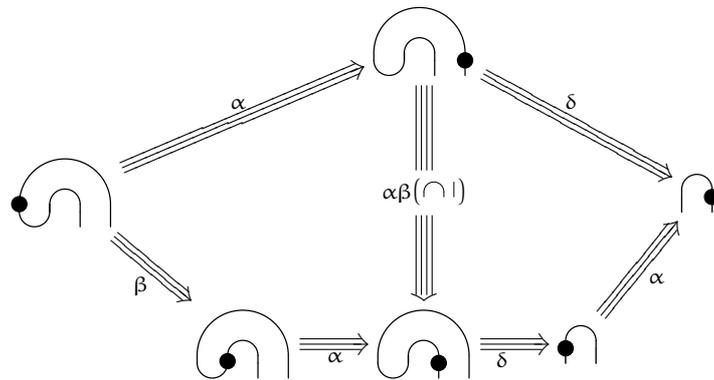
for every \bullet in \mathbb{N}_0 and n in \mathbb{N} .

5. The case of 3-polygraphs

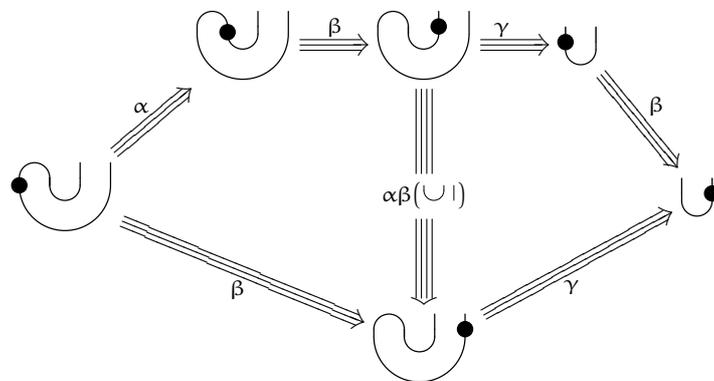
Now we check that, for each one of these 2-cells, the corresponding critical branching $\alpha\beta_{\text{cell}}$ is confluent. Let us note that, for the first three, there are several possible confluence diagrams, because they also contain regular critical branchings of Σ . For $\text{cell} = \cup$, we choose the following one:



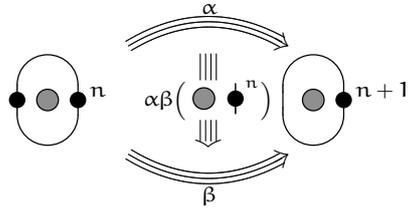
For $\text{cell} = \cap$:



For $\text{cell} = \cup$:



Finally, for $\text{⊔} = \text{⊙} \text{⋈}^n$:



5.5.4. Homotopy basis. The 3-polygraph Σ is convergent and right-indexed. Thus, Theorem 5.3.3 tells us that the following 4-cells form a homotopy basis of Σ^\top :

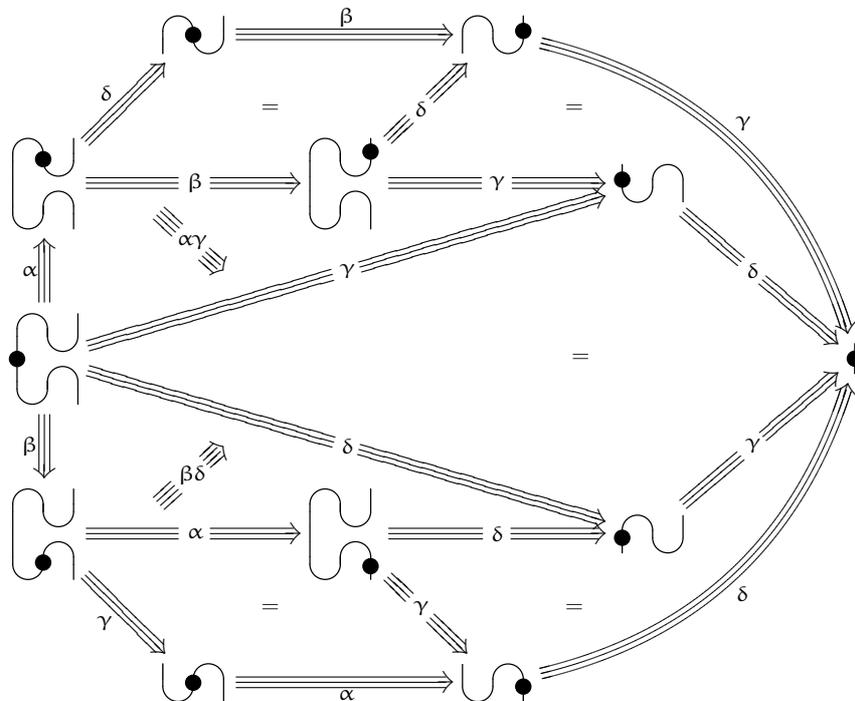
$$\gamma\delta, \delta\gamma, \alpha\gamma, \beta\delta, \alpha\beta \left(\text{⊔} \right), \alpha\beta \left(\text{⋈} \right), \alpha\beta \left(\text{⊔} \right),$$

