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On asymptotic normality of sequential LS-estimates of unstable autoregressive processes. *

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Abstract

For estimating the unknown parameters in an unstable autoregressive AR(p), the paper proposes sequential least squares estimates with a special stopping time defined by the trace of the observed Fisher information matrix. The limiting distribution of the sequential LSE is shown to be normal for the parameter vector lying both inside the stability region and on some part of its boundary in contrast to the ordinary LSE. The asymptotic normality of the sequential LSE is provided by a new property of the observed Fisher information matrix which holds both inside the stability region of AR(p) process and on the part of its boundary. The asymptotic distribution of the stopping time is derived.

AMS 1991 Subject Classification : 62L10, 62L12.

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1 Introduction

Consider the autoregressive $AR(p)$ model

$$x_n = \theta_1 x_{n-1} + \ldots + \theta_p x_{n-p} + \varepsilon_n, \; n = 1, 2, \ldots, \tag{1.1}$$

where (x_n) is the observation, (ε_n) is the noise which is a sequence of independent identically distributed (i.i.d.) random variables with $\mathbf{E}\varepsilon_1 = 0$ and $0 < \mathbf{E}\varepsilon_1^2 = \sigma^2 < \infty$, σ^2 is known, $x_0 = x_{-1} = \ldots = x_{1-p} = 0$; parameters of the model $\theta_1, \ldots, \theta_p$ are unknown. This model can be expressed in vector form as

$$X_n \;=\; AX_{n-1} \;+\; \xi_n \,, \tag{1.2}$$

where $X_n = (x_n, x_{n-1}, \ldots, x_{n-p+1})', \xi = (\varepsilon_n, 0, \ldots, 0)'$,

$$A \;=\; \left(\begin{array}{cc} \theta_1 \ldots & \theta_p \\ I_{p-1} & 0 \end{array} \right), \tag{1.3}$$

the prime denotes the transposition.

A commonly used estimate of the parameter vector $\theta = (\theta_1, \ldots, \theta_p)'$ is the least squares estimate (LSE)

$$\theta(n) = M_n^{-1} \sum_{k=1}^{n} X_{k-1} x_k, \quad M_n = \sum_{k=1}^{n} X_{k-1} X'_{k-1}, \tag{1.4}$$

where M_n^{-1} denotes the inverse of matrix M_n if $\det M_n > 0$ and $M_n^{-1} = 0$ otherwise. Let

$$\mathcal{P}(z) = z^p - \theta_1 z^{p-1} - \ldots - \theta_p \tag{1.5}$$

denote the characteristic polynomial of the autoregressive model (1.1). The process (1.1) is said to be stable (asymptotically stationary) if all roots $z_i = z_i(\theta)$ of the characteristic polynomial (1.5) lie inside the unit circle, that is the parameter vector $\theta = (\theta_1, \ldots, \theta_p)'$ belongs to the parametric stability region Λ_p defined as

$$\Lambda_p = \{\theta \in R^p : |z_i(\theta)| < 1, i = 1, \ldots, p\} \,. \tag{1.6}$$

The process (1.1) is called unstable if the roots of $\mathcal{P}(z)$ lie on or inside the unit circle, that is, $\theta \in [\Lambda_p]$; $[\Lambda_p]$ denotes the closure of the stability region Λ_p.

It is well known (see,e.g. Anderson (1971), Th.5.5.7) that the LSE $\theta(n)$ is asymptotically normal for all $\theta \in \Lambda_p$, that is

$$\sqrt{n}(\theta(n) - \theta) \stackrel{\mathcal{L}}{\Longrightarrow} \mathcal{N}(0, F), \text{ as } n \to \infty,$$

where $F = F(\theta)$ is a positive definite matrix, $\stackrel{\mathcal{L}}{\Longrightarrow}$ indicates convergence in law. It should be noted that the asymptotic normality of $\theta(n)$ is provided by the following asymptotic property of the observed Fisher information matrix

$$\lim_{n\to\infty} M_n/n \;=\; F \text{ a.s.} \tag{1.7}$$

for all $\theta \in \Lambda_p$. On the boundary $\partial\Lambda_p$ of the stability region Λ_p, this property does not hold and the distribution of $\theta(n)$ is no longer asymptotically normal. The investigation of the asymptotic distribution of LSE $\theta(n)$ when x_n is unstable goes back to the late fifties with the paper of White (1958) (see also Ahtola and Tiao (1987), Dickey and Fuller (1979), Rao (1978), Sriram (1987),(1988)) who considered the AR(1) model with i.i.d. $\mathcal{N}(0,\sigma^2)$ random errors ε_n and $\theta_1 = 1$ and established that

$$n(\theta(n)-1) \stackrel{\mathcal{L}}{\Longrightarrow} (W^2(1)-1)/\int_0^1 W^2(t)dt\,,$$

where $W(t)$ is a standard brownian motion. Subsequently the research of the limiting distribution of $\theta(n)$ for unstable AR(p) processes has been receiving considerable attention due to important applications in time series analysis, in modeling economic and financial data and in system identification and control. We can not go into the detail here and refer the reader to the paper by Chan and Wei (1988) who derived the limiting distribution of LSE $\theta(n)$ for the general unstable AR(p) model. By making use of the functional central limit theorem approach, Chan and Wei expressed the limiting distribution of LSE $\theta(n)$ in terms of functionals of standard brownian motions. However, the closed forms of the distribution functions of these functionals are not known and that may cause difficulties in practice (see section 4 in Chan and Wei).

For the unstable AR(1) model with i.i.d. random errors and $-1 \le \theta_1 \le 1$, Lai and Siegmund (1983) proposed to use the sequential least squares estimate for θ_1 which is obtained from the LSE

$$\theta_1(n) \;=\; \sum_{k=1}^{n} x_{k-1}x_k/\sum_{k=1}^{n} x_{k-1}^2$$

by replacing n with a special stopping time τ based on the observed Fisher information. They proved that, in contrast with the ordinary LSE $\theta_1(n)$, the sequential LSE is asymptotically normal uniformly in $\theta \in [-1,1]$. For the unstable AR(2) model, Galtchouk and Konev (2006) applied the sequential LSE with a particular stopping time and established that it is asymptotically normal not only inside the stability rigion Λ but also for its boundary points

θ corresponding to a pair of conjugate complex roots $z_1 = e^{i\phi}, z_2 = e^{-i\phi}$ of the polynomial (1.5).

In this paper, for the case of unstable AR(p) process, we propose a sequential LSE for θ and find the conditions on θ (see Conditions 1-3 in the next section) ensuring its asymptotic normality. The set $\tilde{\Lambda}_p$ of the points θ , satisfying these conditions includes the stability region Λ_p and some part of its boundary. It is shown that the convergence of the sequential LSE to the normal distribution is uniform in $\theta \in K$ for any compact set $K \in \tilde{\Lambda}_p$ (see Theorem 2.1). The extension of the property of asymptotic normality of the sequential estimate to the part of the boundary $\partial\Lambda_p$ is achieved by making use of a new property of observed Fisher information matrix M_n, which holds in a broader subset of $[\Lambda_p]$ as compared with (1.7) (see Lemma 3.3).

The remainder of this paper is arranged as follows. In Section 2 we introduce a sequential procedure for estimating the parameter vector $\theta = (\theta_1, \ldots, \theta_p)'$ in (1.1) and study its properties. Section 3 gives a new property of the observed Fisher information matrix and establishes some technical results. In Section 4 we prove Theorem 2.2 from Section 2 on asymptotic distribution of the stopping time.

2 Sequential least squares estimate. Uniform asymptotic normality.

In this section we consider the sequential least squares estimate and study its asymptotic properties. We define the sequential LSE for the parameter vector $\theta = (\theta_1, \ldots, \theta_p)'$ in model (1.1) as

$$\theta(\tau(h)) = M^{-1}_{\tau(h)} \sum_{k=1}^{\tau(h)} X_{k-1} x_k, \tag{2.1}$$

where

$$\tau(h) = \inf\left\{n \geq 1 : tr\, M_n \geq h\sigma^2\right\}, \ \inf\{\emptyset\} = +\infty, \tag{2.2}$$

is stopping time, h is a positive number (threshold).

Assume that the parameter vector $\theta = (\theta_1, \ldots, \theta_p)'$ in (1.1) satisfies the following Conditions.

Condition 1. Parameter $\theta = (\theta_1, \ldots, \theta_p)'$ is such that all roots $z_i = z_i(\theta)$ of the characteristic polynomial (1.5) lie inside or on the unite circle.

Condition 2. All the roots $z_i = z_i(\theta)$ of $\mathcal{P}(z)$, which are equal to one in modulus, are simple.

Condition 3. The system of linear equations with respect to $Y_1, \ldots, Y_{p-1}$

$$\begin{cases} Y_1 - \sum_{l=2}^{p} \theta_l Y_{l-1} = \theta_1 \\ -\sum_{k=1}^{j-1} \theta_{j-k} + Y_j - \sum_{k=1}^{p-j} \theta_{k+j} Y_k = \theta_j, \\ 2 \leq j \leq p-1, \end{cases} \tag{2.3}$$

has a unique solution $(Y_1, \ldots, Y_{p-1}), Y_i = \kappa_i(\theta), 1 \leq i \leq p-1$, and the matrix

$$L(\theta_1, \ldots, \theta_p) = \begin{pmatrix} 1 & \kappa_1(\theta) & \kappa_2(\theta) & \ldots & \kappa_{p-1}(\theta) \\ \kappa_1(\theta) & 1 & \kappa_1(\theta) & \ldots & \kappa_{p-2}(\theta) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa_{p-1}(\theta) & \kappa_{p-2}(\theta) & \ldots & \kappa_1(\theta) & 1 \end{pmatrix} \tag{2.4}$$

is positive definite.

Let $\overset{\circ}{\Lambda}_p$ denote all $\theta = (\theta_1, \ldots, \theta_p)'$ in (1.1) which satisfy Conditions 1,2, and $\tilde{\Lambda}_p$– all $\theta = (\theta_1, \ldots, \theta_p)'$ satisfying all Conditions 1-3.

