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▶ **To cite this version:**

Frédéric Klopp, Konstantin Pankrashkin. Localization on quantum graphs with random edge lengths. 2008. hal-00326964v1

HAL Id: hal-00326964

https://hal.science/hal-00326964v1

Preprint submitted on 6 Oct 2008 (v1), last revised 14 Jan 2009 (v2)

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Localization on quantum graphs with random edge lengths

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Abstract

The spectral properties of the Laplacian on a class of quantum graphs with random metric structure are studied. Namely, we consider quantum graphs spanned by the simple $\mathbb{Z}^d$-lattice with δ-type boundary conditions at the vertices, and we assume that the edge lengths are randomly independently identically distributed. Under the assumption that the coupling constant at the vertices does not vanish, we show that the operator exhibits the Anderson localization at the bottom of the spectrum almost surely. We also study the case of other spectral edges.

1 Introduction

The present paper deals with a class of random operators on quantum graphs, the so-called random length model. We consider the Laplacian on a metric graph spanned by the simple lattice $\mathbb{Z}^d$ with δ-type boundary conditions where the edge lengths are i.i.d. random variables. Under certain technical assumptions we show that the operator exhibits the Anderson localization at the bottom of the spectrum, i.e. that the bottom of the spectrum is pure point with exponentially decaying eigenfunctions.

The random length model on metric graphs was studied in [1] for regular trees, where stability of the absolutely continuous spectrum was found, and in [14] for radial trees, where Anderson localization at all energies was proved. Also the random necklace model studied in the earlier paper [17] can be viewed as a special random length model. Random length models for metric graphs spanned by $\mathbb{Z}^d$ were studied in [12] where existence of the integrated density of states was discussed, but, to our knowledge, no paper studied the Anderson localization in this situation so far. For discussion of localization in other random models on quantum graphs see also [7, 13, 16].

The localization in the model we are studying can be proved by an iterative procedure called the multiscale analysis, see [28], which works in rather abstract situations as far as some input data are available (see also the paper [7] which is concerned specifically with metric graphs). So we are concentrating on obtaining the necessary ingredients. Our main construction is the reduction of the study of finite volume operators on metric graphs to a family of random discrete operators. It turns out that the discrete operators obtained are closely related to the random hopping model considered in [15], so we are able to use similar construction to obtain the two most important estimates, the Wegner estimate and the initial scale estimate.

2 Random length model on a quantum graph lattice

We recall here some basic constructions for quantum graphs. For the general theory see e.g. the reviews [10, 19, 20] and the collections of papers [3, 6, 8, 18].

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a countable directed graph with $\mathcal{V}$ and $\mathcal{E}$ being the sets of vertices and edges, respectively. For an edge $e \in \mathcal{E}$, we denote by ιe its initial vertex and by τe its terminal vertex. For $e \in \mathcal{E}$ and $v \in \mathcal{V}$ we write $v \sim e$ or $e \sim v$ if $v \in \{\iota e, \tau e\}$. The degree of a vertex is the number $\deg v := \#\{e \in \mathcal{E} : e \sim v\}$.

For $0 < l_{\min} < l_{\max} < \infty$ consider a function $l : \mathcal{E} \to [l_{\min}, l_{\max}]$. Sometimes we will write l_e instead of $l(e)$; this number will be interpreted as the length of the edge e. Replacing each edge e by a copy of the segment $\big[0, l_e\big]$ in such a way that 0 corresponds to ιe and l_e to τe and, thus, obtains a so-called metric graph. Our aim is to study a special type of differential operator on such a structure.

In the space $\mathcal{H} := \bigoplus_{e\in\mathcal{E}} \mathcal{H}_e$, $\mathcal{H}_e = L^2[0, l_e]$ consider an operator H acting as $f =: (f_e) \mapsto (-f_e'') =: Hf$ on the domain consisting of the functions $f \in \bigoplus_{e\in\mathcal{E}} H^2[0, l_e]$ satisfying the Kirchhoff boundary conditions, i.e. for any $v \in \mathcal{V}$ one has

$$f_e(l_e) = f_b(0) =: f(v), \quad \tau e = \iota b = v, \tag{1}$$

and

$$f'(v) = \alpha f(v), \quad f'(v) := \sum_{e:\iota e=v} f_e'(0) - \sum_{e:\tau e=v} f_e'(l_e), \tag{2}$$

where α is a real number, the so-called coupling constant (for simplicity, we assume that the coupling constants are the same for all vertices). It is known, that the operator thus obtained is self-adjoint. We denote this operator by $H(\Gamma, l, \alpha)$.

