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► **To cite this version:**

Haydar Abdelhamid, Marie-Françoise Bidaut-Véron. Correlation between two quasilinear elliptic problems with a source term involving the function or its gradient. 2008. hal-00326563v1

**HAL Id: hal-00326563**

**<https://hal.science/hal-00326563v1>**

Preprint submitted on 3 Oct 2008 (v1), last revised 19 Nov 2008 (v2)

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# Correlation between two quasilinear elliptic problems with a source term involving the function or its gradient

Haydar Abdelhamid\*

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## Abstract

Thanks to a change of unknown we compare two elliptic quasilinear problems with Dirichlet data in a bounded domain of  $\mathbb{R}^N$ . The first one, of the form  $-\Delta_p u = \beta(u) |\nabla u|^p + \lambda f(x)$ , where  $\beta$  is nonnegative, involves a gradient term with natural growth. The second one, of the form  $-\Delta_p v = \lambda f(x)(1 + g(v))^{p-1}$  where  $g$  is nondecreasing, presents a source term of order 0. The correlation gives new results of existence, nonexistence and multiplicity for the two problems.

## Résumé

**Corrélation entre deux problèmes quasilineaires elliptiques avec terme de source relatif à la fonction ou à son gradient** A l'aide d'un changement d'inconnue nous comparons deux problèmes elliptiques quasilineaires avec conditions de Dirichlet dans un domaine borné  $\Omega$  de  $\mathbb{R}^N$ . Le premier, de la forme  $-\Delta_p u = \beta(u) |\nabla u|^p + \lambda f(x)$ , où  $\beta$  est positif, comporte un terme de gradient à croissance critique. Le second, de la forme  $-\Delta_p v = \lambda f(x)(1 + g(v))^{p-1}$  où  $g$  est croissante, contient un terme de source d'ordre 0. La comparaison donne des résultats nouveaux d'existence, nonexistence et multiplicité pour les deux problèmes.

## Version française abrégée

Soit  $\Omega$  un domaine borné régulier de  $\mathbb{R}^N$  ( $N \geq 2$ ) et  $1 < p < N$ . Dans cette note nous comparons deux problèmes quasilineaires. Le premier comporte un terme de source d'ordre 1:

$$-\Delta_p u = \beta(u) |\nabla u|^p + \lambda f(x) \quad \text{dans } \Omega, \quad u = 0 \quad \text{sur } \partial\Omega, \quad (1)$$

où  $\beta \in C^0([0, L])$ ,  $L \leq \infty$ , à valeurs  $\geq 0$ ,  $\lambda > 0$  et  $f \in L^1(\Omega)$ ,  $f \geq 0$  p.p. dans  $\Omega$ . Le second problème comporte un terme de source d'ordre 0:

$$-\Delta_p v = \lambda f(x)(1 + g(v))^{p-1} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \quad (2)$$

où  $g \in C^1([0, \Lambda])$ ,  $\Lambda \leq \infty$ ,  $g(0) = 0$  et  $g$  est croissante.

Le changement d'inconnue

$$v(x) = \Psi(u(x)) = \int_0^{u(x)} e^{\gamma(\theta)/(p-1)} d\theta, \quad \text{où } \gamma(t) = \int_0^t \beta(\theta) d\theta,$$

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conduit formellement du problème (1) au problème (2), et  $\beta$  et  $g$  sont liés par la relation  $\beta(u) = (p-1)g'(v)$ . En particulier  $\beta$  est croissant si et seulement si  $g$  est convexe. Nous montrons que le changement d'inconnue inverse formel est donné explicitement par

$$u(x) = H(v(x)) = \int_0^{v(x)} \frac{ds}{1+g(s)}.$$

Toutefois dans la transformation peuvent s'introduire des mesures. Notons  $M_b^+(\Omega)$  l'espace des mesures de Radon positives bornées sur  $\Omega$ , et  $M_s^+(\Omega)$  le sous-ensemble des mesures concentrées sur un ensemble de  $p$ -capacité 0. Nous établissons une correspondance exacte entre les deux problèmes:

**Théorème 1** *Soit  $u$  une solution renormalisée du problème*

$$-\Delta_p u = \beta(u) |\nabla u|^p + \lambda f(x) + \alpha_s \quad \text{dans } \Omega, \quad u = 0 \quad \text{sur } \partial\Omega, \quad (3)$$

où  $\alpha_s \in M_s^+(\Omega)$ , et  $0 \leq u(x) < L$  p.p. dans  $\Omega$ . Alors il existe  $\mu_s \in M_s^+(\Omega)$ , concentrée sur l'ensemble  $\{u = L\}$ , telle que  $v = \Psi(u)$  est solution renormalisée du problème

$$-\Delta_p v = \lambda f(x)(1+g(v))^{p-1} + \mu_s \quad \text{dans } \Omega, \quad v = 0 \quad \text{sur } \partial\Omega. \quad (4)$$

Réciproquement soit  $v$  une solution renormalisée de (4), telle  $0 \leq v(x) < \Lambda$  p.p. dans  $\Omega$ , où  $\mu_s \in M_s^+(\Omega)$ . Alors il existe  $\alpha_s \in M_s^+(\Omega)$  concentrée sur l'ensemble  $\{v = \Lambda\}$ , telle que  $u = H(v)$  est solution renormalisée de (3). De plus, si  $\mu_s = 0$ , alors  $\alpha_s = 0$ . Si  $L = \infty$  et  $\beta \in L^1((0, \infty))$ , alors  $\mu_s = e^{\gamma(\infty)} \alpha_s$ . Si  $L < \infty$ , ou  $L = \infty$  et  $\beta \notin L^1((0, \infty))$ , et  $\alpha_s \neq 0$  alors (3) n'a pas de solution. Si  $\Lambda < \infty$  et  $\mu_s \neq 0$ , alors (4) n'a pas de solution.

Dans le cas  $\beta$  constant, les résultats suivants généralisent ceux de [1] relatifs au cas  $p = 2$ :

**Théorème 2** *On suppose que  $\beta(u) \equiv p-1$ , donc  $v = \Psi(u) = e^u - 1$  et  $g(v) = v$ , et que*

$$\lambda_1(f) = \inf \left\{ \left( \int_{\Omega} |\nabla w|^p dx \right) / \left( \int_{\Omega} f |w|^p dx \right) : w \in W_0^{1,p}(\Omega) \setminus \{0\} \right\} > 0$$

Si  $\lambda > \lambda_1(f)$ , ou  $\lambda = \lambda_1(f)$  et  $f \in L^{N/p}(\Omega)$ , alors (1) et (2) n'ont pas de solution renormalisée. Si  $0 < \lambda < \lambda_1(f)$  alors (2) a une solution unique  $v_0 \in W_0^{1,p}(\Omega)$ , et (1) a une solution unique  $u_0 \in W_0^{1,p}(\Omega)$  telle que  $e^{u_0} - 1 \in W_0^{1,p}(\Omega)$ . Si de plus  $f \in L^r(\Omega)$  avec  $r > N/p$ , alors  $u$  et  $v \in L^\infty(\Omega)$ ; et pour toute mesure  $\mu_s \in M_s^+(\Omega)$ , (4) a une solution renormalisée  $v_s$ , et donc (1) a une infinité de solutions  $u_s \in W_0^{1,p}(\Omega)$  moins régulières que  $u_0$ .

Le théorème 1 et l'utilisation du problème (1) nous permettent de déduire un résultat important pour le problème (2), étendant un résultat classique de [2] dans le cas  $p = 2$ :

**Théorème 3** *On suppose que  $\Lambda = \infty$ ,  $\lim_{s \rightarrow \infty} g(s)/s = \infty$ ,  $g$  est convexe à l'infini, et  $f \in L^r(\Omega)$  avec  $r > N/p$ . Alors il existe  $\lambda^* > 0$  tel que pour tout  $\lambda \in (0, \lambda^*)$  le problème (2) a une solution minimale bornée  $\underline{v}_\lambda$ , et pour tout  $\lambda > \lambda^*$  il n'a aucune solution renormalisée.*

Nous étudions aussi les propriétés de la fonction extrême  $v^* = \sup_{\lambda \nearrow \lambda^*} \underline{v}_\lambda$  étendant certains résultats de [3], [9], [12]. Dans le cas où  $g$  est à croissance limitée par l'exposant de Sobolev, nous obtenons des résultats d'existence d'une seconde solution variationnelle, nouveaux même dans le cas  $p = 2$ , étendant ceux de [1] et de [5]:

**Théorème 4** *Sous les hypothèses du Théorème 3, on suppose que*

$$\limsup_{\tau \rightarrow \infty} g^{p-1}(\tau)/\tau^Q < \infty \quad \text{avec } Q \in (1, p^* - 1),$$

et  $f \in L^r(\Omega)$  avec  $(Q+1)r' < p^*$ . Alors il existe  $\lambda_0 > 0$  tel que pour tout  $\lambda \leq \lambda_0$ , le problème (2) a au moins deux solutions  $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ . Si de plus  $p = 2$ ,  $g$  est convexe, ou si  $g$  satisfait la condition d'Ambrosetti-Rabinowitz et  $f \in L^\infty(\Omega)$  le résultat est valable avec  $\lambda_0 = \lambda^*$ .