Example 2.1. For AR(2) process, one finds

$$\Lambda_2 = \{\theta = (\theta_1, \theta_2)' : -1 + \theta_2 < \theta_1 < 1 - \theta_2, |\theta_2| < 1\},$$

$$[\Lambda_2] = \{\theta = (\theta_1, \theta_2)' : -1 + \theta_2 \leq \theta_1 \leq 1 - \theta_2, |\theta_2| \leq 1\},$$

$$\overset{\circ}{\Lambda}_p = [\Lambda_2] \setminus \{(-2, -1), (2, -1)\},$$

$$\tilde{\Lambda}_2 = \{\theta = (\theta_1, \theta_2)' : -1 + \theta_2 < \theta_1 < 1 - \theta_2, -1 \leq \theta_2 < 1\},$$

$$L(\theta_1, \theta_2) = \begin{pmatrix} 1 & \theta_1/(1-\theta_2) \\ \theta_1/(1-\theta_2) & 1 \end{pmatrix}.$$

Example 2.2. By numerical calculation for AR(3) process, one can check that Conditions 1-3 are satisfied, for example, for the values of $\theta = (\theta_1, \theta_2, \theta_3)$ such that $z_1(\theta) = e^{i\phi}$, $z_2(\theta) = e^{-i\phi}$ with $3\pi/10 \leq \phi \leq 3\pi/5$ and $-1 \leq z_3(\theta) \leq -0.5$.

As is shown in Lemma 3.3 (Section 3), Conditions 1-3, imposed on the parameter $\theta = (\theta_1, \ldots, \theta_p)'$ in (1.1), provide the convergence of the ratio $M_n / \sum_{k=1}^{n} x_{k-1}^2$ to the matrix $L(\theta_1, \ldots, \theta_p)$ given in (2.4). This property can be viewed as an extension of (1.7) outside the stability region (1.6).

Remark 2.1 *It will be observed that* $\Lambda_p \subset \tilde{\Lambda}_p$ *and, for all* $\theta \in \tilde{\Lambda}_p$*, one has*

$$\lim_{n\to\infty} \frac{M_n}{\sum_{k=1}^{n} x_{k-1}^2} = \frac{pF}{tr\, F} = \Lambda(\theta_1, \ldots, \theta_p) \text{ a.s.}, \tag{2.5}$$

where F is the same as in (1.7). Indeed, by making use of the identity

$$\sum_{k=1}^{n} x_{k-1}^2 = \frac{1}{p}\sum_{k=1}^{n} \|X_{k-1}\|^2 + \frac{1}{p}\sum_{i=2}^{p}\sum_{l=n-i+2}^{n} x_{l-1}^2 \,, \tag{2.6}$$

one obtains

$$\frac{M_n}{\sum_{k=1}^{n} x_{k-1}^2} = \frac{M_n}{n}\left(\frac{1}{p}tr\frac{M_n}{n}(1+(\sum_{k=1}^{n}\|X_{k-1}\|^2)^{-1}\sum_{i=2}^{p}\sum_{l=n-i+2}^{n} x_{l-1}^2)\right)^{-1}.$$

Limiting $n \to \infty$, one comes to (2.5), in view of (1.7).

Theorem 2.1 *Suppose that in the AR(p) model (1.1), (ε_n) is a sequence of i.i.d. random variables with $\mathbf{E}\varepsilon_n = 0$ and $\mathbf{E}\varepsilon_n^2 = \sigma^2 < \infty$ and the parameter vector $\theta = (\theta_1, \ldots, \theta_p)'$ satisfies Conditions 1-3. Then for any compact set $K \subset \tilde{\Lambda}_p$*

$$\lim_{h\to\infty}\sup_{\theta\in K}\sup_{t\in R^p}\left|\mathbf{P}_\theta\left(M_{\tau(h)}^{1/2}(\theta(\tau(h)) - \theta) \le t\right) - \Phi_p(\frac{t}{\sigma})\right| = 0\,, \tag{2.7}$$

where $\Phi_p(t) = \Phi(t_1)\cdots\Phi(t_p), \Phi$ is the standard normal distribution function, $\tilde{\Lambda}_p$ is defined in Condition 3.

Proof. Substituting (1.1) in (2.1) yields

$$M_{\tau(h)}^{1/2}(\theta(\tau(h) - \theta) = M_{\tau(h)}^{-1/2}\sum_{k=1}^{\tau(h)} X_{k-1}\varepsilon_k = \sqrt{h}M_{\tau(h)}^{-1/2}L^{1/2}(\theta_1, \ldots, \theta_p)Y_h, \tag{2.8}$$

where

$$Y_h = \frac{1}{\sqrt{h}}\sum_{k=1}^{\tau(h)} L^{-1/2}(\theta_1, \ldots, \theta_p)X_{k-1}\varepsilon_k \tag{2.9}$$

and $L(\theta_1, \ldots, \theta_p)$ is given in (2.4). Denote

$$\mathcal{G}_{\tau(h)} = L^{-1/2}(\theta_1, \ldots, \theta_p)M_{\tau(h)}L^{-1/2}(\theta_1, \ldots, \theta_p).$$

One can easily verify that

$$\left\|L^{-1/2}(\theta_1, \ldots, \theta_p)\frac{M_{\tau(h)}^{1/2}}{\sqrt{h}} - I_p\right\|^2 = \left\|\frac{1}{\sqrt{h}}\mathcal{G}_{\tau(h)}^{1/2} - I_p\right\|^2$$

$$\le \|h^{-1}\mathcal{G}_{\tau(h)} - I_p\|^2 \le tr\, L^{-1}(\theta_1, \ldots, \theta_p)\|h^{-1}M_{\tau(h)} - L(\theta_1, \ldots, \theta_p)\|^2\,.$$

From here, by making use of Lemma 3.4 from Section 3, one gets, for any compact set $K \subset \tilde{\Lambda}_p$ and $\delta > 0$,

$$\lim_{h\to\infty} \sup_{\theta\in K} \mathbf{P}_\theta \left(\|\sqrt{h} M_{\tau(h)}^{-1/2} L^{1/2}(\theta_1, \ldots, \theta_p) - I_p\| > \delta \right) = 0 \,. \tag{2.10}$$

Now we prove that for any compact set $K \subset \tilde{\Lambda}_p$ and for each constant vector $v \in R^p$ with $\|v\| = 1$

$$\lim_{h\to\infty} \sup_{\theta\in K} \sup_{t\in R} |\mathbf{P}_\theta(v' Y_h \le t) - \Phi(t)| = 0 \,. \tag{2.11}$$

In view of (2.9), one has

$$v' Y_h = \frac{1}{\sqrt{h}} \sum_{k=1}^{\tau(h)} g_{k-1}\varepsilon_k, \;\; g_{k-1} = v' L^{-1/2}(\theta_1, \ldots, \theta_p) X_{k-1}.$$

For each $h > 0$, we define an auxiliary stopping time as

$$\tau_0 = \tau_0(h) = \inf\{n \ge 1 : \sum_{k=1}^{n} g_{k-1}^2 \ge h\}, \, \inf\{\emptyset\} = +\infty.$$

Further we make use of the representation

$$v' Y_h = \frac{1}{\sqrt{h}} \sum_{k=1}^{\tau_0(h)} g_{k-1}\varepsilon_k + \eta(h) + \Delta(h) \,,$$

where $\Delta(h) = \sum_{i=1}^{4} \Delta_i(h)$,

$$\Delta_1(h) = h^{-1/2} I_{(\tau(h)=1)} g_0 \varepsilon_1, \Delta_2(h) = h^{-1/2} g_{\tau(h)-1} \varepsilon_{\tau(h)} \,,$$

$$\Delta_3(h) = -h^{-1/2} I_{(\tau_0(h)=1)} g_0 \varepsilon_1, \Delta_4(h) = h^{-1/2} g_{\tau_0(h)-1} \varepsilon_{\tau_0(h)} \,,$$

$$\eta(h) = \frac{1}{\sqrt{h}} \sum_{k=1}^{\tau(h)-1} g_{k-1}\varepsilon_k - \frac{1}{\sqrt{h}} \sum_{k=1}^{\tau_0(h)-1} I_{(\tau_0(h)>1)} g_{k-1}\varepsilon_k \,.$$

Now we show that

$$\lim_{h\to\infty} \sup_{\theta\in K} \sup_{t\in R} |\mathbf{P}_\theta(\frac{1}{\sqrt{h}} \sum_{k=1}^{\tau_0(h)} g_{k-1}\varepsilon_k \le t) - \Phi(t)| = 0 \tag{2.12}$$

and for any $\delta > 0$

$$\lim_{h\to\infty} \sup_{\theta\in K} \mathbf{P}_\theta(|\eta(h)| > \delta) = 0 \,, \tag{2.13}$$

$$\lim_{h\to\infty}\sup_{\theta\in K}\mathbf{P}_\theta(|\Delta(h)|>\delta)=0\,. \tag{2.14}$$

The proof of (2.12) is based on Proposition 3.1 from the paper by Lai and Siegmund (1983). Actually one needs to check only the condition A_6, that is, for each $\delta>0$,

$$\lim_{m\to\infty}\sup_{\theta\in K}\mathbf{P}_\theta(g_n^2\geq\delta\sum_{k=1}^n g_{k-1}^2 \text{ for some } n\geq m)=0\,. \tag{2.15}$$

Conditions A_1-A_5 are evidently satisfied. It will be noted that

$$\sum_{k=1}^n g_{k-1}^2=\left(v'L^{-1/2}(M_n/\sum_{k=1}^n x_{k-1}^2-L)L^{-1/2}v+1\right)\sum_{k=1}^n x_{k-1}^2.$$

Proceeding from this equality one gets the inclusion

$$\{g_n^2\geq\delta\sum_{k=1}^n g_{k-1}^2 \text{ for some } n\geq m\}\subseteq$$

$$\subseteq\{\|X_n\|^2\geq\delta_1\sum_{k=1}^n g_{k-1}^2 \text{ for some } n\geq m\}$$

$$=\{\|X_n\|^2\geq\delta_1\sum_{k=1}^n x_{k-1}^2[1+v'L^{-1/2}(M_n/\sum_{k=1}^n x_{k-1}^2-L)L^{-1/2}v] \text{ for some } n\geq m\}$$

$$\subseteq\{\|X_n\|^2\geq\delta_1\sum_{k=1}^n x_{k-1}^2[1-\|L^{1/2}v\|^2\|M_n/\sum_{k=1}^n x_{k-1}^2-L\|] \text{ for some } n\geq m\}$$

$$\subseteq\{\|X_n\|^2\geq\delta_1\sum_{k=1}^n x_{k-1}^2[1-a^*\|M_n/\sum_{k=1}^n x_{k-1}^2-L\|] \text{ for some } n\geq m\}$$

$$\subset\{\|M_n/\sum_{k=1}^n x_{k-1}^2-L\|\geq(2a^*)^{-1} \text{ for some } n\geq m\}$$

$$\cup\{\|X_n\|^2\geq\frac{\delta_1}{2}\sum_{k=1}^n x_{k-1}^2 \text{ for some } n\geq m\}\,,$$

where $\delta_1=\delta/a^*$, $a^*=\sup_{\theta\in K}\|v'L^{-1/2}\|^2$.

