Hyperdeterminantal computation for the Laughlin wave function

Adrien Boussicault, Christophe Tollu, Jean-Gabriel Luque

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A. Boussicault, J.-G. Luque and C. Tollu

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Abstract

The decomposition of the Laughlin wave function in the Slater orthogonal basis appears in the discussion on the second-quantized form of the Laughlin states and is straightforwardly equivalent to the decomposition of the even powers of the Vandermonde determinants in the Schur basis. Such a computation is notoriously difficult and the coefficients of the expansion have not yet been interpreted. In our paper, we give an expression of these coefficients in terms of hyperdeterminants of sparse tensors. We use this result to construct an algorithm allowing to compute one coefficient of the development without computing the others. Thanks to a program in `C`, we performed the calculation for the square of the Vandermonde up to an alphabet of eleven lettres.

1 Introduction

When submitted to a magnetic field orthogonal to their motion, electrons experience the Lorentz force which generates an asymmetric distribution of the charge density in the conductor perpendicularly to both the line of sight path of the current and the magnetic field. The resulting voltage, called the Hall voltage, is proportional to both the current and the magnetic flux density. To extreme low temperature, in a strong magnetic field and for a two-dimensional electron system, the Hall conductance admits quantized values which are integer or fractional multiples of $\frac{e^2}{h}$. In the aim to explain this phenomenon, Laughlin [11] proposed quantum wave functions indexed by fractional fillings of the lowest Landau level[1]. In the simplest cases [5, 6], Fermi statistics require a fractional filling $\frac{1}{2k+1}$ (k being integer) and the corresponding Laughlin wavefunction reads

$$\begin{aligned}\Psi^{n,k}_{\text{Laughlin}}(z_1,\dots,z_n) &= V(z_1,\dots,z_n)^{2k+1}\exp\{-\tfrac{1}{2}\sum_{i=1}^{N}|z_i|^2\}\\ &= V(z_1,\dots,z_n)^{2k}\Psi^0_{\text{Laughlin}}(z_1,\dots,z_n),\end{aligned} \tag{1}$$

where $V(z_1,\dots,z_n)=\prod_{i<j}(z_i-z_j)$ is the Vandermonde determinant. Dunne [6][2] and Di Francesco *et al.* [5] studied, independently, the expansion of the Laughlin wave function as a linear combination of Slater wavefunctions for n particules

$$\Psi^{\lambda}_{\text{Slater}} := \frac{1}{\sqrt{n!\pi^n\prod_{i=1}^{n}\lambda_i!}}\exp\{-\frac{1}{2}\sum_{i=1}^{N}|z_i|^2\}\begin{vmatrix} z_1^{\lambda_1} & z_1^{\lambda_2} & \dots & z_1^{\lambda_n}\\ z_2^{\lambda_1} & z_2^{\lambda_2} & \dots & z_2^{\lambda_n}\\ \vdots & \vdots & & \vdots\\ z_n^{\lambda_1} & z_n^{\lambda_2} & \dots & z_n^{\lambda_n}\end{vmatrix}. \tag{2}$$

It is easy to show that this problem is equivalent to the expansion of a power of the discriminant in the Schur basis [5, 6, 8, 24]. Indeed, it suffices to factorize the Slater wave function $\Psi^{\lambda}_{\text{Slater}}$ by the Schur function S_λ

$$\Psi^{\lambda}_{\text{Slater}} = \frac{1}{\sqrt{n!\pi^n\prod_{i=1}^{n}\lambda_i!}}S_\lambda\Psi^0_{\text{Laughlin}}.$$

A short time after the study of Di Francesco *et al.*, Sharf *et al.* [24] proposed several algorithms to compute this expansion. In particular, they performed it until $n=9$ for the square of the Vandermonde determinant and showed that a conjecture (referred to as the admissibility condition) of [5] about the characterization of the partitions having a non-null contribution in the expansion fails for $n=8$. Note that King *et al.* showed [8] that the conjecture becomes true if one considers the q-discriminant instead of the discriminant. In the same paper, they gave other methods for computing the expansion and perform it until $n=9$ in the case of the q-discriminant. In [25], the reader can found the expansion of $V^{2k}(z_1,\dots,z_n)$ until $n=10$ for $k=1$ and until $n=6$

[1]Energy levels of a particule in a constant uniform magnetic field [9]

[2]Dunne discussed the second-quantized form of the Laughlin states for the fractional quantum Hall effect by decomposing the Laughlin wavefunctions into the n-particle Slater basis and gives a general formula for the expansion coefficients in terms of the characters of the symmetric group.

for $k = 2$. In the present paper, we give an expression of each coefficient as a hyperdeterminant (a natural generalization of the determinant for higher order tensors). As an application, we propose a new algorithm to compute each coefficient independently from the others. The interest of such a result is twofold: First the calculation can be distributed on several computers and the computation being essentially numerical, the algorithm can be implement in many programming languages. Second this method being based on the Laplace expansion of hyperdeterminants, it allows us to write new recurrence formulae.

