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Reducibility of quasiperiodic cocycles in linear Lie groups

C.Chavaudret

Abstract: Let G be a linear Lie group. We define the G-reducibility of a continuous or discrete cocycle modulo N. We show that a G-valued continuous or discrete cocycle which is $GL(n,\mathbb{C})$-reducible is in fact G-reducible modulo 2 if $G = GL(n,\mathbb{R}), SL(n,\mathbb{R}), Sp(n,\mathbb{R})$ or $O(n)$ and modulo 1 if $G = U(n)$.

Introduction

Let G be a Lie subgroup of $GL(n,\mathbb{C})$ and $\mathcal{G}$ its Lie algebra. Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $N\mathbb{T}^d = \mathbb{R}^d/(N\mathbb{Z})^d$ for $N \in \mathbb{N} \setminus \{0\}$. Let us consider the equation

$$\forall t \in \mathbb{R}, \forall \theta \in \mathbb{T}^d, \frac{d}{dt}X(t,\theta) = A(\theta + t\omega)X(t,\theta) \tag{1}$$

where $A : \mathbb{T}^d \to \mathcal{G}$ is continuous and $\omega \in \mathbb{R}^d$ is rationally independant. Let $X : (t,\theta) \mapsto X^t(\theta)$ be the associated continuous cocycle, i.e the map from $\mathbb{R} \times \mathbb{T}^d$ to G satisfying (1) such that for all $\theta \in \mathbb{T}^d$, $X^0(\theta) = Id$. The terminology comes from the fact that X satisfies the cocycle relation

$$\forall t, s \in \mathbb{R}, \forall \theta \in \mathbb{T}^d,\ X^{t+s}(\theta) = X^t(\theta + s\omega)X^s(\theta) \tag{2}$$

As X is continuous in the variable t, $X^t(\theta)$ remains in the connected component of the identity for all t, θ, so we can suppose G is connected.

Definition: Let X be a G-valued continuous cocycle. We say that X is G-reducible modulo $N \in \mathbb{N} \setminus \{0\}$ if there exists $Z : N\mathbb{T}^d \to G$ continuous and $B \in \mathcal{G}$ such that for all $t \in \mathbb{R}$ and $\theta \in \mathbb{T}^d$,

$$X^t(\theta) = Z(\theta + t\omega)^{-1}e^{tB}Z(\theta) \tag{3}$$

We say X is reducible if it is reducible modulo 1.

Remark: For a continuous cocycle, reducibility implies that for all θ,

$$\partial_\omega Z(\theta) = BZ(\theta) - Z(\theta)A(\theta) \tag{4}$$

where $\partial_\omega Z(\theta) := \frac{d}{dt}Z(\theta + t\omega)_{|t=0}$.

We shall prove the following theorems for continuous cocycles before adapting them to discrete cocycles:

Theorem 1 *Let X be a continuous cocycle with values in $GL(n,\mathbb{R})$; if it is $GL(n,\mathbb{C})$-reducible, then it is $GL(n,\mathbb{R})$-reducible modulo 2.*

Theorem 2 *Let X be a G-valued continuous cocycle, where G is either the symplectic group $Sp(n,\mathbb{R})$[1], the group $SL(n,\mathbb{R})$ of matrices with determinant 1, the orthogonal group $O(n)$, or the unitary group $U(n)$. Suppose X is $GL(n,\mathbb{C})$-reducible. Then it is G-reducible modulo 2 if $G = Sp(n,\mathbb{R}), SL(n,\mathbb{R})$ or $O(n)$ and modulo 1 if $G = U(n)$.*

Definition: Assume $(\omega, 1)$ is rationally independant. A discrete G-valued cocycle is a map $X : \mathbb{Z} \times \mathbb{T}^d \to G$ such that for all $n, m \in \mathbb{Z}$ and all $\theta \in \mathbb{T}^d$,

$$X^{n+m}(\theta) = X^n(\theta + m\omega)X^m(\theta) \tag{5}$$

Definition: A discrete cocycle X is G-reducible modulo N if there exists a continuous $Z : N\mathbb{T}^d \to G$ and $A \in G$ such that

$$\forall n \in \mathbb{Z}, \forall \theta \in \mathbb{T}^d,\ X^n(\theta) = Z(\theta + n\omega)^{-1}A^n Z(\theta)$$

This is equivalent to the fact that $X^1(\theta) = Z(\theta+\omega)^{-1}AZ(\theta)$ for all θ. A discrete cocycle is reducible if it is reducible modulo 1.

Theorems 1 and 2 also hold for discrete cocycles. Adapting their proofs to the discrete case, one gets:

Theorem 3 *Let X be a G-valued discrete cocycle with G in $GL(n,\mathbb{R})$,$SL(n,\mathbb{R})$, $Sp(n,\mathbb{R}), O(n)$ or $U(n)$, and assume it is $GL(n,\mathbb{C})$-reducible. Then X is G-reducible modulo χ_G with*

$$\chi_G = \begin{cases} 2 & \text{if } G = GL(n,\mathbb{R}), SL(n,\mathbb{R}), G = Sp(n,\mathbb{R}) \text{ or } G = O(n) \\ 1 & \text{if } G = U(n) \end{cases}$$

In [5], H.He and J.You have solved a conjecture from [4] showing that if ω is diophantine, if X_λ is the cocycle which is solution of

$$\frac{d}{dt}X^t(\theta,\lambda) = (A(\lambda) + F_\epsilon(\theta,\lambda))X^t(\theta,\lambda) \tag{6}$$

where F_ϵ is sufficiently small and $A(\lambda)$ satisfies non-degeneracy conditions on an interval $\Lambda \subset \mathbb{R}$, then X_λ is $GL(n,\mathbb{C})$-reducible for almost all $\lambda \in \Lambda$.
Applying theorems 1 and 2 to this result, we get that if $X^t(\theta,\lambda)$ is G-valued, with G in $GL(n,\mathbb{R})$,$SL(n,\mathbb{R})$, $Sp(n,\mathbb{R}), O(n), U(n)$, then for almost all $\lambda \in \Lambda$, $X^t(\theta,\lambda)$ is G-reducible modulo 2 if $G = GL(n,\mathbb{R}), Sp(n,\mathbb{R}), SL(n,\mathbb{R}), O(n)$ and modulo 1 if $G = U(n)$. This completes R.Krikorian's result (see [8]): let $A(\lambda)$ be a generic one-parameter family taking its values in the Lie algebra of a compact semi-simple group G; then the system (6) is G-reducible for almost every λ modulo some integer χ_G depending only on G, and $\chi_G = 1$ if $G = U(n)$. Now we know that $\chi_G = 2$ if $G = O(n)$.

[1] In this case, n is even

So, when G is real, there is a loss of periodicity. In the periodic case ($d = 1$), this is a well-known phenomenon. However, it seems that there exists a large class of real cocycles that are reducible in a subgroup of $GL(n, \mathbb{R})$ without loss of periodicity. For instance, in [7] (proposition 2.2.4), R. Krikorian has showed when G is a compact semi-simple group that if a discrete cocycle X is G-reducible modulo m to a constant cocycle $n \mapsto e^{nB}$, then there exists a subset $S \subset G$ of Haar measure 1 such that if $e^B \in S$, then X is reducible modulo 1. This tells us that loss of periodicity is quite rare, at least in the compact case. We shall prove the following:

Proposition 1 *If a continuous G-valued cocycle X, with $G = GL(n, \mathbb{R}), SL(n, \mathbb{R}), Sp(n, \mathbb{R})$ or $O(n)$, is $GL(n, \mathbb{C})$-reducible to a cocycle $t \mapsto e^{tB}$ such that no eigenvalue of B is in $\mathbb{R} + i\pi\langle \mathbb{Z}^d, \omega\rangle \setminus \{0\}$, then X is G-reducible.*

There is a natural question: considering a generic one-parameter family of cocycles which are solution of (6), where $A(\lambda)$ satisfies non-degeneracy conditions, is it true that for almost all λ such that the cocycle X_λ is reducible to a constant cocycle $t \mapsto e^{tB_\lambda}$, no eigenvalue of B_λ is in $\mathbb{R} + i\pi\langle \mathbb{Z}^d, \omega\rangle \setminus \{0\}$?
If it were true, the already mentioned result of [5] and proposition 1 would imply G-reducibility almost everywhere, without loss of periodicity, for generic one-parameter families of cocycles of type (6).

Remark: All the results which we shall prove also hold in higher regularity classes: defining "C^r-reducibility" in the same way as reducibility, but with Z in C^r and not only continuous, it is easy to check that we get theorems 1, 2, 3 and proposition 1 with "C^r-reducibility" instead of "reducibility".

Sketch of the proof

We shall define notions of invariant subbundle and of Jordan subbundle as families parametrized by $\mathbb{T}^d$ and with values in the subspaces of $\mathbb{C}^n$, satisfying a continuity condition and some invariance properties. In order to prove theorem 1, we shall first study the properties of the decomposition of $\mathbb{C}^n$ into Jordan subbundles given by the $GL(n, \mathbb{C})$-reducibility of X to a cocycle $t \mapsto e^{tB}$; we shall decompose $\mathbb{R}^n$ into two reducible invariant subbundles, one of them, say W, modulo 2 and having a basis with real exponents, the other, say W', modulo 1 and having a basis with no exponent in $\mathbb{R} + i\pi\langle \mathbb{Z}^d, \omega\rangle$, and such that the gap between the imaginary parts of two exponents cannot be in $2\pi\langle \mathbb{Z}^d, \omega\rangle$ (this is called a non-resonance condition). This gives theorem 1 as a corollary, but it also facilitates the proof of theorem 2 for the orthogonal and the symplectic group, since it is then easy to construct real global bases for the cocycle's invariant subbundles, which are respectively orthonormal and symplectic.

If in equation (3), no eigenvalue of B has its imaginary part in $\pi\langle \mathbb{Z}^d, \omega\rangle \setminus \{0\}$, then the first of these two subbundles, W, is trivial, so we can have real reducibility without doubling the period, and consequently, we can also have G-reducibility without loss of periodicity, if $G = SL(n, \mathbb{R}), Sp(n, \mathbb{R})$ or $O(n)$, whence proposition 1.

In order to get theorem 2 for $SL(n,\mathbb{R})$, we will just apply theorem 1, then show that the determinant of Z is constant, so we can assume it equal to 1. Notice that no condition on the exponents of the subbundles is used.

In the case where $G = U(n)$, we shall start from the decomposition of $\mathbb{C}^n$ into complex Jordan subbundles with non-resonant exponents, and construct a global complex orthonormal basis. As $U(n)$ is not a real Lie group, we do not need to double the period.

To prove theorem 3, we can make exactly the same proof as for theorems 1 and 2, simply adapting the first lemma to the discrete case, i.e considering integer translations instead of continuous translations in the direction of ω. The dynamics are not modified by the fact that the time is discrete.

For a particular class of discrete cocycles, there is another way of proving G-reducibility:

Definition: A discrete cocycle X is called G-exponential if there exists a continuous $A : \mathbb{T}^d \to \mathcal{G}$ such that for all $\theta \in \mathbb{T}^d$, $X^1(\theta) = e^{A(\theta)}$.

To prove theorem 3 for G-exponential cocycles, we can also construct a suspension of X on a torus of greater dimension, taking its values in G, using the function A from the definition of a G-exponential cocycle. We will obtain a continuous cocycle over $(\omega, 1)$, which is possible since $(\omega, 1)$ is assumed to be rationally independant. We then show that if X is $GL(n,\mathbb{R})$-reducible, then so is its suspension. Using theorems 1 and 2, we obtain G-reducibility for the suspension modulo 1 or 2. Restricting to integer time and to a subtorus, we finally obtain G-reducibility for X modulo 1 or 2.

Notations

For a vector $v \in \mathbb{R}^n$, denote by $Re\ v$ and $Im\ v$ its real and imaginary parts. The euclidean scalar product is denoted by $\langle , \rangle$ for a real vector space, and $\langle , \rangle_{\mathbb{C}}$ for a complex vector space (we shall take it semilinear in the second variable); euclidean distance is denoted by $d(,)$. Also, we shall write M^* for the adjoint of a matrix M; if M is real, M^* is simply the transpose of M. Matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is denoted by J. Finally, $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.

1 $GL(n,\mathbb{R})$-reducibility

In this section, we shall assume that X is a real cocycle.

1.1 Preliminary lemmas

Lemma 1 *1. Let $\omega \in \mathbb{R}^d$ rationally independant, $\beta \in \mathbb{R}$ and $N \in \mathbb{N}^*$. Suppose that for any real sequence $t_j \to \infty$,*

$$t_j\omega \to 0 \in N\mathbb{T}^d \Rightarrow t_j\beta \to 0 \in 2\pi\mathbb{T} \tag{7}$$

Then there exists $k \in \mathbb{Z}^d$ such that $\beta = 2\pi\langle k, \frac{\omega}{N}\rangle$.

2. *Let $\omega \in \mathbb{R}^d$ such that $(\omega, 1)$ is rationally independant and $N \in \mathbb{N}^*$. Suppose that for any integer sequence $t_j \to \infty$,*

$$t_j\omega \to 0 \in N\mathbb{T}^d \Rightarrow t_j\beta \to 0 \in 2\pi\mathbb{T} \tag{8}$$

Then there exists $k \in \mathbb{Z}^d$ such that $\beta = 2\pi\langle k, \frac{(\omega,1)}{N}\rangle$.