# 1 Introduction and main results

Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) and  $1 < p < N$ . In this Note we compare two quasilinear problems. The first one presents a source gradient term with a natural growth:

$$-\Delta_p u = \beta(u) |\nabla u|^p + \lambda f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where  $\beta \in C^0([0, L])$ ,  $L \leq \infty$ , and  $\beta$  is nonnegative, and  $\lambda > 0$  is a given real, and  $f \in L^1(\Omega)$ ,  $f \geq 0$  a.e. in  $\Omega$ . Here  $\beta$  can have an asymptote at point  $L$ , and is not supposed to be nondecreasing.

The second problem involves a source term of order 0, with the same  $\lambda$  and  $f$ :

$$-\Delta_p v = \lambda f(x)(1 + g(v))^{p-1} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \quad (1.2)$$

where  $g \in C^1([0, \Lambda])$ ,  $\Lambda \leq \infty$ ,  $g(0) = 0$  and  $g$  is nondecreasing.

Problems of type (1.1) have been intensively studied the last twenty years. The main questions are existence, according to the regularity of  $f$  and the value of  $\lambda$ , regularity and multiplicity of the solutions. Problems of type (1.2) are also the object of a very rich litterature, in particular when  $\Lambda = \infty$ ,  $g$  is superlinear, and convex. Here a main question is to find the range of  $\lambda$  for which there exists a variational solution, or for which there exists a minimal bounded solution, and to get regularity properties of their limit, called extremal solution. Another question is the existence of a second variational solution when  $g^{p-1}$  is subcritical with respect to the Sobolev exponent. A last question is the existence with possible measure data.

It is well known that the change of unknown in (1.1)

$$v(x) = \Psi(u(x)) = \int_0^{u(x)} e^{\gamma(\theta)/(p-1)} d\theta, \quad \text{where } \gamma(t) = \int_0^t \beta(\theta) d\theta,$$

leads formally to problem (1.2), where  $\Lambda = \Psi(L)$  and  $g$  is given by  $g(v) = e^{\gamma(u)/(p-1)} - 1$ . This is a way for studying problem (1.1) from the knowledge of problem (1.2). What is **remarkable**, and apparently new, even in the case  $p = 2$ , is the reverse correlation: **for any function**  $g$  nondecreasing on  $[0, \Lambda]$ , the substitution in (1.2)

$$u(x) = H(v(x)) = \int_0^{v(x)} \frac{ds}{1 + g(s)}$$

leads formally to problem (1.1), where  $\beta$  is defined on  $[0, L]$  with  $L = H(\Lambda)$ ; indeed  $H = \Psi^{-1}$ . And  $\beta$  is linked to  $g$  by relation  $\beta(u) = (p-1)g'(v)$ . In particular  $\beta$  is nondecreasing if and only if  $g$  is convex; and  $L$  is finite if and only if  $1/(1+g) \notin L^1(0, \Lambda)$ .

## Some examples with $p = 2$ .

1.  $\beta(u) = 1$  and  $1 + g(v) = 1 + v$ .
2.  $\beta(u) = q/(1 + (1 - q)u)$ ,  $q \in (0, 1)$ , and  $1 + g(v) = (1 + v)^q$ .
3.  $\beta(u) = 1 + e^u$  and  $1 + g(v) = (1 + v)(1 + \ln(1 + v))$ .
4.  $\beta(u) = q/(1 - (q - 1)u)$ ,  $q > 1$  and  $1 + g(v) = (1 + v)^q$ .
5.  $\beta(u) = 1/(1 - u)$  and  $1 + g(v) = e^v$ .
6.  $\beta(u) = q/(1 - (q + 1)u)$ ,  $q > 0$  and  $1 + g(v) = 1/(1 - v)^q$ .

It had been observed in [6] that the correspondence between  $u$  and  $v$  is more complex, because some measures can appear, in particular in the equation in  $v$ . Our first main result is to make precise the link between the two problems. We denote by  $M_b^+(\Omega)$  the set of nonnegative bounded Radon measures, and by  $M_s^+(\Omega)$  the subset of measures concentrated on a set of  $p$ -capacity 0.