2 The Laughlin wavefunction and the admissibility conditions

Di Francesco *et al.* [5] defined *admissible partitions* as the partitions which *can appear* when one expands $V(z_1, \ldots, z_n)^{2k}$ on the Schur basis. That is the partition arising as the dominant exponents when one expands $V(z_1, \ldots, z_n)^{2k+1}$ on the monomials without simplifying. In other words, a partition λ is admissible if and only if $z^\lambda := z_1^{\lambda_1} \ldots z_n^{\lambda_n}$ appears with a nonvanishing coefficient in the expansion of $\prod_{i<j}(z_i + z_j)^{2k+1}$,

$$\prod_{i<j}(z_i + z_j)^{2k+1} = \cdots + \alpha_\lambda z^\lambda + \ldots ..$$

For a given pair of integers n and k, the set of admissible partitions is the interval for the dominance order (*i.e.* $\lambda \geq \mu$ if and only if for each $0 \leq i \leq n$, $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$) whose upper bound is $[2k(n-1), \ldots, 2k, 0]$ and lower bound is $[k(n-1), \ldots, k(n-1)]$.
In [5], Di Francesco *et al.* conjectured that admissibility is a necessary and sufficient condition for non-nullity of the coefficient $g_\lambda^{n,k}$. The first counter example appears for $n = 8$ and $k = 1$ and was given by Scharf *et al.* [24] who computed all the coefficients up to $n = 9$, for $k = 1$.

3 Hyperdeterminants

3.1 Definition and basics properties

The birth of hyperdeterminants dates back to 1843, when Cayley gave a lecture at the Cambridge Philosophical Society, about functions that are reducible to sums of determinants. Actually, Cayley used the same name of *hyperdeterminant* to define several polynomials extending the notion of determinants to higher order tensors. The polynomial which we use here, can be considered as the simplest one because of its definition extending in a natural way the expression of the determinant as an alternated sum. Let $\mathrm{M} = (M_{i_1,\dots,i_p})_{1\le i_1,\dots,i_p\le n}$ be a tensor with p indices, the *hyperdeterminant* of M is the alternated sum over p copies of the symmetric group $\mathfrak{S}_n$,

$$\mathrm{Det}(\mathrm{M}) := \frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_p\in\mathfrak{S}_n}\mathrm{sign}(\sigma_1\dots\sigma_p)\prod_{i=1}^{n}M_{\sigma_1(i)\dots\sigma_p(i)}. \tag{3}$$

For example, if $p=4$ and $n=2$,

$$\mathrm{Det}(\mathrm{M}) = -M_{2,1,1,1}M_{1,2,2,2} + M_{2,1,1,2}M_{1,2,2,1} + M_{2,1,2,1}M_{1,2,1,2}$$
$$-M_{2,1,2,2}M_{1,2,1,1}+M_{2,2,1,1}M_{1,1,2,2}-M_{2,2,1,2}M_{1,1,2,1}-M_{2,2,2,1}M_{1,1,1,2}+M_{2,2,2,2}M_{1,1,1,1}$$

Straightforwardly, Det is the zero polynomial when p is odd. Hence, we will suppose that $p=2k$ is even.

We will consider a special kind of hyperdeterminants: the *Hankel hyperdeterminants*, whose entries depends only on the sums of the indices,

$$\mathrm{H}^f := (f(i_1+\dots+i_{2k}))_{0\le i_1,\dots,i_{2k}\le n-1}. \tag{4}$$

The Hankel hyperdeterminants appear in the literature in the work of Lecat [13] (see also [12, 13]), but few properties have been considered. More recently, one of the authors with Jean-Yves Thibon [16, 17] and two of the authors with Hacene Belbachir [2] investigated the links between these polynomials and the Selberg integral and the Jack polynomials.

More generally, one defines a *shifted Hankel hyperdeterminant* depending on $2k$ decreasing vectors $\lambda^{(1)},\dots,\lambda^{(2k)}\in\mathbb{Z}^n$ as the hyperdeterminant of the *shifted Hankel tensor*

$$\mathrm{H}^f_{\lambda^{(1)},\dots,\lambda^{(2k)}} := (f(\lambda^{(1)}_{i_1}+\dots+\lambda^{(2k)}_{i_{2k}}+i_1+\dots+i_{2k}))_{0\le i_1,\dots,i_{2k}\le n-1}. \tag{5}$$

3.2 Minors of hypermatrices

We will denote by $\mathrm{M}\begin{bmatrix} I_1 \\ \vdots \\ I_{2k} \end{bmatrix}$ the minor of a tensor $\mathrm{M} = (M_{i_1,\dots,i_{2k}})_{1\leq i_1,\dots,i_{2k}\leq n}$ obtained by choosing the elements indexed by the $2k$ increasing m-vectors $I_1, \dots, I_{2k}$, *i.e.*,

$$\mathrm{M}\begin{bmatrix} I_1 \\ \vdots \\ I_{2k} \end{bmatrix} := \left(M_{j^{(1)}_{i_1},\dots,j^{(2k)}_{i_{2k}}} \right)_{1\leq i_1,\dots,i_{2k}\leq m},$$

if $I_1 = (j^{(1)}_1 \leq \cdots \leq j^{(1)}_m)$, $\dots$, $I_{2k} = (j^{(2k)}_1 \leq \cdots \leq j^{(2k)}_m)$.