This, in view of Lemmas 3.1,3.3, yields (2.15). It will be observed that (2.15) enables one to show (by the same argument as in Lemma 3.5) that, for any compact set $K \subset \tilde{\Lambda}_p$ and $\delta > 0$

$$\lim_{h\to\infty} \sup_{\theta\in K} \mathbf{P}_\theta(g^2_{\tau_0-1}/\sum_{k=1}^{\tau_0-1} g^2_{k-1} \geq \delta) = 0\,. \tag{2.16}$$

Now we check (2.13). One can easily verify that

$$\mathbf{E}_\theta \eta^2(h) = \mathbf{E}_\theta u(h), u(h) = \frac{1}{h}\left|\sum_{k=1}^{\tau-1} g^2_{k-1} - \sum_{k=1}^{\tau_0-1} g^2_{k-1}\right|.$$

The random variable $u(h)$ is uniformly bounded from above uniformly in $\theta \in K$ because

$$u(h) \leq \frac{1}{h}\sum_{k=1}^{\tau-1} g^2_{k-1} + 1 = \frac{1}{h} v' L^{-1/2} M_{\tau(h)-1} L^{-1/2} v + 1$$

$$\leq \frac{a^*}{h}\sum_{k=1}^{\tau-1} \|X_{k-1}\|^2 + 1 \leq a^* + 1\,.$$

Therefore, it suffices to show that for each $\delta > 0$

$$\lim_{h\to\infty} \sup_{\theta\in K} \mathbf{P}_\theta(u(h) \geq \delta) = 0\,. \tag{2.17}$$

To this end, one can use the following estimate

$$u(h) = \frac{1}{h}\left| v' L^{-1/2} M_{\tau(h)-1} L^{-1/2} v - \sum_{k=1}^{\tau_0-1} g^2_{k-1}\right|$$

$$= h^{-1}\sum_{k=1}^{\tau-1} x^2_{k-1} v' L^{-1/2}\left(M_{\tau(h)-1}/\sum_{k=1}^{\tau-1} x^2_{k-1} - L\right) L^{-1/2} v$$

$$+ h^{-1}\sum_{k=1}^{\tau-1} x^2_{k-1} - h^{-1}\sum_{k=1}^{\tau_0-1} g^2_{k-1}$$

$$\leq a^* \|M_{\tau(h)-1}/\sum_{k=1}^{\tau(h)-1} x^2_{k-1} - L\| + x^2_{\tau(h)-1}/\sum_{k=1}^{\tau(h)-1} x^2_{k-1} + g^2_{\tau_0(h)-1}/\sum_{k=1}^{\tau_0(h)-1} g^2_{k-1}\,.$$

From here, by making use of (2.16) and Lemmas 3.4,3.5 one comes to (2.17) which implies (2.13). By a similar argument, one can check (2.14). This completes the proof of (2.11). Combining (2.10) and (2.11) one arrives at (2.7). Hence Theorem 2.1. $\square$

Now we will study the properties of the stopping time $\tau(h)$ defined by (2.2). Further we need the following functionals

$$\begin{aligned}
&J_1(x;t) = \int_0^t x^2(s)ds, \\
&J_2(x,y;t) = \int_0^t (x^2(s) + y^2(s))ds, \\
&J_3(x,y;t) = \int_0^t (x^2(s) + \mu_1 y^2(s))ds, \\
&J_4(x,y,z;t) = J_2(x,y;t) + \mu_2 J_1(z;t), \\
&J_5(x,y,z,u;t) = J_2(x,y;t) + \mu_3 J_1(z;t) + \mu_4 J_1(u;t),
\end{aligned} \tag{2.18}$$

where $\mu_i, i = \overline{1,4}$ are defined by (4.17),(4.27) and (4.28). For the set $\tilde{\Lambda}_p$ of the parameter vector $\theta = (\theta_1, \ldots, \theta_p)'$ satisfying Conditions 1-3, we introduce the following subsets belonging to its boundary $\partial\tilde{\Lambda}_p$

$$\begin{aligned}
&\Gamma_1(p) = \{\theta \in \partial\tilde{\Lambda}_p : z_1(\theta) = -1, |z_k(\theta)| < 1, k = \overline{2,p}\} \\
&\Gamma_2(p) = \{\theta \in \partial\tilde{\Lambda}_p : z_1(\theta) = 1, |z_k(\theta)| < 1, k = \overline{2,p}\} \\
&\Gamma_3(p) = \{\theta \in \partial\tilde{\Lambda}_p : z_1(\theta) = e^{i\phi}, z_2(\theta) = e^{-i\phi}, \phi \in (0,\pi), |z_k(\theta)| < 1, k = \overline{3,p}\} \\
&\Gamma_4(p) = \{\theta \in \partial\tilde{\Lambda}_p : z_1(\theta) = -1, z_2(\theta) = 1, |z_k(\theta)| < 1, k = \overline{3,p}\} \\
&\Gamma_5(p) = \{\theta \in \partial\tilde{\Lambda}_p : z_1(\theta) = -1, z_2(\theta) = e^{i\phi}, z_3(\theta) = e^{-i\phi}, \phi \in (0,\pi), \\
&|z_k(\theta)| < 1, k = \overline{4,p}\}, \\
&\Gamma_6(p) = \{\theta \in \partial\tilde{\Lambda}_p : z_1(\theta) = 1, z_2(\theta) = e^{i\phi}, z_3(\theta) = e^{-i\phi}, \phi \in (0,\pi), \\
&|z_k(\theta)| < 1, k = \overline{4,p}\} \\
&\Gamma_7(p) = \{\theta \in \partial\tilde{\Lambda}_p : z_1(\theta) = -1, z_2(\theta) = 1, z_3(\theta) = e^{i\phi}, z_4(\theta) = e^{-i\phi}, \phi \in (0,\pi), \\
&|z_k(\theta)| < 1, k = \overline{5,p}\},
\end{aligned} \tag{2.19}$$

where $z_k(\theta)$ are roots of the characteristic polynomial (1.5).

It will be noted that all these sets will be used only for the AR(p) model (1.1) with $p \geq 5$. In the case when $p \leq 4$, it is obious which of the sets $\Gamma_i(p)$ are odd and how to amend the remaining subsets $\Gamma_i(p)$.

Theorem 2.2 *Suppose that in the AR(p) model (1.1), $(\varepsilon_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with $\mathbf{E}\varepsilon_n = 0$, $0 < \mathbf{E}\varepsilon_n^2 = \sigma^2 < \infty$ and the parameter vector $\theta = (\theta_1, \ldots, \theta_p)$ satisfies Conditions 1-3. Let $\tau(h)$ be defined by (2.2). Then, for each $\theta \in \Lambda_p$,*

$$\mathbf{P}_\theta - \lim_{h\to\infty} \frac{\tau(h)}{h} = \frac{\sigma^2}{trF}. \tag{2.20}$$

Moreover, for each $\theta \in \partial\tilde{\Lambda}_p$, as $h \to \infty$,

$$\frac{\tau(h)}{b_i\sqrt{h}} \xrightarrow{\mathcal{L}} \nu_i, \text{ if } \theta \in \Gamma_i(p), 1 \leq i \leq 7\,, \tag{2.21}$$

where Λ_p *is given in (1.6);*

$$\begin{aligned}
\nu_1 &= \inf\{t \ge 0 : J_1(W_1;t) \ge 1\}\,, \\
\nu_2 &= \inf\{t \ge 0 : J_1(W_2;t) \ge 1\}\,, \\
\nu_3 &= \inf\{t \ge 0 : J_2(W_1,W_2;t) \ge 1\}\,, \\
\nu_4 &= \inf\{t \ge 0 : J_3(W_1,W_2;t) \ge 1\}\,, \\
\nu_i &= \inf\{t \ge 0 : J_4(W_1,W_2,W_3;t) \ge 1\}\,,\ i=5,6, \\
\nu_7 &= \inf\{t \ge 0 : J_5(W_1,W_2,W_3,W_4;t) \ge 1\}\,;
\end{aligned}$$

$b_1,\ldots,b_7$ *are defined by (4.7),(4.8),(4.10),(4.17), (4.27) and (4.28), respectively;* $W_1(t),\ldots,W_4(t)$ *are independent standard brownian motions.*

The proof of Theorem 2.2 is given in the Appendix.

3 Auxiliary propositions.

In this Section we establish some properties of the process (1.1) and the observed Fisher information matrix M_n used in Section 2.

We need some notations. Let $z_1(\theta),\ldots,z_q(\theta)$ denote all the distinct roots of the characteristic polynomial (1.5), $m_i(\theta)$ be the multiplicity of $z_i(\theta)$, $(\sum_{i=1}^q m_i(\theta)=p)$. Let

$$\rho(\theta)=\begin{cases} \max(m_i(\theta) : |z_i(\theta)|=1) & \text{if } \max|z_i(\theta)|=1, \\ 0 & \text{if } \max|z_i(\theta)|\ne 1 \text{ for all } i=\overline{1,q}. \end{cases}$$

Formally the set $\overset{\circ}{\Lambda}_p$ introduced in Condition 3 can be written as

$$\overset{\circ}{\Lambda}_p = \Lambda_p \cup \{\theta : \max_{1\le i\le q}|z_i(\theta)|=1,\ \rho(\theta)=1\}. \tag{3.1}$$

It includes both the stability region Λ_p and the points θ of its boundary for which all the roots of the polynomial (1.5), lying on the unit circle, are simple.