We are going to study a special case of underlying configuration, a periodic lattice. Let $\mathcal{V} = \mathbb{Z}^d$, $d \geq 1$. Let h_j, $j = 1, \dots, d$, be canonical basis of $\mathbb{Z}^d$, and set

$$\mathcal{E}_d := \{(m, m + h_j), \quad m \in \mathbb{Z}^d, \quad j = 1, \dots, d\},$$

where for an edge $e = (v, v') \in \mathbb{Z}^d$, $v, v' \in \mathbb{Z}^d$, one has $\iota e := v$, $\tau e := v'$. For this graph $\Gamma^d := (\mathbb{Z}^d, \mathcal{E}_d)$ consider the operator $H(l, \alpha) := H(\Gamma^d, l, \alpha)$. In the present paper we are going to study some spectral properties of such operators under assumption that l_e are random independent identically distributed variables.

Namely, on $(\Omega, \mathbb{P})$, a probability space, let $\left(l_e^\omega\right)_{e \in \mathcal{E}}$ be a family of independent identically distributed (i.i.d.) random variables whose common distribution has a Lipschitz continuous density ρ with support $[l_{\min}, l_{\max}]$. By a random Hamiltonian on the quantum graph, we mean a family of operators $H^\omega(\alpha) := H(l^\omega, \alpha)$.

By rescaling $t \mapsto l_e^\omega t$ on each edge, one obtains a unitary equivalent operator acting on the same space $\oplus_e L^2[0, 1]$ (independent of ω), which allows one to use the standard results from the theory of random operators, the existence of an almost sure spectrum and of almost sure spectral components (see e.g. [5, 27]), i.e. the existence of closed subsets $\Sigma_\bullet = \Sigma_\bullet(\alpha) \subset \mathbb{R}$ and a subset $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that $\operatorname{spec}_\bullet H^\omega(\alpha) = \Sigma_\bullet$, $\bullet \in \{\mathrm{pp}, \mathrm{ac}, \mathrm{sc}\}$, for any $\omega \in \Omega'$. Let $\Sigma(\alpha) = \Sigma_{\mathrm{pp}} \cup \Sigma_{\mathrm{ac}} \cup \Sigma_{\mathrm{sc}}$ be the almost sure spectrum of $H^\omega(\alpha)$. Let us describe this set. For $l > 0$ and $\beta \in \mathbb{R}$ denote by $P_{l,\beta}$ the periodic Krönig-Penney operator acting in $L^2(\mathbb{R})$,

$$P_{u,\beta} := -\frac{d^2}{dx^2} + \beta \sum_{k \in \mathbb{Z}} \delta(\cdot - ku).$$

The operator can be correctly defined through its quadratic form; it is unitarily equivalent to the operator $H(l_u, \beta)$ for $d = 1$, where l_u is the constant function, $l_u(e) \equiv u$ for all $e \in \mathcal{E}$. It is well known [2] that

$$\operatorname{spec} P_{u,\beta} = \left\{k^2 : k \in \mathbb{R} \cup i\mathbb{R},\ \cos ku + \frac{\beta}{2k} \sin ku \in [-1, 1]\right\}. \tag{3}$$

In particular, each Dirichlet eigenvalue $(\pi n)^2/u^2$, $n = 1, 2, \dots$, is a spectral edge, and the bands depend continuously on both α and u. As follows from the general constructions [5, 27], one has

$$\Sigma := \bigcup_{u \in [l_{\min}, l_{\max}]} \operatorname{spec} H(l_u, \alpha). \tag{4}$$

On the other hand, one has $\operatorname{spec} H(l_u, \alpha) := \operatorname{spec} P_{u,\alpha/d}$, see [25]. Hence, an elementary analysis shows that the almost sure spectrum $\Sigma(\alpha)$ of $H^\omega(\alpha)$ is a union of bands, and the bottom of the spectrum is given by

$$\inf \Sigma(\alpha) = \begin{cases} k^2 :\ k \in (0, \pi/l_{\max}) \text{ and } \cos kl_{\max} + \dfrac{\alpha}{2kd} \sin kl_{\max} = 1, & \alpha > 0, \\ 0, & \alpha = 0, \\ -k^2 : k > 0 \text{ and } \cosh kl_{\min} + \dfrac{\alpha}{2kd} \sinh kl_{\min} = 1, & \alpha < 0. \end{cases}$$

Define the set

$$\Delta := \bigcup_{n\in\mathbb{Z}} \Big[\frac{\pi^2 n^2}{l_{\max}^2}, \frac{\pi^2 n^2}{l_{\min}^2}\Big]. \tag{5}$$

The set consists of the spectra of the operator $H(l_u)$ with Dirichlet boundary condition (which formally corresponds to $\alpha = \infty$) at each vertex when u ranges over the support of the random variables $(l_e^\omega)_{e\mathcal{E}}$ and the point 0. Our main result is