plus, for every ⊙ in \mathbb{N}_0 and n in \mathbb{N} , the 4-cell

$$\alpha\beta \left(\text{⊙} \text{⋈}^n \right).$$

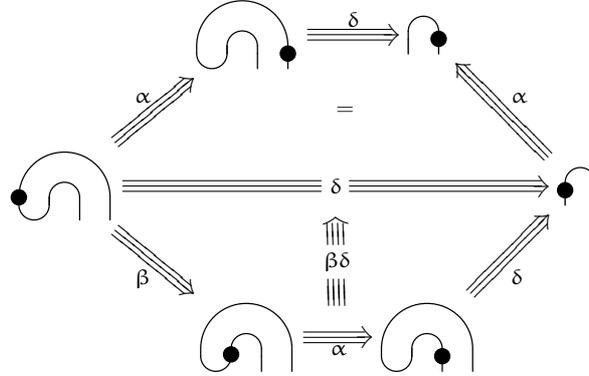
In fact, the 4-cells $\alpha\beta \left(\text{⊔} \right)$, $\alpha\beta \left(\text{⋈} \right)$ and $\alpha\beta \left(\text{⊔} \right)$ are superfluous. Indeed, their source and target are the boundary of 4-cells in $\Sigma^\top (\{\alpha\gamma, \beta\delta\})$, as proved thereafter.

For $\alpha\beta \left(\text{⊔} \right)$:

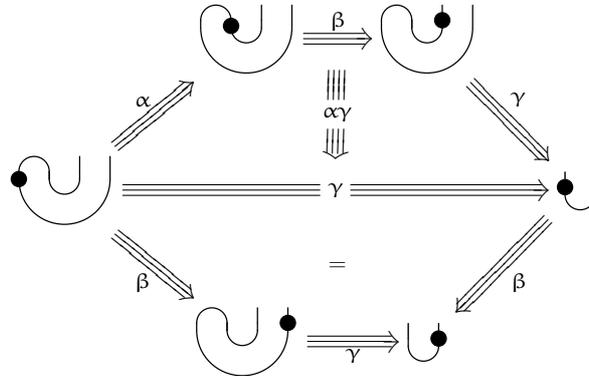


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For $\alpha\beta(\cap |)$:



And, for $\alpha\beta(\cup |)$:



We denote by Γ_0 the family made of the 4-cells $\gamma\delta$, $\delta\gamma$, $\alpha\gamma$ and $\beta\delta$. Then, for every natural number n , one defines:

$$\Gamma_{n+1} = \Gamma_n \cup \{ \alpha\beta(\bullet \blacklozenge^n), \bullet \in \mathbb{N}_0 \} .$$

Thus, the following set of 4-cells is a homotopy basis of Σ :

$$\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n .$$

For every natural number n , we denote by ξ_n the 4-cell $\alpha\beta(\blacklozenge^n)$.

5.5.5. Lemma. *Let n be a natural number. There is no 4-cell of $\Sigma^\top(\Gamma_0)$ which is parallel to ξ_n , i.e.:*

$$s\xi_n \not\approx_{\Gamma_0} t\xi_n .$$

Proof. Let us assume, on the contrary, that there exists a 4-cell Φ in $\Sigma^\top(\Gamma_0)$ such that $s\Phi = s\xi_n$ and $t\Phi = t\xi_n$ hold. We build the derivation d of Σ^* into the trivial Σ^* -module that takes the following values on the generating 3-cells:

$$d\alpha = 1, \quad d\beta = -1, \quad d\gamma = 0, \quad d\delta = 0 .$$

Then d induces a derivation of Σ^\top into the trivial Σ^\top -module: we define $d(f^{-1}) = -d(f)$ and we verify that, with those values, d is compatible with the inverse relations.

Then, we check that, for any 4-cell Ψ of $\Sigma^\top(\Gamma_0)$, the extension of d satisfies $d\Psi = 0$. Since d is a derivation, it is sufficient to check this on the generating 4-cells:

- $d(\gamma\delta) = d\gamma - d\delta = 0$.
- $d(\delta\gamma) = d\delta - d\gamma = 0$.
- $d(\alpha\gamma) = d\alpha + d\beta + d\gamma - d\gamma = 1 - 1 = 0$.
- $d(\beta\delta) = d\beta + d\alpha + d\delta - d\delta = -1 + 1 = 0$.

Thus, since Φ is in $\Sigma^\top(\Gamma_0)$, one must have $d\Phi = 0$. However, one has:

$$d\Phi = d\alpha - d\beta = 1 - (-1) = 2.$$

This proves that Φ cannot be in $\Sigma^\top(\Gamma_0)$. □

5.5.6. Lemma. *Let n be a natural number. There is no 4-cell of $\Sigma^\top(\Gamma_n)$ which is parallel to ξ_n , i.e.:*

$$s\xi_n \not\approx_{\Gamma_n} t\xi_n.$$

Proof. On the contrary, let us assume that Φ is a 4-cell of $\Sigma^\top(\Gamma_n)$ such that $s\Phi = s\xi_n$ and $t\Phi = t\xi_n$ hold.