A shifted Hankel tensor is nothing but a minor of the infinite Hankel tensor

$$\mathrm{H}^f_\infty := (f(i_1 + \cdots + i_{2k})_{-\infty<i_1,\dots,i_{2k}<\infty}.$$

Hence, the property *to be a shifted Hankel tensor* is closed for the operation *extracting a minor.*

More generally, consider the generic infinite tensor

$$\mathrm{M}_\infty := (M_{i_1,\dots,i_{2k}})_{-\infty<i_1,\dots,i_{2k}<\infty},$$

and set for each $2k$-tuple of decreasing vectors $\lambda^{(1)}, \dots, \lambda^{(2k)} \in \mathbb{Z}^n$,

$$\mathrm{M}_{\lambda^{(1)},\dots,\lambda^{(2k)}} = \left(M_{n-\lambda^{(1)}_{i_1}+1+i_1,\dots,n-\lambda^{(2k)}_{i_{2k}}+1+i_{2k}} \right)_{1\leq i_1,\dots,i_{2k}\leq n}.$$

The tensor $\mathrm{M}_{\lambda^{(1)},\dots,\lambda^{(2k)}}$ is obviously a minor of M_∞ and conversely, each minor $\mathrm{M}_\infty\begin{bmatrix} I_1 \\ \vdots \\ I_{2k} \end{bmatrix}$ of M_∞ is equal to some $\mathrm{M}_{\lambda^{(1)},\dots,\lambda^{(2k)}}$. Hence, each minor of $\mathrm{M}_{\lambda^{(1)},\dots,\lambda^{(2k)}}$ is again a minor of M_∞ and can be written in the form $\mathrm{M}_{\mu^{(1)},\dots,\mu^{(2k)}}$.

More precisely, one has the following property.

Proposition 3.1 *(Compositions of minors)*
Let $\lambda^{(1)}, \dots, \lambda^{(2k)} \in \mathbb{Z}^n$ be $2k$ decreasing vectors and $J_1, \dots, J_{2k} \subset \{1, \dots, n\}$

be $2k$ subsets of $\{1,\dots,n\}$ with the same cardinality m, $0\le m\le n$. Then the minor

$$\mathrm{M}_{\lambda^{(1)},\dots,\lambda^{(2k)}}\left[\begin{array}{c}\{1,\dots,n\}\setminus J_1\\ \vdots\\ \{1,\dots,n\}\setminus J_{2k}\end{array}\right]=\mathrm{M}_{\nu^{(1)},\nu^{(2)},\dots,\nu^{(2k)}},$$

where

$$\nu^{(p)}:=[\lambda_1^{(p)}+m,\dots,\lambda_{n-j_m}^{(p)}+m,\lambda_{n-j_m+2}^{(p)}+m-1,\dots,\lambda_{n-j_{m-1}}^{(p)}+m-1,\lambda_{n-j_{m-1}+2}^{(p)}+m-2,\\ \dots,\lambda_{n-j_1}^{(p)}+1,\lambda_{n-j_1+2}^{(p)},\dots,\lambda_n^{(p)}]$$

if $J_p=\{j_1\le\cdots\le j_m\}\subset\{1,\dots,n\}$.

Proof It suffices to understand the case of the vectors (*i.e.* the tensors with only one indice). A straightforward induction on the size of J_1 allows us to conclude. □

3.3 A generalization of the Laplace expansion

In the general case, there is no efficient algorithm for computing an hyperdeterminant. Nevertheless, we will use a generalization of the Laplace expansion for the hyperdeterminant due to Zajaczkowski [26][3].

Theorem 3.2 *(Generalized Laplace) Zajaczkowski [26], Gegenbauer [7] Consider a tensor $\mathrm{M}=(M_{i_1,\dots,i_{2k}})_{1\le i_1,\dots,i_{2k}\le n}$, $0\le m\le n$ and $I_1=\{j_1^{(1)}\le\cdots\le j_m^{(1)}\}\subset\{1\dots n\}$. The hyperdeterminant of M can be expanded as a alternated sum of $\binom{n}{m}^{2k-1}$ products of two minors,*

$$\mathrm{Det}(\mathrm{M})=\sum_{I_2,\dots,I_{2k}}\pm\mathrm{Det}\left(\mathrm{M}\left[\begin{array}{c}I_1\\ \vdots\\ I_{2k}\end{array}\right]\right)\mathrm{Det}\left(\mathrm{M}\left[\begin{array}{c}\{1,\dots,n\}\backslash I_1\\ \vdots\\ \{1,\dots,n\}\backslash I_{2k}\end{array}\right]\right)\qquad(6)$$

where the sum runs over the m-uplets

$$I_2=[j_1^{(2)},\dots,j_m^{(2)}],\ \dots,\ I_{2k}=[j_1^{(2k)},\dots,j_m^{(2k)}]\in\{1,\dots,n\}^m$$

[3] Armenante [1] gave the first generalization of the Laplace formula for cubic hyperdeterminants whose first index is not alternating. Few years after Zajaczkowski, Gegenbauer [7] stated a Laplace formula for general hyperdeterminants with alternating and not alternating indices.

and $\pm$ denotes the sign of the product of the permutations σ_i bringing the indices of I_i followed by the indices of $\{1,\ldots,n\}\backslash I_i$ into the original order.