Proof: 1. It is enough to prove the assertion for $N = 1$: if it is true for $N = 1$ and that for every sequence t_j, $t_j\omega \to 0 \in N\mathbb{T}^d \Rightarrow t_j\beta \to 0 \in 2\pi\mathbb{T}$, then $t_j\frac{\omega}{N} \to 0 \in \mathbb{T}^d \Rightarrow t_j\beta \to 0 \in 2\pi\mathbb{T}$, and applying the case $N = 1$ with $\frac{\omega}{N}$ instead of ω, we get $\beta = 2\pi\langle k, \frac{\omega}{N}\rangle$ for some $k \in \mathbb{Z}^d$.

First notice that $(\frac{\beta}{2\pi}, \omega)$ is rationally dependant, since the orbit of the translation in the direction $(\frac{\beta}{2\pi}, \omega)$ is not dense in $\mathbb{T}^{d+1}$: otherwise, there would exist a sequence (t_j) satisfying $t_j(\frac{\beta}{2\pi}, \omega) \to (\frac{1}{2}, 0) \in \mathbb{T}^{d+1}$, which would contradict the assumption. So there exists $k = (k_1, \dots k_d) \in \mathbb{Z}^d, p \in \mathbb{Z}$ such that (p, k) is primitive (i.e the greatest common divisor of k_i and p is 1) and

$$\langle k, \omega\rangle + p\frac{\beta}{2\pi} = 0 \tag{9}$$

Notice that this is the only possible resonance (i.e (p, k) is unique up to a scalar). For if there existed a (p', k') independant from (p, k) and such that $\langle k', \omega\rangle + p'\frac{\beta}{2\pi} = 0$, then

$$\begin{cases} p\beta + 2\pi\langle k, \omega\rangle = 0 \\ p'\beta + 2\pi\langle k', \omega\rangle = 0 \end{cases} \tag{10}$$

would hold, so $pk' - p'k = 0 \in \mathbb{R}^d$ since ω is rationally independant, which would contradict the assumption that (p, k) and (p', k') are independant.

Now let us show that $p = \pm 1$. By contraposition, suppose $|p| \geq 2$.
Let V the subspace of $\mathbb{R}^{d+1}$ generated by (p, k) . Let $m_1, \dots m_d \in \mathbb{Z}^{d+1}, m_i = (m_{i,1}, \dots m_{i,d+1})$ such that the $(d + 1) \times (d + 1)$-matrix

$$C := \begin{pmatrix} (p, k) \\ m_1 \\ \vdots \\ m_d \end{pmatrix} \tag{11}$$

has determinant 1. Such a matrix exists, according to [1], corollary 3 p.14. Form the following commuting diagram:

$$\begin{array}{ccc} \mathbb{R}^{d+1} & \xrightarrow{C} & \mathbb{R}^{d+1} \\ \Pi \downarrow & & \Pi \downarrow \\ \mathbb{T}^{d+1} & \xrightarrow{\bar{C}} & \mathbb{T}^{d+1} \end{array} \tag{12}$$

where Π is the canonical projection from $\mathbb{R}^{d+1}$ onto $\mathbb{T}^{d+1}$. As C has determinant 1, $\bar{C}$ is a homeomorphism. So the orbit of $\Pi\begin{pmatrix}\frac{\beta}{2\pi}\\ {}^t\omega\end{pmatrix}$ is dense is $\Pi(V^\perp)$ if the orbit of $\Pi(C\begin{pmatrix}\frac{\beta}{2\pi}\\ {}^t\omega\end{pmatrix})$ is dense in $\Pi(C(V^\perp))$. Now $\Pi(C(V^\perp)) \subset \{0\}\times\mathbb{T}^d$ and

$$\Pi(C\begin{pmatrix}\frac{\beta}{2\pi}\\ {}^t\omega\end{pmatrix}) = \begin{pmatrix}0\\ \langle m_1, (\frac{\beta}{2\pi},\omega)\rangle\\ \vdots\\ \langle m_d, (\frac{\beta}{2\pi},\omega)\rangle\end{pmatrix}$$

Moreover, assume that

$$\sum_i a_i\langle m_i, (\frac{\beta}{2\pi},\omega)\rangle = 0 \Leftrightarrow \sum_i\sum_{j\neq 1} a_i m_{i,j}\omega_{j-1} + \sum_i a_i m_{i,1}\frac{\beta}{2\pi} = 0 \tag{13}$$

then, as the resonance is unique, $(\sum_i a_i m_{i,1}, \cdots \sum_i a_i m_{i,d+1}) = \sum_i a_i m_i$ is a multiple of (p,k), which is impossible since m_i are independant from (p,k) by definition. So $(\langle m_1, (\frac{\beta}{2\pi},\omega)\rangle, \ldots \langle m_d, (\frac{\beta}{2\pi},\omega)\rangle)$ is rationally independant and its orbit is dense in $\mathbb{T}^d$. Therefore, the orbit of $\Pi\begin{pmatrix}\frac{\beta}{2\pi}\\ {}^t\omega\end{pmatrix}$ is dense in $\Pi(V^\perp)$.

Let $m\in\mathbb{Z}^d$ such that $\frac{\langle k,m\rangle}{p}$ is not an integer (it exists, since (p,k) is primitive and $|p|\geq 2$). Then ${}^t(\frac{\langle k,m\rangle}{p}, -m)\in V^\perp$. As $\Pi\begin{pmatrix}\frac{\beta}{2\pi}\\ {}^t\omega\end{pmatrix}$ dans $\Pi(V^\perp)$ has a dense orbit, there exists an unbounded sequence t_j such that $\Pi\begin{pmatrix}t_j\frac{\beta}{2\pi}\\ {}^t t_j\omega\end{pmatrix} \to \Pi\begin{pmatrix}\frac{\langle k,m\rangle}{p}\\ -{}^t m\end{pmatrix} =: \begin{pmatrix}\alpha\\ 0\end{pmatrix} \neq \begin{pmatrix}0\\ 0\end{pmatrix}$, which contradicts our assumption.

2. Again, we can assume that $N=1$. Let t_j be a real unbounded sequence such that $t_j(\omega,1)\to 0\in\mathbb{T}^{d+1}$. For all j, let $n_j\in\mathbb{Z}$ and $r_j\in[0,1[$ such that $t_j = n_j + r_j$. In particular, $t_j\to 0\in\mathbb{T}$, so $r_j\to 0\in\mathbb{T}$. Since $t_j\omega\to 0\in\mathbb{T}^d$ and $r_j\omega\to 0\in\mathbb{T}^d$, then $n_j\omega\to 0\in\mathbb{T}^d$. By assumption, this implies that $n_j\beta\to 0\in 2\pi\mathbb{T}$. But $r_j\beta\to 0\in 2\pi\mathbb{T}$, so $t_j\beta\to 0\in 2\pi\mathbb{T}$. By 1., this implies that $\beta\in 2\pi\langle\mathbb{Z}^d,(\omega,1)\rangle$. $\square$

Lemma 2 *Let W be a subspace of $\mathbb{C}^n$. Let $(W\cap\mathbb{R}^n)\otimes\mathbb{C}$ be the complex vector space generated by $W\cap\mathbb{R}^n$. Then*

1. *$W=\bar{W}\Leftrightarrow (W\cap\mathbb{R}^n)\otimes\mathbb{C} = W$;*
2. *Let V be a subspace of W such that $V\oplus\bar{V} = W$ and $z_1,\ldots z_k$ a basis of V, then $Rez_1, Imz_1, \ldots Rez_k, Imz_k$ is a basis of $W\cap\mathbb{R}^n$.*

Proof: 1. $\Rightarrow$: Let $w_1,\ldots w_l$ be a basis of W; as $W=\bar{W}$, then $\bar{w_1},\ldots\bar{w_l}$ are also in W. So, for all $j\in\{1,\ldots l\}$, $Rew_j = \frac{1}{2}(w_j+\bar{w_j})$ and $Imw_j = \frac{1}{2i}(w_j-\bar{w_j})$ are in $W\cap\mathbb{R}^n$. So $w_j = Rew_j + iImw_j\in(W\cap\mathbb{R}^n)\otimes\mathbb{C}$ and so $W\subset(W\cap\mathbb{R}^n)\otimes\mathbb{C}$. The other inclusion is obvious, therefore $W=(W\cap\mathbb{R}^n)\otimes\mathbb{C}$.

$\Leftarrow$: Let $w \in (W \cap \mathbb{R}^n) \otimes \mathbb{C}$; then $w = \sum_{j=1}^{l} a_j v_j$ where $a_j \in \mathbb{C}$ and $v_j \in (W \cap \mathbb{R}^n)$. So $\bar{w} = \sum_{j=1}^{l} \bar{a_j} \bar{v_j} = \sum_{j=1}^{l} \bar{a_j} v_j \in (W \cap \mathbb{R}^n) \otimes \mathbb{C}$ and so $W = \bar{W}$.

2. $Rez_1, Imz_1, \ldots Rez_k, Imz_k$ generate W and are real, so they generate $W \cap \mathbb{R}^n$. With complex coefficients, they generate $(W \cap \mathbb{R}^n) \otimes \mathbb{C}$, and by 1., this is equal to W. As W has dimension $2k$, they form a basis of $W \cap \mathbb{R}^n$. $\square$

1.2 Subbundles, invariant subbundles and Jordan subbundles

Definitions:

- A real (resp. complex) subbundle is a family $V = \{V(\theta), \theta \in \mathbb{T}^d\}$ of subspaces of $\mathbb{R}^n$ (resp.$\mathbb{C}^n$) which is continuous in θ, i.e such that for all $\theta_0 \in \mathbb{T}^d$, there exists an open subset $\mathcal{U}$ containing θ_0 and, for all $\theta \in \mathcal{U}$, a basis $\{x_1(\theta), \ldots x_k(\theta)\}$ of $V(\theta)$ which is continuous in θ on $\mathcal{U}$.
- The dimension of $V(\theta)$ is automatically independant of θ; this number is called the dimension of the subbundle V and is denoted by $\dim V$.
- A real (resp. complex) invariant subbundle for the cocycle X is a real (resp. complex) subbundle such that for all t, θ, $X^t(\theta)V(\theta) = V(\theta + t\omega)$. In what follows, we shall omit to mention the cocycle X, as no other cocycle is involved.

Remark: A real invariant subbundle does not always have a basis which is continuous on $\mathbb{T}^d$.

Example: Consider the discrete 1-periodic cocycle $X^1(\theta) := \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$ acting on $\mathbb{R}^2$. Let $z_1(\theta) := \begin{pmatrix} \cos \pi\theta \\ \sin \pi\theta \end{pmatrix}$. Then $\mathrm{Vect}_{\mathbb{R}}(z_1(\theta))$ is an invariant subbundle for $X^n(\theta)$, since $z_1(\theta + 1) = -z_1(\theta)$ for all θ. But z_1 is continuous on $2\mathbb{T}$ and not on $\mathbb{T}$. Moreover, if z is another function such that for all θ, $z(\theta)$ generates $\mathrm{Vect}_{\mathbb{R}}(z_1(\theta))$, then there exists a continuous function λ bounded away from 0 and such that for all θ, $z(\theta) = \lambda(\theta)z_1(\theta)$. So $z(\theta + 1) = \lambda(\theta + 1)z_1(\theta + 1) = -\lambda(\theta + 1)z_1(\theta)$, and so $z(\theta)$ is continuous on $\mathbb{T}$ if and only if for all θ, $-\lambda(\theta + 1) = \lambda(\theta)$. But this implies that the function λ changes sign, so it takes the value 0 since it is continuous, which is impossible.

Remark: The intersection of two (real or complex) subbundles is not necessarily a subbundle. For instance, in $\mathbb{R}^2$, for all $\theta \in \mathbb{T}$, let $V(\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ et $W(\theta) = \begin{pmatrix} \cos 2\pi\theta \\ \sin 2\pi\theta \end{pmatrix}$, then $V(\theta) \cap W(\theta) = V(\theta)$ if $\theta = 0$ or $\frac{1}{2}$ mod 1 and $\{0\}$ otherwise, so the dimension of the intersection is not independant of θ. However, the following proposition holds:

Proposition 2 *The intersection of two real or complex invariant subbundles is an invariant subbundle.*

Proof: Let U, V be two invariant subbundles, then for all t, θ,

$$X^t(\theta)(U(\theta) \cap V(\theta)) = X^t(\theta)U(\theta) \cap X^t(\theta)V(\theta) = U(\theta + t\omega) \cap V(\theta + t\omega)$$

so the intersection is invariant.