**Theorem 1.1** *Let  $u$  be any renormalized solution of problem*

$$-\Delta_p u = \beta(u) |\nabla u|^p + \lambda f(x) + \alpha_s \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.3)$$

where  $\alpha_s \in M_s^+(\Omega)$  and such that  $0 \leq u(x) < L$  a.e. in  $\Omega$ . Then there exists  $\mu_s \in M_s^+(\Omega)$ , concentrated on the set  $\{u = L\}$ , such that  $v = \Psi(u)$  is a renormalized solution of problem

$$-\Delta_p v = \lambda f(x)(1 + g(v))^{p-1} + \mu_s \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \quad (1.4)$$

Conversely let  $v$  be any renormalized solution of (1.4), where  $\mu_s \in M_s^+(\Omega)$  and such that  $0 \leq v(x) < \Lambda$  a.e. in  $\Omega$ . Then there exists  $\alpha_s \in M_s^+(\Omega)$  concentrated on the set  $\{v = \Lambda\}$ , such that  $u = H(v)$  is a renormalized solution of (1.3).

Moreover if  $\mu_s = 0$ , then  $\alpha_s = 0$ . If  $L = \infty$  and  $\beta \in L^1((0, \infty))$ , then  $\mu_s = e^{\gamma(\infty)} \alpha_s$ . If  $L < \infty$  or if  $L = \infty$  and  $\beta \notin L^1((0, \infty))$ , and  $\alpha_s \neq 0$ , then (1.3) has no solution. If  $\Lambda < \infty$ , and  $\mu_s \neq 0$ , then (1.4) has no solution.

This theorem extends in particular the results of [1] where  $p = 2$  and  $L = \infty$ . The nonexistence result when  $\beta \notin L^1((0, \infty))$ , and  $\alpha_s \neq 0$ , answers to an open question of [10].

First we apply to the case  $\beta$  constant, which means  $g$  linear.

**Theorem 1.2** *Assume that  $\beta(u) \equiv p - 1$ , thus  $v = \Psi(u) = e^u - 1$  and  $g(v) = v$ . Suppose that*

$$\lambda_1(f) = \inf \left\{ \left( \int_{\Omega} |\nabla w|^p dx \right) / \left( \int_{\Omega} f |w|^p dx \right) : w \in W_0^{1,p}(\Omega) \setminus \{0\} \right\} > 0 \quad (1.5)$$

If  $\lambda > \lambda_1(f)$ , or  $\lambda = \lambda_1(f)$  and  $f \in L^{N/p}(\Omega)$ , then (1.1) and (1.2) admit no renormalized solution.

If  $0 < \lambda < \lambda_1(f)$  there exists a unique solution  $v \in W_0^{1,p}(\Omega)$  to (1.2), thus a **unique** solution  $u \in W_0^{1,p}(\Omega)$  to (1.1) such that  $e^u - 1 \in W_0^{1,p}(\Omega)$ . Moreover, if  $f \in L^r(\Omega)$  with  $r > N/p$ , then  $u$  and  $v \in L^\infty(\Omega)$ , and for any measure  $\mu_s \in M_s^+(\Omega)$ , there exists a renormalized solution  $v_s$  of (1.4), thus there exists an **infinity** of less regular solutions  $u_s \in W_0^{1,p}(\Omega)$  of (1.1).

**Remark 1.3** *Under the assumption (1.5), most of these existence results extend to general  $g$  such that  $\Lambda = \infty$  and  $\limsup_{\tau \rightarrow \infty} g(\tau)/\tau < \infty$ . They extend to the case*

$$\limsup_{\tau \rightarrow \infty} g(\tau)/\tau^q < \infty \quad \text{for some } q \in (1, N/(N-p)) \quad (1.6)$$

if moreover  $f \in L^r(\Omega)$  with  $qr' < N/(N-p)$ .