For general hypermatrices, the algorithm induced by this theorem is not more efficient than the direct expansion but we will use it to compute hyperdeterminants of sparse tensors.

4 Computing the coefficients $g_\lambda^{n,k}$

4.1 Hyperdeterminantal expression

One can write some multiple integrals involving products of determinants as hyperdeterminants.

Proposition 4.1 *(Generalized Heine identity)*
Let $(f_j^{(i)})_{\substack{1\leq i\leq 2k\\ 1\leq j\leq n}}$ be a family of functions $\mathbb{C}\to\mathbb{C}$, and μ be any mesure on $\mathbb{C}$ such that the integrals appearing in equality (7) are defined. Then one has

$$\frac{1}{n!}\int\cdots\int \det(f_j^{(1)}(z_i))\ldots\det(f_j^{(2k)}(z_i))d\mu(z_1)\ldots d\mu(z_n) = \mathrm{Det}\left(\int f_{i_1}^{(1)}(z)\ldots f_{i_{2k}}^{(2k)}(z)d\mu(z)\right)_{1\leq i_1,\ldots,i_{2k}\leq n}. \quad (7)$$

Proof Straightforward, expanding the left and right hand sides of equality (7). □

In particular, if one applies Proposition 4.1 to the product of a Schur function and a power of the discriminant, one obtains a shifted Hankel hyperdeterminant whose entries are the moments of the measure μ.

Corollary 4.2 *Let μ be any mesure on $\mathbb{C}$ such that the integrals appearing in equality (7) are defined. One has*

$$\frac{1}{n!}\int\cdots\int s_\lambda(z_1,\ldots,z_n)V(z_1,\ldots,z_n)^{2k}d\mu(z_1)\ldots d\mu(z_n) = \mathrm{Det}(c_{\lambda_{n-i_1+1}+i_1+\cdots+i_{2k}-2k}) \quad (8)$$

where $c_n=\int z^n d\mu(z)$ denotes the nth moment of the measure μ.

Proof It suffices to remark that

$$s_\lambda(z_1,\ldots,z_n)V(z_1,\ldots,z_n)^{2k} = \det(z_i^{\lambda_{n-j+1}+j-1})\det(z_i^{j-1})^{2k-1},$$

and to apply (7).□
Let $\lambda^{(1)}, \dots, \lambda^{(2k)}$ be $2k$ decreasing vectors of $\mathbb{Z}^n$. One defines the tensor

$$\Delta_{\lambda^{(1)},\dots,\lambda^{(2k)}} := \left(\delta_{\lambda^{(1)}_{n-i_1+1}+\dots+\lambda^{(2k)}_{n-i_{2k}+1}+i_1+\dots+i_{2k},(2k-1)n+1}\right)_{1\leq i_1,\dots,i_{2k}\leq n},$$

and

$$\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}} := \mathrm{Det}\left(\Delta_{\lambda^{(1)},\dots,\lambda^{(2k)}}\right),$$

its hyperdeterminant.
The following property gives an expression of the coefficient $g_\lambda^{n,k}$ in terms of hyperdeterminants.

Corollary 4.3 *One has*

$$g_\lambda^{n,k} = (-1)^{\frac{n(n-1)}{2}} \mathfrak{D}_{\lambda,\underbrace{[0^n],\dots,[0^n]}_{2k+1\times}} \tag{9}$$

Proof This is a direct consequence of (8) and the definition of $\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}}$. The complete discussion appears in [2]. □

4.2 Basic properties of the hyperdeterminants $\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}}$

Let us list some straightforward properties of such hyperdeterminants.

Proposition 4.4 *Let σ be any permutation of $\mathfrak{S}_{2k}$, then*

$$\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}} = \mathfrak{D}_{\lambda^{(\sigma(1))},\dots,\lambda^{(\sigma(2k))}}.$$

Proposition 4.5 *Let $m_1, m_2, \dots, m_{2k-1} \in \mathbb{Z}$ be $2k-1$ integers. One has,*

$$\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}} = \mathfrak{D}_{[\lambda^{(1)}_1+m_1,\dots,\lambda^{(1)}_n+m_1],\dots,[\lambda^{(2k)}_1+m_{2k},\dots,\lambda^{(2k)}_n+m_{2k}]},$$

where $m_{2k} = -m_1 - \dots - m_{2k-1}$.