Let us show that it has constant dimension. Let $\mathcal{U}$ be an open subset of the torus such that there exists $(u_1, \dots u_k)$ and $(v_1, \dots v_l)$ continuous on $\mathcal{U}$ and such that for all $\theta \in \mathcal{U}$, $(u_1(\theta), \dots u_k(\theta))$ is a basis of $U(\theta)$ and $(v_1(\theta), \dots v_l(\theta))$ a basis of $V(\theta)$. For all $\theta \in \mathcal{U}$, let

$$M(\theta) := \left[\begin{array}{cccccc} u_1(\theta) & \dots u_k(\theta) & v_1(\theta) & \dots & v_l(\theta) \end{array} \right]$$

the $n \times (k+l)$-matrix whose columns are the vectors from the two bases. Let r be the rank of $M(\theta_0)$ for a fixed θ_0 in $\mathcal{U}$. Then there exists a $r \times r$-submatrix of $M(\theta_0)$ with non-zero determinant. This determinant is continuous in θ, so it is non-zero on a neighbourhood $\mathcal{V}$ of θ_0, so, on this neighbourhood, the rank of $M(\theta)$ is greater than, or equal to r. Therefore, if $d(\theta)$ is the dimension of $U(\theta) \cap V(\theta)$, then $d(\theta) \leq d(\theta_0)$ for all $\theta \in \mathcal{V}$.

Let $\theta_0, \phi_0 \in \mathbb{T}^d$. As we have just seen, there exists a neighbourood $\mathcal{U}_0$ of θ_0 and a neighbourhood $\mathcal{V}_0$ of ϕ_0 such that $d(\theta) \leq d(\theta_0)$ for all $\theta \in \mathcal{U}_0$ and $d(\phi) \leq d(\phi_0)$ for all $\phi \in \mathcal{V}_0$. As the orbits of $t \mapsto \theta_0 + t\omega$ and $t \mapsto \phi_0 + t\omega$ are dense on the torus, there exists $t, t' \in \mathbb{R}$ such that $\theta_0 + t\omega \in \mathcal{V}_0$ and $\phi_0 + t'\omega \in \mathcal{U}_0$. Invariance and invertibility of $X^t(\theta_0)$ imply that $d(\theta_0 + t\omega) = \dim(U \cap V)(\theta_0 + t\omega) = \dim X^t(\theta_0)(U \cap V)(\theta_0) = \dim(U \cap V)(\theta_0) = d(\theta_0)$, and analogously $d(\phi_0) = d(\phi_0 + t'\omega)$. Moreover, as $\theta_0 + t\omega \in \mathcal{V}_0$, $d(\theta_0 + t\omega) \leq d(\phi_0)$, and analogously $d(\phi_0 + t'\omega) \leq d(\theta_0)$. Therefore, $d(\theta_0) = d(\phi_0)$. As θ_0 et ϕ_0 are arbitrarily chosen, the dimension of $U \cap V$ is constant on $\mathbb{T}^d$.

Let us now define a local basis of $U \cap V$. Let $\mathcal{U}$ be a sufficiently small neighbourhood of θ_0 in $\mathbb{T}^d$ and $(u_1, \dots u_k)$ and $(v_1, \dots v_l)$ two bases for U and V which are continuous on $\mathcal{U}$. In the neighbourhood of θ_0, up to a permutation of the bases, there exists $l' \leq l$ such that $u_1(\theta), \dots u_k(\theta), v_1(\theta), \dots v_{l'}(\theta)$ is a basis of $U(\theta) + V(\theta)$. the integer l' does not depend on θ since the dimension of $U(\theta) \cap V(\theta)$ is independant of θ. So, for all l'', $l' \leq l'' \leq l$, there exists $a_1, \dots a_k, b_1, \dots b_{l'}$ which are continuous on a neignbourhood of θ_0 such that $v_{l''}(\theta) = \sum_{i=1}^k a_i(\theta)u_i(\theta) + \sum_{i=1}^{l'} b_i(\theta)v_i(\theta)$. Let $\bar{v}_{l''}(\theta) := \sum_{i=1}^k a_i(\theta)u_i(\theta)$, then $\bar{v}_{l''}(\theta) \in U(\theta) \cap V(\theta)$, $\bar{v}_{l''}$ is continuous on a neighbourhood of θ_0. The vectors $(\bar{v}_{l'+1}(\theta_0), \dots \bar{v}_l(\theta_0))$ form a basis of $U(\theta_0) \cap V(\theta_0)$, therefore $\bar{v}_{l'+1}(\theta), \dots \bar{v}_l(\theta)$ form a basis of $U(\theta) \cap V(\theta)$ in a neighbourhood of θ_0. $\square$

Definition: A Jordan subbundle of rank k modulo N is a complex invariant subbundle having a basis $(z_1, \dots z_k)$ which is continuous on $N\mathbb{T}^d$ and such that there exists $\alpha + i\beta \in \mathbb{C}$ satisfying for all θ, t,

$$\begin{aligned} X^t(\theta)z_1(\theta) &= e^{t(\alpha+i\beta)}z_1(\theta + t\omega) \\ X^t(\theta)z_2(\theta) &= e^{t(\alpha+i\beta)}z_2(\theta + t\omega) + te^{t(\alpha+i\beta)}z_1(\theta + t\omega) \\ &\dots \\ X^t(\theta)z_k(\theta) &= e^{t(\alpha+i\beta)}\sum_{i=1}^k \frac{t^{k-i}}{(k-i)!}z_i(\theta + t\omega) \end{aligned} \tag{14}$$

A Jordan subbundle is a Jordan subbundle modulo 1. The family of functions $(z_1, \dots z_k)$ is called a Jordan basis, it is not unique. If it is real for all θ, it is called a real Jordan basis (for a complex Jordan subbundle). The number $\alpha + i\beta$ is called an exponent of the Jordan subbundle, and also the exponent of the Jordan basis $(z_1, \dots z_k)$.

Remark: An exponent of a Jordan subbundle is not unique, but the exponent of a Jordan basis is.
If unnecessary, we shall omit to mention the rank of a Jordan subbundle. Notice that the rank is not supposed to be maximal: if $k \geq 2$, a Jordan subbundle of rank k contains another Jordan subbundle of rank $k-1$.

Definition: An invariant subbundle W of dimension k is reducible modulo N if there exists a basis $(z_1, \dots z_k)$ of W which is continuous on $N\mathbb{T}^d$ and a constant matrix A of dimension $k \times k$ such that $X^t(\theta)[z_1(\theta) \dots z_k(\theta)] = [z_1(\theta + t\omega) \dots z_k(\theta + t\omega)]e^{tA}$ for all t, θ.
Remark: A Jordan subbundle is a particular type of reducible invariant subbundle and $GL(n, \mathbb{R})$-reducibility is equivalent to the existence of a decomposition of $\mathbb{R}^n$ into invariant reducible subbundles.

Proposition 3 *Let V be a Jordan subbundle modulo N.*
i) If $\alpha + i\beta$ is an exponent for V, then for all $m \in \mathbb{Z}^d$, $\alpha + i\beta + 2i\pi\langle \omega, \frac{m}{N} \rangle$ is an exponent for V.
ii) If $\alpha + i\beta$ and $\alpha' + i\beta'$ are two exponents for V, then $\alpha = \alpha'$ and $\beta - \beta' \in 2\pi\langle \mathbb{Z}^d, \frac{\omega}{N} \rangle$.

Proof: i) Suppose $(z_1, \dots z_k)$ is a Jordan basis of V with exponent $\alpha + i\beta$.
Let $m \in \mathbb{Z}^d$. For all $1 \leq j \leq k$ and all $\theta \in N\mathbb{T}^d$, let $z'_j(\theta) = e^{-2i\pi\langle \frac{\theta}{N}, m \rangle} z_j(\theta)$. Then the vectors $z'_j(\theta)$ form a global basis of V which is continuous on $N\mathbb{T}^d$ and for all θ, t and all $j \leq k$,

$$X^t(\theta) z'_j(\theta) = e^{t(\alpha + i\beta + 2i\pi\langle m, \frac{\omega}{N} \rangle)} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!} z'_i(\theta + t\omega) \tag{15}$$

so $\alpha + i\beta + 2i\pi\langle m, \frac{\omega}{N} \rangle$ is also an exponent of V.

ii) Let $(v_1, \dots v_k)$ and $(v'_1, \dots v'_k)$ be Jordan bases of V with respective exponents $\alpha + i\beta$ and $\alpha' + i\beta'$.
For all $\theta \in N\mathbb{T}^d$, let $v'_1(\theta) = \sum_{j=1}^{k} \gamma_j(\theta) v_j(\theta)$ where γ_j are continuous on $N\mathbb{T}^d$. Then for all t,

$$\sum_{j=1}^{k} \gamma_j(\theta) e^{t(\alpha + i\beta)} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!} v_i(\theta + t\omega) = e^{t(\alpha' + i\beta')} \sum_{j=1}^{k} \gamma_j(\theta + t\omega) v_j(\theta + t\omega)$$

As the $v_j(\theta + t\omega)$ are linearly independant, in particular

$$\gamma_k(\theta) e^{t(\alpha + i\beta)} = e^{t(\alpha' + i\beta')} \gamma_k(\theta + t\omega)$$

Suppose $\gamma_k(\theta) \neq 0$ for some $\theta \in N\mathbb{T}^d$. As γ_k is bounded, then $\alpha = \alpha'$. Let t_m be an unbounded real sequence such that $t_m \omega \to 0 \in N\mathbb{T}^d$. Then as $m \to \infty$, since $\gamma_k(\theta) \neq 0$,

$t_m(\beta-\beta') \to 0 \in 2\pi\mathbb{T}$. By lemma 1, there exists $K \in \mathbb{Z}^d$ such that $\beta-\beta' = 2\pi\langle K, \frac{\omega}{N}\rangle$. If γ_k is identically zero, then

$$\gamma_{k-1}(\theta)e^{t(\alpha+i\beta)} = e^{t(\alpha'+i\beta')}\gamma_{k-1}(\theta+t\omega)$$

and we deduce in the same way that $\beta-\beta' = 2\pi\langle K, \frac{\omega}{N}\rangle$ for some $K \in \mathbb{Z}^d$. Otherwise, we repeat the argument until we find a non zero $\gamma_j(\theta)$ and deduce that for some $K \in \mathbb{Z}^d$, $\beta-\beta' = 2\pi\langle K, \frac{\omega}{N}\rangle$. $\square$

Remark: • Thus, the exponent of a Jordan subbundle modulo N is well defined modulo $2i\pi\langle\mathbb{Z}^d, \frac{\omega}{N}\rangle$. In particular, if $\beta \in 2\pi\langle\mathbb{Z}^d, \frac{\omega}{N}\rangle$, then we can assume that $\beta = 0$.

• The term "Jordan subbundle" comes from the fact that if (3) holds for some B in Jordan normal form, then the columns of $Z(\theta)^{-1}$ whose indices are the same as those of the first columns of a Jordan block of B with eigenvalue $\alpha+i\beta$ form a Jordan basis with exponent $\alpha+i\beta$.

Lemma 3 *$GL(n,\mathbb{C})$-reducibility modulo N is equivalent to the existence of a decmposition of $\mathbb{C}^n$ into Jordan subbundles modulo N. The existence of a decomposition of $\mathbb{R}^n$ into Jordan subbundles modulo N with a real Jordan basis implies $GL(n,\mathbb{R})$-reducibility modulo N.*

Proof: By definition, $GL(n,\mathbb{C})$-reducibility of X is the existence of a matrix $B = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & \ddots \end{pmatrix}$, where each B_j is a Jordan block with exponent $\alpha_j+i\beta_j$, and of a continuous function $Z : \mathbb{T}^d \to GL(n,\mathbb{C})$ such that for all θ, t,

$$X^t(\theta) = Z(\theta+t\omega)^{-1}e^{tB}Z(\theta) \tag{16}$$

If $z_1(\theta), \dots z_n(\theta)$ are the columns of $Z(\theta)^{-1}$, then (16) is equivalent to the fact that for all θ, t, j, if $l_1, \dots l_{k_j}$ are the indices of the columns containing B_j,

$$\begin{aligned} &X^t(\theta)z_{l_1}(\theta) = e^{t(\alpha_j+i\beta_j)}z_{l_1}(\theta+t\omega) \\ &X^t(\theta)z_{l_2}(\theta) = e^{t(\alpha_j+i\beta_j)}z_{l_2}(\theta+t\omega) + te^{t(\alpha_j+i\beta_j)}z_{l_1}(\theta+t\omega) \\ &\dots \\ &X^t(\theta)z_{l_{k_j}}(\theta) = e^{t(\alpha_j+i\beta_j)}\sum_{i=1}^{k_j}\frac{t^{k_j-i}}{(k_j-i)!}z_i(\theta+t\omega) \end{aligned} \tag{17}$$

which is also equivalent to the fact that if for all j, $V_j(\theta) = \mathrm{Vect}_{\mathbb{C}}(z_{l_1}(\theta), \dots z_{l_{k_j}}(\theta))$, then V_j is a Jordan subbundle with exponent $\alpha_j+i\beta_j$. Moreover, $V_j(\theta)$ are in direct sum since $Z(\theta)^{-1}$ is invertible.

In the preceding argument, it is clear that X is in fact $GL(n,\mathbb{R})$-reducible if all V_j have a real global basis. $\square$

Remark: • Decomposition into Jordan subbundles is not always unique. For instance, if for all θ, t, $X^t(\theta) = Z(\theta + t\omega)^{-1} e^{t\alpha Id} Z(\theta)$, then for any invertible matrix P, $X^t(\theta) = Z(\theta + t\omega)^{-1} P e^{t\alpha Id} P^{-1} Z(\theta)$, so $\mathbb{C}^n$ decomposes into Jordan subbundles of rank 1 generated by the columns of $Z(\theta)^{-1}P$, where P is arbitrarily chosen.