Theorem 1. *Let $\alpha \neq 0$. Let E_0 be a edge of Σ that is not contained in Δ. Then, there exists $\varepsilon > 0$ such that the spectrum of H^ω in $(E_0 - \varepsilon, E_0 + \varepsilon) \cap \Sigma$ is almost surely dense pure point with exponentially decaying eigenfunctions.*

In particular, we get

Theorem 2. *Let $\alpha \neq 0$. There exists $\varepsilon > 0$ such that the spectrum of H^ω in $(\inf \Sigma, \inf \Sigma + \varepsilon)$ is almost surely dense pure point with exponentially decaying eigenfunctions.*

Remark 3. It is quite easy to see that, if $l_{\max} - l_{\min}$ is sufficiently small, there are band edges of Σ outside Δ. This can be obtained using (3), (4) and the asymptotics given for $\mathrm{spec} P_{L,\beta}$ for large k.

Our results do not establish localization for the most important case $\alpha = 0$. In this case, it is quite easy to see that both the Wegner estimate near $\inf \Sigma = 0$ and the initial scale estimate (see theorems 4 and 5 below) fail to hold. The operator $H(l^\omega, 0)$ is analogous to the acoustic operator (see e.g. [22] and references therein); for this operator, at least in dimension one, it is known that localization does not hold in its strongest form at the bottom of the spectrum (see [11]). In the case $\alpha = 0$, the reduced operator we use to study the random quantum graph is a discrete version of the acoustic operator.

As in [16], the assumptions of theorems 1 and 2 also imply dynamical localization (see remark 7 in [16]).

3 Multiscale analysis and finite-volume operators

Let Λ be a subset of edges from $\mathcal{E}_d$. Denote $\mathcal{V}_\Lambda := \{\iota e : e \in \Lambda\} \cup \{\tau e : e \in \Lambda\}$ and consider the graph $\Gamma_\Lambda^d := \big(\mathcal{V}_\Lambda, \Lambda\big)$. Note that this graph has no isolated vertices. We will call the operator $H_\Lambda(l, \alpha) := H(\Gamma_\Lambda^d, l, \alpha)$ the finite-volume Hamiltonian associated to Λ. For random operators with random length functions l^ω, we write $H_\Lambda^\omega(\alpha) := H_\Lambda(l^\omega, \alpha)$

In what follows, we consider Hamiltonians associated with finite cubes $\Lambda = \Lambda(n)$ constructed as follows: take $n \in \mathbb{N}$ and denote by $\Lambda(n)$ the set of edges e such that at least one of the vertices $v \in \{\iota e, \tau e\}$ satisfies $|v| \leq n$; for the corresponding set of vertices, we write $\mathcal{V}(n) := \mathcal{V}_{\Lambda(n)}$.

In order to prove Anderson localization using a multiscale analysis, it is sufficient to verify two groups of conditions. The first one consists of rather abstract properties, namely:

- the finite-volume Hamiltonians corresponding to non-intersecting sets of edges are independent,

- the finite-volume operator obeys a Weyl estimate for the eigenvalues,
- there exists a geometric resolvent inequality relating operators corresponding to different finite-volume subsets,
- a generalized spectral theorem holds, i.e.n one can determine the spectrum through generalized eigenfunctions.

Note that the first condition is trivially satisfied in our case. The second and the third one can be obtained by minor modification of the arguments of [7], where equilateral lattices were considered. The last condition is satisfied due to the results of [4] (see also [21] for generalizations).

Below, we are concerned with the second group of estimates, which crucially depend on the way randomness enters the system. They consist in the Wegner estimate and the initial scale estimate; they are proved in sections 4 and 5, respectively. Using a multiscale analysis, they imply by the multiscale analysis theorem 1 (see e.g. [7].

Theorem 4 (The Wegner estimate). *Let I be an interval such that $\bar{I} \cap \Delta = \emptyset$, Then there exists a $C = C(I) > 0$ such that for any interval $J \subset I$, and any cube $\Lambda = \Lambda(n)$ there holds*

$$\mathbb{P}\big\{ \operatorname{spec} H^\omega_\Lambda(\alpha) \cap J \neq \emptyset \big\} \leq C|\Lambda|\,|J|. \tag{6}$$

Theorem 5 (The initial scale estimate). *Let E be a spectral edge of $H^\omega(\alpha)$ which is not contained in the set Δ from (5), then for each $\xi > 0$ and $\beta \in (0,1)$ there exists $n^* = n^*(\xi,\beta) > 0$ such that*

$$\mathbb{P}\{\operatorname{dist}(\operatorname{spec} H^\omega_{\Lambda(n)}(\alpha), E) \leq n^{\beta-1}\} \leq n^{-\xi}$$

for $n \geq n^*$.

