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Regularization and finiteness of the Lorentzian loop quantum gravity vertices

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We give an explicit form for the Lorentzian vertices recently introduced for possibly defining the dynamics of loop quantum gravity. As a result of doing so, a natural regularization of the vertices is suggested. The regularized vertices are then proven to be finite. An interpretation of the regularization in terms of a gauge fixing is also given.

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I. INTRODUCTION

In the search for quantum gravity, loop quantum gravity (LQG) [1] has provided a well-understood kinematical framework, arising from standard quantization methods, with the assumption that parallel transports have well-defined operator analogs in the quantum theory. Spin foams have been proposed as an approach to the dynamics of the theory that retains manifest space-time covariance (for reviews and some useful original papers, see [2,3]). Through works of the past year [4–11], progress was made in modifying the more traditional Barrett-Crane model [12,13], by addressing the issue of the simplicity constraints—which turn the topological theory known as BF theory into gravity [3,14]—with more care. As a result, the kinematics of the models [5,6,10,11], covering all values of the Immirzi parameter in both Euclidean and Lorentzian signatures, exactly matched those of loop quantum gravity.[1]

However, the issue of the finiteness of the Lorentzian LQG spin-foam vertices was not addressed in the papers [10,11]. We address the issue in this present paper. In this paper we show that Lorentzian LQG spin-foam vertices possess an overall multiplicative factor equal to the volume of $SL(2,\mathbb{C})$, so that without regularization, the vertices are indeed infinite. However, in computing expectation values of quantities, these volume factors will just cancel. Therefore, one can simply drop the overall volume factor in the vertex. Such a regularization can be independently justified via a gauge-fixing interpretation, similar to but different from that in [15]; this is discussed in Appendix C. We prove that with this regularization, the vertex is finite. In both proposing the regularization and proving finiteness, an explicit form of the LQG vertices derived in Sec. II is key. This new form has formal similarities to the Barrett-Crane Lorentzian vertex, which allows some of the reasoning of [16] to be used also for proving the finiteness of the LQG vertices. Adaptations of the relevant arguments from [16] are summarized in the form of a lemma and two theorems in Appendix A. The rest of the proof of finiteness of the LQG vertices is then presented in Sec. IV in the main text.

For both the Euclidean and Lorentzian LQG models, there still remains the issue of finiteness of the full state sum for a fixed triangulation. We leave this issue for future work. We would like to remark that the analysis presented here does not directly apply to the Lorentzian model proposed in [7]; nevertheless it could provide a foundation for analysis of the Lorentzian model in [7].

The paper is organized as follows. We start with a brief review of the Lorentzian LQG spin-foam models for finite γ, starting from a triangulation with spacelike tetrahedra; in so doing, a more concrete and explicit approach is taken than that originally taken in [11]. This then aides in suggesting the regularization, proposed in the subsequent section. In the last section we prove finiteness, in part reusing reasoning from the paper [16]. Finally, we close with a summary and brief discussion. Appendix C presents the gauge-fixing interpretation of the regularization.

II. REVIEW OF THE MODELS

Here we review the classical discrete theory, canonical quantization of the constraints, and path-integral dynamics from [11], deriving a new expression for the vertex in the process.

A. Classical discrete theory

The starting point is a Regge-like discretization of first order gravity in the Plebanski formulation, in which gravity is obtained by adding the so-called simplicity constraints to BF theory [3,14]. The continuum variables are an $SL(2,\mathbb{C})$ connection ω and an $\mathfrak{sl}(2,\mathbb{C})$-valued two-form B on space-time; the simplicity constraints ensure that B is of the form $B = {}^\star e \wedge e$ for some cotetrad e.

To construct the Regge-like discretization, introduce a triangulation Δ of space-time $\mathcal{M}$ by (oriented) 4-simplices. We will denote typical 4-simplicies, tetrahedra,

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[1]This is not true for all the models proposed in the above works; see [7,9].

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and triangles in Δ respectively by v, t, and f. The basic variables then consist of an $SL(2,\mathbb{C})$ group element $V_{vt}(\equiv V_{tv}^{-1})$ for each 4-simplex v and tetrahedron t therein, and an $\mathfrak{sl}(2,\mathbb{C})$ algebra element $B_f(t)$ for each tetrehedron t and triangle f therein. V_{vt} is to be heuristically understood as the parallel transport map, determined by ω, from the tetrahedron t to the center of the 4-simplex v. $B_f(t)$ is to be understood as the integral of $B = {}^\star e \wedge e$ on the triangle f, in the frame at t. (For details, see [6,9,11].)

It is convenient to furthermore define, for each triangle f and each pair of tetrahedra t, $t' \in \mathrm{Link}(f)$,

$$U_f(t,t') := V_{tv_1}V_{v_1t_1}V_{t_1v_2}\cdots V_{v_nt'},$$

where the product is around the link in the clockwise direction from t' to t. The constraints on the variables are then

(1) $U_f(t,t')B_f(t') = B_f(t)U_f(t,t') \quad \forall\ f$ and $t,\ t' \in \mathrm{Link}(f)$.