• If a Jordan subbundle has a real Jordan basis with exponent $\alpha + i\beta$ $(\mathrm{mod} 2i\pi\langle\mathbb{Z}^d, \omega\rangle)$, then $\beta = 0$ $(\mathrm{mod} 2\pi\langle\mathbb{Z}^d, \omega\rangle)$. But there exists Jordan subbundles with real exponent $(\mathrm{mod} 2i\pi\langle\mathbb{Z}^d, \omega\rangle)$ but without a real Jordan basis. A trivial example is the constant Jordan subbundle generated by $\begin{pmatrix} 1 \\ i \end{pmatrix}$ with exponent 0 $(\mathrm{mod} 2i\pi\langle\mathbb{Z}^d, \omega\rangle)$ for the identity cocycle. However, the following lemma holds:

Lemma 4 *Let W be a Jordan subbundle modulo N without a real Jordan basis, $z_1, \dots z_k$ a Jordan basis of W with real exponent α, and for all $j \leq k$, u_j the real part of z_j and v_j its imaginary part. Then $U := \mathrm{Vect}_{\mathbb{R}}(u_1, \dots u_k)$ and $V := \mathrm{Vect}_{\mathbb{R}}(v_1, \dots v_k)$ are Jordan subbundles modulo N with exponent α and there exists $l, m \leq k$ such that $u_l, \dots u_k$ is a Jordan basis of U and $v_m, \dots v_k$ a Jordan basis of V. Moreover, either l or m is equal to 1.*

Proof: For all j, $t \in \mathbb{R}, \theta \in N\mathbb{T}^d$, as $X^t(\theta)$ is real, then

$$X^t(\theta) u_j(\theta) = e^{t\alpha} \sum_{j' \leq j} \frac{t^{j-j'}}{(j-j')!} u_{j'}(\theta + t\omega)$$

Suppose there exists $j \geq 1, \theta_0$ and $\lambda_1, \dots \lambda_{j-1} \in \mathbb{C}$ such that $u_j(\theta_0) = \sum_{i \leq j-1} \lambda_i u_i(\theta_0)$. Then for all t,

$$\begin{aligned} 0 &= X^t(\theta_0)(u_j(\theta_0) - \sum_{i \leq j-1} \lambda_i u_i(\theta_0)) \\ &= e^{t\alpha} \sum_{j' \leq j} \frac{t^{j-j'}}{(j-j')!} u_{j'}(\theta_0 + t\omega) - \sum_{i \leq j-1} \lambda_i \sum_{j' \leq i} \frac{t^{i-j'}}{(i-j')!} u_{j'}(\theta_0 + t\omega) \end{aligned} \tag{18}$$

so, dividing by $e^{t\alpha} t^{j-1}$, for all $t \neq 0$,

$$0 = \sum_{j' \leq j} \frac{t^{-j'+1}}{(j-j')!} u_{j'}(\theta_0 + t\omega) - \sum_{i \leq j-1} \lambda_i \sum_{j' \leq i} \frac{t^{i-j'-j+1}}{(i-j')!} u_{j'}(\theta_0 + t\omega)$$

Let θ be any point of $N\mathbb{T}^d$. Let t_s be an unbounded real sequence satisfying $t_s\omega \to \theta - \theta_0$ in $N\mathbb{T}^d$. Then, as s tends to infinity,

$$\frac{1}{(j-1)!} u_1(\theta) = 0$$

Assume by induction that $u_{j''}$ is identically 0 for all j'' strictly inferior to some $J \leq j$. Then, dividing equation (18) by $e^{t\alpha} t^{j-J}$, for all $t \neq 0$,

$$0 = e^{t\alpha} \sum_{J \leq j' \leq j} \frac{t^{J-j'}}{(j-j')!} u_{j'}(\theta_0 + t\omega) - \sum_{i \leq j-1} \lambda_i \sum_{J \leq j' \leq i} \frac{t^{J-j+i-j'}}{(i-j')!} u_{j'}(\theta_0 + t\omega)) \tag{19}$$

so, with the sequence t_s above defined, if $t = t_s$ and taking the limit as $s \to \infty$,

$$\frac{1}{(j-J)!} u_J(\theta) = 0$$

and so $u_J(\theta) = 0$ for all θ. Therefore, for all θ and all $j' \leq j$, $u_{j'}(\theta) = 0$.

Thus, we have shown that there exists $l \leq k$ so that the functions $u_1, \dots u_{l-1}$ are identically 0 if $l \geq 2$ and $(u_l, \dots u_k)$ form a global basis of U, which is then a Jordan basis. We proceed exactly in the same way to show that there exists $m \leq k$ such that $v_1, \dots v_{m-1}$ are identically 0 if $m \geq 2$ and $(v_m, \dots v_k)$ form a Jordan basis of V. Moreover, as u_1 and v_1 cannot be 0 at the same time, then either l or m is equal to 1. $\square$

1.3 Properties of Jordan subbundles with a real Jordan basis

Let $\{u_j, 1 \leq j \leq k\}$ be a real Jordan basis of a Jordan subbundle U modulo N of rank k and real exponent α.

Sublemma 1 *Every invariant subbundle contained in U is a Jordan subbundle modulo N generated by $(u_1, \dots u_j)$ for some $j \leq k$.*

Proof: Let W be a non zero invariant subbundle contained in U and $u_1, \dots u_k$ as above. For some θ_0, let j be the maximal integer lower than k such that there exists $\sum_{j' \leq j} a_{j'} u_{j'}(\theta_0)$ in $W(\theta_0)$ with $a_j \neq 0$.
As W is invariant, for all $t \in \mathbb{R}$, $X^t(\theta_0) \sum_{j' \leq j} a_{j'} u_{j'}(\theta_0) \in W(\theta_0 + t\omega)$. Now this vector is equal to $e^{t\alpha} \sum_{j' \leq j} a_{j'} \sum_{i=1}^{j'} \frac{t^{j'-i}}{(j'-i)!} u_i(\theta_0 + t\omega)$. Dividing by $e^{t\alpha} t^{j-1}$, for all $t \neq 0$, the vector $\sum_{j' \leq j} a_{j'} \sum_{i=1}^{j'} \frac{t^{j'-j+1-i}}{(j'-i)!} u_i(\theta_0 + t\omega)$ is in $W(\theta_0 + t\omega)$. Let $\theta \in N\mathbb{T}^d$ and (t_k) an unbounded real sequence such that $t_k \omega \to \theta - \theta_0$ in $N\mathbb{T}^d$. Then, taking the limit as $k \to \infty$,

$$\sum_{j' \leq j} a_{j'} \sum_{i=1}^{j'} \frac{t_k^{j'-j+1-i}}{(j'-i)!} u_i(\theta_0 + t_k\omega) \to \frac{1}{(j-1)!} a_j u_1(\theta) \in W(\theta)$$

So for all θ, $u_1(\theta) \in W(\theta)$. Suppose that for all $1 \leq j'' \leq j'$, $u_{j''}(\theta) \in W(\theta)$ for all θ. Then $\sum_{i=j'+1}^{j} a_i u_i(\theta_0) \in W(\theta_0)$ so by invariance of W, for all k,

$$\sum_{i=j'+1}^{j} e^{t_k \alpha} \sum_{j''=1}^{i} a_i \frac{t_k^{i-j''}}{(i-j'')!} u_{j''}(\theta_0 + t_k\omega) \in W(\theta_0 + t_k\omega)$$

and so

$$\sum_{i=j'+1}^{j} e^{t_k \alpha} \sum_{j''=j'+1}^{i} a_i \frac{t_k^{i-j''}}{(i-j'')!} u_{j''}(\theta_0 + t_k\omega) \in W(\theta_0 + t_k\omega)$$

Dividing by $e^{t_k\alpha}t_k^{j-j'-1}$, we get

$$\sum_{i=j'+1}^{j} a_i \sum_{j''=j'+1}^{i} \frac{t_k^{i-j''+j'-j+1}}{(i-j'')!} u_{j''}(\theta_0+t_k\omega) \in W(\theta_0+t_k\omega)$$

and taking the limit as k goes to infinity, as $a_j \neq 0$, $u_{j'+1}(\theta) \in W(\theta)$. Eventually, $\mathrm{Vect}_{\mathbb{C}}(u_1,\dots u_j)$ is contained in W; therefore, since we assumed j is maximal, $\mathrm{Vect}_{\mathbb{C}}(u_1,\dots u_j)$ is equal to W. $\square$

Sublemma 2 *Let W' be an invariant subbundle such that for all $\theta \in \mathbb{T}^d$, $W'(\theta) = \bigoplus_{i=1}^m W^i(\theta)$ where each W^i is a Jordan subbundle modulo N with a real Jordan basis $(w_1^i,\dots w_{l_i}^i)$ with exponent α and suppose $u_j(\theta_0) = \sum_{i=1}^m \sum_{l=1}^{l_i} \lambda_i^l w_l^i(\theta_0)$ for some θ_0, then for all $\theta \in N\mathbb{T}^d$ and all $j' \leq j$, $u_{j'}(\theta) = \sum_{i=1}^m \sum_{1\leq j''\leq \min(j',l_i-j+j')} \lambda_i^{j-j'+j''} w_{j''}^i(\theta)$. In particular, $u_1(\theta) = \sum_{i=1}^m \lambda_i^j w_1^i(\theta)$.*

Proof: For all t,

$$\begin{aligned} 0 &= X^t(\theta_0)(u_j(\theta_0) - \sum_{i=1}^m \sum_{l=1}^{l_i} \lambda_i^l w_l^i(\theta_0)) \\ &= e^{t\alpha} \sum_{j'\leq j} \frac{t^{j-j'}}{(j-j')!} u_{j'}(\theta_0+t\omega) - \sum_{i=1}^m \sum_{l=1}^{l_i} \lambda_i^l e^{t\alpha} \sum_{l'=1}^{l} \frac{t^{l-l'}}{(l-l')!} w_{l'}^i(\theta_0+t\omega) \end{aligned} \tag{20}$$

Dividing by $e^{t\alpha}$, we get for all t

$$0 = \sum_{i\leq j} \frac{t^{j-j'}}{(j-j')!} u_{j'}(\theta_0+t\omega) - \sum_{i=1}^m \sum_{l=1}^{l_i} \lambda_i^l \sum_{l'=1}^{l} \frac{t^{l-l'}}{(l-l')!} w_{l'}^i(\theta_0+t\omega) \tag{21}$$

Let L be the greatest power of t in this expression. Let θ be any point of $N\mathbb{T}^d$. Take a sequence $t_k \to \infty$ such that $t_k\omega \to \theta-\theta_0 \in N\mathbb{T}^d$ as $k \to \infty$. Suppose first that $L \geq j$. Then, dividing (21) by t^L, and making k go to infinity,

$$\sum_{i=1}^m \sum_{l=L+1}^{l_i} \lambda_i^l \frac{1}{L!} w_{l-L}^i(\theta) = 0 \tag{22}$$

Since $w_{l-L}^i(\theta)$ are linearly independant, $\lambda_i^l = 0$ if $l \geq L+1$. Consequently, (21) can be rewritten

$$0 = \sum_{j'\leq j} \frac{t^{j-j'}}{(j-j')!} u_{j'}(\theta_0+t\omega) - \sum_{i=1}^m \sum_{l=1}^{L} \lambda_i^l \sum_{l'=1}^{l} \frac{t^{l-l'}}{(l-l')!} w_{l'}^i(\theta_0+t\omega) \tag{23}$$

But this contradicts the definition of L, so the assumption under which $L \geq j$ is false. Therefore, (21) can be rewritten

$$0=\sum_{j'\leq j}\frac{t^{j-j'}}{(j-j')!}u_{j'}(\theta_0+t\omega)-\sum_{i=1}^{m}\sum_{l=1}^{\min(j,l_i)}\lambda_i^l\sum_{l'=1}^{l}\frac{t^{l-l'}}{(l-l')!}w_{l'}^i(\theta_0+t\omega) \tag{24}$$

Dividing (24) by $t^{j-1},\dots t$, replacing t by t_k and making k go to ∞, we see that for all $1\leq j'\leq j$ and all $\theta\in N\mathbb{T}^d$,

$$\begin{aligned}u_{j'}(\theta)&=\sum_{i=1}^{m}\sum_{l-l'=j-j'}\lambda_i^l w_{l'}^i(\theta)=\sum_{i=1}^{m}\sum_{l=j-j'+1}^{\min(j,l_i)}\lambda_i^l w_{l-j+j'}^i(\theta)\\&=\sum_{i=1}^{m}\sum_{l=1}^{\min(j',l_i-j+j')}\lambda_i^{l+j-j'}w_l^i(\theta)\ \square\end{aligned} \tag{25}$$

Remark: Coefficients $\lambda_i^{j-j'+j''}$ do not depend on θ.

Lemma 5 *Let W' be an invariant subbundle such that for all $\theta\in\mathbb{T}^d$, $W'(\theta)=\bigoplus_{i'=1}^m W^{i'}(\theta)$ where $W^{i'}$ are Jordan subbundles modulo N with a real basis $(w_1^{i'},\dots w_{l_{i'}}^{i'})$ with exponent α. Then $W'+U$ is a direct sum of Jordan subbundles modulo N with a real basis.*

Proof: If $U(\theta)\cap W'(\theta)=\{0\}$ for all θ, this is trivial.
Let us now suppose that this intersection is non trivial. It is then equal to some non trivial invariant subbundle. By sublemma 1, it is generated by $u_1,\dots u_j$ for some $j\leq k$.