Remark 6. As easily seen, for $\alpha \neq 0$ the initial scale estimate is satisfied near the bottom of the spectrum $\inf \Sigma$, which is outside Δ.

As for $\alpha = 0$, one can easily see that independently of the random variables l^ω_e and the set Λ the constant function $f \equiv 1$ satisfies $H^\omega_\Lambda f = 0$. Hence, both the Wegner estimate and the initial scale estimate fail for the energy $E = 0$, and this is the only spectral edge (the almost sure spectrum is the positive half-line.) Moreover, for $d = 1$ the operator H^ω is unitary equivalent to the free Laplacian. Hence, one has a certain similarity to the Schrödinger operators with random vector potentials, where only localization near internal spectral edges is proved so far [9].

We will prove both estimates, theorems 4 and 5 by considering a certain family of discrete operators which we introduce now. A similar approach was used in [16] for quantum graphs with random coupling constants and the details of the reduction can be found there.

Denote by D^ω_e the positive Laplacian with the Dirichlet boundary conditions in $L^2[0, l^\omega_e]$ and set $D^\omega_\Lambda := \bigoplus_{e\in\Lambda} D^\omega_e$. Clearly,

$$\operatorname{spec} D^\omega_\Lambda = \bigcup_{e\in\Lambda} \operatorname{spec} D_e, \quad \operatorname{spec} D^\omega_e = \left\{ \left(\frac{\pi k}{l^\omega_e}\right)^2 : \, k = 1, 2, \dots \right\}.$$

For $E \notin \operatorname{spec} D^\omega_\Lambda$ consider the operators $M_\Lambda(l^\omega, z)$ acting on $\ell^2(\mathcal{V}_\Lambda)$,

$$M_\Lambda(l^\omega, E)\varphi(v) = \sqrt{E}\Big(\sum_{e\in\Lambda:\iota e=v} \frac{1}{\sin l^\omega_e\sqrt{E}}\,\varphi(\tau e) + \sum_{e\in\Lambda:\tau e=v} \frac{1}{\sin l^\omega_e\sqrt{E}}\,\varphi(\iota e) - \sum_{e\in\Lambda:v\sim e} \cot l^\omega_e\sqrt{E}\,\varphi(v)\Big). \quad (7)$$

For the complete graph, $\Lambda = \mathcal{E}^d$, we write simply $M(l^\omega, E)$ instead of $M_\Lambda(l^\omega, E)$. The map $E \mapsto M_\Lambda(l^\omega, E)$ is obviously analytic outside $\operatorname{spec} D^\omega$. As shown in [24], a value $E \notin \operatorname{spec} D^\omega_\Lambda$ is in the spectrum of $H^\omega_\Lambda(\alpha)$ if and only if $\alpha \in \operatorname{spec} M_\Lambda(l^\omega, E)$ and, moreover, for each such E there holds $\dim\ker\big(H^\omega_\Lambda(\alpha) - E\big) = \dim\ker\big(M_\Lambda(l^\omega, E) - \alpha\big)$. The relation will be the key to our analysis below. We note that similar relations between quantum graphs and discrete operators exist for more general boundary conditions at the vertices, but the corresponding reduced operators $M(E)$ become much more complicated, see [26].

4 Proof of theorem 4 (Wegner estimate)

As noted previously,

$$\mathbb{P}\big\{\operatorname{spec} H^\omega_\Lambda(\alpha)\cap J \neq \emptyset\big\} = \mathbb{P}\big\{\exists E\in J:\ \alpha\in\operatorname{spec} M_\Lambda(l^\omega, E)\big\}. \quad (8)$$

Note also that, for any $E_* \in I$, one can write $M_\Lambda(l^\omega, E) = M_\Lambda(l^\omega, E_*) + (E - E_*)_\Lambda B_\Lambda(l^\omega, E, E_*)$. Due to analyticity, one can find a constant $b > 0$ such that $\|B_\Lambda(l^\omega, E, E_*)\| \le b$ for all $E_*, E \in I$ and $\Lambda \subset \mathcal{E}$ and almost all $\omega \in \Omega$.

On the other hand, the condition $\alpha \in \operatorname{spec} M_\Lambda(l^\omega, E)$ implies the existence of a vector $\varphi \in \ell^2(\mathcal{V}_\Lambda)$, $\|\varphi\| = 1$, such that $\big(M_\Lambda(l^\omega, E) - \alpha\big)\varphi = 0$. Let E_J be the center of J. Estimating, for $E \in J$,

$$\big\|\big(M_\Lambda(l^\omega, E_J) - \alpha\big)\varphi\big\| \le \big\|\big(M_\Lambda(l^\omega, E) - \alpha\big)\varphi\big\| + |E - E_J|\cdot\big\|B_\Lambda(l^\omega, E, E_J)\varphi\big\| \le b|J|,$$

yields the inequality

$$\mathbb{P}\big\{\operatorname{spec} H^\omega_\Lambda(\alpha)\cap J \neq \emptyset\big\} \le \mathbb{P}\Big\{\operatorname{dist}\big(\operatorname{spec} M_\Lambda(l^\omega, E_J), \alpha\big) \le b|J|\Big\}. \quad (9)$$

In what follows, we denote E_J simply by E to alleviate the notation.