First, we prove that Φ cannot contain any occurrence of a generating 4-cell $\alpha\beta \left(\begin{array}{c} \bullet \\ \downarrow^k \end{array} \right)$ or its inverse, with $k < n$. For that, we define the derivation d of Σ_2^* into the module $\mathcal{M}(X, *, G)$ given thereafter:

- The 2-functor $X : \Sigma_2^* \rightarrow \mathbf{Set}$ is generated by the values:

$$X(|) = \mathbb{N}, \quad X(\cap) = (0, 0), \quad X(\downarrow^k)(i) = i + 1.$$

- The abelian group G is freely generated by the set \mathbb{N} of natural numbers. The natural number n , seen as a generator of G , is denoted by a_n .
- The derivation d is given by:

$$d(\cap) = 0, \quad d(\cup)(i, j) = a_j, \quad d(\downarrow^k)(i) = 0.$$

Now, let us assume that there exist 4-cells Ψ_1, Ψ_2 in $\Sigma^\top(\Gamma_n)$, a context C of Σ^\top , an ε in $\{-1, 1\}$, a 2-cell \bullet and a k in $\{0, \dots, n-1\}$ such that the 4-cell Φ decomposes this way:

$$\Phi = \Psi_1 \star_3 C \left[\alpha\beta \left(\begin{array}{c} \bullet \\ \downarrow^k \end{array} \right)^\varepsilon \right] \star_3 \Psi_2.$$

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Then we have $s_2\Phi = (s_2C) [s_2\alpha\beta (\bullet \downarrow^k)]$. In particular, both 2-cells are sent by d to the same element of G . But, on the one hand, by hypothesis, one gets:

$$s_2\Phi = s_2\xi_n = \bullet \downarrow^n ,$$

so that the following holds:

$$d(s_2\Phi) = \alpha_n.$$

And, on the other hand, one has:

$$(s_2C) [s_2\alpha\beta (\bullet \downarrow^k)] = \text{Diagram}$$

Then, using the fact that d is a derivation, one computes:

$$d \left(\text{Diagram} \right) = d \left(\text{Diagram} \right) + d \left(\bullet \downarrow^k \right) + d \left(\bullet \right) = d(f) + \alpha_k ,$$

where $f = (s_2C) [\bullet]$. Thus, we have $\alpha_n = \alpha_k + d(f)$, with $k < n$ and f in Σ_2^* . This is impossible because G is freely generated and d sends any 2-cell to an element of G written using the α_i 's with positive coefficients.

Thus, we conclude that the 4-cell Φ must be built using the 4-cells of Γ_0 and their inverses, *i.e.* Φ is a 4-cell of $\Sigma^\top(\Gamma_0)$. However, this would contradict the previous result. \square

5.5.7. Theorem. The 3-polygraph Σ does not have finite derivation type.

Proof. On the contrary, let us assume that Σ does have finite derivation type. Then, by application of Proposition 3.2.3, there exists a finite subfamily Γ' of Γ which is a homotopy basis of Σ^\top .

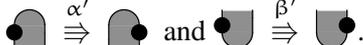
Since Γ' is finite, there exists some natural number n such that Γ' is contained in Γ_n . In particular, the 4-cell ξ_n is not in Γ' . However, since Γ' is a homotopy basis and since Γ' is contained in Γ_n , we have:

$$s\xi_n \approx_{\Gamma_n} t\xi_n.$$

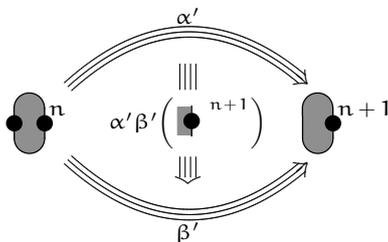
We have seen that this is not possible, thus contradicting the fact that one can extract a finite homotopy basis from Γ . As a consequence, the 3-polygraph Σ does not have finite derivation type. \square

5.5.8. A variant of the counterexample. In the previous 3-polygraph, one can think that the problem comes from the complicated normal forms, especially from the fact that one can find normal forms \bullet of N_0 everywhere in a given 2-cell. Here we give another example, similar to the first one but with more simple normal forms. It is a bit more contrived, which led us to prefer the other one for the main exposition.

Let Ξ be the 3-polygraph with the following generating cells:

- Two 0-cells, denoted by ξ and η and, in the diagrammatic representations, respectively pictured by a white background and by a gray one.
- Two 1-cells $\xi \xrightarrow{p} \eta$ and $\eta \xrightarrow{q} \xi$. By abuse, both are pictured by a wire, leaving the backgrounds discriminate them.
- Four 2-cells .
- Two 3-cells  $\xRightarrow{\alpha'}$  and  $\xRightarrow{\beta'}$ .

Following the same reasoning steps as for the previous example one proves that the finite 3-polygraph Ξ is convergent and that it does not have finite derivation type. Indeed, the following family of 4-cells, indexed by the natural number n , form a homotopy basis of Ξ^\top and cannot be reduced further:



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