Assume first that $\dim U\leq\dim W_i$ for all i.
By sublemma 2, there exists $\lambda_1,\dots\lambda_m$ such that for all $\theta\in N\mathbb{T}^d$,

$$u_1(\theta)=\sum_{i=1}^{m}\lambda_i w_1^i(\theta)$$

Let $u_1'=u_2-\sum_{i=1}^m\lambda_i w_2^i,\dots u_{n-1}'=u_n-\sum_{i=1}^m\lambda_i w_n^i$. If $u_1',\dots u_{n-1}'$ are a basis, since for all $j\leq n-1$

$$\begin{aligned}X^t(\theta)u_j'(\theta)&=X^t(\theta)(u_{j+1}(\theta)-\sum_{i=1}^{m}\lambda_i w_{j+1}^i(\theta))\\&=\sum_{j'\leq j+1}\frac{t^{j-j'}}{(j-j')!}(u_{j'}(\theta+t\omega)-\sum_{i=1}^{m}\lambda_i w_{j'}^i(\theta+t\omega))\\&=\sum_{j'\leq j}\frac{t^{j-j'}}{(j-j')!}u_{j'}'(\theta+t\omega)\end{aligned} \tag{26}$$

this means that they are a Jordan basis.
If u_1' is in the space generated by $u_2',\dots u_{n-1}'$, then we carry out the same construction. After finitely many steps, we have defined a Jordan basis for $U+\bigoplus_i W_i$.

Let now U be of any dimension. We shall proceed by induction.

- If U has dimension 1, it is included in W' so the conclusion immediatly follows.
- Suppose now that the conclusion holds for any U of dimension $\leq n-1$. If now U has dimension n, write $W' = W_1 \oplus W_2$ où $W_1 = \bigoplus_{dimW_i<n} W_i$ and $W_2 = \bigoplus_{dimW_i \geq n} W_i$. By the above, $W_2 + U$ is the direct sum of Jordan subbundles modulo N with a real basis. Then, we add one by one the W_i with dimension $< n$, and by induction hypothesis we still get a direct sum of Jordan subbundles modulo N with a real basis. $\square$

1.4 Decomposition into invariant subbundles

Suppose X is $GL(n, \mathbb{C})$-reducible. Then by lemma 3,

$$\forall \theta \in \mathbb{T}^d,\ \mathbb{C}^n = W_1(\theta) \oplus \cdots \oplus W_r(\theta)$$

where each W_j is the sum of all the Jordan subbundles with exponent $\alpha_j + i\beta_j \pmod{2i\pi\langle \mathbb{Z}^d, \omega\rangle}$.

Lemma 6 *For all $1 \leq j \leq r$, there exists $1 \leq j' \leq r$ such that $\bar{W}_j = W_{j'}$. Moreover, $W_j = \bar{W}_j$ iff $\beta_j \in \pi\langle \mathbb{Z}^d, \omega\rangle$.*

Proof: Let $v(\theta) \in W_j(\theta)$ generating a complex invariant subbundle of dimension 1, then for all $\theta \in \mathbb{T}^d, t \in \mathbb{R}$,

$$X^t(\theta)v(\theta) = e^{t(\alpha_j + i\beta_j)} v(\theta + t\omega)$$

and

$$X^t(\theta)\bar{v}(\theta) = e^{t(\alpha_j - i\beta_j)} \bar{v}(\theta + t\omega)$$

Write $\bar{v}(\theta) = \sum_{l=1}^r \gamma_l(\theta) w_l(\theta)$ with $(w_l(\theta))_{l=1\ldots r}$ a Jordan basis of $\mathbb{C}^n$ and γ_l continuous and $\mathbb{C}$-valued. Then there exist polynomials $\{P_l(t), l = 1, \ldots r\}$ such that for all t,

$$X^t(\theta)\bar{v}(\theta) = \sum_{l=1}^r \gamma_l(\theta) e^{t(\alpha_l + i\beta_l)} P_l(t) w_l(\theta + t\omega)$$

So

$$e^{(\alpha_j - i\beta_j)t} \sum_{l=1}^r \gamma_l(\theta) w_l(\theta + t\omega) = \sum_{l=1}^r \gamma_l(\theta) e^{(\alpha_l + i\beta_l)t} P_l(t) w_l(\theta + t\omega)$$

Since $w_l(\theta + t\omega)$ are linearly independant, for all l,

$$e^{(\alpha_j - i\beta_j)t} \gamma_l(\theta) w_l(\theta + t\omega) = \gamma_l(\theta) e^{(\alpha_l + i\beta_l)t} P_l(t) w_l(\theta + t\omega)$$

so if $\gamma_l(\theta) \neq 0$, then for all t

$$w_l(\theta + t\omega) = e^{(\alpha_l - \alpha_j + i(\beta_j + \beta_l))t} P_l(t) w_l(\theta + t\omega)$$

This implies that $\alpha_l = \alpha_j$, P_l is constant equal to 1 and $\beta_j = -\beta_l$.
Let j' be such that $w_l(\theta) \in W_{j'}(\theta)$ for all $\theta \in \mathbb{T}^d$; then $\bar{W}_j = W_{j'}$.

Suppose now that $W_j = \bar{W}_j$. Let $V_1, \dots V_{R_j}$ be the Jordan subbundles contained in W_j, and for each V_s, $u_1^s + iv_1^s, \dots u_{k_s}^s + iv_{k_s}^s$ a global basis with exponent $\alpha + i\beta$. Write for all θ the decomposition $u_1^s(\theta) - iv_1^s(\theta) = \sum_{s' \leq r, j \leq k_{s'}} a_j^{s'}(\theta)(u_j^{s'}(\theta) + iv_j^{s'}(\theta))$, then let $X^t(\theta)$ act on each side; then for all t,

$$\begin{aligned}
&X^t(\theta)(u_1^s(\theta) - iv_1^s(\theta)) = e^{t(\alpha - i\beta)}(u_1^s(\theta + t\omega) - iv_1^s(\theta + t\omega)) \\
&= e^{t(\alpha - i\beta)} \sum_{s' \leq r, j \leq k_{s'}} a_j^{s'}(\theta + t\omega)(u_j^{s'}(\theta + t\omega) + iv_j^{s'}(\theta + t\omega)) \\
&= \sum_{s' \leq r, j \leq k_{s'}} a_j^{s'}(\theta) X^t(\theta)(u_j^{s'}(\theta) + iv_j^{s'}(\theta)) \qquad (27) \\
&= \sum_{s' \leq r, j \leq k_{s'}} a_j^{s'}(\theta) e^{t(\alpha + i\beta)} \sum_{j' \leq j} \frac{t^{j-j'}}{(j-j')!}(u_{j'}^{s'}(\theta + t\omega) + iv_{j'}^{s'}(\theta + t\omega))
\end{aligned}$$

as $u_{k_s}^s(\theta + t\omega) + iv_{k_s}^s(\theta + t\omega)$ is linearly independant from the rest, then

$$e^{t(\alpha - i\beta)} a_{k_s}^s(\theta + t\omega) = a_{k_s}^s(\theta) e^{t(\alpha + i\beta)}$$

whence, by lemma 1, the fact that $2\beta = 2\pi\langle m, \omega\rangle$ for some $m \in \mathbb{Z}^d$.

Conversely, if $2\beta = 2\pi\langle m, \omega\rangle$ for some $m \in \mathbb{Z}^d$, then W_j is it own complex conjugate. □

1.5 Main result

We get to the proof of theorem 1.

Proposition 4 *Assume that the continuous cocycle X is $GL(n, \mathbb{C})$-reducible. Then there exists a decomposition of $\mathbb{R}^n$ into two invariant subbundles $\mathcal{W}$ and $\mathcal{W}'$ such that:*

- *$\mathcal{W}$ is a reducible subbundle modulo 2, generated by a basis $(z_1, \dots z_s)$ such that for all $(\theta, t) \in 2\mathbb{T}^d \times \mathbb{R}$, $X^t(\theta)[z_1(\theta) \dots z_s(\theta)] = [z_1(\theta + t\omega) \dots z_s(\theta + t\omega)]e^{A_1 t}$ where A_1 has a real spectrum;*
- *$\mathcal{W}'$ is a reducible subbundle modulo 1 with a basis $(z_{s+1}, \dots z_n)$ such that for all $(\theta, t) \in \mathbb{T}^d \times \mathbb{R}$,*

$$X^t(\theta)[z_{s+1}(\theta) \dots z_n(\theta)] = [z_{s+1}(\theta + t\omega) \dots z_n(\theta + t\omega)]e^{A_2 t}$$

  with $\sigma(A_2) \cap (\mathbb{R} + i\pi\langle \mathbb{Z}^d, \omega\rangle \setminus \{0\}) = \emptyset$ and if $\alpha_1 + i\beta_1, \alpha_2 + i\beta_2 \in \sigma(A_2)$, then $\beta_1 - \beta_2$ is not in $2\pi\langle \mathbb{Z}^d, \omega\rangle \setminus \{0\}$.

Proof: From lemma 3, we get a decomposition of $\mathbb{C}^n$ into complex Jordan subbundles. Let us keep the notations introduced in section 1.4.

By lemma 6, there exists a decomposition $\mathbb{C}^n = W \oplus W'$ where W is the direct sum of all W_j which are their own complex conjugate, $W = \bigoplus_{j=1}^{r'} W_j$, and W' the direct sum of all the others: $W' = \bigoplus_{j=r'+1}^{r} W_j$.

1. • By lemma 6, W contains exactly the Jordan subbundles whose exponent is in $\mathbb{R} + i\pi\langle\mathbb{Z}^d, \omega\rangle$. Decompose again W into $W_{\mathbb{R}}$ and $W_{\mathbb{C}}$ where $W_{\mathbb{R}}$ is the sum of the Jordan subbundles having exponent 0 *mod* $2i\pi\langle\mathbb{Z}^d, \omega\rangle$, and $W_{\mathbb{C}}$ is the sum of the Jordan subbundles whose exponent is in $i\pi\langle\mathbb{Z}^d, \omega\rangle \setminus 2i\pi\langle\mathbb{Z}^d, \omega\rangle$.

For $W_{\mathbb{R}}$, we can find real Jordan bases with real exponent which are continuous on $\mathbb{T}^d$.

Proposition 3 implies that we can find real exponents for $W_{\mathbb{C}}$, but for bases which are continuous on $2\mathbb{T}^d$ and not on $\mathbb{T}^d$ anymore.

• We will show by induction that there is a decomposition of each $W_j \subset W_{\mathbb{C}}$ into Jordan subbundles with a real Jordan basis.
Let $V_1, \dots V_{R_j}$ be the Jordan subbundles included in W_j. According to lemmas 4 and 5, $(V_1 + \bar{V}_1) \cap \mathbb{R}^n$ is the direct sum of two Jordan subbundles modulo 2 with a real basis, since it is the sum of the Jordan subbundle modulo 2 generated by the real parts of the vectors in the basis of V_1, and of the Jordan subbundle modulo 2 generated by their imaginary parts.

Let $\bar{W}$ and $\bar{W}'$ be invariant subbundles such that there exists $k \geq 2$ with $\bar{W} = (V_k + \bar{V}_k) \cap \mathbb{R}^n$ and that $\bar{W}'$ is a direct sum of Jordan subbundles modulo 2 with a real basis.
By lemma 5, $\bar{W}$ is the direct sum of two Jordan subbundles modulo 2, U and V.
Using lemma 5 again, $U + \bar{W}'$ is the direct sum of Jordan subbundles modulo 2 with a real Jordan basis.
Finally, by lemma 5, $\bar{W} + \bar{W}' = V + U + \bar{W}'$ is the direct sum of Jordan subbundles modulo 2 with a real basis, which ends the induction.

2. In W', choose for all $r' + 1 \leq j \leq r$ and for each Jordan subbundle $V_s^j, s \leq R_j$ contained in some $W_j \subset W'$, a Jordan basis with exponent $\alpha_j + i\beta_j$ such that for all j, j', $\beta_j - \beta_{j'}$ is not in $2\pi\langle\mathbb{Z}^d, \omega\rangle \setminus \{0\}$. We have already showed that for all j, β_j is not in $\pi\langle\mathbb{Z}^d, \omega\rangle$.
Let W'' be a sum of Jordan subbundles such that $W' = W'' \oplus \bar{W}''$. If $(u_1 + iv_1, \dots u_{\frac{s}{2}} + iv_{\frac{s}{2}})$ is the global basis of W'' which is the union of all those Jordan bases, then lemma 2 implies that $(u_1(\theta), v_1(\theta), \dots u_{\frac{s}{2}}(\theta), v_{\frac{s}{2}}(\theta))$ form a basis of $W'(\theta) \cap \mathbb{R}^n$ for all θ. Moreover,

$$X^t(\theta)\big[u_1(\theta)\ v_1(\theta)\ \dots u_{\frac{s}{2}}(\theta)\ v_{\frac{s}{2}}(\theta)\big] = \big[u_1(\theta+t\omega)\ v_1(\theta+t\omega)\ \dots u_{\frac{s}{2}}(\theta+t\omega)\ v_{\frac{s}{2}}(\theta+t\omega)\big]e^{tA_2}$$

where $\sigma(A_2) = \{\alpha_j + i\beta_j, r' + 1 \leq j \leq r\}$.