For $e \in \mathcal{E}$, introduce the operators P^e_1, P^e_2, I_e acting on $\ell^2(\mathcal{V}_\Lambda)$:

$$P^e_1 f(v) = \begin{cases} \frac{1}{2}\big(f(\iota e) + f(\tau e)\big), & v \in \{\iota e, \tau e\},\\ 0, & \text{otherwise,}\end{cases} \quad (10)$$

$$P^e_2 f(v) = \begin{cases} \frac{1}{2}\big(f(\iota e) - f(\tau e)\big), & v = \iota e,\\ \frac{1}{2}\big(f(\tau e) - f(\iota e)\big), & v = \tau e,\\ 0, & \text{otherwise,}\end{cases} \quad (11)$$

$$I^e f(v) = \begin{cases} f(v), & v \in \{\iota e, \tau e\},\\ 0, & \text{otherwise.}\end{cases} \quad (12)$$

In terms of these operators, one has

$$M_\Lambda(l^\omega, E) = \sum_{e\in\Lambda} \Big(\frac{\sqrt{E}}{\sin l_e^\omega \sqrt{E}} \, (P_1^e - P_2^e) - \sqrt{E} \cot l_e^\omega \sqrt{E} \, I^e \Big)$$

and

$$\frac{\partial M_\Lambda(l^\omega, E)}{\partial l_e^\omega} = -\frac{E \cos l_e^\omega \sqrt{E}}{\sin^2 l_e^\omega \sqrt{E}} \, (P_1^e - P_2^e) + \frac{E}{\sin^2 l_e^\omega \sqrt{E}} \, I^e.$$

Consider first the case $\bar{I} \subset (0, +\infty)$. As $\|P_1^e - P_2^e\| = 1$ and $P_j^e I^e = P_j^e$ for $j \in \{1, 2\}$, one has

$$-\cos l_e^\omega \sqrt{E} \, (P_1^e - P_2^e) + I^e \geq \big(1 - |\cos l_e^\omega \sqrt{E}|\big) I^e.$$

As I does not meet Δ, there exist constants $c_1, c_2 > 0$ such that

$$1 - |\cos l_e \sqrt{E}| \geq c_1 \text{ and } \frac{E}{\sin^2 l_e^\omega \sqrt{E}} \geq c_2 \quad \text{for all } e \in \mathcal{E} \text{ and } E \in I \text{ and a.e. } \omega \in \Omega,$$

hence

$$\frac{\partial M_\Lambda(l^\omega, E)}{\partial l_e^\omega} \geq c_1 c_2 I^e \quad \text{for all } e \in \mathcal{E} \text{ and } E \in I$$

which gives

$$\sum_{e\in\mathcal{E}} \frac{\partial M_\Lambda(l^\omega, E)}{\partial l_e^\omega} \geq c_1 c_2 \sum_{e\in\mathcal{E}} I^e \geq \beta \, \mathrm{id}, \quad \beta = c_1 c_2 > 0$$

or

$$D_\Lambda M_\Lambda(l^\omega, E) \geq \beta \, \mathrm{id} \text{ with } D_\Lambda := \sum_{e\in\Lambda} \frac{\partial}{\partial l_e}.$$

Let $E_\Lambda^\omega(a, b)$ denote the spectral projection of $M_\Lambda(l^\omega, E)$ onto the interval (a, b). There holds

$$\begin{aligned} &\#\left[\operatorname{spec} M_\Lambda(l^\omega, E) \cap \big(\alpha - b|J|, \alpha + b|J|\big)\right] \\ &\qquad = \operatorname{tr} E_\Lambda^\omega\big(\alpha - b|J|, \alpha + b|J|\big) = \operatorname{tr}\left[\int_{\alpha - b|J|}^{\alpha + b|J|} \partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E_J) - \lambda) d\lambda\right]. \quad (13) \end{aligned}$$

On the other hand, one has

$$\begin{aligned} &-\operatorname{tr}\left[D_\Lambda \chi_{(-\infty,0]}\big(M_\Lambda(l^\omega, E) - \lambda\big)\right] \\ &\quad = \operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E) - \lambda) D_\Lambda M_\Lambda(l^\omega, E)\right] \geq \beta \operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}\big(M_\Lambda(l^\omega, E) - \lambda\big)\right]. \end{aligned}$$