Proposition 4.6 *If $\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}} \neq 0$ then*

$$\sum \lambda_i^{(j)} = (k-1)n(n-1).$$

Proof If $\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}} \neq 0$ then there exist $\sigma_1,\dots,\sigma_{2k} \in \mathfrak{S}_n$ such that

$$\prod_{i=1}^{n} \delta_{\lambda^{(1)}_{n-\sigma_1(i)+1}+\cdots+\lambda^{(2k)}_{n-\sigma_{2k}(i)+1}+\sigma_1(i)+\cdots+\sigma_{2k}(i),(2k-1)n+1} \neq 0.$$

This implies

$$\sum_i \lambda^{(1)}_{n-\sigma_1(i)+1}+\cdots+\lambda^{(2k)}_{n-\sigma_{2k}(i)+1}+\sigma_1(i)+\cdots+\sigma_{2k}(i) = n((2k-1)n+1).$$

But the left hand side is nothing but $\sum_{i,j}\lambda^{(i)}_j + kn(n+1)$. The result follows. $\square$

4.3 Minors of the matrices $\Delta_{\lambda^{(1)},\dots,\lambda^{(2k)}}$

Consider the sets defined by

$$\Gamma_{k,n} := \{\Delta_{\lambda^{(1)},\dots,\lambda^{(2k)}} | \lambda^{(1)},\dots,\lambda^{(2k)} \text{ are decreasing vectors of } \mathbb{Z}^n\}.$$

Proposition 4.7 *Let $\lambda^{(1)},\dots,\lambda^{(2k)} \in \mathbb{Z}^n$ be $2k$ decreasing vectors and $J_1,\dots,J_{2k} \subset \{1,\dots,n\}$ be $2k$ subsets of $\{1,\dots,n\}$ with the same cardinality m, $0 \leq m \leq n$. Hence the minor*

$$\Delta_{\lambda^{(1)},\dots,\lambda^{(n)}} \begin{bmatrix} \{1,\dots,n\}\setminus J_1 \\ \vdots \\ \{1,\dots,n\}\setminus J_{2k} \end{bmatrix}$$

belongs to $\Gamma_{k,n-m}$.

Proof From Proposition 3.1, one obtains

$$\Delta_{\lambda^{(1)},\dots,\lambda^{(2k)}} \begin{bmatrix} \{1,\dots,n\}\setminus J_1 \\ \vdots \\ \{1,\dots,n\}\setminus J_{2k} \end{bmatrix} =$$
$$\left(\delta_{\nu^{(1)}_{n-i_1+1}+\cdots+\nu^{(2k)}_{n-i_{2k}+1}+i_1+\cdots+i_{2k},(2k-1)(n-m+1)+1}\right)_{1\leq i_1,\dots,i_{2k}\leq n-m}$$

where

$$\nu^{(p)} := [\lambda_1^{(p)}+m,\dots,\lambda_{n-j_m}^{(p)}+m,\lambda_{n-j_m+2}^{(p)}+m-1,\dots,\lambda_{n-j_{m-1}}^{(p)}+m-1,\lambda_{n-j_{m-1}+2}^{(p)}+m-2, \\ \dots,\lambda_{n-j_1}^{(p)}+1,\lambda_{n-j_1+2}^{(p)},\dots,\lambda_n^{(p)}]$$

if $J_p=\{j_1\le\cdots\le j_m\}\subset\{1,\dots,n\}$. Furthermore,

$$\delta_{\nu_{n-i_1+1}^{(1)}+\cdots+\nu_{n-i_{2k}+1}^{(2k)}+i_1+\cdots+i_{2k},(2k-1)(m+1)+1}=\delta_{\nu'^{(1)}_{n-i_1+1}+\cdots+\nu_{n-i_{2k}+1}^{(2k)}+i_1+\cdots+i_{2k},(2k-1)(n-m)+1},$$

where $\nu'^{(1)}$ is the decreasing sequence

$$\nu'^{(1)} := [\nu_1{}^{(1)}-m(2k-1),\dots,\nu_n{}^{(1)}-m(2k-1)]. \tag{10}$$

Hence,

$$\Delta_{\lambda^{(1)},\dots,\lambda^{(2k)}}\left[\begin{array}{c}\{1,\dots,n\}\setminus J_1\\ \vdots\\ \{1,\dots,n\}\setminus J_{2k}\end{array}\right]=\Delta_{\nu'^{(1)},\nu^{(2)},\dots,\nu^{(2k)}}\in\Gamma_{k,n-m}.$$