Lemma 3.1 *Let* $(x_n)_{n\ge0}$ *be an autoregressive process defined by (1.1). Then for any compact set* $K\subset\overset{\circ}{\Lambda}_p$ *and* $\delta>0$

$$\lim_{m\to\infty}\sup_{\theta\in K}\mathbf{P}_\theta(\max_{0\le i\le p-1}x_{n-i}^2 \ge \delta\sum_{k=1}^n x_{k-1}^2 \text{ for some } n\ge m)=0\,.$$

Proof. Taking into account the equation (1.2), it suffices to show that for any compact set $K \subset \overset{\circ}{\Lambda}_p$ and $\delta > 0$

$$\lim_{m\to\infty} \sup_{\theta\in K} \mathbf{P}_\theta(B_m(\delta)) = 0\,,$$

where

$$B_m(\delta) = \{\|X_n\|^2 \ge \delta \sum_{k=1}^{n} \|X_{k-1}\|^2 \text{ for some } n \ge m\}.$$

Now we estimate the ratio $\|X_n\|^2/\sum_{k=1}^n \|X_{k-1}\|^2$ from above. For each $1 \le s < n$, we introduce the quantity

$$l_s = \min\{1 \le i \le s : \min_{1\le j\le s} \|X_{k-j}\|^2 = \|X_{k-i}\|^2\}$$

and have the inequality

$$\sum_{k=1}^{n} \|X_{k-1}\|^2 \ge \sum_{k=1}^{s} \|X_{n-k}\|^2 \ge s\|X_{n-l_s}\|^2\,. \tag{3.2}$$

On the other hand, it follows from (1.1) that

$$X_n = A^{l_s} X_{n-l_s} + \sum_{i=0}^{l_s-1} A^i \xi_{n-i}$$

and, therefore, one gets

$$\|X_n\|^2 \le 2\|A^{l_s}\|^2\|X_{n-l_s}\|^2 + 2\|\sum_{i=0}^{l_s-1} A^i\xi_{n-i}\|^2$$

$$\le 2\|A^{l_s}\|^2\|X_{n-l_s}\|^2 + 2\left(\sum_{i=0}^{s-1}\|A^i\xi_{n-i}\|\right)^2 \le 2\|A^{l_s}\|^2\|X_{n-l_s}\|^2 + 2s\sum_{i=1}^{s-1}\|A^i\xi_{n-i}\|^2$$

$$\le 2\|A^{l_s}\|^2\|X_{n-l_s}\|^2 + 2s\sum_{i=1}^{s-1}\|A^i\|^2\varepsilon_{n-i}^2\,. \tag{3.3}$$

Further it will be observed that, for every compact set $K \subset \overset{\circ}{\Lambda}_p$, there exists a positive number κ such that

$$\sup_{\theta\in K}\|A^n\|^2 \le \kappa,\ n \ge 1\,. \tag{3.4}$$

Indeed, we express A in its Jordan normal form

$$A = S\mathcal{D}S^{-1}\,, \tag{3.5}$$

where $\mathcal{D} = \mathrm{diag}(J_1, \ldots, J_q)$, J_l is the $m_l \times m_l$ submatrix of the form

$$J_l = \begin{pmatrix} z_l & 1 & 0 & \ldots & 0 \\ 0 & z_l & 1 & \ldots & 0 \\ 0 & \ldots & & z_l & 1 \\ 0 & \ldots & & & z_l \end{pmatrix}$$

if z_l is a multiple root with multiplicity $m_l \geq 2$, and $J_l = z_l$ if z_l is a simple root.

By direct computation with (3.5) one finds $A^n = S\mathcal{D}^n S^{-1}$, $\mathcal{D}^n = \mathrm{diag}(J_1^n, \ldots, J_q^n)$, where the powers of the matrix J_l are equal to z_l^n for a simple root z_l and consist of the elements (see, R.Varga (2000))

$$< J_l^n >_{ij} = \begin{cases} 0, & j < i\,, \\ \binom{n}{j-i} z_l^{n-j+i}, & i \leq j \leq \min(m_l, n+i)\,, \\ 0, & n+i < j \leq m_l\,, \end{cases}$$

for the roots z_l with multiplicity $m_l \geq 2$, $\binom{n}{j-i}$ is the binomial coefficient. From here, in view of the definition (3.1), one comes to (3.4). By making use of (3.3) and (3.4), one obtains

$$\|X_n\|^2 \leq 2\kappa \|X_{n-l_s}\|^2 + 2s\kappa \sum_{i=0}^{s-1} \varepsilon_{n-i}^2\,.$$

Combining this inequality and (3.2) yields

$$\frac{\|X_n\|^2}{\sum_{k=1}^n \|X_{k-1}\|^2} \leq \frac{2\kappa}{s} + 2s\kappa \frac{\sum_{i=0}^{s-1} \varepsilon_{n-i}^2}{\sum_{k=1}^n \|X_{k-1}\|^2}\,.$$

It remains to use elementary inequality

$$\sum_{i=0}^{n-1} \varepsilon_k^2 \leq 2(1 + \|A\|^2) \sum_{k=1}^n \|X_{k-1}\|^2\,,$$

which follows from (1.2), to derive the desired estimate for the ratio

$$\frac{\|X_n\|^2}{\sum_{k=1}^n \|X_{k-1}\|^2} \leq \frac{2\kappa}{s} + 2s\kappa \frac{2(1 + \|A\|^2) \sum_{i=0}^{s-1} \varepsilon_{n-i}^2}{\sum_{k=1}^{n-1} \varepsilon_k^2}\,.$$

This inequality implies the inclusion

$$B_m(\delta) \subset \{2\kappa/s > \delta/2 \text{ for some } n \geq m\}$$

$$\cup\{4(1+\|A\|^2)s\kappa \sum_{i=0}^{s-1} \varepsilon_{i-1}^2 / \sum_{k=1}^{n-1} \varepsilon_k^2 > \frac{\delta}{2} \text{ for some } n \geq m\}\,.$$

Therefore, for sufficiently large s, one gets

$$\sup_{\theta \in K} \mathbf{P}_\theta(B_m(\delta) \leq \mathbf{P}\{\nu \sum_{i=0}^{s-1} \varepsilon_{i-1}^2 / \sum_{k=1}^{n-1} \varepsilon_k^2 > \frac{\delta}{2} \text{ for some } n \geq m\}\,,$$

where $\nu = \sup_{\theta \in K} 4(1+\|A\|^2)s\kappa$. Limiting $m \to \infty$ and applying the law of large numbers one comes to the assertion of Lemma 3.1.

Lemma 3.2 *Let $(x_n)_{n\geq 0}$ be an autoregressive process defined by (1.1). Then for any compact set $K \subset \overset{\circ}{\Lambda}_p$ and each $l = \overline{1, p-1}$*

$$\lim_{m\to\infty} \sup_{\theta \in K} \mathbf{P}_\theta(\left|\sum_{k=1}^{n} x_{k-l}\varepsilon_k\right| \geq \delta \sum_{k=1}^{n} x_{k-1}^2 \text{ for some } n \geq m) = 0\,, \tag{3.6}$$

where $\overset{\circ}{\Lambda}_p$ is given in (3.1)

Proof. We will apply Lemma 2.2 from the paper by Lai and Siegmund (1983)) given in the Appendix. Let $c_n = n^{3/4}$. For the set of interest one has the following inclusions

$$\left\{|\sum_{k=1}^{n} x_{k-l}\varepsilon_k| > \delta \sum_{k=1}^{n} x_{k-1}^2 \text{ for some } n \geq m\right\}$$

$$\subseteq \left\{|\sum_{k=1}^{n} x_{k-l}\varepsilon_k| > \delta \sum_{k=1}^{n} x_{k-l}^2 \text{ for some } n \geq m\right\}$$

$$= \left\{\frac{|\sum_{k=1}^n x_{k-l}\varepsilon_k|}{(\sum_{k=1}^n x_{k-l}^2)^{2/3} \vee c_n} \cdot \frac{(\sum_{k=1}^n x_{k-l}^2)^{2/3} \vee c_n}{\sum_{k=1}^n x_{k-l}^2} > \delta \text{ for some } n \geq m\right\}$$

$$\subseteq \left\{\frac{|\sum_{k=1}^n x_{k-l}\varepsilon_k|}{(\sum_{k=1}^n x_{k-l}^2)^{2/3} \vee c_n} > \sqrt{\delta} \text{ for some } n \geq m\right\} \cup$$

$$\left\{(\sum_{k=1}^{n} x_{k-l}^2)^{-1/3} \vee c_n(\sum_{k=1}^{n} x_{k-l}^2)^{-1} > \sqrt{\delta} \text{ for some } n \geq m\right\}$$

$$\subset \left\{ |\sum_{k=1}^{n} x_{k-l}\varepsilon_k| \left((\sum_{k=1}^{n} x_{k-l}^2)^{2/3} \vee c_n \right)^{-1} > \sqrt{\delta} \text{ for some } n \geq m \right\} \cup$$

$$\cup \left\{ (\sum_{k=1}^{n} x_{k-l}^2)^{-1} > \delta^{3/2} \text{ for some } n \geq m \right\} \cup$$

$$\cup \left\{ c_n(\sum_{k=1}^{n} x_{k-l}^2)^{-1} > \sqrt{\delta} \text{ for some } n \geq m \right\}$$

$$\subset \left\{ |\sum_{k=1}^{n} x_{k-l}\varepsilon_k| > \delta(\sum_{k=1}^{n} x_{k-l}^2)^{2/3} \vee c_n \text{ for some } n \geq m \right\}$$

$$\cup \left\{ c_n(\sum_{k=1}^{n} x_{k-l}^2)^{-1} > \sqrt{\delta} \wedge \delta^{3/2} \text{ for some } n \geq m \right\}.$$

From here, it follows that

$$\mathbf{P}_\theta \left(|\sum_{k=1}^{n} x_{k-l}\varepsilon_k| > \delta \sum_{k=1}^{n} x_{k-l}^2 \text{ for some } n \geq m \right)$$

$$\leq \mathbf{P}_\theta \left(|\sum_{k=1}^{n} x_{k-l}\varepsilon_k| > \delta \left(\sum_{k=1}^{n} x_{k-l}^2 \right)^{2/3} \vee c_n \text{ for some } n \geq m \right)$$

$$+\mathbf{P}_\theta \left\{ n^{3/4}(\sum_{k=1}^{n} x_{k-l}^2)^{-1} > \sqrt{\delta} \wedge \delta^{3/2} \text{ for some } n \geq m \right\}. \tag{3.7}$$

By making use of (1.1) and the elementary inequalities, one obtains

$$\sum_{k=1}^{n} \varepsilon_k^2 = \sum_{k=1}^{n} (x_k - \theta_1 x_{k-1} - \ldots - \theta_p x_{k-p})^2$$

$$\leq (p+1) \left(\sum_{k=1}^{n} x_k^2 + \sum_{j=1}^{p} \theta_j^2 \sum_{k=1}^{n} x_{k-j}^2 \right)$$

$$\leq (p+1) \max_{1 \leq j \leq p} \theta_j^2 (\sum_{k=1}^{n} x_k^2 + \sum_{j=1}^{p} \sum_{k=1}^{n} x_{k-j}^2)$$

$$\leq (p+1)^2 \mu_k \sum_{k=1}^{n} x_{k-l}^2 \left(1 + \sum_{k=n-l+1}^{n} x_k^2 / \sum_{k=1}^{n} x_{k-l}^2 \right),$$

where $\mu_k = \sup_{\theta\in K} \|\theta\|^2$.