(The last estimate is possible as both operators under the trace sign are non-negative.) Hence,

$$\operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}\big(M_\Lambda(l^\omega, E) - \lambda\big)\right] \leq \beta^{-1} \operatorname{tr}\left[\sum_{e\in\mathcal{E}} -\partial_e \chi_{(-\infty,0]}\big(M_\Lambda(l^\omega, E) - \lambda\big)\right],$$

where we denoted for brevity $\partial_e := \partial/\partial l_e^\omega$, and

$$\operatorname{tr}\left[E_\Lambda^\omega\big(\alpha - b|J|, \alpha + b|J|\big)\right] \leq \beta^{-1} \int_{\alpha-b|J|}^{\alpha+b|J|} \sum_{e\in\mathcal{E}} \operatorname{tr}\left[-\partial_e \chi_{(-\infty,0]}\big(M_\Lambda(l^\omega, E) - \lambda\big)\right] d\lambda.$$

Taking the expectation, one obtains

$$\mathbb{E}\operatorname{tr} E_\Lambda\big(\alpha - b|J|, \alpha + b|J|\big) \leq \beta^{-1} \sum_{e\in\mathcal{E}} \int_{l_{\min}}^{l_{\max}} \prod_{e'\neq e} \rho(l_{e'}^\omega) dl_{e'}^\omega \int_{\alpha-b|J|}^{\alpha+b|J|} G_e(E, \lambda, \omega) d\lambda \qquad (14)$$

where we denote

$$G_e(E, \lambda, \omega) = -\int_{l_{\min}}^{l_{\max}} \rho(l)\partial_l \operatorname{tr}\left[\chi_{(-\infty,0]}\big(M_{\Lambda,e}(l^\omega, l, E) - \lambda\big)\right] dl$$

and $M_{\Lambda,e}(l^\omega, l, E)$ is the operator $M_\Lambda(l^\omega, E)$ with l_e^ω replaced by l. As the density ρ is Lipschitz continuous by assumption, one can integrate by parts and obtain

$$G_e(E, \lambda, \omega) = -\rho(l)F_e(l, E, \lambda, \omega)\big|_{l=l_{\min}}^{l=l_{\max}} + \int_{l_{\min}}^{l_{\max}} \rho'(l)F_e(l, E, \lambda, \omega) dl,$$

where

$$F_e(l, E, \lambda, \omega) := \operatorname{tr}\Big[\chi_{(-\infty,0]}\big(M_{\Lambda,e}(l^\omega, l, E) - \lambda\big) - \chi_{(-\infty,0]}\big(M_{\Lambda,e}(l^\omega, l_{\min}, E) - \lambda\big)\Big].$$

As $\partial_e M_\Lambda(l^\omega, E)$ is a rank-two operator, the functions $F_e(l, E, \lambda, \omega)$ are uniformly bounded by 2. Hence, the functions G_e are bounded as well, say $|G_e| \leq G$, $G > 0$. Plugging this estimate into (14), one obtains

$$\mathbb{E}\operatorname{tr} E_\Lambda\big(\alpha - b|J|, \alpha + b|J|\big) \leq G\beta^{-1} \sum_{e\in\mathcal{E}} \int_{\alpha-b|J|}^{\alpha+b|J|} d\lambda = C|\Lambda||J|, \quad C := \frac{2bG}{\beta} > 0.$$

It remains to observe that

$$\mathbb{P}\Big\{\operatorname{dist}\big(\operatorname{spec} M_\Lambda(l^\omega, E), \alpha\big) \leq b|J|\Big\} \leq \mathbb{E}\operatorname{tr} E_\Lambda\big(\alpha - b|J|, \alpha + b|J|\big). \qquad (15)$$

Now, let us consider the remaining case $\bar{I} \subset (-\infty, 0)$. For $E < 0$, it is more convenient to rewrite

$$M_\Lambda(l^\omega, E) = \sum_{e\in\Lambda} \Big(\frac{\sqrt{-E}}{\sinh l_e^\omega\sqrt{-E}}\,(P_1^e - P_2^e) - \sqrt{-E}\coth l_e^\omega\sqrt{-E}\, I^e\Big).$$

Then,

$$\partial_e M_\Lambda(l^\omega, E) = \frac{|E|}{\sinh^2 l_e^\omega\sqrt{-E}}\Big[I_e - \cosh l_e^\omega\sqrt{-E}(P_1^e - P_2^e)\Big],$$

and one has

$$F_\Lambda M_\Lambda(l^\omega, E) = M_\Lambda(l^\omega, E) + K_\Lambda(l^\omega, E)$$

with

$$F_\Lambda := -\frac{1}{\sqrt{-E}} \sum_{e\in\Lambda} \tanh l_e^\omega \sqrt{-E}\, \frac{\partial}{\partial l_e^\omega}, \quad K_\Lambda(l^\omega, E) := \frac{1}{\sqrt{-E}} \sum_{e\in\Lambda} \tanh l_e^\omega \sqrt{-E}\, I^e.$$