(2) (Discrete simplicity constraint)
$\exists$ is an assignment of a timelike n_t^I to each t, such that

$$C_{ft}^I := n_{tJ}({}^\star B_f)^{JI} \approx 0 \quad \forall\ f \in t. \tag{1}$$

Note that the simplicity constraint (1) implies *both* of the more traditional simplicity constraints, the so-called diagonal simplicity constraint $\mathrm{tr}[({}^\star B_f(t))B_f(t)] \approx 0$ and cross-simplicity constraints $\mathrm{tr}[({}^\star B_f(t))B_{f'}(t)] \approx 0\ \forall\ f,\ f' \in t$.[2] In the prior literature [5,6,11], the fact that (1) implies the diagonal simplicity constraint as well the cross-simplicity constraint was overlooked, so that diagonal simplicity was imposed separately. We will see that these observations extend to the quantum theory as well: the quantum version of (1), appropriately understood, will be seen to already contain within it the quantum diagonal simplicity constraint. Additionally, as noted in [11], (1) is able to distinguish between the $B = \pm{}^\star e \wedge e$ and $B = \pm e \wedge e$ sectors of the Plebanski theory, selecting only the first of these sectors.

The above constraints are incorporated as follows: (1) is imposed prior to varying the action, while (2) is first solved canonically and then the result inserted in the path integral.[3]

Next consider a 3-surface Σ consisting of tetrahedra in the triangulation Δ; call this triangulation Δ_3. Let γ_Σ denote the graph dual to Δ_3. We will denote typical links and nodes in γ_Σ by ℓ, n, respectively. The canonical phase space Γ_Σ associated with Σ is then labeled by the basic variables $B_\ell(n) \in \mathfrak{so}(3,1)$, $U_\ell(n,n') \in SL(2,\mathbb{C})$. Define the array of $\mathfrak{sl}(2,\mathbb{C})$ matrices $\tau^{IJ} = -\tau^{JI}$ by

$$\tau^{i0} = \frac{1}{2}\sigma^i, \qquad \tau^{ij} = \frac{-i}{2}\epsilon^{ij}{}_k\sigma^k. \tag{2}$$

Define

$$J_\ell(n) = \frac{1}{16\pi G}\left(B_\ell(n) + \frac{1}{\gamma}{}^\star B_\ell(n)\right).$$

The nonzero Poisson brackets are then given by

$$\{J_\ell(n')^{IJ}, U_\ell(n,n')\} = U_\ell(n,n')\tau^{IJ}, \tag{3}$$

$$\{J_\ell(n)^{IJ}, U_\ell(n,n')\} = \tau^{IJ}U_\ell(n,n'), \tag{4}$$

$$\{J_\ell(n)^{IJ}, J_\ell(n)^{KL}\} = \lambda^{[IJ][KL]}{}_{[MN]}J_\ell(n)^{MN}, \tag{5}$$

where $\lambda^{[IJ][KL]}{}_{[MN]}$ denotes the structure constants in the basis τ^{IJ}.

B. Quantization

The quantization of Γ_Σ leads us to the kinematical space of states

$$\mathcal{H}_\Sigma = L^2(SL(2,\mathbb{C})^{|L(\gamma_\Sigma)|}). \tag{6}$$

Let $\hat{J}_\ell(n)^{IJ}$ denote the right-invariant vector fields, determined by the basis τ^{IJ} of $\mathfrak{sl}(2,\mathbb{C})$, on the copy of $SL(2,\mathbb{C})$ associated with the link ℓ, with orientation such that the node n is the source of ℓ. The $B_\ell(n)$'s are then represented by

$$\hat{B}_\ell(n) := 16\pi G\left(\frac{\gamma^2}{\gamma^2+1}\right)\left(\hat{J}_\ell(n) - \frac{1}{\gamma}{}^\star\hat{J}_\ell(n)\right). \tag{7}$$

As in [11], to solve the simplicity constraint, we gauge fix the normal n_t^I in (1) to be $n_t^I \equiv n^I := (1,0,0,0)$. The simplicity constraints (1) are then imposed by appropriately quantizing the "master constraint"

$$M_{n\ell} := \sum_{i=1}^{3}[({}^\star B_f(t))^{0i}]^2 \tag{8}$$

associated to each node n and incident link ℓ. To quantize and solve this, it is convenient to introduce a basis of $\mathcal{H}_\Sigma$ adapted to the constraint. First, we recall that if $\mathcal{H}_{N,\rho}$ is the carrying space for a Lorentz irreducible representation (irrep) (N,ρ) in the principal series, one can decompose $\mathcal{H}_{N,\rho}$ into irreps of the $SU(2)$ subgroup preserving n^I, arriving at

$$\mathcal{H}_{N,\rho} = \oplus_{k\geq N/2}\mathcal{H}_k, \tag{9}$$

where $\mathcal{H}_k$ is the carrying space for the spin k $SU(2)$ irrep appearing in the decomposition. Using this, we construct a basis of generalized $SL(2,\mathbb{C})$ spin networks; specifically, these will be the projected spin networks of [17] with the

[2]Thanks to Laurent Freidel for pointing out this fact about the classical theory.