This completes the proof. □

4.4 A recursive formula for $\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}}$

As a consequence of the preceding sections, one has

Corollary 4.8 *Let $1\le p\le m$, one has*

$$\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}}=\sum_I(-1)^{i_1+\cdots+i_{2k}}\mathfrak{D}_{\mu_I^{(1)},\dots,\mu_I^{(2k)}}, \tag{11}$$

where the sum is over the $2k$-tuples, $I=[i_1,i_2,\dots,i_{2k}]\in\{1,\dots,n\}^{2k}$ verifying

$$\lambda_{n-i_1+1}^{(1)}+\cdots+\lambda_{n-i_{2k}+1}^{(2k)}+i_1+\cdots+i_{2k}=(2k-1)n+1,$$

and the decreasing vectors $\mu_I^{(1)},\dots,\mu_I^{(2k)}$ are defined by

$$\mu_I^{(1)}=[\lambda_1^{(1)}-2(k-1),\dots,\lambda_{n-i_1}^{(1)}-2(k-1),\lambda_{n-i_1+2}^{(1)}-2(k-1)-1,\dots,\lambda_n^{(1)}-2(k-1)-1]$$
$$\mu_I^{(2)}=[\lambda_1^{(2)}+1,\dots,\lambda_{n-i_2}^{(2)}+1,\lambda_{n-i_2+2}^{(2)},\dots,\lambda_n^{(2)}]$$
$$\vdots$$
$$\mu_I^{(2k)}=[\lambda_1^{(2k)}+1,\dots,\lambda_{n-i_{2k}}^{(2k)}+1,\lambda_{n-i_{2k}+2}^{(2k)},\dots,\lambda_n^{(2k)}]. \tag{12}$$

Proof Setting $I_1 = \{i_1\}$ in Theorem 3.2, from the definition of $\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}}$, one gets,

$$\mathfrak{D}_{\lambda^{(1)},\dots,\lambda^{(2k)}} = \sum_{I} (-1)^{i_1+\cdots+i_{2k}} \mathrm{Det}\left(\Delta_{\lambda^{(1)},\dots,\lambda^{(k)}}\begin{bmatrix} \{1,\dots,n\}\setminus i_1 \\ \vdots \\ \{1,\dots,n\}\setminus i_{2k}\end{bmatrix}\right)$$

where the sum is over the $2k$-tuples, $I = [i_1, i_2, \dots, i_{2k}] \in \{1,\dots,n\}^{2k}$ verifying

$$\lambda^{(1)}_{n-i_1+1} + \cdots + \lambda^{(2k)}_{n-i_{2k}+1} + i_1 + \cdots + i_{2k} = (2k-1)n + 1.$$

Furthermore, one has

$$\Delta_{\lambda^{(1)},\dots,\lambda^{(2k)}}\begin{bmatrix} \{1,\dots,n\}\setminus i_1 \\ \vdots \\ \{1,\dots,n\}\setminus i_{2k}\end{bmatrix} = \Delta_{\mu^{(1)},\dots,\mu^{(2k)}}$$

where the partitions $\mu^{(i)}$ are defined by (12). The result follows. $\square$

Example 4.9 Suppose that we want to compute $\mathfrak{D}_{[211][100][100][000]}$. That is to compute the hyperdeterminant of

$$\Delta_{[211][100][100][000]} = \begin{array}{c|ccccccccc|} {}_{i_3,i_4}\backslash^{i1,i2} & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\ \hline 11 & & & & & & & & & \\ 12 & & & & & & 1 & & 1 & \\ 13 & & & 1 & & & & 1 & & \\ 21 & & & & & & 1 & & 1 & \\ 22 & & & 1 & & & & 1 & & \\ 23 & & & & & 1 & & & & \\ 31 & & & & & 1 & & & & \\ 32 & & 1 & & 1 & & & & & \\ 33 & 1 & & & & & & & & \end{array}.$$

If one sets $i_1 = 1$, the only indices (i_1, i_2, i_3, i_4) such that the corresponding entries of $\Delta_{[211][100][100][000]}$ do not vanish are $(1,3,2,2), (1,2,3,2), (1,1,3,3)$ and $(1,3,1,3)$. Furthermore

$$\Delta_{[211][100][100][000]}\begin{bmatrix} \{2,3\} \\ \{1,2\} \\ \{1,3\} \\ \{1,3\}\end{bmatrix} = \begin{array}{c|cccc|} {}_{i_3,i_4}\backslash^{i1,i2} & 11 & 12 & 21 & 22 \\ \hline 11 & & & & \\ 12 & & & 1 & \\ 21 & & 1 & & \\ 22 & & & & \end{array} = \Delta_{[0-1][00][20][10]}.$$

On the same way, one has

$$\Delta_{[211][100][100][000]}\begin{bmatrix} \{2,3\} \\ \{1,3\} \\ \{1,2\} \\ \{1,3\}\end{bmatrix} = \Delta_{[0-1][20][00][10]},$$

$$\Delta_{[211][100][100][000]}\left[\begin{array}{c}\{2,3\}\\ \{2,3\}\\ \{1,2\}\\ \{1,2\}\end{array}\right]=\Delta_{[0-1][21][00][00]}$$

and

$$\Delta_{[211][100][100][000]}\left[\begin{array}{c}\{2,3\}\\ \{1,2\}\\ \{2,3\}\\ \{1,2\}\end{array}\right]=\Delta_{[0-1][00][21][00]}.$$

Hence,

$$\mathfrak{D}_{[211][100][100][000]}=\mathfrak{D}_{[0-1][00][20][10]}+\mathfrak{D}_{[0-1][20][00][10]}+\mathfrak{D}_{[0-1][21][00][00]}+\mathfrak{D}_{[0-1][00][21][00]}.$$

A straightforward computation gives

$$\mathfrak{D}_{[0-1][20][00][10]}=\mathfrak{D}_{[0-1][00][20][10]}=1$$

and

$$\mathfrak{D}_{[0-1][21][00][00]}=\mathfrak{D}_{[0-1][00][21][00]}=2.$$

from what it follows that $\mathfrak{D}_{[211][100][100][000]}=6$.