Therefore the second summand in the right-hand side of (3.6) can be estimated as

$$\mathbf{P}_\theta \left\{ n^{3/4}(\sum_{k=1}^{n} x_{k-l}^2)^{-1} > \sqrt{\delta} \wedge \delta^{3/2} \text{ for some } n \geq m \right\}$$

$$\leq \mathbf{P}_\theta \left\{ 2(p+1)^2 \mu_k n^{3/4} (\sum_{k=1}^{n} \varepsilon_k^2)^{-1} > \sqrt{\delta} \wedge \delta^{3/2} \text{ for some } n \geq m \right\}$$

$$+\mathbf{P}_\theta \left\{ \sum_{k=n-l+1}^{n} x_k^2 / \sum_{k=1}^{n} x_{k-l}^2 \geq 1 \text{ for some } n \geq m \right\}.$$

Combining this and (3.7) and applying Lemma 3.1, and Lemma 2.2 by Lai and Siegmund (see Section 4) yield (3.6).

This completes the proof of Lemma 3.2. □

Lemma 3.3 *Let parameters $\theta_1, \ldots, \theta_p$ in the equation (1.1) satisfy Conditions 1-3 and $p \times p$ matrix M_n be given by (1.3). Then, for any compact set $K \subset \tilde{\Lambda}_p$ and each $\delta > 0$,*

$$\lim_{m\to\infty} \sup_{\theta\in K} \mathbf{P}_\theta \left(\| \frac{M_n}{\sum_{k=1}^n x_{k-1}^2} - L(\theta_1, \ldots, \theta_p) \| \geq \delta \text{ for some } n \geq m \right) = 0,$$

where $L(\theta_1, \ldots, \theta_p)$ is defined in (2.4).

Proof. Each diagonal element of the matrix M_n can be expressed through $\sum_{l=1}^n x_{l-1}^2$ as

$$< M_n >_{ii} = \sum_{k=i}^{n} x_{k-i}^2 = \sum_{l=1}^{n-i+1} x_{l-1}^2 = \sum_{l=1}^{n} x_{l-1}^2 - \sum_{l=n-i+2}^{n} x_{l-1}^2, \ 2 \leq i \leq p\,. \quad (3.8)$$

Further it will be observed that each element $< M_n >_{ij}$, $2 \leq i < j \leq p$, of M_n standing above the principal diadonal and below the first row can be expressed through some element of the first row as

$$< M_n >_{ij} = \sum_{k=1}^{n} x_{k-i} x_{k-j} = \sum_{l=1}^{n} x_{l-1} x_{l-1+i-j} - \sum_{l=n-i+2}^{n} x_{l-1} x_{l-1+i-j} \quad (3.9)$$

$$= < M_n >_{i,j-i+1} - \sum_{t=n-i+2}^{n} x_{t-1+i-j}\,.$$

Now we derive the equations relating the elements of the first row, that is, $< M_n >_{1s},\ 2 \le s \le p$. Making use of the equation (1.1), one gets

$$< M_n >_{1s} = \sum_{k=1}^{n} x_{k-1} x_{k-s} = \sum_{k=1}^{n} \left(\sum_{l=1}^{p} \theta_l x_{k-l-1} + \varepsilon_{k-1} \right) x_{k-s}$$

$$= \sum_{l=1}^{p} \theta_l \sum_{k=1}^{n} x_{k-l-1} x_{k-s} + \sum_{k=1}^{n} x_{k-s} \varepsilon_{k-1} .$$

Since

$$\sum_{k=1}^{n} x_{k-l-1} x_{k-s} = \begin{cases} < M_n >_{ss}, & \text{if } l = s-1, \\ < M_n >_{s,l+1}, & \text{if } l \ge s, \\ < M_n >_{l+1,s}, & \text{if } l \le s-2, \end{cases}$$

this implies the following system of equations

$$< M_n >_{12} = \theta_1 < M_n >_{22} + \sum_{l=2}^{p} \theta_l < M_n >_{2,l+1} + \sum_{k=1}^{n} x_{k-2} \varepsilon_{k-1} ,$$

$$< M_n >_{1s} = \sum_{l=2}^{s-2} \theta_l < M_n >_{l+1,s} + \theta_{s-1} < M_n >_{ss}$$

$$+ \sum_{l=s}^{p} \theta_l < M_n >_{s,l+1} + \sum_{k=1}^{n} x_{k-s} \varepsilon_{k-1},\ 3 \le s \le p.$$

Taking into account (3.8),(3.9) one can represent this system as

$$< M_n >_{12} = \theta_1 \sum_{k=1}^{n} x_{k-1}^2 + \sum_{l=2}^{p} \theta_l < M_n >_{1,l} + \eta_{1,2}(n), \tag{3.10}$$

$$< M_n >_{1s} = \sum_{l=1}^{s-2} \theta_l < M_n >_{1,s-l} + \theta_{s-1} \sum_{k=1}^{n} x_{k-1}^2 + \sum_{l=s}^{p} \theta_l < M_n >_{1,l-s+2} + \eta_{1,s}(n),$$

where $3 \le s \le p$,

$$\eta_{1,2}(n) = -\theta_1 x_{n-1}^2 - \sum_{l=2}^{p} \theta_l x_{n-1} x_{n-l} + \sum_{k=1}^{n} x_{k-2} \varepsilon_{k-1},$$

$$\eta_{1,s}(n) = -\sum_{l=1}^{s-2} \theta_l \sum_{t=n-l+1}^{n} x_{t-1} x_{t+l-s} - \theta_{s-1} \sum_{l=n-s+2}^{n} x_{l-1}^2$$

$$-\sum_{l=s}^{p}\theta_l\sum_{t=n-s+2}^{n}x_{t-1}x_{t+s-l-2}+\sum_{k=1}^{n}x_{k-s}\varepsilon_{k-1}.$$

Denote

$$z_i(n)=\frac{<M_n>_{1,i+1}}{\sum_{k=1}^{n}x_{k-1}^2},\ \tilde{\eta}_{1,i+1}=\frac{\eta_{1,i+1}(n)}{\sum_{k=1}^{n}x_{k-1}^2},\ i=\overline{1,p-1}.$$

Then the system of equations (3.10) takes the form

$$\begin{cases} z_1(n)-\sum_{l=2}^{p}\theta_l z_{l-1}(n)=\theta_1+\tilde{\eta}_{1,2}, \\ -\sum_{k=1}^{j-1}\theta_{j-k}z_k(n)+z_j(n)-\sum_{k=1}^{p-j}\theta_{k+j}z_k(n)=\theta_j+\tilde{\eta}_{1,j+1}, \\ j=\overline{2,p-1} \end{cases} \tag{3.11}$$

In virtue of Lemmas 3.1,3.2, for any compact set $K\subset\tilde{\Lambda}_p$ and $\delta>0$, one has

$$\lim_{m\to\infty}\sup_{\theta\in K}\mathbf{P}_\theta(\max_{2\le s\le p}|\tilde{\eta}_{1,s}(n)|>\delta \text{ for some } n\ge m)=0.$$

From here and the Condition 3, which holds for each vector $\theta\in K$, it follows that the solution of the system (3.11) converges, as $n\to\infty$, to the unique solution of system (2.3) uniformly in $\theta\in K$, that is,

$$\lim_{m\to\infty}\sup_{\theta\in K}\mathbf{P}_\theta(\max_{1\le i\le p-1}|z_i(n)-\kappa_i(\theta)|>\delta \text{ for some } n\ge m)=0.$$

This, in view of (3.9) and Lemma 3.1, implies the desired convergence of the remaining elements of the matrix M_n. Hence Lemma 3.3. $\square$

Lemma 3.4 *Let $M_n,\tau(h)$ and $L(\theta)=L(\theta_1,\dots,\theta_p)$ be given by (1.4), (2.2) and (2.4), respectively. Then, for any compact set $K\subset\mathring{\Lambda}_p$ and $\delta>0$,*

$$\lim_{h\to\infty}\sup_{\theta\in K}\mathbf{P}_\theta\left(\|\frac{M_{\tau(h)}}{h}-L(\theta_1,\dots,\theta_p)\|>\delta\right)=0, \tag{3.12}$$

where $\mathring{\Lambda}_p$ is defined in Condition 3.

Proof. By making use of the equality

$$\frac{M_{\tau(h)}}{h}-L(\theta)=\frac{M_{\tau(h)}}{\sum_{k=1}^{\tau(h)}x_{k-1}^2}-L(\theta)+M_{\tau(h)}\left(\frac{1}{h}-\frac{1}{\sum_{k=1}^{\tau(h)}x_{k-1}^2}\right)$$

one gets the estimate

$$\|\frac{M_{\tau(h)}}{h}-L(\theta)\|\le\|\frac{M_{\tau(h)}}{\sum_{k=1}^{\tau(h)}x_{k-1}^2}-L(\theta)\|$$

$$+\left(\|M_{\tau(h)}\|/\sum_{k=1}^{\tau(h)}x_{k-1}^2\right)\left(\sum_{k=1}^{\tau(h)}x_{k-1}^2-h\right)/h.$$

Therefore

$$\left\{\|\frac{M_{\tau(h)}}{h}-L(\theta)\|>\delta\right\}\subset\left\{\|\frac{M_{\tau(h)}}{\sum_{k=1}^{\tau(h)}x_{k-1}^2}-L(\theta)\|>\delta/2\right\}\tag{3.13}$$

$$\cup\left\{\|\frac{M_{\tau(h)}}{\sum_{k=1}^{\tau(h)}x_{k-1}^2}\|\frac{\left(\sum_{k=1}^{\tau(h)}x_{k-1}^2-h\right)}{h}>\delta/2\right\}.$$

Further one has the inclusions

$$\left\{\|\frac{M_{\tau(h)}}{\sum_{k=1}^{\tau(h)}x_{k-1}^2}-L(\theta)\|>\delta/2\right\}\subseteq\{\tau(h)\le m\}$$

$$\cup\left\{\|\frac{M_n}{\sum_{k=1}^{n}x_{k-1}^2}-L(\theta)\|>\delta/2\text{ for some }n\ge m\right\},$$

$$\left\{\|\frac{M_{\tau(h)}}{\sum_{k=1}^{\tau(h)}x_{k-1}^2}\|\left(\sum_{k=1}^{\tau(h)}x_{k-1}^2-h\right)/h>\delta/2\right\}\subset\{\tau(h)\le m\}$$

$$\cup\left\{\|\frac{M_n}{\sum_{k=1}^{n}x_{k-1}^2}-L(\theta)\|>\delta/2\text{ for some }n\ge m\right\},$$

$$\left\{\|\frac{M_{\tau(h)}}{\sum_{k=1}^{\tau(h)}x_{k-1}^2}\|\left(\sum_{k=1}^{\tau(h)}x_{k-1}^2-h\right)/h>\delta/2\right\}\subset\{\tau(h)\le m\}$$

$$\cup\left\{\frac{\|M_n\|}{\sum_{k=1}^{n}x_{k-1}^2}\cdot\frac{x_{n-1}^2}{\sum_{k=1}^{n-1}x_{k-1}^2}>\delta/2\text{ for some }n\ge m\right\}.$$