[3]The closure constraint, $\sum_{f\in t}B_f(t) = 0$, which is dealt with more directly in other presentations, is exactly recovered only at the quantum level, by the integration over connection variables in the path-integral dynamics.

normal gauge fixed to be n^I. Given an assignment of a Lorentz irrep (N_ℓ, ρ_ℓ) in the principal series to each link, an $SU(2)$ spin $k_{n\ell}$ for each specification of a node and incident link, and an $SU(2)$ intertwiner i_n among the four $SU(2)$ irreps $\{k_{n\ell}\}_{\ell\in n}$ at n, we define

$$\Psi_{\{N_\ell,\rho_\ell;k_{n\ell},i_n\}}(U_\ell) \equiv \langle U_\ell \mid \{N_\ell, \rho_\ell; k_{n\ell}, i_n\}\rangle := (\bigotimes_\ell D^{(N_\ell,\rho_\ell)}(U_\ell) \cdot \bigotimes_n [(\otimes_{\ell\in n} P_{k_{n\ell}}) \otimes i_n]), \quad (10)$$

where P_k is the projector onto the spin-k component in the decomposition (9). Note in this expression that at each node $[(\otimes_{\ell\in n} P_{k_{n\ell}}) \otimes i_n]$ is a tensor in $\otimes_{\ell\in n}\mathcal{H}_{(N_\ell,\rho_\ell)}$; the role of the labels $\{k_{\ell n}\}_{\ell\in n}$ and i_n at each node is to specify a tensor among the four Lorentz irreps on the adjacent edges. Contracting these all together gives the desired generalized $SL(2,\mathbb{C})$ spin network (10). Note, in particular, that even though $SL(2,\mathbb{C})$ representations in the principal series are infinite dimensional, the incorporation of the projection operators P_k in (9) ensures that all contractions involve effectively only finite sums, so that the right-hand side of (10) is guaranteed to be finite. Finally, let $\hat{L}^i_{\ell n} := \frac{1}{2}\epsilon^i{}_{jk}\hat{J}^{jk}_\ell(n)$. The basis (10) then diagonalizes the operators

$$(\hat{J}_\ell \cdot \hat{J}_\ell)\Psi_{\{N_\ell,\rho_\ell;k_{n\ell},i_n\}} = \tfrac{1}{2}(N_\ell^2 - \rho_\ell^2 - 4)\Psi_{\{N_\ell,\rho_\ell;k_{n\ell},i_n\}},$$
$$(\hat{J}_\ell \cdot {}^\star\hat{J}_\ell)\Psi_{\{N_\ell,\rho_\ell;k_{n\ell},i_n\}} = N_\ell\rho_\ell\Psi_{\{N_\ell,\rho_\ell;k_{n\ell},i_n\}},$$
$$\hat{L}^2_{n\ell}\Psi_{\{N_\ell,\rho_\ell;k_{n\ell},i_n\}} = k_{n\ell}(k_{n\ell}+1)\Psi_{\{N_\ell,\rho_\ell;k_{n\ell},i_n\}}.$$

In terms of this basis, the master constraint (8), quantized as in [11], is

$$\hat{M}_{n\ell}\Psi_{\{N_\ell,\rho_\ell;k_{n\ell},i_n\}} = \left[\left(1+\frac{1}{\gamma^2}\right)(k^2_{n\ell} - (N_\ell/2)^2) + \frac{1}{4\gamma^2}(\rho_\ell - \gamma N_\ell)^2\right]\Psi_{\{N_\ell,\rho_\ell;k_{n\ell},i_n\}}. \quad (11)$$

As $k_{n\ell} \geq N_\ell/2$, solving this constraint forces both of the terms on the right-hand side to separately vanish. In all, simplicity thus implies

$$k_{\ell_-\ell} = \frac{N_\ell}{2} = \frac{\rho_\ell}{2\gamma} = k_{\ell_+\ell} \quad (12)$$

for all ℓ, and where ℓ_-, ℓ_+ denotes, respectively, the source and target of ℓ. Because $k_{n\ell}$ is the quantum number for the non-Lorentz scalar quantity $\hat{L}^2_{n\ell}$, this is not an $SL(2,\mathbb{C})$ invariant equation. This lack of $SL(2,\mathbb{C})$ invariance derives from the gauge fixing of n^I and will be relevant in Appendix C.

C. Path-integral dynamics

Consider the case when Δ consists of a single 4-simplex, and let γ denote its boundary graph. The vertex amplitude is derived as the amplitude for a generalized spin-network state on the boundary. We begin by writing down the amplitude for the BF theory, reflecting the flatness equation of motion present in the BF theory:

$$A[U_{tt'}] = \int dV_{vt}\Pi_{tt'}\delta(U_{tt'}V_{t'v}V_{vt}), \quad (13)$$

multiplying by a generalized spin network (10) and integrating over the $U_{tt'}$ with the Haar measure leads to the amplitude for a generalized spin network on the boundary of a single 4-simplex v

$$A_v[N_f, \rho_f; k_{tf}, i_t] = \int dV_{vt}\Psi^{4\text{-simplex}}_{\{N_f,\rho_f;k_{tf},i_t\}}(V_{tv}V_{vt'}) = (P^{\text{gauge}}_{SL(2,\mathbb{C})}\Psi^{4\text{-simplex}}_{\{N_f,\rho_f;k_{tf},i_t\}})|_{\text{Triv.Conn.}} \equiv 15 j_{SL(2,\mathbb{C})}(N_f, \rho_f; I^t(k_{tf}, i_t)), \quad (14)$$

where $I^t(k_{tf}, i_t) := P^t_{SL(2,\mathbb{C})}[(\otimes_{f\in t}P_{k_{ft}}) \otimes i_t]$, and where $P^t_{SL(2,\mathbb{C})}$ denotes a formal group averaging over $SL(2,\mathbb{C})$ gauge transformations at t. At each t, $I^t(k_{tf}, i_t)$ is thus formally an $SL(2,\mathbb{C})$ intertwiner.