Example 4.10 Here one illustrates the fact that the recurrence (11) provides an algorithm to compute the coefficient $g_\lambda^{k,n}$. Suppose that one wants to compute the coefficient of s_{411} in the square of the Vandermonde determinant for an alphabet of size 3. One needs to compute the value of $\mathfrak{D}_{[4,1,1],[0,0,0],[0,0,0],[0,0,0]}$. Applying the Laplace expansion, one finds that this can be written as a sum involving 27 hyperdeterminants

$$\mathfrak{D}_{[4,1,1],[0,0,0],[0,0,0],[0,0,0]}=\alpha_{3111}\mathfrak{D}_{[2,-1],[0,0],[0,0],[0,0]}+\alpha_{3112}\mathfrak{D}_{[2,-1],[0,0],[0,0],[1,0]}+\\ \cdots+\alpha_{3333}\mathfrak{D}_{[2,-1],[1,1],[1,1],[1,1]}.$$

But for only three of them the coefficient α_I does not vanish

$$\mathfrak{D}_{[4,1,1],[0,0,0],[0,0,0],[0,0,0]}=\alpha_{3211}\mathfrak{D}_{[2,-1],[1,0],[0,0],[0,0]}+\alpha_{3112}\mathfrak{D}_{[2,-1],[0,0],[0,0],[1,0]}\\ +\alpha_{3121}\mathfrak{D}_{[2,-1],[0,0],[1,0],[0,0]}.$$

One has $\alpha_{3112}=\alpha_{3121}=\alpha_{3211}=-1$ and for reason of symmetry

$$\mathfrak{D}_{[2,-1],[1,0],[0,0],[0,0]}=\mathfrak{D}_{[2,-1],[0,0],[0,0],[1,0]}=\mathfrak{D}_{[2,-1],[0,0],[1,0],[0,0]}.$$

It remains to compute $\mathfrak{D}_{[2,-1],[1,0],[0,0],[0,0]}$. Using again the Laplace expansion, one finds that this can be written as the sum of 8 hyperdeterminants, of which only one gives a nonvanishing α_I,

$$\mathfrak{D}_{[2,-1],[1,0],[0,0],[0,0]} = \alpha_{2111}\mathfrak{D}_{[0],[0],[0],[0]} = -1$$

Hence, $g^{1,3}_{411} = 3$.

4.5 Factorisation formulæ

Proposition 4.11 *Let $\lambda^{(1)}, \ldots, \lambda^{2k}$ such that there exists an integer $0 < m < n$ verifying*

$$\lambda^{(1)}_1 + \cdots + \lambda^{(1)}_m + \lambda^{(2)}_{n-m} + \cdots + \lambda^{(2)}_n + \cdots + \lambda^{(2k)}_{n-m} + \cdots + \lambda^{(2k)}_n = (k-1)m(m-1)$$

then $\mathfrak{D}_{\lambda^{(1)},\ldots,\lambda^{(2k)}}$ factorizes as

$$\mathfrak{D}_{\lambda^{(1)},\ldots,\lambda^{(2k)}} = \pm\mathfrak{D}_{\mu^{(1)},\ldots,\mu^{(2k)}}\mathfrak{D}_{\nu^{(1)},\ldots,\nu^{(2k)}},$$

where

$$\mu^{(1)} := [\lambda^{(1)}_1 - 2(k-1)m, \ldots, \lambda^{(1)}_m - 2(k-1)m],\ \nu^{(1)} := [\lambda^{(1)}_{m+1}, \ldots, \lambda^{(1)}_n]$$
$$\mu^{(2)} := [\lambda^{(2)}_{n-m+1}, \ldots, \lambda^{(2)}_n],\ \nu^{(2)} := [\lambda^{(2)}_1, \ldots, \lambda^{(2)}_{n-m}]$$
$$\vdots$$
$$\mu^{(2k)} := [\lambda^{(2k)}_{n-m+1}, \ldots, \lambda^{(2k)}_n],\ \nu^{(2k)} := [\lambda^{(2k)}_1, \ldots, \lambda^{(2k)}_{n-m}]$$

Proof It is a direct consequence of the generalized Laplace expansion. □

Corollary 4.12 *Let λ be such that it exists an integer $0 < m < n$ verifying*

$$\lambda_1 + \cdots + \lambda_m = km(m-1)$$

then

$$g^{n,k}_\lambda = g^{n-m,k}_\mu g^{m,k}_\nu,$$

where

$$\mu := [\lambda_1 - 2k(m-1), \ldots, \lambda_m - 2k(m-1)],\ \textit{and}\ \nu := [\lambda_{m+1}, \ldots, \lambda_n].$$