From here and (3.13), it follows that

$$\mathbf{P}_\theta\left\{\|\frac{M_{\tau(h)}}{h}-L(\theta)\|>\delta\right\}\le 2\mathbf{P}_\theta\{\tau(h)\le m\}\tag{3.14}$$

$$+\mathbf{P}_\theta\left\{\|\frac{M_n}{\sum_{k=1}^{n}x_{k-1}^2}-L(\theta)\|>\delta/2\text{ for some }n\ge m\right\}$$

$$+\mathbf{P}_\theta\left\{\frac{\|M_n\|}{\sum_{k=1}^{n}x_{k-1}^2}\cdot\frac{x_{n-1}^2}{\sum_{k=1}^{n-1}x_{k-1}^2}>\delta/2\text{ for some }n\ge m\right\}.$$

By the definition of $\tau(h)$ in (2.2)

$$\{\tau(h) < m\} = \left\{ \sum_{k=1}^{m} (x_{k-1}^2 + \cdots + x_{k-p}^2) > h \right\}$$

$$= \left\{ \sum_{k=1}^{m} (x_{k-1}^2 + \cdots + x_{k-p}^2) > h, \ \max_{1 \le j \le m} (x_{j-1}^2 + \cdots + x_{j-p}^2) < l \right\}$$

$$+ \left\{ \sum_{k=1}^{m} (x_{k-1}^2 + \cdots + x_{k-p}^2) > h, \ \max_{1 \le j \le m} (x_{j-1}^2 + \cdots + x_{j-p}^2) \ge l \right\}$$

$$\subset \{ml > h\} \cup \cup_{j=1}^{m} \{(x_{j-1}^2 + \cdots + x_{j-p}^2) \ge l\}. \tag{3.15}$$

This yields

$$\mathbf{P}_\theta \{\tau(h) < m\} \le I_{(ml>h)} + \sum_{k=1}^{m} \mathbf{P}_\theta \{x_{k-1}^2 + \cdots + x_{k-p}^2 \ge l\}.$$

Consider the last term in (3.14). By the inequality

$$\frac{\|M_n\|}{\sum_{k=1}^n x_{k-1}^2} \le \| \frac{M_n}{\sum_{k=1}^n x_{k-1}^2} - L(\theta)\| + \|L(\theta)\|$$

one has

$$\mathbf{P}_\theta \left\{ \frac{\|M_n\|}{\sum_{k=1}^n x_{k-1}^2} \cdot \frac{x_{n-1}^2}{\sum_{k=1}^{n-1} x_{k-1}^2} > \delta/2 \text{ for some } n \ge m \right\}$$

$$\le \mathbf{P}_\theta \left\{ \| \frac{M_n}{\sum_{k=1}^n x_{k-1}^2} - L(\theta)\| > \sqrt{\delta/4} \text{ for some } n \ge m \right\} \tag{3.16}$$

$$+ \mathbf{P}_\theta \left\{ L_k^* \frac{x_{n-1}^2}{\sum_{k=1}^{n-1} x_{k-1}^2} > \sqrt{\delta/4} \text{ for some } n \ge m \right\}$$

$$+ \mathbf{P}_\theta \left\{ L_k^* \frac{x_{n-1}^2}{\sum_{k=1}^{n-1} x_{k-1}^2} > \delta/4 \text{ for some } n \ge m \right\},$$

where $L_k^* = \sup_{\theta \in K} \|L(\theta)\|$.

Combining (3.14)-(3.16) yields

$$\sup_{\theta \in K} \mathbf{P}_\theta \left(\| \frac{M_{\tau(h)}}{h} - L(\theta)\| > \delta \right)$$

$$\leq 2I_{(ml>h)} + 2\sum_{k=1}^{m} \sup_{\theta\in K} \mathbf{P}_\theta\{\|X_{k-1}\|^2 \geq l\}$$

$$+\mathbf{P}_\theta\left\{\|\frac{M_n}{\sum_{k=1}^{n} x_{k-1}^2} - L(\theta)\| > \frac{1}{2}(\delta\wedge\sqrt{\delta}) \text{ for some } n\geq m\right\}$$

$$+2\mathbf{P}_\theta\left\{L_k^* \frac{x_{n-1}^2}{\sum_{k=1}^{n-1} x_{k-1}^2} > \frac{1}{2}(\delta\wedge\sqrt{\delta}) \text{ for some } n\geq m\right\}.$$

Limiting $h\to\infty,\ l\to\infty,\ m\to\infty$ and taking into account Lemma 3.3, one comes to (3.12). Hence Lemma 3.4. $\square$

Lemma 3.5 *Let x_k and $\tau(h)$ be defined by (1.1) and (2.2). Then for any compact set $K\subset \mathring{\Lambda}_0$ and $\delta>0$*

$$\lim_{h\to\infty} \sup_{\theta\in K} \mathbf{P}_\theta\left(x_{\tau(h)-1}^2 / \sum_{k=1}^{\tau(h)-1} x_{k-1}^2 > \delta\right) = 0. \tag{3.17}$$

Proof. In view of the inclusion

$$\left\{\frac{x_{\tau(h)-1}^2}{\sum_{k=1}^{\tau(h)-1} x_{k-1}^2} > \delta\right\} \subset \{\tau(h)\leq m\} \bigcup \left\{\frac{x_n^2}{\sum_{k=1}^{n-1} x_{k-1}^2} > \delta \text{ for some } n\geq m\right\}.$$

one has

$$\sup_{\theta\in K} \mathbf{P}_\theta\left\{\frac{x_{\tau(h)-1}^2}{\sum_{k=1}^{\tau(h)-1} x_{k-1}^2} > \delta\right\} \leq I_{(ml>h)} + \sum_{j=1}^{m} \sup_{\theta\in K} \mathbf{P}_\theta\left\{\|X_{j-1}\|^2 \geq l\right\}$$

$$+\sup_{\theta\in K} \mathbf{P}_\theta\left\{\frac{x_n^2}{\sum_{k=1}^{n-1} x_{k-1}^2} > \delta \text{ for some } n\geq m\right\}.$$

Limiting $h\to\infty$, $l\to\infty$, $m\to\infty$ and applying Lemma 3.1 lead to (3.17). This completes the proof of Lemma 3.5. $\square$

4 Appendix.

In this Section we cite the probabilistic result from the paper of Lai and Siegmund (1983) used in Section 3 and give the proof of Theorem 2.2.

Lemma 2.2 (by Lai and Siegmund (1983)). Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration on a measurable space $(\Omega,\mathcal{F})$, $(x_n)_{n\geq 0}$ and $(\varepsilon_n)_{n\geq 0}$ be sequences of random variables adapted to $(\mathcal{F}_n)_{n\geq 0}$. Let $(\mathbf{P}_\theta, \theta\in\Theta)$ be a family of probability measures on $(\Omega, \mathcal{F})$ such that under every $\mathbf{P}_\theta$ ε_n is independent of $\mathcal{F}_{n-1}$ for each $n \geq 1$. Then, for each $\gamma > 1/2, \delta > 0$, and increasing sequence of positive constants $c_n \to \infty$,

$$\sup_{\theta \in \Theta} \mathbf{P}_\theta \left(|\sum_{i=1}^{n} x_{i-1}\varepsilon_i| \geq \delta \max(c_n, (\sum_{i=1}^{n} x_{i-1}^2)^\gamma) \text{ for some } n \geq m \right) \to 0\,,$$

as $m \to \infty$.

Proof of Theorem 2.2. Assertion (2.20) easely follows from Lemma 3.12 in [6]. For the points θ belonging to $\partial\Lambda_p$ we decompose the original time series (1.2) into several components depending on the number of the roots of the characteristic polynomial (1.5) lying on the unit circle and their values. To this end, the characteristic polynomial (1.5) is represented as

$$\mathcal{P}(z) = (z+1)^{\delta_1}(z-1)^{\delta_2}(z^2 - 2z\cos\phi\, + 1)^{\delta_3}\varphi(z),$$

where δ_i are either zero or 1 with $\delta_1 + \delta_2 + \delta_3 \geq 1$, $\varphi(z)$ is the polynomial of order $r = p - \delta_1 - \delta_2 - 2\delta_3$ which has all roots inside the unit circle. Assuming (without loss of generality) that $r \geq 1$, one has

$$\varphi(z) = z^r + \beta_1 z^{r-1} + \ldots + \beta_r\,.$$

By applying the backshift operator q^{-1} (i.e. $q^{-1}x_n = x_{n-1}$) one can write down (1.1) as

$$q^{-\delta_1}(q+1)^{\delta_1}q^{-\delta_2}(q-1)^{\delta_2}q^{-2\delta_3}(q^2 - 2q\cos\phi + 1)^{\delta_3}q^{-r}\varphi(q)x_n = \varepsilon_n \qquad (4.1)$$

Let $\theta \in \Gamma_1(\rho)$. Then this equation, in view of (2.19), takes the form

$$q^{-1}(q+1)q^{-p+1}\varphi(q)x_n = \varepsilon_n\,.$$

Denote

$$u_n = q^{-p+1}\varphi(q)x_n\,,$$
$$v_n = q^{-1}(q+1)x_n\,,$$

that is

$$\begin{aligned} u_n &= x_n + \beta_1 x_{n-1} + \ldots + \beta_{p-1}x_{n-p+1}\,, \\ v_n &= x_n + x_{n-1}\,. \end{aligned} \qquad (4.2)$$

Introducing the vector $V_n = (v_n, \ldots, v_{n-p+1})'$ and the matrix

$$Q = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \ldots & \beta_{p-1} \\ 1 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & & \ldots & 1 & 1 \end{pmatrix} \qquad (4.3)$$

one can rewrite equations (4.2) in the vector form

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = Q_1 X_n,\ X_n = (x_n, \dots, x_{n-p+1})' . \tag{4.4}$$

The processes u_n and v_n satisfy the equations

$$u_n = -u_{n-1} + \varepsilon_n,\ v_n + \beta_1 v_{n-1} + \dots + \beta_{p-1} v_{n-p+1} = \varepsilon_n .$$