Combining with the simplicity constraints, we obtain an $SU(2)$ LQG spin-foam model with partition function

$$Z_{\text{GR}} = \sum_{j_f,i_t}\prod_f(2j_f)^2(1+\gamma^2)\prod_v A_v(j_f, i_t) \quad (15)$$

with $j_f \in \mathbb{N}/2$ and

$$A_v(j_f, i_t) := A_v[2j_f, 2\gamma j_f; j_f, i_t] = 15 j_{SL(2,\mathbb{C})}(2j_f, 2\gamma j_f; I^t(j_f, i_t)). \quad (16)$$

This is the vertex amplitude for general γ, as in [11]. Setting $\gamma = 0$ gives the flipped Lorentzian model [10].

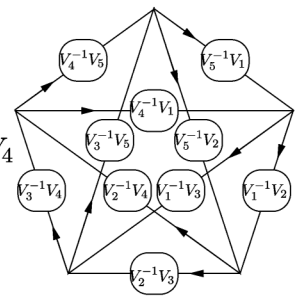
III. REGULARIZATION

Is the vertex (16) and/or (14) finite? The answer is no. However, it is not hard to see that an observation similar to that in [13] can be used to regularize it: the vertex consists of an integral over five copies of the group, but one of these is redundant. That is, if we perform any four of the five integrals, the result is independent of the fifth integration variable, so that the last integral is redundant.[4]

To demonstrate, number the tetrahedra 1, 2, 3, 4, 5, and label the 5 group integration variables as V_1, V_2, V_3, V_4, V_5. Dropping the fifth integration, we symbolically write

[4]This comes from a $SL(2,\mathbb{C})$ gauge invariance acting at each vertex; see Appendix C.

$$A_v^{\mathrm{reg}}[N_f, \rho_f; k_{tf}, i_t] := \int dV_1 dV_2 dV_3 dV_4$$



$$. \tag{17}$$

Performing the change of variables

$$V_1 \mapsto \tilde{V}_1 = V_5^{-1} V_1, \tag{18}$$

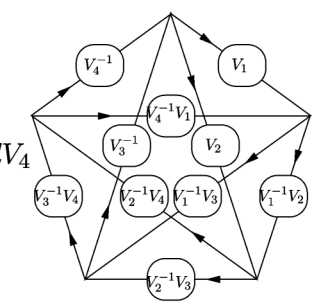
$$V_2 \mapsto \tilde{V}_2 = V_5^{-1} V_2, \tag{19}$$

$$V_3 \mapsto \tilde{V}_3 = V_5^{-1} V_3, \tag{20}$$

$$V_4 \mapsto \tilde{V}_4 = V_5^{-1} V_4, \tag{21}$$

using the right invariance of the Haar measures, and noting that for $i, j = 1, \ldots, 4$, $\tilde{V}_i^{-1}\tilde{V}_j = V_i^{-1}V_j$, the expression (17) simplifies to

$$A_v^{\mathrm{reg}}[N_f, \rho_f; k_{tf}, i_t] = \int dV_1 dV_2 dV_3 dV_4$$



$$, \tag{22}$$

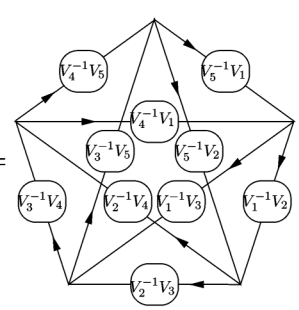
which is manifestly independent of the unintegrated group element V_5. Thus, the last integral, when performed, simply introduces a factor equal to the volume of $SL(2, C)$, which is infinite. We regularize it by simply dropping the last integral, which, depending on how the V's are numbered, could be any one of the five integrals. The result is independent of which one you drop. Incorporating the simplicity constraints in the form (12), we thus propose

$$A_v^{\mathrm{reg}}(j_f, i_t) := A_v^{\mathrm{reg}}[2j_f, 2\gamma j_f; j_f, i_t]. \tag{23}$$

IV. PROOF OF FINITENESS

For each set of labels $\{\rho_f, N_f; k_{tf}, i_t\}$ determining a projected spin network on the 4-simplex boundary graph, let $\tilde{F}_{\rho_f, N_f; k_{tf}, i_t}\colon SL(2, \mathbb{C})^5 \to \mathbb{C}$ denote the integrand in the definition of the associated vertex (17). That is, define

$$\tilde{F}_{\rho_f, N_f; k_{tf}, i_t}(V_1, V_2, V_3, V_4, V_5) :=$$



$$\tag{24}$$

where the schematic on the right-hand side represents the projected spin network with its 10 $SL(2,\mathbb{C})$ arguments. Next, every element $V \in SL(2,\mathbb{C})$ can be decomposed

$$V = B(x)R \tag{25}$$

for some $R \in SU(2)$ and some boost $B(x)$. We here parametrize the boosts by a point x in the hyperboloid H of future directed unit timelike vectors in Minkowski space, with $B(x)$ denoting the unique boost mapping $e := (1,0,0,0)$ to x. Decomposing each of the arguments of $\tilde{F}$ in the manner (25), and using the $SU(2)$ invariance of the projected spin network on the right-hand side of (24) to drop the rotations,

$$\tilde{F}_{\rho_f,N_f;k_{tf},i_t}(V_1,\ldots,V_5) = \tilde{F}_{\rho_f,N_f;k_{tf},i_t}(B(x_1),\ldots,B(x_5)). \tag{26}$$