Let $\mathcal{W} = W_{\mathbb{C}} \cap \mathbb{R}^n$ and $\mathcal{W}' = W_{\mathbb{R}} \cap \mathbb{R}^n \oplus W' \cap \mathbb{R}^n$. We have shown the existence of the required bases $(z_1, \dots z_s)$ for $\mathcal{W}$ and $(z_{s+1}, \dots z_n)$ for $\mathcal{W}'$. $\square$

Corollary 1 *With the notations of the proposition 4, let* $Z(\theta) = \big(z_1(\theta) \dots z_n(\theta)\big)$. *Then for all* θ, t,

$$X^t(\theta) = Z(\theta + t\omega)e^{t\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}}Z(\theta)^{-1}$$

This proves theorem 1.

2 Reducibility in other Lie groups

We now give the proof of the reducibility theorem for the groups $SL(n,\mathbb{R})$, $Sp(n,\mathbb{R})$, $O(n)$ and $U(n)$.

2.1 $SL(n,\mathbb{R})$-reducibility

Proposition 5 *Let X be a continuous $SL(n,\mathbb{R})$-valued cocycle which is $GL(n,\mathbb{R})$-reducible modulo N to a cocycle $t \mapsto e^{tB}$. Then $B \in sl(n,\mathbb{R})$ and there exists $\tilde{Z} : N\mathbb{T}^d \to SL(n,\mathbb{R})$ such that for all t, θ,*

$$X^t(\theta) = \tilde{Z}(\theta + t\omega)^{-1} e^{tB} \tilde{Z}(\theta) \tag{28}$$

so $X^t(\theta)$ is $SL(n,\mathbb{R})$-reducible modulo N.

Proof: Let $\tilde{Z}(\theta) := \frac{1}{\det Z(\theta)} Z(\theta)$. By construction, $\tilde{Z} \in C^0(N\mathbb{T}^d, SL(n,\mathbb{R}))$ and for all θ, t,

$$e^{tB} = Z(\theta + t\omega) X^t(\theta) Z(\theta)^{-1} \tag{29}$$

so

$$\frac{\det Z(\theta)}{\det Z(\theta + t\omega)} e^{tB} = \tilde{Z}(\theta + t\omega) X^t(\theta) \tilde{Z}(\theta)^{-1} \tag{30}$$

Thus the left-hand side has determinant 1. So

$$\forall t,\ \mathrm{tr}(\ln \frac{\det Z(\theta)}{\det Z(\theta + t\omega)} I + tB) = 0 \tag{31}$$

As $\ln \frac{\det Z(\theta)}{\det Z(\theta + t\omega)}$ is bounded, then $\mathrm{tr}(B) = 0$ and $\det Z$ is constant. Therefore, for all θ, t,

$$X^t(\theta) = Z(\theta + t\omega)^{-1} \det Z e^{tB} \frac{Z(\theta)}{\det Z} = \tilde{Z}(\theta + t\omega)^{-1} e^{tB} \tilde{Z}(\theta) \tag{32}$$

Corollary 2 *Let X be a $GL(n,\mathbb{C})$-reducible cocycle. If it is $SL(n,\mathbb{R})$-valued then it is $SL(n,\mathbb{R})$-reducible modulo 2.*

Proof: Apply proposition 4, then proposition 5. □

This proves theorem 2 when $G = SL(n,\mathbb{R})$.

2.2 Symplectic reducibility

Proposition 6 *If X is $Sp(2n,\mathbb{R})$-valued and $GL(2n,\mathbb{C})$-reducible, then it is $Sp(2n,\mathbb{R})$-reducible modulo 2.*

Proof: Let $\mathbb{R}^n = \mathcal{W} \oplus \mathcal{W}'$ as in proposition 4 and Z as in corollary 1: for all θ, $Z(\theta) = [z_1(\theta) \ \dots z_n(\theta)]$.
Write $X^t(\theta) = Z(\theta + t\omega)Ce^{tB}C^{-1}Z(\theta)^{-1}$ with B in Jordan normal form.
Let $Y(\theta) = C^*Z(\theta)^*JZ(\theta)C$. Then the coefficients $y_{j,k}(\theta)$ of $Y(\theta)$ satisfy $y_{j,k}(\theta) = \langle z_j(\theta), Jz_k(\theta)\rangle_{\mathbb{C}}$ where $z_j(\theta)$ is the j-th column of $Z(\theta)C$. Since $X^t(\theta)^*JX^t(\theta) = J$, then for all θ, t,

$$y_{j,k}(\theta) = \langle X^t(\theta)z_j(\theta), JX^t(\theta)z_k(\theta)\rangle_{\mathbb{C}} \tag{33}$$

Three cases are to be considered:

1. $y_{j,k}$ is continuous on $\mathbb{T}^d$;
2. z_j is continuous on $2\mathbb{T}^d$ and z_k is continuous on $\mathbb{T}^d$;
3. z_j and z_k are only continuous on $2\mathbb{T}^d$.

Case 1: z_j and z_k are in $\mathcal{W}'$. Then for some r_j, r_k,

$$\begin{aligned} y_{j,k}(\theta) &= \langle e^{t(\alpha_j+i\beta_j)} \sum_{i=r_j}^{j} \frac{t^{j-i}}{(j-i)!} z_i(\theta+t\omega), Je^{t(\alpha_k+i\beta_k)} \sum_{i=r_k}^{k} \frac{t^{k-i}}{(k-i)!} z_i(\theta+t\omega)\rangle_{\mathbb{C}} \\ &= e^{t(\alpha_j+\alpha_k+i\beta_j-i\beta_k)} \sum_{i=r_j}^{j} \sum_{i'=r_k}^{k} \frac{t^{j-i}}{(j-i)!} \frac{t^{k-i'}}{(k-i')!} \langle z_i(\theta+t\omega), Jz_{i'}(\theta+t\omega)\rangle_{\mathbb{C}} \\ &= e^{t(\alpha_j+i\beta_j+\alpha_k-i\beta_k)} \sum_{i=r_j}^{j} \sum_{i'=r_k}^{k} \frac{t^{j+k-i-i'}}{(j-i)!(k-i')!} y_{i,i'}(\theta+t\omega) \end{aligned} \tag{34}$$

In particular, if $j = r_j$ and $k = r_k$,

$$y_{j,k}(\theta) = e^{t(\alpha_j+i\beta_j+\alpha_k-i\beta_k)} y_{j,k}(\theta+t\omega) \tag{35}$$

Developing into Fourier series, since $y_{j,k}$ is continuous on $\mathbb{T}^d$, for all $m \in \mathbb{Z}^d$,

$$\hat{y}_{j,k}(m) = e^{t(\alpha_j+i\beta_j+\alpha_k-i\beta_k)} \hat{y}_{j,k}(m) e^{2i\pi\langle m,t\omega\rangle} \tag{36}$$

Thus, either $\hat{y}_{j,k}(m) = 0$, or $e^{t(\alpha_j+i\beta_j+\alpha_k-i\beta_k+2i\pi\langle m,\omega\rangle)} = 1$ for all t, and then $\alpha_j + i\beta_j + \alpha_k - i\beta_k + 2i\pi\langle m,\omega\rangle = 0$. But if $m \neq 0$, this is impossible since $\beta_j - \beta_k$ is not in $2\pi\langle\mathbb{Z}^d,\omega\rangle \setminus \{0\}$. Therefore, $y_{j,k}$ is constant.
For any j, k, it is possible to show, using equations (34) in the appropriate order, that $y_{j,k}$ is constant: equation (34), once developed in Fourier series, gives for all $m \in \mathbb{Z}^d$,

$$\hat{y}_{j,k}(m) = e^{t(\alpha_j + i\beta_j + \alpha_k - i\beta_k)} \sum_{i=r_j}^{j} \sum_{i'=r_k}^{k} \frac{t^{j+k-i-i'}}{(j-i)!(k-i')!} \hat{y}_{i,i'}(m) e^{2i\pi\langle m, t\omega\rangle} \tag{37}$$

Assume $y_{i,i'}$ is constant for all (i, i') such that $i < j$ or $i = j, i' < k$. Then, if $m \neq 0$,

$$\hat{y}_{j,k}(m) = e^{t(\alpha_j + i\beta_j + \alpha_k - i\beta_k)} \hat{y}_{j,k}(m) e^{2i\pi\langle m, t\omega\rangle} \tag{38}$$

which again implies that $y_{j,k}$ is constant.

Case 2: z_j is in $\mathcal{W}$ and z_k in $\mathcal{W}'$. Then for some r_j, r_k,

$$y_{j,k}(\theta) = e^{t(\alpha_j + \alpha_k - i\beta_k)} \sum_{i=r_j}^{j} \sum_{i'=r_k}^{k} \frac{t^{j+k-i-i'}}{(j-i)!(k-i')!} y_{i,i'}(\theta + t\omega) \tag{39}$$

In particular, if $z_j(\theta)$ and $z_k(\theta)$ generate Jordan subbundles of rank 1, for all θ, t,

$$y_{j,k}(\theta) = e^{t(\alpha_j + \alpha_k - i\beta_k)} y_{j,k}(\theta + t\omega) \tag{40}$$

Developing this into Fourier series, since $y_{j,k}$ is continuous on $2\mathbb{T}^d$, for all $m \in \mathbb{Z}^d$,

$$\hat{y}_{j,k}(m) = e^{t(\alpha_j + \alpha_k - i\beta_k)} \hat{y}_{j,k}(m) e^{2i\pi\langle m, t\frac{\omega}{2}\rangle} \tag{41}$$

So, either $\hat{y}_{j,k}(m) = 0$, or $e^{t(\alpha_j + \alpha_k - i\beta_k + i\pi\langle m,\omega\rangle)} = 1$ for all t, which implies that $\alpha_j + \alpha_k - i\beta_k + i\pi\langle m, \omega\rangle = 0$. Since β_k is not in $\pi\langle \mathbb{Z}^d, \omega\rangle \setminus \{0\}$, this is impossible if $m \neq 0$, so $y_{j,k}$ is constant.
For other j, k, (39) implies that $y_{j,k}$ is constant.

Case 3: z_j and z_k are in $\mathcal{W}$. Thus they are in a Jordan basis with real exponent, continuous on $2\mathbb{T}^d$.
If $z_{r_j}, \dots z_j$ generate a Jordan subbundle with exponent α_j and $z_{r_k}, \dots z_k$ generate a Jordan subbundle with exponent α_k, then for all θ, t, (39) holds, but with $\beta_k = 0$.
In particular, if z_j and z_k generate Jordan subbundles of rank 1, for all θ, t,

$$y_{j,k}(\theta) = e^{t(\alpha_j + \alpha_k)} y_{j,k}(\theta + t\omega) \tag{42}$$

Developing into Fourier series again, since $y_{j,k}$ is continuous on $2\mathbb{T}^d$,

$$\hat{y}_{j,k}(m) = e^{t(\alpha_j + \alpha_k)} \hat{y}_{j,k}(m) e^{2i\pi\langle m, t\frac{\omega}{2}\rangle} \tag{43}$$

Thus $y_{j,k}$ is constant.
More generally, for arbitrary j, k, for all $m \in \mathbb{Z}^d$ and all t,

$$\hat{y}_{j,k}(m) = e^{t(\alpha_j + \alpha_k)} \sum_{i=r_j}^{j} \sum_{i'=r_k}^{k} \frac{t^{j-i}}{(j-i)!} \frac{t^{k-i'}}{(k-i')!} \hat{y}_{i,i'}(m) e^{2i\pi\langle m, t\omega\rangle} \tag{44}$$

and we can use these equations in the appropriate order to show that all the coefficients of Y are constant, so Y is constant. This implies that $Z(\theta)^* JZ(\theta)$ does not depend on θ.

- Let $\bar{Z}(\theta) = Z(\theta)Z(0)^{-1}$. Then

$$\bar{Z}(\theta)^* J\bar{Z}(\theta) = (Z(0)^{-1})^* Z(\theta)^* JZ(\theta)(Z(0)^{-1}) = J \tag{45}$$

since Z^*JZ is constant. Moreover, $\bar{Z}$ is real, so it is $Sp(2n, \mathbb{R})$-valued. It is continuous on $2\mathbb{T}^d$. Finally, for all θ, t,

$$X^t(\theta) = Z(\theta + t\omega)e^{tA}Z(\theta)^{-1}$$

where $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ thus

$$X^t(\theta) = \bar{Z}(\theta + t\omega)e^{tZ(0)AZ(0)^{-1}}\bar{Z}(\theta)^{-1} \tag{46}$$

and therefore, X is $Sp(2n, \mathbb{R})$-reducible modulo 2. $\square$

This proves theorem 2 when $G = Sp(2n, \mathbb{R})$.