Proof It is a direct consequence of Proposition 4.11.□

Note that Corollary 4.12 can also be obtained as a straightforward consequence of the factorization

$$\Delta(x_1,\dots,x_n)=\Delta(x_1,\dots,x_m)\prod_{i=1}^{m}\prod_{j=m+1}^{n}(x_i-x_j)\Delta(x_{m+1},\dots,x_n).$$

Example 4.13 *To calculate the coefficient $g^{1,5}_{77420}$, one may compute the hyperdeterminant*

$$\mathfrak{D}_{[77420],[0000],[0000],[0000]}.$$

From Proposition 4.11, it factorizes as

$$\pm\mathfrak{D}_{[420],[000],[000],[000]}\mathfrak{D}_{[33],[00],[00],[00]}..$$

Hence,

$$g^{1,5}_{77420}=g^{1,3}_{420}g^{1,2}_{33}.$$

5 Results

The rules explained in the previous sections enables to write an algorithm computing the coefficients $g^{k,n}_{\lambda}$. The calculations being completely numerical, they can be implemented in a programming language such as `C` which allows us to optimize runtime and memory management. A program written in `C` can be downloadeed from [15]. All calculations have been performed on a personal computer[4], with the only exception of the case $k=1$ and $n=11$, for which a 8-processors cluster with 32 Go Ram was used. In the most general case, computing a hyperdeterminant using the generalized Laplace theorem is possible only for very small dimensions. Here, as we consider only very sparse tensors, the computation can be achieved for reasonably large alphabets.

Table 1 contains the list of the cases which have been computed with this program. The results can be downloaded from [15]. As expected, there are fewer nonvanishing partitions than admissible partitions. Tables 2 and 3 contain respectively the number of admissible partitions and the number of vanishing admissible partitions.

[4]Intel Pentium processor 1.86Ghz, 1Go Ram.

k	n max
1	up to 11
2	up to 7
3	up to 6
4	up to 5
5	up to 5

Table 1: List of the case for which the computation have been performed for all admissible partitions.

	$n=2$	3	4	5	6	7	8	9	10	11
$k=1$	2	5	16	59	247	1111	5302	28376	135670	716542
$k=2$	3	13	76	521	3996	32923				
$k=3$	4	25	213	2131	23729					
$k=4$	5	41	459	6033	88055					
$k=5$	6	61	846	13771						

Table 2: Number of admissible partitions

6 Conclusion

We have described an algorithm which computes each coefficient appearing in the expansion of the Laughlin wave functions in the Slater basis without computing the others, which allows to distribute easily the computation. This algorithm is based on an interpretation of each coefficient as an hyperdeterminant. This approach being completely numerical our algorithm can be implemented in various languages (such as `C`). The principal limitation of our method is that the generalization to the q-deformation is not easy. In particular, one has to construct an analogue of the (multi)-antisymmetrizer. One possible approach would consist in searching for the latter operator in the double affine Hecke algebra. Indeed, in previous articles, two of the authors gave q-deformations [3, 18] which can be written as symmetric Macdonald functions indexed by rectangular or staircase partitions for some specializations of the parameters (which made us think that the Hecke algebra may play a rôle). We have not identified the operator yet.
The method can also be adapted to write the powers of the discriminant in the monomial basis. In this case, one has to compute the hypedeterminant

	$n=2$	3	4	5	6	7	8	9	10	11
$k=1$	0	0	0	0	0	0	8	66	389	1671
$k=2$	0	0	0	0	6	46	?	?	?	?
$k=3$	0	0	0	2	14	?	?	?	?	?
$k=4$	0	0	0	16	?	?	?	?	?	?
$k=5$	0	0	0	0	?	?	?	?	?	?

Table 3: Number of vanishing admissible partitions

with non alternating indices of a sparse tensor using the more general version of the Gegenbauer-Laplace expansion theorem [7]. Nevertheless, the tensor considered are bigger (with an odd number of indices). Furthermore, several others methods exist to perform this computation (see *e.g.* [24]) and we do not know whether ours is very efficient in this case.

It is also worth noting that Physicists use another and more efficient method to carry out these calculations. They proceed by diagonalization of the unphysical model Hamiltonian for which the power of the Vandermonde is the exact ground state (see *e.g.* [20, 21, 23]). The drawback of that algorithm is that one cannot obtain one coefficient without computing the others. Another advantage of our method is that it is based on a combinatorial description of some hyperdeterminants (after recoding them, one only uses the vectors which index them). Giving new relations, this can be used to understand the very difficult problem of the characterization of the partitions which have a nonvanishing contribution. One can follow two tracks to solve this problem. The first one consists in understanding the combinatorics of these hyperdeterminants. The second, more algebraic and geometric, consists in characterizing the varieties defined by the vanishing of a hyperdeterminant.

Finally, as the powers of the Vandermonde are special cases of the Read-Rezayi states [22], one can naturally ask the question of the generalization of our method to other cases.

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