Next, for each ρ, N, $k \geq N/2$, and $m_\pm \in \{-k, -k+1, \ldots, k\}$, define

$$K^{\rho,N}_{km,k'm'}(x_1,x_2) := D^{\rho,N}_{km,k'm'}(B(x_1)^{-1}B(x_2)). \tag{27}$$

Let us label faces f in v by the two tetrahedra (tt') they bound, so that each f is labeled by an unordered pair of numbers 1, 2, 3, 4, 5. Furthermore, let us label pairs $(t,f) \in v$ by *ordered* pairs of tetrahedra $(t,(tt')) =: tt'$, and hence ordered pairs of numbers 1, 2, 3, 4, 5. For each assignment of $\rho_{(tt')}$, $N_{(tt')}$ to faces $f \in v$ and $k_{tt'}$, $m_{tt'}$ to pairs $(t,f) \equiv (t,(tt')) \in v$, define

$$[F_{\rho_{(tt')},N_{(tt')};k_{tt'}}]_{\{m_{tt'}\}} := \int dx_1 dx_2 dx_3 dx_4 \times \prod_{\substack{t,t'\in\{1,2,3,4,5\}\\ t<t'}} K^{\rho_{(tt')},N_{(tt')}}_{k_{tt'}m_{tt'},k_{t't}m_{t't}}(x_t,x_{t'}), \tag{28}$$

where dx denotes the volume form on the hyperboloid. One can check that in terms of the decomposition (25), the Haar measure decomposes as

$$dV = dRdx, \tag{29}$$

where dR is the Haar measure on $SU(2)$. Using this equation, one sees the vertex is equal to

$$A^{\rm reg}_v[\rho_f,N_f;k_{ft},i_t] = [\otimes_{t\in v} i_t]\cdot[F_{\rho_f,N_f;k_{ft}}]. \tag{30}$$

Because the contraction sums on the right-hand side are all finite, it is sufficient to prove finiteness of the elements $[F_{\rho_f,N_f;k_{ft}}]_{\{m_{(tf)}\}}$ in order to prove finiteness of the vertex. We shall do this, using arguments very similar to [16].

Let us now look at the boosts entering (28). First we rewrite the composition of two boosts $B(x_1)^{-1}$, $B(x_2)$ as

$$B(x_1)^{-1}B(x_2) = R(x_1,x_2)B_z(r(x_1,x_2))R'(x_1,x_2) \tag{31}$$

for some two rotations $R(x_1,x_2)$, $R(x_1,x_2)$ and boost $B_z(r(x_1,x_2))$ in the z direction, where $r(x_1,x_2)$ denotes the rapidity of the boost. We can always choose this decomposition such that $r(x_1,x_2)$ is positive, and we do so. $r(x_1,x_2)$ is in fact the hyperbolic distance between x_1 and x_2. To see this, we recall that the hyperbolic distance, or hyperbolic angle, between two points x_1, $x_2 \in H$ is defined by

$$d(x_1,x_2) := \cosh^{-1}(x_1,x_2), \tag{32}$$

where $(\cdot,\cdot)$ denotes the Minkowski metric. We thus have

$$\begin{aligned}\cosh d(x_1,x_2) &= (x_1,x_2) = (B(x_1)e, B(x_2)e)\\ &= (e, B(x_1)^{-1}B(x_2)e)\\ &= (e, R(x_1,x_2)B_z(r(x_1,x_2))R'(x_1,x_2)e)\\ &= (R(x_1,x_2)^{-1}e, B_z(r(x_1,x_2))R'(x_1,x_2)e)\\ &= (e, B_z(r(x_1,x_2))e) = \cosh r(x_1,x_2)\end{aligned} \tag{33}$$

so that $r(x_1,x_2) = d(x_1,x_2)$, proving $r(x_1,x_2)$ is the hyperbolic distance, as claimed.

Now, let us consider the matrix elements of $B_z(r)$ in a given representation (ρ,N) in the principal series. We use the canonical basis for the carrying space, i.e., the basis diagonalizing $\hat{L}^2$ and $\hat{L}_z$: $\hat{L}^2|k,m\rangle = k(k+1)|k,m\rangle$, $\hat{L}_z|k,m\rangle = m|k,m\rangle$. Because the generator $K_z := J^{03}$ of z boosts commutes with $\hat{L}_z$, we have

$$D^{\rho,N}_{km,k'm'}(B_z(r)) = \delta_{mm'}d^{\rho,N}_{kk'm}(r) \tag{34}$$

for some function $d^{\rho,N}_{kk'm}(r)$. As shown in Appendix B, the behavior of $d^{\rho,N}_{kk'm}(r)$ in the $r\to\infty$ limit is of the form