As $(l_e^\omega)_e$ are bounded and bounded away from 0 and as $\bar{I} \subset (-\infty, 0)$, there exists $\gamma > 0$ such that $K_\Lambda(l^\omega, E) \geq \gamma\,\mathrm{id}$ for all Λ, $E \in I$, and a.e. ω. One arrives at

$$\begin{aligned}
&-\operatorname{tr}\left[F_\Lambda \chi_{(-\infty,0]}\big(M_\Lambda(l^\omega, E) - \lambda\big)\right] \\
&\qquad = \operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E) - \lambda) F_\Lambda M_\Lambda(l^\omega, E)\right] \\
&= \operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E) - \lambda) M_\Lambda(l^\omega, E)\right] + \operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E) - \lambda) K_\Lambda(l^\omega, E)\right] \\
&\qquad \geq \operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E) - \lambda) M_\Lambda(l^\omega, E)\right] + \gamma \operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E) - \lambda)\right],
\end{aligned}$$

and

$$\begin{aligned}
&\gamma \operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E) - \lambda)\right] \\
&\qquad \leq \operatorname{tr}\left[\sum_{e\in\Lambda} \frac{\tanh l_e^\omega \sqrt{-E}}{\sqrt{-E}}\, \frac{\partial}{\partial l_e^\omega} \chi_{(-\infty,0]}\big(M_\Lambda(l^\omega, E) - \lambda\big)\right] \\
&\qquad\qquad - \operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E) - \lambda) M_\Lambda(l^\omega, E)\right].
\end{aligned}$$

Using this inequality and (13) one can estimate

$$\begin{aligned}
\gamma \mathbb{E} \operatorname{tr}\left[E_\Lambda(\alpha - b|J|, \alpha + b|J|)\right] &= \gamma \operatorname{tr}\left[\int_{\alpha-b|J|}^{\alpha+b|J|} \partial_\lambda \chi_{(-\infty,0]}\big(M_\Lambda(l^\omega, E) - \lambda\big)\, d\lambda\right] \\
&\leq \int_{\alpha-b|J|}^{\alpha+b|J|} \sum_{e\in\Lambda} \frac{\tanh l_e^\omega \sqrt{-E}}{\sqrt{-E}}\, \frac{\partial}{\partial l_e^\omega} \operatorname{tr}\left[\chi_{(-\infty,0]}\big(M_\Lambda(l^\omega, E) - \lambda\big)\right] d\lambda \\
&\qquad + \int_{\alpha-b|J|}^{\alpha+b|J|} \left|\operatorname{tr}\left[\partial_\lambda \chi_{(-\infty,0]}(M_\Lambda(l^\omega, E) - \lambda) M_\Lambda(l^\omega, E)\right]\right| d\lambda =: A_1 + A_2.
\end{aligned}$$

The first term A_1 can be estimated as previously by $C'|\Lambda|\,|J|$ with some positive constant C' uniformly in $E \in I$ and Λ. To estimate the second term A_2, we note that $M_\Lambda(l^\omega, E)$ is a finite-dimensional operator acting on $\ell^2(\mathcal{V}_\Lambda\})$; hence, it has at most $|\mathcal{V}_\Lambda| \leq 2|\Lambda|$ eigenvalues counting multiplicities and that one can estimate the norm $\|M_\Lambda(l^\omega, E)\| \leq m$ (and, therefore, the absolute value of each eigenvalue) with some $m > 0$ uniformly in $E \in I$ and a.e. ω. Therefore, uniformly in $E \in I$, one can estimate the trace under the integral sign by $2m|\Lambda|$, which implies the estimate $A_2 \leq C''|\Lambda|\,|J|$ with some $C'' > 0$. Now, it remains to use again the inequality (15) to complete the proof of theorem 4. □

5 Proof of theorem 5 (initial scale estimate)

As in the proof of theorem 4, one can show that, for some $b > 0$, there holds

$$\mathbb{P}\{\operatorname{dist}(\operatorname{spec} H_{\Lambda(n)}(l_\omega, \alpha), E) \leq n^{\beta-1}\} \leq \mathbb{P}\{\operatorname{dist}(\operatorname{spec} M_{\Lambda(n)}(l_\omega, E), \alpha) \leq bn^{\beta-1}\}. \tag{16}$$