Substituting (4.4) in (1.4) yields

$$\frac{tr\ M_n}{n^2} = tr(Q_1 Q_1')^{-1} \begin{pmatrix} n^{-2}\sum_{k=1}^n u_{k-1}^2 & n^{-2}\sum_{k=1}^n u_{k-1} V_{k-1}' \\ n^{-2}\sum_{k=1}^n u_{k-1} V_{k-1} & n^{-2}\sum_{k=1}^n V_{k-1} V_{k-1}' \end{pmatrix} . \tag{4.5}$$

By Theorem 3.4.2. in Chan and Wei (1988)

$$n^{-2}\sum_{k=1}^n u_{k-1} V_{k-1}' \xrightarrow{\mathbf{P}} 0,\ \text{as } n \to \infty.$$

Since the process V_n is stable

$$\lim_{n\to\infty} n^{-2}\sum_{k=1}^n V_{k-1} V_{k-1}' = 0 \text{ a.s.}$$

By Donsker's theorem

$$n^{-2}\sum_{k=1}^{[nt]} u_{k-1}^2 \overset{\mathcal{L}}{\Longrightarrow} \sigma^2 \int_0^t W_1^2(s) ds,\ 0 \le t \le 1 ,$$

as $n \to \infty$. By making use of these limiting relations in (4.5), one get

$$\frac{tr\ M_{[nt]}}{n^2} \overset{\mathcal{L}}{\Longrightarrow} \kappa_{11} \int_0^t W_1^2(s) ds,\ 0 \le t \le 1 , \tag{4.6}$$

where

$$\kappa_{11} = \sigma^2 < (Q_1 Q_1')^{-1} >_{11} .$$

Now by definition of $\tau(h)$ in (2.2) one has

$$\mathbf{P}_\theta\left(\frac{\tau(h)}{b_1\sqrt{h}} \le t\right) = \mathbf{P}_\theta(tr\ M_{[tb_1\sqrt{h}]} \ge h) = \mathbf{P}_\theta\left(\frac{tr\ M_{[tb_1\sqrt{h}]}}{b_1^2 h} b_1^2 \ge 1\right) .$$

This and (4.6) imply the validity of (2.21) for $\theta \in \Gamma_1(\rho)$ with

$$b_1^2 = 1/\kappa_{11} . \tag{4.7}$$

By a similar argument, one check (2.21) for $\theta \in \Gamma_2(\rho)$ with

$$b_2^2 = 1/(\sigma^2 < (Q_2 Q_2')^{-1} >_{11})\,, \tag{4.8}$$

where

$$Q_2 = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \dots & \beta_{p-1} \\ 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & & & 0 \\ 0 & & \dots & 1 & -1 \end{pmatrix}.$$

Assume that $\theta \in \Gamma_3(\rho)$. Then using the equation

$$q^{-2}(q^2 - 2q\cos\phi - 1)q^{-p+2}\varphi(q)x_n = \varepsilon_n\,,$$

we decompose the process (1.2) into two processes

$$U_n = (u_n, u_{n-1})',\ V_n = (v_n, \dots, v_{n-p+3})'\,,$$

which obey the equations

$$U_n = \begin{pmatrix} 2\cos\phi & -1 \\ 1 & 0 \end{pmatrix} U_{n-1} + \begin{pmatrix} \varepsilon_n \\ 0 \end{pmatrix}, \tag{4.9}$$

$$v_n + \beta_1 v_{n-1} + \dots + \beta_{p-2} v_{n-p+2} = \varepsilon_n\,.$$

These processes are related with $X_n = (x_n, \dots, x_{n-p+1})'$ by the following transformation

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = Q_3 X_n$$

with

$$Q_3 = \begin{pmatrix} 1 & \beta_1 & \dots & & \beta_{p-2} & 0 \\ 0 & 1 & \beta_1 & \dots & & \beta_{p-2} \\ 1 & -2\cos\phi & 1 & 0 & \dots & 0 \\ 0 & 1 & -2\cos\phi & 1 & 0 & \dots \\ \vdots & & & & & \\ 0 & \dots & 0 & 1 & -2\cos\phi & 1 \end{pmatrix}.$$

Further, by the same argument, one shows that

$$n^{-2}\, tr\, M_{[nt]} \stackrel{\mathcal{L}}{\Longrightarrow} tr(Q_3 Q_3')^{-1} \begin{pmatrix} H_t & 0 \\ 0 & 0 \end{pmatrix}, 0 \le t \le 1,$$

where

$$H_t = \frac{\sigma^2}{4\sin^2\phi} \int_0^t (W_1^2(s) + W_2^2(s))ds \begin{pmatrix} 1 & \cos\phi \\ \cos\phi & 1 \end{pmatrix},$$

that is,

$$n^{-2}\, tr\, M_{[nt]} \stackrel{\mathcal{L}}{\Longrightarrow} \frac{r^2}{4\sin^2\phi}\int_0^t (W_1^2(s)+W_2^2(s))ds\,,$$

where

$$r^2 = \sigma^2\, tr(Q_3Q_3')^{-1}\begin{pmatrix} 1 & \cos\phi \\ \cos\phi & 1 \end{pmatrix}.$$

From here one comes to (2.21) with

$$b_3 = (2\sin\phi)/r\,. \tag{4.10}$$

Now assume that $\theta \in \Gamma_4(\rho)$. In this case the process (1.2) is decomposed into two vector processes

$$U_n = (u_n, u_{n-1})',\ V_n = (v_n, \ldots, v_{n-p+3})'\,,$$

which are defined by the formulae

$$u_n = x_n + \beta_1 x_{n-1} + \ldots + \beta_{p-2}x_{n-p+2},\ v_n = x_n - x_{n-2}$$

and satisfy the equations

$$u_n = u_{n-2} + \varepsilon_n,\ v_n + \beta_1 v_{n-1} + \ldots + \beta_{p-2}v_{n-p+2} = \varepsilon_n\,.$$

These processes are related to the original process (1.2) by the transformation

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = Q_4 X_n\,, \tag{4.11}$$

where

$$Q_4 = \begin{pmatrix} 1 & \beta_1 & \ldots & & \beta_{p-2} & 0 \\ 0 & 1 & \beta_1 & \ldots & & \beta_{p-2} \\ 1 & 0 & -1 & 0 & \ldots & 0 \\ \vdots & & & & & \\ 0 & \ldots & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Further we represent U_n as

$$U_n = T^{-1}\begin{pmatrix} y_n \\ z_n \end{pmatrix}, \tag{4.12}$$

where processes y_n and z_n satisfy the equations

$$y_n = y_{n-1} + \varepsilon,\ z_n = -z_{n-1} + \varepsilon\,, \tag{4.13}$$

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

By making use of (4.11)-(4.12) one gets

$$\frac{tr\, M_n}{n^2} = tr(Q_4 Q_4')^{-1} \begin{pmatrix} n^{-2}\sum_{k=1}^n U_{k-1}U_{k-1}' & n^{-2}\sum_{k=1}^n U_{k-1}V_{k-1}' \\ n^{-2}\sum_{k=1}^n V_{k-1}U_{k-1}' & n^{-2}\sum_{k=1}^n V_{k-1}V_{k-1}' \end{pmatrix} , \tag{4.14}$$

$$n^{-2}\sum_{k=1}^n U_{k-1}U_{k-1}' = T^{-1} \begin{pmatrix} n^{-2}\sum_{k=1}^n y_{k-1}^2 & n^{-2}\sum_{k=1}^n y_{k-1}z_{k-1} \\ n^{-2}\sum_{k=1}^n y_{k-1}z_{k-1} & n^{-2}\sum_{k=1}^n z_{k-1}^2 \end{pmatrix} (T^{-1})' .$$

By Theorems 3.4.1 and 2.3 in Chan and Wei (1988), one has, as $n \to \infty$,

$$n^{-2}\sum_{k=1}^n U_{k-1}V_{k-1}' \xrightarrow{\mathbf{P}} 0,\ n^{-2}\sum_{k=1}^n y_{k-1}z_{k-1} \xrightarrow{\mathbf{P}} 0 ,$$

$$\sigma^{-2}\left(n^{-2}\sum_{k=1}^{[nt]} y_{k-1}^2, n^{-2}\sum_{k=1}^{[nt]} z_{k-1}^2 \right) \overset{\mathcal{L}}{\Longrightarrow} (J_1(W_1;t), J_1(W_2;t)) ,$$

where $0 \le t \le 1$. From here and (4.11), it follows that

$$n^{-2} tr\, M_{[nt]} \overset{\mathcal{L}}{\Longrightarrow} \sigma^2 tr(Q_4Q_4')^{-1}\mathcal{D}_t , \tag{4.15}$$

where

$$\mathcal{D}_t = \begin{pmatrix} S_t & 0 \\ 0 & 0 \end{pmatrix} , S_t = T^{-1} \begin{pmatrix} \int_0^t W_1^2(s)ds & 0 \\ 0 & \int_0^t W_2^2(s)ds \end{pmatrix} (T^{-1})' .$$

It easy to check that

$$\sigma^2 tr(Q_4Q_4')^{-1}\mathcal{D}_t = \int_0^t \left(r_1 W_1^2(s) + r_2 W_2^2(s) \right) ds , \tag{4.16}$$

where

$$r_1 = \frac{\sigma^2}{4}\sum_{i=1}^2\sum_{j=1}^2 < (Q_4Q_4')^{-1} >_{ij}, r_2 = \frac{\sigma^2}{4}\sum_{i=1}^2\sum_{j=1}^2 (-1)^{i+j} < (Q_4Q_4')^{-1} >_{ij} \ .$$

Denote

$$b_4^2 = 1/\mu_1,\ \mu_1 = r_2/r_1 . \tag{4.17}$$

It remains to note that (4.15),(4.16) imply

$$\mathbf{P}_\theta(\tau(h) \le t b_4 \sqrt{h}) \to \mathbf{P}_\theta(J_3(W_1, W_2) \ge 1),\ \text{as } h \to \infty ,$$

where J_3 is defined in (2.18).