$$d^{\rho,N}_{kk'm}(r) \propto e^{-\lambda_{N,m}r}, \tag{35}$$

where

$$\lambda_{N,m} = 1 + \left|m + \frac{N}{2}\right| \geq 1. \tag{36}$$

This, in particular, implies that for any $\epsilon > 0$,

$$\lim_{r\to\infty} e^{(1-\epsilon)r}d^{\rho,N}_{kk'm}(r) = 0. \tag{37}$$

Because $e^{(1-\epsilon)r}d^{\rho,N}_{kk'm}(r)$ is furthermore continuous, we know $e^{(1-\epsilon)r}d^{\rho,N}_{kk'm}(r)$ is bounded on $r\in[0,\infty)$, so that there exists $C^{\rho,N,\epsilon}_{kk'm} \in \mathbb{R}^+$ such that

$$e^{(1-\epsilon)r}d^{\rho,N}_{kk'm}(r) < C^{\rho,N,\epsilon}_{kk'm} \Rightarrow d^{\rho,N}_{kk'm}(r) < C^{\rho,N,\epsilon}_{kk'm}e^{-(1-\epsilon)r} \tag{38}$$

for all $r\in[0,\infty)$.

Consider next the matrix elements of the rotations in (31). From p. 63 in [18], for R a rotation,

$$D^{\rho,N}_{km,k'm'}(R) = \delta_{kk'} D^{k}_{mm'}(R) \tag{39}$$

where $D^{k}_{mm'}(R)$ are the matrix elements in the spin-k representation of $SU(2)$. Because matrix elements on the right-hand side are in a *unitary* representation, and we are using an orthonormal basis, all of these matrix elements have an absolute value less than or equal to 1.[5]

Let us put the above observations together. From (31), (34), and (39), we have

$$D^{\rho,N}_{km,k'm'}(B(x_1)^{-1}B(x_2)) = \sum_{m''\in\{-\min\{k,k'\},-\min\{k,k'\}+1,\ldots,\min\{k,k'\}\}} D^{k}_{mm''}(R) d^{\rho,N}_{kk'm''}(r(x_1,x_2)) D^{k'}_{m''m'}(R'), \tag{40}$$

where the one sum has been made explicit. We then have

$$\begin{aligned} |D^{\rho,N}_{km,k'm'}(B(x_1)^{-1}B(x_2))| &\le \sum_{m''} |D^{k}_{mm''}(R) d^{\rho,N}_{kk'm''}(r(x_1,x_2)) D^{k'}_{m''m'}(R')| = \sum_{m''} |D^{k}_{mm''}(R)||d^{\rho,N}_{kk'm''}(r(x_1,x_2))||D^{k'}_{m''m'}(R')| \\ &\le \sum_{m''} |d^{\rho,N}_{kk'm''}(r(x_1,x_2))| < \left(\sum_{m''} C^{\rho,N,\epsilon}_{kk'm''}\right) e^{-(1-\epsilon)r}. \end{aligned} \tag{41}$$

Defining $C^{\rho,N,\epsilon}_{kk'} := \sum_m C^{\rho,N,\epsilon}_{kk'm}$, which is finite because the sum is finite, we thus have

$$|K^{\rho,N}_{kk',mm'}| = |D^{\rho,N}_{km,k'm'}(B(x_1)^{-1}B(x_2))| < C^{\rho,N,\epsilon}_{kk'} e^{-(1-\epsilon)r} \tag{42}$$

for all $r \in [0,\infty)$. This bound (42), given the expression (28) for $[F_{\rho_f,N_f;k_{ft}}]_{m_{ft}}$, allows us to adapt the arguments of Baez and Barrett in [16] to show that $[F_{\rho_f,N_f;k_{ft}}]_{m_{ft}}$ is finite.

Let us summarize how the arguments of Baez and Barrett can be used. The bound (42) is the analog of Lemma 1 in [16]. Lemma 2, 3, and 4 in [16] can be used again without change. One can prove the analog of Lemma 5, and Theorems 2 and 3 in [16] using logic analogous to that in [16]. For completeness, we present these analogs in Appendix A. The desired finiteness of $[F_{\rho_f,N_f;k_{ft}}]_{m_{ft}}$ then comes as a corollary. As was already noted, this in turn is then sufficient to prove that the vertex amplitude (30) is finite for all labels $\{\rho_f, N_f, k_{ft}, i_t\}$ on the 4-simplex graph. Note this finiteness of the vertex holds *even prior* to imposing the simplicity constraints (12); nevertheless it is the case when (12) is satisfied that ultimately concerns us.[6] Note this proves finiteness of the vertex for all finite gamma, as well as for the flipped case.

[5]To see that this is true for a general unitary matrix U, and orthonormal basis x_i, from $UU^\dagger = \mathbb{1}$ we have $\langle x_i, UU^\dagger x_j\rangle = \delta_{ij}$, so that for $i = j$ we have

$$\begin{aligned} 1 = \langle x_i, UU^\dagger x_i\rangle &= \sum_k \langle x_i, Ux_k\rangle\langle x_k, U^\dagger x_i\rangle \\ &= \sum_k \langle x_i, Ux_k\rangle\overline{\langle x_i, Ux_k\rangle} = \sum_k |\langle x_i, Ux_k\rangle|^2 \end{aligned}$$

from which $|\langle x_i, Ux_k\rangle|^2 \le 1$.

[6]The finiteness in the unconstrained case may be useful for defining a Lorentzian BF-theory model; but in a BF-theory model, one will be summing over all possible k_{tf} in the state sum, which is an infinite sum for each pair (tf).