2.3 Orthogonal group

Proposition 7 *Let X be a $GL(n, \mathbb{C})$-reducible cocycle. If it is $O(n)$-valued, then it is $O(n)$-reducible modulo 2.*

Proof: It is possible to carry out exactly the same proof as for proposition 6, but defining $Y(\theta)$ as $C^*Z(\theta)^*Z(\theta)C$ and not as $C^*Z(\theta)^*JZ(\theta)C$ anymore. This way, its coefficients are $y_{j,k}(\theta) = \langle z_j(\theta), z_k(\theta)\rangle_{\mathbb{C}} = \langle X^t(\theta)z_j(\theta), X^t(\theta)z_k(\theta)\rangle_{\mathbb{C}}$; since X is bounded, all the Jordan subbundles have rank 1, thus the coefficients $y_{j,k}$ satisfy equations (35), (40) and (42) with $\alpha_j = \alpha_k = 0$. We show in exactly the same way that they are constant, then define a function $\bar{Z}$ which is continuous on $2\mathbb{T}^d$ and $O(n)$-valued and such that $X^t(\theta) = \bar{Z}(\theta + t\omega)e^{tA}\bar{Z}(\theta)^{-1}$ for some constant matrix A and for all t, θ. $\square$

This proves theorem 2 when $G = O(n)$.

2.4 $U(n)$-reducibility

Proposition 8 *Assume that the continuous cocycle X is $U(n)$-valued and $GL(n, \mathbb{C})$-reducible. Then X is $U(n)$-reducible.*

Proof: By lemma 3, there is a decomposition of $\mathbb{C}^n$ into Jordan subbundles. Since the cocycle X is $U(n)$-valued, it is bounded, so all Jordan subbundles have rank 1 and a purely imaginary exponent. Let $z_1, \dots z_n$ be continuous on $\mathbb{T}^d$, each one generating a Jordan subbundle, chosen in such a way that the difference of two exponents cannot be in $2i\pi\langle\mathbb{Z}^d, \omega\rangle \setminus \{0\}$. Let $Z(\theta)$ be the matrix whose columns are $z_1(\theta), \dots z_n(\theta)$; then there is a diagonal matrix D with coefficients $i\beta_1, \dots i\beta_n$ such that for all θ, t,

$$X^t(\theta) = Z(\theta + t\omega)e^{tD}Z(\theta)^{-1}$$

Let $Y(\theta) = Z(\theta)^* Z(\theta)$, then the coefficients $y_{j,k}$ of Y satisfy

$$y_{j,k}(\theta) = e^{it(\beta_j - \beta_k)} y_{j,k}(\theta + t\omega) \tag{47}$$

Developing into Fourier series, for all $n \in \mathbb{Z}^d$,

$$\hat{y}(n)_{j,k} = e^{it(\beta_j - \beta_k)} \hat{y}(n)_{j,k} \tag{48}$$

By construction, $\beta_j - \beta_k$ is either 0 or is not in $2\pi\langle \mathbb{Z}^d, \omega \rangle$, so Y is constant equal to $Z(0)^* Z(0)$. Thus, if $\bar{Z}(\theta) := Z(\theta)Z(0)^{-1}$, then $\bar{Z}(\theta) \in U(n)$ and

$$X^t(\theta) = \bar{Z}(\theta + t\omega)e^{tZ(0)DZ(0)^{-1}}\bar{Z}(\theta)^{-1} \tag{49}$$

Therefore X is $U(n)$-reducible. $\square$

This completes the proof of theorem 2.

3 Discrete cocycles

We now want to adapt these results to discrete cocycles. In all this section, we shall assume that $(\omega, 1)$ is rationally independant.

Definition: Let $\bar{\omega} \in \mathbb{R}^D$; X is a continuous (resp. discrete) cocycle over $\bar{\omega}$ if it is defined on $\mathbb{T}^D \times \mathbb{R}$ (resp. $\mathbb{T}^D \times \mathbb{Z}$) and for all $\bar{\theta} \in \mathbb{T}^D$, $t, s \in \mathbb{R}$ (resp. $t, s \in \mathbb{Z}$), $X^{t+s}(\bar{\theta}) = X^t(\bar{\theta} + s\bar{\omega})X^s(\bar{\theta})$.

Remark: The cocycles we studied in the previous sections are all over ω. But to talk about a discrete cocycle over ω, it is necessary to assume that $(\omega, 1)$ is rationally independant. Notice that if a continuous cocycle X over $(\omega, 1)$ is G-reducible, then its restriction to integer time and to the d-dimensional subtorus $\mathcal{T} := \{(\theta, 0), \theta \in \mathbb{T}^d\}$ is a discrete cocycle over ω which is G-reducible. Indeed, let $Z : \mathbb{T}^d \to G$ and $B \in \mathcal{G}$ such that

$$X^t(\theta) = Z(\theta + t(\omega, 1))^{-1}e^{tB}Z(\theta) \tag{50}$$

It is enough to restrict this expression to integer time and to the subtorus $\mathcal{T}$ to get G-reducibility for the discrete cocycle $(n, \theta) \mapsto X^n(\theta, 0)$.

3.1 G-exponential discrete cocycles

Given a discrete G-valued cocycle X, we want to define a suspension of X, i.e a continuous G-valued cocycle whose restriction to integer times and possibly to a subtorus coincides with the initial cocycle. But this cannot be done if X takes its values in two different connected components of G, nor if $\theta \mapsto X^1(\theta)$ is not homotopic to the identity in G (since the suspension would be a homotopy). However, if there is a $\mathcal{G}$-valued function A which

is continuous on $\mathbb{T}^d$ such that for all θ, $X^1(\theta) = e^{A(\theta)}$, then we can define a continuous G-valued cocycle whose restriction to integer time and to a subtorus coincides with X: this will be done in the following proposition. Recall the definition:

Definition: A discrete cocycle X is called G-exponential if there exists a $\mathcal{G}$-valued function A, continuous on $\mathbb{T}^d$, such that $X^1(\theta) = e^{A(\theta)}$ for all θ.

Proposition 9 *Let X be a discrete G-exponential cocycle over ω. Then there exists a continuous cocycle $\tilde{X} : \mathbb{R} \times \mathbb{T}^d \times \mathbb{T} \to G, (t, \theta, \theta') \mapsto X^t(\theta, \theta')$ over $(\omega, 1)$ whose restriction to $t \in \mathbb{Z}$ and $\{\theta' = 0\}$ coincides with X.*

Proof: By assumption, there exists a $\mathcal{G}$-valued function A, continuous on $\mathbb{T}^d$, such that for all θ, $X^1(\theta) = e^{A(\theta)}$.
For all $(\theta, \theta_{d+1}) \in \mathbb{T}^d \times [0, 1[$, let $B(\theta, \theta_{d+1}) = \phi(\theta, \theta_{d+1})A(\theta - \theta_{d+1}\omega)$ where ϕ is a real function continuous on $\mathbb{T}^d \times [0, 1[$ with support contained in $\mathbb{T}^d \times [\frac{1}{4}, \frac{3}{4}]$ such that

$$\int_0^1 \phi(\theta + s\omega, \theta_{d+1} + s)ds = 1$$

and for all $n \in \mathbb{Z}$, $B(\theta, \theta_{d+1} + n) = B(\theta, \theta_{d+1})$. So defined, B is continuous on $\mathbb{T}^d \times \mathbb{R}$ and periodic in θ_{d+1}. Let $\bar{B}$ be the continuous function on $\mathbb{T}^{d+1}$ which we obtain by taking the quotient.
Let $(t, \theta, \theta_{d+1}) \mapsto \tilde{X}^t(\theta, \theta_{d+1})$ be the continuous cocycle satisfying

$$\frac{d}{dt}\tilde{X}^t(\theta, \theta_{d+1}) = \bar{B}(\theta + t\omega, \theta_{d+1} + t)\tilde{X}^t(\theta, \theta_{d+1})$$

This cocycle is G-valued. Since $\int_0^t \bar{B}(\theta + s\omega, \theta_{d+1} + s)ds$ commutes with $\bar{B}(\theta + t\omega, \theta_{d+1} + t)$ for all θ, t, we can compute $\tilde{X}^t(\theta, \theta_{d+1})$:

$$\forall t, \theta, \theta_{d+1},\ \tilde{X}^t(\theta, \theta_{d+1}) = \exp(\int_0^t \phi(\theta + s\omega, \theta_{d+1} + s)dsA(\theta - \theta_{d+1}\omega))$$

Thus, for all $\theta \in \mathbb{T}^d$,

$$\tilde{X}^1(\theta, 0) = \exp(A(\theta)) = X^1(\theta)$$

and for $n \in \mathbb{N}, n \geq 1$,

$$\tilde{X}^n(\theta, 0) = \tilde{X}^1(\theta + (n-1)\omega, n-1) \ldots \tilde{X}^1(\theta, 0) = X^1(\theta + (n-1)\omega) \ldots X^1(\theta) = X^n(\theta)$$

and for $n \in \mathbb{Z}, n \leq -1$,

$$\tilde{X}^n(\theta, 0) = \tilde{X}^{-n}(\theta + n\omega, n)^{-1} = \tilde{X}^{-n}(\theta + n\omega, 0)^{-1} = X^{-n}(\theta + n\omega)^{-1} = X^n(\theta)$$

whence the proposition. $\square$

Remark: It is possible to show that if $\theta \mapsto X^1(\theta)$ is homotopic to the identity, which is weaker than supposing that X is G-exponential, then there exists a continuous cocycle

whose restriction to integer time coincides with X. However, this cocycle is not G-valued anymore.

Definition: The continuous cocycle $\tilde{X}$ defined this way is called a suspension of X.

We shall show that $GL(N,\mathbb{C})$-reducibility of a discrete G-exponential cocycle implies $GL(N,\mathbb{C})$-reducibility of its suspension.

Proposition 10 *Let $\tilde{X}$ be the suspension of a discrete cocycle X which is $GL(N,\mathbb{C})$-reducible. Then $\tilde{X}$ is $GL(N,\mathbb{C})$-reducible.*

Proof: Let $Z \in C^0(\mathbb{T}^d, GL(N,\mathbb{C}))$ and $A \in GL(N,\mathbb{C})$ such that

$$X^n(\theta) = Z(\theta + n\omega)^{-1} A^n Z(\theta)$$

for all θ, n. There exists $B \in gl(N,\mathbb{C})$ such that

$$X^n(\theta) = Z(\theta + n\omega)^{-1} e^{nB} Z(\theta)$$

Let us define, for all $\theta \in \mathbb{T}^d$, $\tilde{Z}(\theta, 0) := Z(\theta)$, and for all $t \in \mathbb{R}$,

$$\tilde{Z}((\theta,0) + t(\omega,1)) = e^{tB}\tilde{Z}(\theta,0)\tilde{X}^t(\theta,0)^{-1}$$

Thus, for all $(\theta, \theta_{d+1}) \in \mathbb{T}^{d+1}$,

$$\tilde{Z}(\theta,\theta_{d+1}) = \tilde{Z}((\theta - \theta_{d+1}\omega, 0) + \theta_{d+1}(\omega,1)) = e^{\theta_{d+1}B} Z(\theta - \theta_{d+1}\omega, 0)\tilde{X}^{\theta_{d+1}}(\theta - \theta_{d+1}\omega, 0)^{-1}$$

The map $(\theta, \theta_{d+1}) \mapsto \tilde{Z}(\theta, \theta_{d+1})$ is periodic in θ and for all θ, θ_{d+1},

$$\begin{aligned}
&\tilde{Z}(\theta,\theta_{d+1}+1) = e^{(\theta_{d+1}+1)B} Z(\theta - (\theta_{d+1}+1)\omega, 0)\tilde{X}^{(\theta_{d+1}+1)}(\theta - (\theta_{d+1}+1)\omega, 0)^{-1} \\
&= e^{\theta_{d+1}B} e^{B} Z(\theta - (\theta_{d+1}+1)\omega, 0)\tilde{X}^{1}(\theta - (\theta_{d+1}+1)\omega, 0)^{-1}\tilde{X}^{\theta_{d+1}}(\theta - \theta_{d+1}\omega, 1)^{-1} \\
&= e^{\theta_{d+1}B} Z(\theta - \theta_{d+1}\omega, 1)\tilde{X}^{\theta_{d+1}}(\theta - \theta_{d+1}\omega, 1)^{-1} \\
&= e^{\theta_{d+1}B} Z(\theta - \theta_{d+1}\omega, 0)\tilde{X}^{\theta_{d+1}}(\theta - \theta_{d+1}\omega, 0)^{-1} = \tilde{Z}(\theta,\theta_{d+1})
\end{aligned} \tag{51}$$

so $\tilde{Z}$ is periodic in θ_{d+1}. Moreover, for all θ, θ_{d+1}, t,

$$\begin{aligned}
&\tilde{X}^t(\theta,\theta_{d+1}) = \tilde{X}^{t+\theta_{d+1}}(\theta - \theta_{d+1}\omega, 0)\tilde{X}^{\theta_{d+1}}(\theta - \theta_{d+1}\omega, 0)^{-1} \\
&= \tilde{Z}((\theta - \theta_{d+1}\omega, 0) + (t+\theta_{d+1})(\omega,1))^{-1} e^{(t+\theta_{d+1})B}\tilde{Z}(\theta - \theta_{d+1}\omega, 0) \\
&\tilde{Z}(\theta - \theta_{d+1}\omega, 0)^{-1} e^{-\theta_{d+1}B}\tilde{Z}((\theta - \theta_{d+1}\omega, 0) + \theta_{d+1}(\omega,1)) \\
&= \tilde{Z}((\theta,\theta_{d+1}) + t(\omega,1))^{-1} e^{tB}\tilde{Z}(\theta,\theta_{d+1})
\end{aligned} \tag{52}$$

whence the $GL(N,\mathbb{C})$-reducibility of $\tilde{X}$. $\square$

Now we can form the analogue, for discrete time, if X is a G-exponential cocycle, of propositions 4, 5, 6, 7 and 8. They come as corollaries of the above.