Next consider problem (1.2) with general  $g$ , and  $f \in L^r(\Omega)$  with  $r > N/p$ . It is easy to prove that for small  $\lambda > 0$  there exists a minimal solution  $\underline{v}_\lambda \in W_0^{1,p}(\Omega)$  such that  $\|\underline{v}_\lambda\|_{L^\infty(\Omega)} < \Lambda$ . Our main result is an extension of a well known result of [2] for  $p = 2$ , and [3] for  $p > 1$ . It is noteworthy that **the proof uses problem (1.1)**:

**Theorem 1.4** *Assume that  $\Lambda = \infty$ , and  $\lim_{s \rightarrow \infty} g(s)/s = \infty$ , and  $g$  is convex near infinity, and  $f \in L^r(\Omega)$  with  $r > N/p$ . There exists a real  $\lambda^* > 0$  such that:*

- (i) for  $\lambda \in (0, \lambda^*)$  problem (1.2) has a minimal **bounded** solution  $\underline{v}_\lambda$ ,
- (ii) for  $\lambda > \lambda^*$  there exists no **renormalized** solution.

When  $g$  is subcritical with respect to the Sobolev exponent  $p^* = Np/(N-p)$ , we obtain **new multiplicity results** for problem (1.2), even in the case  $p = 2$ , extending [1] and [5] :

**Theorem 1.5** Under the assumptions of Theorem 1.4, assume that

$$\limsup_{\tau \rightarrow \infty} g^{p-1}(\tau)/\tau^Q < \infty \text{ for some } Q \in (1, p^* - 1), \quad (1.7)$$

and  $f \in L^r(\Omega)$  with  $(Q+1)r' < p^*$ . Then there exists  $\lambda_0 > 0$  such that for any  $\lambda < \lambda_0$ , there exists at least **two solutions**  $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$  of (1.2). Moreover if  $p = 2$ ,  $g$  is convex, or  $g$  satisfies the Ambrosetti-Rabinowitz growth condition and  $f \in L^\infty(\Omega)$ , the same result holds with  $\lambda_0 = \lambda^*$ .

Concerning the extremal solution, we get the following, extending some results of [3], [12]:

**Theorem 1.6** Under the assumptions of Theorem 1.4, the extremal function  $v^* = \sup_{\lambda \nearrow \lambda^*} \underline{v}_\lambda$  is a renormalized solution of (1.2) with  $\lambda = \lambda^*$ . If  $N < p(1+p')/(1+p'/r)$  then  $v^* \in W_0^{1,p}(\Omega)$ . Moreover  $v^* \in L^\infty(\Omega)$  in any of the following conditions:

- (i)  $N$  is arbitrary and (1.7) holds and  $(Q+1)r' < p^*$ ,
- (ii)  $N$  is arbitrary and (1.6) holds and  $qr' < N/(N-p)$ ,
- (iii)  $N < pp'/(1+1/(p-1)r)$ .

**Remark 1.7** Using Theorems 1.1, 1.4 and 1.5, we deduce existence and nonexistence results for problem 1.1. In Theorem 1.1, function  $f$  can depend on  $u$  or  $v$ , which strongly extends the range of applications. For example, taking  $g(v) = v$ , and  $f = u^b$ ,  $b > 0$ , problem  $-\Delta_p u = (p-1)|\nabla u|^p + \lambda u^b$  relative to  $u$  leads to  $-\Delta_p v = \lambda(1+v)^{p-1} \ln^b(1+v)$  relative to  $v$ . Then for small  $\lambda$  the problem in  $u$  has an infinity of solutions  $u \in W_0^{1,p}(\Omega)$ , two of them being bounded.

**Remark 1.8** A part of our results is based on a growth assumption on  $g$ . Returning to problem (1.1), this condition is not always easy to verify. When  $L = \infty$ , all the "usual" functions  $\beta$ , even with a strong growth, satisfy  $\limsup_{\tau \rightarrow \infty} g(\tau)/\tau^q < \infty$  for any  $q > 0$ , see [1]. However the conjecture that this condition always holds is **wrong**: let  $F \in C^0([0, \infty))$  be any strictly convex function, with  $\lim_{s \rightarrow \infty} F(s) = \infty$ . Then there exists an increasing function  $\beta$  such that  $\lim_{t \rightarrow \infty} \beta(t) = \infty$  and the corresponding  $g$  satisfies  $\limsup_{\tau \rightarrow \infty} g(\tau)/F(\tau) = \infty$ .