V. DISCUSSION

In this paper, by writing the Lorentzian vertex of [11] in a more concrete manner, we were able to see a natural way to regularize the vertex. We then proved the vertex, so regularized, is finite.

We close with a remark concerning the finiteness of the state sum. In order to prove finiteness of the state sum, one would need the explicit evaluation of the constants $C^{\rho,N,\epsilon}_{kk'}$ in (42) as functions of the representation labels (see Appendix B). However, we leave this for further investigation.

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APPENDIX A: PROOF OF FINITENESS FROM THE MATRIX ELEMENT BOUND

Throughout this Appendix we will use the notion of integrability of what we call a labeled graph. Given a graph Γ, we assign a principal series representation (ρ_ℓ, N_ℓ) to each link ℓ, and to each pair (n, ℓ) of a node and incident link, we assign an $SU(2)$ spin $k_{n\ell}$ and a half-integer $m_{n\ell} \in \{-k_{n\ell}, -k_{n\ell}+1, \ldots, k_{n\ell}\}$. The graph Γ, together with the labels ρ_ℓ, N_ℓ, $k_{n\ell}$, and $m_{n\ell}$ which we collectively denote by Ξ, is what we call a " labeled graph." Given such a labeled graph (Γ, Ξ), choose an arbitrary node t_* in Γ, and number the nodes in Γ, starting with $t_*, 1, \ldots, M$ for convenience. As in Sec. IV, denote links by the unordered pair (ij) of numbers corresponding to the nodes at either end, and let ordered pairs ij of adjacent nodes denote the choice of a

node i and link (ij) incident on it. Then (Γ, Ξ) is said to be an integrable graph if the following quantity is finite:

$$F^{\Gamma}(\Xi) := \left[\prod_{i=1}^{N}\int_{x_i\in H} dx_i\right]\prod_{\substack{i,j\in\{1,\ldots,M\}\\ i<j}} K^{\rho_{(ij)},N_{(ij)}}_{k_{ij}m_{ij},k_{ji}m_{ji}}(x_i, x_j), \quad \text{(A1)}$$

where $K^{\rho_{(ij)},N_{(ij)}}_{k_{ij}m_{ij},k_{ji}m_{ji}}(x_i, x_j)$ is defined as in (27). Note Eq. (28) is a special case of Eq. (A1) when Γ is the boundary of a 4-simplex.

We prove in this Appendix the analogs of Lemma 5 and Theorem 2 of [16]. Although not all of the analog of Theorem 3 of [16] is needed for this paper, we state it in full as well, for completeness, though without proof, as the proof is an immediate adaptation of that in [16].[7]

The importance of Lemma 5 is twofold. In the first place, it is important in the proof of Theorem 2, which states that the tetrahedron graph is integrable. Secondly, and more importantly, it guarantees that, given an integrable graph, every other graph constructed from it by adding a node with at least three legs will also be integrable. This is the first part of Theorem 3. These two conclusions then imply that the 4-simplex graph is integrable, which we state as a corollary. Notice the full content of Theorem 3 in fact proves integrability for a much larger class of graphs. The integrability of these more general graphs may be useful, e.g., for defining versions of the new spin-foam models in which polyhedra more general than 4-simplices are allowed.

(Analog of) Lemma 5.—If $n \geq 3$, the integral

$$J := \int_H dx |K^{\rho_1,N_1}_{k_1k_1',m_1m_1'}(x, x_1)| \cdots |K^{\rho_n,N_n}_{k_nk_n',m_nm_n'}(x, x_n)|$$

converges and for any $0 < \epsilon < 1/3$ there exists $C^{\epsilon}(\{\rho_i, N_i, k_i, k_i'\})$, $i = 1 \cdots n$, function of the representation labels, such that for any $(x_1, \ldots, x_n)$,

$$J \leq C^{\epsilon}(\{\rho_i, N_i, k_i, k_i'\}) \exp\left(-\frac{n-2-n\epsilon}{n(n-1)}\sum_{i<j} r_{ij}\right),$$

where $r_{ij} := d(x_i, x_j)$.

Proof.—First, using (42) one has

$$|K^{\rho_1,N_1}_{k_1k_1',m_1m_1'}(x, x_1)| \cdots |K^{\rho_n,N_n}_{k_nk_n',m_nm_n'}(x, x_n)| \leq \left(\prod_{i=1}^{n} C^{\rho_i,N_i,\epsilon}_{k_ik_i'}\right) e^{-(1-\epsilon)\sum r_i}, \quad \text{(A2)}$$

where $r_i := d(x, x_i)$. Define $\tilde{C} := \prod_{i=1}^{n} C^{\rho_i,N_i,\epsilon}_{k_ik_i'}$, and then one has

$$J \leq 4\pi\tilde{C}\int_0^{\infty} \sinh^2 r dr e^{-(1-\epsilon)\sum r_i}, \quad \text{(A3)}$$

where r is defined as the distance of x from the bary center of the points $(x_1, \ldots, x_n)$. The fact that it exists is the object of Lemma 4 in [16]. From the same lemma, one has

$$\sum r_i \geq nr. \quad \text{(A4)}$$

In addition, defining

$$M := \frac{1}{n}\min_x \sum_i r_i(x), \quad \text{(A5)}$$

one has

$$\sum r_i \geq nM. \quad \text{(A6)}$$

Both inequalities can be used to prove the following bound for $\mathbf{J}$:

$$J \leq 4\pi\tilde{C}C' e^{-(n-2-n\epsilon)M}, \quad \text{(A7)}$$

for some positive constant C' depending only on ϵ and n. From the triangle inequality, one has

$$\sum r_i \geq \frac{1}{n-1}\sum_{i<j} r_{ij}, \quad \text{(A8)}$$

and

$$M \geq \frac{1}{n(n-1)}\sum_{i<j} r_{ij}, \quad \text{(A9)}$$

which then implies the lemma with $C = 4\pi\tilde{C}C'$. ■

(Analog of) Theorem 2.— The tetrahedron graph, with any labeling, is integrable.