Proposition 11 *Let X a discrete G-exponential cocycle where G is within $GL(N,\mathbb{R})$, $SL(N,\mathbb{R})$, $Sp(N,\mathbb{R}), SO(N), SU(N)$ and $GL(N,\mathbb{C})$-reducible. Then X is G-reducible modulo χ_G, with*

$$\chi_G = \begin{cases} 2 & \text{if } G = GL(N,\mathbb{R}), SL(N,\mathbb{R}), Sp(N,\mathbb{R}) \text{ or } G = SO(N) \\ 1 & \text{if } G = SU(N) \end{cases}$$

Proof: Let $\tilde{X}$ be a suspension of X. By proposition 10, $\tilde{X}$ is $GL(N,\mathbb{C})$- reducible. Moreover, $\tilde{X}$ is G-valued, so by propositions 4, 5, 6, 7 and 8, $\tilde{X}$ is G-reducible modulo χ_G with $\chi_G = 2$ if $G = GL(N,\mathbb{R}), SL(N,\mathbb{R}), Sp(N,\mathbb{R})$ or $SO(N)$ and $\chi_G = 1$ if $G = SU(N)$. Thus, X is G-reducible modulo χ_G. $\square$

3.2 General case

It is possible to extend theorems 1 and 2 to all discrete cocycles, without even assuming that their values are in a connected Lie group, because the proof of the theorems 1 and 2 does not essentially use the fact that time is continuous.

The definition of a subbundle is the same as in section 1.2. If X is a discrete cocycle, an invariant subbundle is a subbundle such that for all $n \in \mathbb{Z}$ and all $\theta \in \mathbb{T}^d$, $X^n(\theta)V(\theta) = V(\theta + n\omega)$. We define a Jordan subbundle of rank k modulo N, a Jordan subbundle and its exponents in the same way as in section 1.2, but now t varies in $\mathbb{Z}$ and not in $\mathbb{R}$ anymore. We show in the same way, using part 2. of lemma 1, that the exponent of a Jordan subbundle modulo N is well-defined modulo $2i\pi\langle\mathbb{Z}^{d+1}, \frac{(\omega,1)}{N}\rangle$. Lemmas 3, 5 and 6 still hold in the discrete case, but lemma 6 will be reformulated as follows:

Lemma 7 *For all $1 \leq j \leq r$, there exists $1 \leq j' \leq r$ such that $\bar{W}_j = W_{j'}$. Moreover, $W_j = \bar{W}_j$ iff $\beta_j \in \pi\langle\mathbb{Z}^d, (\omega,1)\rangle$.*

Proposition 4 can be reformulated in an analogous way:

Proposition 12 *If X is a real discrete cocycle which is $GL(n,\mathbb{C})$-reducible, then there is a decomposition $\mathbb{R}^n = W \oplus W'$ where*

- *W is reducible subbundle modulo 2 with a basis $z_1, \dots z_r$ such that for all $(\theta,t) \in \mathbb{T}^d \times \mathbb{Z}$, $X^t(\theta)[z_1(\theta) \ \dots z_r(\theta)] = [z_1(\theta + t\omega) \ \dots z_r(\theta + t\omega)]e^{tA_1}$ where A_1 is a matrix with real spectrum;*

- *W' is a reducible subbundle modulo 1 with a basis $z_{r+1}, \dots z_n$ such that for all θ, t, $X^t(\theta)[z_{r+1}(\theta) \ \dots z_n(\theta)] = [z_{r+1}(\theta + t\omega) \ \dots z_n(\theta + t\omega)]e^{tA_2}$ where $\sigma(A_2) \cap \mathbb{R} + i\pi\langle\mathbb{Z}^{d+1}, (\omega,1)\rangle \setminus \{0\} = \emptyset$ and if $\alpha_1 + i\beta_1, \alpha_2 + i\beta_2 \in \sigma(A_2)$, then $\beta_1 - \beta_2$ is not in $2\pi\langle\mathbb{Z}^{d+1}, (\omega,1)\rangle \setminus \{0\}$.*

The proof is exactly the same as in proposition 4.

Proposition 13 *If X is a discrete $SL(n,\mathbb{R})$-valued cocycle which is $GL(n,\mathbb{R})$-reducible, then it is $SL(n,\mathbb{R})$-reducible modulo 2.*

Again, the proof is the same as in proposition 5, except that t varies in $\mathbb{Z}$ and not in $\mathbb{R}$ anymore.

Proposition 14 *If X is a discrete $Sp(n,\mathbb{R})$-valued (resp. $O(n)$-valued) cocycle which is $GL(n,\mathbb{C})$-reducible (resp. $GL(n,\mathbb{C})$-reducible), then it is $Sp(n,\mathbb{R})$-reducible (resp. $O(n)$-reducible) modulo 2.*

The proof is exactly as in propositions 6 and 7, because the fact that t varies in $\mathbb{Z}$ and not in $\mathbb{R}$ does not change the conclusions (we use the second part of lemma 1).

Proposition 15 *If X is a discrete $U(n)$-valued cocycle which is $GL(n,\mathbb{C})$-reducible, then X is $U(n)$-reducible.*

The proof is essentially the same as in the continuous case.

Propositions 12, 13, 14 and 15 together give theorem 3.

4 Applications

The preceding sections enable us to complete some other results on the full-measure reducibility of a generic one-parameter family of cocycles.

Definition: $\omega \in \mathbb{R}^d$ is diophantine with constant κ and exponent τ, denoted by $\omega \in DC(\kappa,\tau)$, if for all $n \in \mathbb{Z}^d$, $|\langle n,\omega\rangle| > \frac{\kappa}{|n|^\tau}$.

Definition: Let Λ be an interval of $\mathbb{R}$ and $A \in C^\infty(\Lambda, gl(n,\mathbb{C}))$ a one-parameter family of matrices; we say A satisfies the non-degeneracy condition $ND(r,\chi)$ on an interval Λ if there exists $r \in \mathbb{Z}^+$ and $\chi > 0$ such that for all $\lambda \in \Lambda$, for all $u \in \mathbb{R}$, $\sup_{l\leq r} |\frac{\partial^l g(\lambda,u)}{\partial\lambda^l}| > \chi$ where

$$g(\lambda,u) = \prod_{\alpha_i(\lambda),\alpha_j(\lambda)\in\sigma(A(\lambda)),i\neq j} (\alpha_i(\lambda) - \alpha_j(\lambda) - iu)$$

Definition: Let Λ an interval of $\mathbb{R}$, denote for $h,\delta > 0$ by $\mathcal{F}$ the set of the functions defined on $\{z \in \mathbb{C}, |Imz| < h\} \times \{x \in \mathbb{R}, d(x,\Lambda) < \delta\}$, holomorphic in the first variable and periodic on the real axis.
Let

$$C^\omega_{h,\delta}(\mathbb{T}^d \times \Lambda) = \{f \in \mathcal{F} \mid |f|_{h,\delta} := \sup_{|Imx|<h, d(z,\Lambda)<\delta} |f(x,z)| < +\infty\}$$

Finally, let $C^\omega_{h,\delta}(\mathbb{T}^d \times \Lambda, \mathcal{G})$ the set of $\mathcal{G}$-valued maps each component of whom is in $C^\omega_{h,\delta}(\mathbb{T}^d \times \Lambda)$.

4.1 Full-measure reducibility in the symplectic case

In [5], H.He and J.You claim the following:

Theorem 4 *Suppose $\omega \in DC(\kappa, \tau)$. Let $A \in C^\infty(\Lambda, gl(n, \mathbb{C}))$ a one-parameter family of matrices satisfying the non-degeneracy condition $ND(r, \chi)$ on an interval Λ. There exists $\epsilon_0 > 0$ depending on κ and τ, and there exists h, δ, such that if $F \in C^\omega_{h,\delta}(\mathbb{T}^d \times \Lambda, gl(n, \mathbb{C})), |F|_{h,\delta} \leq \epsilon_0$, then for almost every $\lambda \in \Lambda$, the cocycle satisfying*

$$\partial_\omega X(\theta) = (A(\lambda) + F(\theta, \lambda))X(\theta)$$

is $GL(n, \mathbb{C})$-reducible.

Let us also assume that $A(\lambda) \in sp(2n, \mathbb{R})$ for all $\lambda \in \Lambda$ and $F \in C^\omega_{h,\delta}(\mathbb{T}^d \times \Lambda, sp(2n, \mathbb{R}))$. Then, as a corollary of proposition 6 and of H.He and J.You's result, we can reformulate the above in the symplectic case:

Corollary 3 *Suppose $\omega \in DC(\kappa, \tau)$. Let $A(\lambda)$ be a one-parameter family of matrices in $sp(2n, \mathbb{R})$ satisfying the non-degeneracy condition $ND(r, \chi)$ on an interval Λ. There exists $\epsilon_0 > 0$ depending on κ, τ, and there exists h, δ, such that if $F \in C^\omega_{h,\delta}(\mathbb{T}^d \times \Lambda, sp(2n, \mathbb{R})), |F|_{h,\delta} \leq \epsilon_0$, then for almost all $\lambda \in \Lambda$, the cocycle satisfying*

$$\partial_\omega X(\theta) = (A(\lambda) + F(\theta, \lambda))X(\theta)$$

is $Sp(2n, \mathbb{R})$-reducible modulo 2.

4.2 Full-measure reducibility in a compact semi-simple group

In [8], R.Krikorian proved the following theorem:

Suppose $\omega \in DC(\kappa, \tau)$. Let A be a generic element of a compact semi-simple group G, $r > 0$ and Λ an interval of $\mathbb{R}$. There exists $\epsilon_0 > 0$ depending on $\kappa, \tau, \Lambda, A, \omega, r$ such that if $F \in C^\omega_r(\mathbb{T}^d, \mathcal{G})$ and $|F|_r \leq \epsilon_0$, then for almost all $\lambda \in \Lambda$, the cocycle satisfying

$$\partial_\omega X(\theta) = (\lambda A + F(\theta))X(\theta)$$

is G-reducible modulo an integer χ_G depending only on G. If $G = U(n)$, then $\chi_G = 1$.

As a corollary of H.He and J.You's result and of proposition 7, we know as well that if $G = O(n)$, then $\chi_G = 2$.

4.3 Does one have full-measure reducibility modulo 1 in any Lie group?

We first point out the following:

Proposition 16 *If X is a continuous G-valued cocycle which is $GL(n, \mathbb{C})$-reducible to a cocycle $t \mapsto e^{tB}$ such that the eigenvalues of B are not in $\mathbb{R} + i\pi\langle \mathbb{Z}^d, \omega \rangle$, then X is G-reducible.*

Proof: In the notations of section 1, there is a decomposition of $\mathbb{R}^n$ into invariant subbundles $W_1 \oplus \cdots \oplus W_r$, each $W_j, j \leq r$ being the sum of all Jordan subbundles with the same exponent. By assumption on the eigenvalues of B, none of the subbundles W_j is its own complex conjugate. For all j, let $(u_1^j + iv_1^j, \ldots u_{k_j}^j + iv_{k_j}^j)$ be a global basis of W_j. Then $(u_1^j, v_1^j, \ldots u_{k_j}^j, v_{k_j}^j)$ is a global basis of $(W_j + \bar{W}_j) \cap \mathbb{R}^n$. For all θ, let $Z(\theta)$ be the matrix whose columns are $(u_1^j(\theta), v_1^j(\theta), \ldots u_{k_j}^j(\theta), v_{k_j}^j(\theta), 1 \leq j \leq r)$, then Z is continuous on $\mathbb{T}^d$ and $GL(n, \mathbb{R})$-valued and for all θ, t, there exists $\tilde{B}$ such that $X^t(\theta) = Z(\theta + t\omega)e^{t\tilde{B}}Z(\theta)^{-1}$, so X is $GL(n, \mathbb{R})$-reducible modulo 1.

If $G = GL(n, \mathbb{R})$, the proof is finished. If $G = SL(n, \mathbb{R}), Sp(n, \mathbb{R})$ or $O(n)$, we do exactly as in the proof of 5, 6 and 7, but since, by assumption, only the case 1 can happen, one gets G-reducibility modulo 1. $\square$

Question: Let $A(\lambda)$ be a $\mathcal{G}$-valued one-parameter family satisfying a non-degeneracy condition for all $\lambda \in \Lambda$ and $F \in C^{\omega}_{h,\delta}(\mathbb{T}^d \times \Lambda)$ sufficiently small. Theorem 4 tells that the cocycle X_λ satisfying

$$X'_\lambda(t, \theta) = (A(\lambda) + F(\theta, \lambda))X_\lambda(t, \theta)$$

is $GL(n, \mathbb{C})$-reducible for almost all λ to $t \mapsto e^{tB_\lambda}$. Is it true that for almost all λ, the eigenvalues of B_λ are not in $\mathbb{R} + i\pi\langle \mathbb{Z}^d, \omega \rangle$? If it were the case, X_λ would be G-reducible modulo 1 for almost every λ.

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