## 2 Sketch of the main proofs

In some proofs we use a regularity Lemma, which up to our knowledge covers the previous results on that subject:

**Lemma 2.1** Let  $1 < p < N$ , and  $F \in L^m(\Omega)$ , and  $\overline{m} = Np/(Np - N + p)$  (thus  $1 < \overline{m} < N/p$ ). Let  $U$  be a renormalized solution of problem

$$-\Delta_p U = F \quad \text{in } \Omega, \quad U = 0 \quad \text{on } \partial\Omega.$$

If  $1 < m < N/p$ , then  $U^{(p-1)} \in L^k(\Omega)$ , with  $k = Nm/(N - pm)$ . If  $m = N/p$ , then  $U^{(p-1)} \in L^k(\Omega)$  for any  $k \geq 1$ . If  $1 < m < N/p$ , then  $U \in L^\infty(\Omega)$ . If  $1 < m < \overline{m}$ , then  $|\nabla U|^{(p-1)} \in L^\tau(\Omega)$ , with  $\tau = Nm/(N - m)$ . If  $m \geq \overline{m}$ , then  $U \in W_0^{1,p}(\Omega)$ .

**Proof of Theorem 1.1** For  $p \neq 2$ , we cannot use approximations of the equations because of the nonuniqueness of the solutions of  $-\Delta_p U = \mu$  with  $\mu \in M_b^+(\Omega)$ . The main idea is to use the equations satisfied in  $\mathcal{D}'(\Omega)$  by the truncated functions  $T_K(u) = \min(K, u)$  and  $T_k(v) = \min(k, v)$  with  $k = \Psi(K)$ , see [4]:

$$\begin{aligned} -\Delta_p T_K(u) &= \beta(T_K(u)) |\nabla T_K(u)|^p + \lambda f \chi_{\{u \leq K\}} + \alpha_K, \\ -\Delta_p T_k(v) &= \lambda f (1 + g(v))^{p-1} \chi_{\{v \leq k\}} + \mu_k, \end{aligned}$$

where  $\mu_k$  and  $\alpha_K$  are two measures concentrated on the same set:  $\{u = K\} = \{v = k\}$ , and explicitly related by  $\mu_k = (1 + g(k))^{p-1} \alpha_K$ , and respectively converging weakly to  $\mu_s$  and  $\alpha_s$ . Then the existence follows from the stability theorem of [4]. The nonexistence results are consequences of some properties of renormalized solutions, also called Inverse Maximum Principle.

**Proof of Theorem 1.2.** The nonexistence is first proved for (1.1), and then for (1.2) by Theorem 1.1. The existence is obtained by iteration and approximation, using again [4]. Uniqueness follows from Picone's identity, adapted to renormalized solutions.

**Proof of Theorem 1.4.** Formally, if  $v$  is a solution of (1.2) for some  $\lambda$ , and  $u = H(v)$ , then  $\bar{u} = (1 - \varepsilon)u$  is a supersolution of (1.1) relative to  $\bar{\lambda} = (1 - \varepsilon)^{p-1}\lambda$ , and  $\bar{v} = \Psi(\bar{u})$  is a supersolution of (1.2) relative to  $\bar{\lambda}$ ; then there exists a solution  $v_1 \leq \bar{v}$ . Using Theorem 1.1 we show that it is not formal, since actually **no measure appears**. In the (best) case  $H(\infty) < \infty$ ,  $\bar{v}$  is bounded, then also  $v_1$  is bounded. Otherwise a bootstrapp using Lemma 2.1 is needed for constructing a bounded solution.

**Proof of Theorem 1.5.** The Euler function  $J_\lambda$  is well defined, and for small  $\lambda$  it has the geometry of Mountain Path near 0, but the Palais-Smale sequences may be unbounded. From [8], (1.2) has a second solution for almost any small  $\lambda$ , and then for a sequence  $\lambda_n \rightarrow \lambda$ , and we are lead to prove that the solutions  $v_{\lambda_n}$  relative to  $\lambda_n$  converge to a solution to (1.2) relative to  $\lambda$ . The usual estimates for the case  $p = 2$ , using an eigenfunction as test function, cannot be extended. We get an estimate of  $-\Delta_p v_{\lambda_n}$  in  $L^1(\Omega)$  in another way, using the convexity of  $g$ . The estimate of  $v_{\lambda_n}$  in  $W_0^{1,p}(\Omega)$  is obtained by contradiction. For larger  $\lambda$ , if  $p = 2$ ,  $J_\lambda$  has still the geometry of mountain path near  $\underline{v}_\lambda$ ; the question is open when  $p \neq 2$ . Under Ambrosetti-Rabinowitz condition we apply some results of [7].

**Proof of Theorem 1.6.** The estimates come from Lemma 2.1 and well known regularity results for quasilinear equations, and from an extension of techniques of [9].

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