Assume that $\theta \in \Gamma_5(\rho)$. Then equation (4.1) has the form

$$q^{-1}(q+1)q^{-2}(q^2 - 2q\cos\phi + 1)q^r\varphi(q)x_n = \varepsilon_n,\ r = p-3.$$

Decompose x_n into three processes

$$\begin{array}{l} u_n = q^{-2}(q^2 - 2q\cos\phi + 1)q^{-r}\varphi(q)x_n = x_n + \gamma_1 x_{n-1} + \ldots + \gamma_{p-1}x_{n-p+1}\,, \\ v_n = q^{-1}(q+1)q^{-r}\varphi(q)x_n = x_n + f_1 x_{n-1} + \ldots + f_{p-2}x_{n-p+2}\,, \\ w_n = q^{-1}(q+1)q^{-2}(q^2 - 2q\cos\phi + 1)x_n = x_n + t_1 x_{n-1} + t_2 x_{n-2} + t_3 x_{n-3}\,, \end{array} \tag{4.18}$$

where γ_i, f_j, t_k are the coefficients of the corresponding polynomials. These processes satisfy the following equations

$$\begin{array}{l} u_n = -u_{n-1} + \varepsilon_n, \\ v_n = 2v_{n-1}\cos\phi - v_{n-2} + \varepsilon_n, \\ w_n = -\beta_1 w_{n-1} - \ldots - \beta_{p-3}w_{n-p+3} + \varepsilon_n\,. \end{array}$$

Introducing vectors $V_n = (v_n, v_{n-1})',\ W_n = (w_n, w_{n-1}, \ldots, w_{n-p+4})'$ and the matrix

$$Q_5 = \left(\begin{array}{cccccccc} 1 & \gamma_1 & \ldots & & & & & \gamma_{p-1} \\ 1 & f_1 & \ldots & & & & f_{p-2} & 0 \\ 0 & 1 & f_1 & \ldots & & & & f_{p-2} \\ 1 & t_1 & t_2 & t_3 & 0 & \ldots & & 0 \\ 0 & 1 & t_1 & t_2 & t_3 & 0 & \ldots & 0 \\ \ldots & & & & & & & \\ 0 & \ldots & & 0 & 1 & t_1 & t_2 & t_3 \end{array} \right), \tag{4.19}$$

we write down (4.18) as

$$\left(\begin{array}{c} U_n \\ V_n \\ W_n \end{array} \right) = Q_5 X_n\,. \tag{4.20}$$

From here and (1.4)

$$n^{-2} tr\, M_{[nt]} = tr(Q_5 Q_5')^{-1} C_{[nt]}/n^2\,,\ 0 \le t \le 1, \tag{4.21}$$

where

$$C_n = \left(\begin{array}{ccc} \sum_{k=1}^n u_{k-1}^2 & \sum_{k=1}^n U_{k-1}V_{k-1}' & \sum_{k=1}^n U_{k-1}W_{k-1}' \\ \sum_{k=1}^n V_{k-1}u_{k-1} & \sum_{k=1}^n V_{k-1}V_{k-1}' & \sum_{k=1}^n V_{k-1}W_{k-1}' \\ \sum_{k=1}^n W_{k-1}u_{k-1} & \sum_{k=1}^n W_{k-1}V_{k-1}' & \sum_{k=1}^n W_{k-1}W_{k-1}' \end{array} \right).$$

By Theorems 3.4.1, 3.4.2 in Chan and Wei (1988),

$$\begin{array}{l} n^{-2}\sum_{k=1}^n V_{k-1}u_{k-1} \xrightarrow{\mathbf{P}} 0\,, \\ n^{-2}\sum_{k=1}^n W_{k-1}u_{k-1} \xrightarrow{\mathbf{P}} 0\,, \\ n^{-2}\sum_{k=1}^n W_{k-1}V'_{k-1} \xrightarrow{\mathbf{P}} 0\,, \end{array} \tag{4.22}$$

as $n \to \infty$. Due to the stability of the process W_k,

$$\lim_{n\to\infty} n^{-2}\sum_{k=1}^{n} W_{k-1}W'_{k-1} = 0 \text{ a.s.} \tag{4.23}$$

By the Theorem 2.3 therein

$$\left(n^{-2}\sum_{k=1}^{[nt]} u_{k-1}^2,\ n^{-2}\sum_{k=1}^{[nt]} V_{k-1}V'_{k-1}\right) \stackrel{\mathcal{L}}{\Longrightarrow} \sigma^2\left(J_2(W_1,W_2;t)H,\ J_1(W_3;t)\right)\,, \tag{4.24}$$

where J_1 and J_2 are defined in (2.18),

$$H = \frac{1}{4\sin^2\phi}\begin{pmatrix} 1 & \cos\phi \\ \cos\phi & 1 \end{pmatrix}.$$

Limiting $n \to \infty$ in (4.21) and taking into account (4.22)-(4.24) one gets

$$n^{-2} tr\, M_{[nt]} \stackrel{\mathcal{L}}{\Longrightarrow} m(t) \text{ as } n \to \infty\,, \tag{4.25}$$

where

$$m(t) = \sigma^2 tr (Q_5Q'_5)^{-1}\begin{pmatrix} S_t & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.26}$$

$$S_t = \begin{pmatrix} J_1(W_3;t) & 0 \\ 0 & J_2(W_1,W_2;t)H \end{pmatrix}.$$

By easy calculation in (4.26) one finds

$$m(t) = \kappa_{11}J_1(W_3;t) + \frac{J_2(W_1,W_2;t)}{4\sin^2\phi}(\kappa_{22}+\kappa_{33}+(\kappa_{23}+\kappa_{32})\cos\phi)\,,$$

where $\kappa_{ij} = \sigma^2 < (Q_5Q'_5)^{-1} >_{ij}$.

Now by the definition of $\tau(h)$ in (2.2) and (4.25) we obtain

$$\mathbf{P}(\tau(h) \le b_5\sqrt{h}t) = \mathbf{P}(tr\, M_{[tb_5\sqrt{h}]} \ge h)\,,$$

$$\mathbf{P}(\frac{tr\, M_{[tb_5\sqrt{h}]}}{hb_5^2}b_5^2 \ge 1) \to \mathbf{P}(m(t)b_5^2 \ge 1)\,.$$

Setting

$$\begin{aligned} b_5^2 &= \{4\sin^2\phi\}/\{\kappa_{22}+\kappa_{33}+(\kappa_{23}+\kappa_{32})\cos\phi\} \\ \mu_2 &= \kappa_{11}b_5^2 \end{aligned} \tag{4.27}$$

yields

$$\mathbf{P}(m(t)b_5^2 \geq 1) = \mathbf{P}(J_4(W_1, W_2, W_3; t) \geq 1)\,,$$

where J_4 is defined in (2.18), that is, (2.21) holds for $\theta \in \Gamma_5(\rho)$.

One can check that for $\theta \in \Gamma_6(\rho)$ the limiting distribution of $\tau(h)$ coincides with that for $\theta \in \Gamma_5(\rho)$ with $b_6 = b_5$.

By a similar argument it can be shown that for $\theta \in \Gamma_7(\rho)$

$$\lim_{h\to\infty} \mathbf{P}(\tau(h) \leq b_7 t\sqrt{h}) = \mathbf{P}(\tilde{m}(t)b_7^2 \geq 1)\,,$$

where

$$\tilde{m}(t) = \tilde{\kappa}_{11}J_1(W_3;t) + \tilde{\kappa}_{22}J_1(W_4;t) +$$

$$+\frac{J_2(W_1, W_2; t)}{4\sin^2\phi}\left(\tilde{\kappa}_{33}+\tilde{\kappa}_{44}+(\tilde{\kappa}_{34}+\tilde{\kappa}_{43})\cos\phi\right)\,,$$

$\tilde{\kappa}_{ij} = \sigma^2 < (Q_7 Q_7')^{-1} >_{ij},\ Q_7$ is corresponding transformation matrix which relates the original process with the decomposed ones.

Setting

$$\begin{aligned} b_7^2 &= (4\sin^2\phi)/\left(\tilde{\kappa}_{33}+\tilde{\kappa}_{44}+(\tilde{\kappa}_{34}+\tilde{\kappa}_{43})\cos\phi\right)\,, \\ \mu_3 &= \tilde{\kappa}_{11}b_7^2,\ \mu_4 = \tilde{\kappa}_{22}b_7^2 \end{aligned} \tag{4.28}$$

one comes to (2.21). This completes the proof of Theorem 2.2. $\square$

References

[1] Ahtola, J.A. and Tiao, G.C. (1987) Distribution of least squares estimators of autoregressive parameters for a process with complex roots on the unit circle. *J. Time Ser. Anal.* **8**, p. 1-14.

[2] Anderson, T.W. (1971) The statistical analysis of time series. *John Wiley Sons*, New York.

[3] Chan, N.H. and Wei, C.Z. (1988) Limiting distribution of least squares estimates of unstable autoregressive processes. *Ann. Statist.*, **16**, p. 367-401.

[4] Dickey, D.A. and Fuller, W.A. (1979) Distribution of the estimators for autoregressive time series with a unit root. *J. Amer. Statist. Assoc.*, **74**, p. 427-431.

[5] Galtchouk, L. and Konev V. (2003) On uniform asymptotic normality of sequential least squares estimators for the parameters in a stable AR(1). *Sequential Anal.* **22**, p. 31-54.

[6] Galtchouk, L. and Konev V. (2004) On uniform asymptotic normality of sequential least squares estimators for the parameters in a stable AR(p). *J. Multivariate Anal.* **91**, p. 119-142.

[7] Galtchouk, L. and Konev V. (2006) Sequential estimation of the parameters in unstable AR(2). *Sequential Anal.* **25**, p. 25-43.

[8] Konev, V.V. and Pergamenshchikov, S.M. (1996) Guaranteed parameter estimation on the basis of a generalized least-squares method. *Theory Probab. Appl.* **41**, p. 765-784.

[9] Lai, T.L. and Siegmund, D. (1983) Fixed-accuracy estimation of an autoregressive parameter. *Ann.Statist.* **11**, p. 478-485.

[10] Lai, T.L. and Wei, C.Z. (1983) Asymptotic properties of general autoregressive models and strong consistency of least squares estimates of their parameters. *J. Multivariate Anal.* **13**, p. 1-23.

[11] Rao, M.M. (1978) Asymptotic distribution of an estimator of the boundary parameter of an unstable process. *Ann.Statist.* **6**, p. 185-190.

[12] Sriram, T.N. (1987) Sequential estimation of the mean of a first order stationary autoregressive process. *Ann.Statist.*, **15**, p. 1079-1090.

[13] Sriram, T.N. (1988) Sequential estimation of the autoregressive parameter in a first order autoregressive process. *Sequential Anal.*, **7**, p. 53-74.

[14] Varga, R. (2000) Matrix iterative analysis, Springer, Berlin.

[15] White, J.S. (1958) The limiting distribution of the serial correlation coefficient in the explosive case. *Ann. Math. Statist.* **29**, p. 1188-1197.

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