Theorem 5 is a consequence of

$$\mathbb{P}\{\mathrm{dist}(\mathrm{spec}\, M_{\Lambda(n)}(l_\omega, E), \alpha) \le bn^{\beta-1}\} \le n^{-\xi}. \tag{17}$$

It is well known, see e.g. Section 2.2 in [28] that, in order to prove the estimates (17), it is sufficient to show the Lifschitz tail behavior for the integrated density of states for the operator $M(l_\omega, E)$. Note that, if E is an edge of the almost sure spectrum of $H^\omega(\alpha)$, then α is an edge of the almost sure spectrum of $M(l^\omega, E)$ (see e.g.[16]). Hence, it is sufficient to study the behavior of the integrated density of states of $M(l^\omega, E)$ at the spectral edges.

The operator $M(l^\omega, E)$ is closely related to the random hopping model considered in [15]; below, we use very the constructions of [15] and [23] to obtain the Lifshitz tails. The integrated density of states is defined by

$$k(t) := \lim_{n\to\infty} \frac{\#\{\lambda \in \mathrm{spec}\, M_{\Lambda(n)}(l_\omega, E) :\ \lambda < t\}}{|\mathcal{V}(n)|}.$$

This limit exists almost surely and $t \mapsto k(t)$ is increasing. Let $[\mu_{\min}, \mu_{\max}]$ be the almost sure spectrum of $M(l^\omega, E)$. Then, $k(t) = 0$ for $t \le \mu_{\min}$ and $k(t) = 1$ for $t \ge \mu_{\max}$. Denote also

$$b := \sup_{l\in[l_{\min}, l_{\max}]} \Big|\frac{\sqrt{E}}{\sin l\sqrt{E}}\Big|.$$

By well-known arguments, see e.g. [28, Section 2.2], in order to prove (17) it is sufficient to show

$$\lim_{\varepsilon\to 0+} \frac{\log\big|\log[1 - k(\mu_{\max} - \varepsilon)]\big|}{\log\varepsilon} \le -\frac{d}{2} \text{ and } \lim_{\varepsilon\to 0+} \frac{\log\big|\log k(\mu_{\min} + \varepsilon)\big|}{\log\varepsilon} \le -\frac{d}{2}. \tag{18}$$

For $n \in \mathbb{N}$ define $M_n^\omega(E) := M(l_n^\omega, E)$ where $l_n^\omega(e) = l_n^\omega(e + \gamma)$ for $\gamma \in (2n+1)\mathbb{Z}^d$. By the Floquet-Bloch theory, the operator $M_n^\omega(E)$ admits a density of states, k_n^ω, satisfying

$$k_n(E) = \frac{1}{(2\pi)^d} \int_{\left[-\frac{\pi}{2n+1}, \frac{\pi}{2n+1}\right]} \#\,\mathrm{spec}\, M_n^\omega(E, \theta) \cap (-\infty, E) d\theta$$

where $M_n^\omega(E, \theta)$ differs from $M_n^\omega(E)$ only by an operator of rank at most Cn^{d-1} with $C > 0$ independent of n. As suggested in [15], in order to obtain (18), it is sufficient to show the analogous estimates with $k(E)$ replaced by $\mathbb{E}k_n^\omega(E)$ uniformly in n for sufficiently large n. In its turn, as noted in [23] and applied in [15], the latter asymptotics can obtained directly from the following local energy estimate which has been proved in [15, Lemma 2.1]. Let $a \in (0, b)$. For $\varphi \in l^2(\mathbb{Z}^d)$ one has

$$\langle \varphi, M_n^\omega \varphi\rangle \ge \langle \varphi, W_n^\omega \varphi\rangle + a\big\langle |\varphi|, H_0|\varphi|\big\rangle$$

where H_0 is the free Laplace operator in $(\mathbb{Z}^d)$,

$$H_0 u(n) = \sum_{n:|m-n|=1} \big(u(n) - u(m)\big),$$

and the potential W_n^ω is given by

$$W_n^\omega(v) = \sum_{e:v\sim v} \beta\Big(\frac{\sqrt{E}}{\sin l_e^\omega \sqrt{E}}\Big) + \sum_{e:v\sim e} \sqrt{E}\cot l_e^\omega \sqrt{E}$$

with

$$\beta(t) := \begin{cases} -|t|, & |t| \geq a, \\ -a, & \text{otherwise.} \end{cases}$$

Then, as in [15], following the computations done in [23], one proves Lifshitz tails for $M(l^\omega, E)$ near $\mu_{\max}$ or $\mu_{\min}$. This completes the proof of theorem 5. □

6 Acknowledgments

The authors thank Olaf Post and Ivan Veselić for motivating discussions. The second named author was supported by the Marie Curie Intra-European Fellowship (PIEF-GA-2008-219641).

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