Proof.—We will show that the following quantity (for any fixed $x_1 \in H$ and independent of it) is finite:

$$I := \int_{H^3} dx_2dx_3dx_4 |K^{\chi_{12}}(x_1, x_2)K^{\chi_{13}}(x_1, x_3)K^{\chi_{14}}(x_1, x_4) \times K^{\chi_{23}}(x_2, x_3)K^{\chi_{24}}(x_2, x_4)K^{\chi_{34}}(x_3, x_4)|, \quad \text{(A10)}$$

where χ_{ij} denotes, for short, the set of labels $(\rho_{(ij)}, N_{(ij)}, k_{ij}, m_{ij}, k_{ji}, m_{ji})$. Start by integrating over x_4 using Lemma 5,

$$I \leq C^{\epsilon}(\{\chi_{ij}\}) \int_{H^2} dx_2dx_3 e^{-(1/6)(1-3\epsilon)(r_{12}+r_{13}+r_{23})} \times |K^{\chi_{12}}(x_1, x_2)K^{\chi_{13}}(x_1, x_3)K^{\chi_{23}}(x_2, x_3)|, \quad \text{(A11)}$$

where $r_{ij} = d(x_i, x_j)$. Next, we integrate over x_3. Consider the quantity

$$L := \int_H dx_3 e^{-(1/6)(1-3\epsilon)(r_{13}+r_{23})} |K^{\chi_{13}}(x_1, x_3)K^{\chi_{23}}(x_2, x_3)|. \quad \text{(A12)}$$

[7] The parts of the argument of [16] involving the mathematical details of the "propagator" [Eq. 3 in [16]] are entirely encapsulated in Lemma 5 and Theorem 2. In proving analogs of the results of [16] for the present case, the only difference is that the relevant propagator is now Eq. (27) of this paper. This is why only the analogs of Lemma 5 and Theorem 2 need to be fully reproven here.

By (42), one has

$$L \leq \tilde{C} \int_H dx_3 e^{-(r_{13}+r_{23})[(7/6)-(3\epsilon/2)]}. \tag{A13}$$

Now, introduce the new coordinate system (k, l, ϕ), where

$$k = \tfrac{1}{2}(r_{13} + r_{23}); \qquad l = \tfrac{1}{2}(r_{13} - r_{23}), \tag{A14}$$

and ϕ is the angle between the plane containing x_1, x_2, x_3 and a given plane containing x_1 and x_2. Their ranges are $k \in [\frac{r_{12}}{2}, \infty)$, $l \in [-\frac{r_{12}}{2}, \frac{r_{12}}{2}]$, and $\phi \in [0, 2\pi)$. The measure dx_3 on H in this coordinate system reads (see the Appendix of [16])

$$dx_3 = 2\frac{\sinh r_{13} \sinh r_{23}}{\sinh r_{12}} dk dl d\phi. \tag{A15}$$

In terms of these new coordinates, we have

$$\begin{aligned} L &\leq \frac{\tilde{C}}{\sinh r_{12}} \int_0^{2\pi} d\phi \int_{-(r_{12}/2)}^{(r_{12}/2)} dl \int_{(r_{12}/2)}^{\infty} dk e^{-k((1/3)-3\epsilon)} \\ &\leq \frac{2\pi r_{12}\tilde{C}}{\sinh r_{12}} e^{-r_{12}((1/6)-(3\epsilon/2))} \end{aligned} \tag{A16}$$

for $\epsilon < 1/9 < 1/3$. Plugging this in the evaluation of I, we get

$$I \leq C' \int dr r \sinh r e^{-r((4/3)-3\epsilon)} \leq C' \int dr r e^{-r((1/3)-3\epsilon)}, \tag{A17}$$

which is finite for $0 < \epsilon < 1/9$ and some constant C' depending on the representation labels $\{\chi_{ij}\}$. ■

(Analog of) Theorem 3.—A graph obtained from an integrable graph by connecting an extra vertex to the existing labeled graph by at least three edges, with arbitrary labeling, is integrable. A graph obtained from an integrable graph by adding extra edges, with arbitrary labeling, is integrable. A graph constructed by joining two disjoint integrable graphs at a vertex is integrable.

Using the analog of Lemma 5 above, the first assertion follows using the same arguments as in [16]. The second and third assertions follow using the same arguments as in [16].

Corollary 1.—The 4-simplex graph, with any labeling, is integrable.

APPENDIX B: USEFUL FACTS ABOUT THE LORENTZ GROUP

Let $V \in SL(2, \mathbb{C})$, then one has the following decomposition:

$$V = Rd(r)R', \tag{B1}$$

where $R, R' \in SU(2)$, and

$$d(r) = B_z(r) = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}. \tag{B2}$$

The Haar measure in this decomposition reads

$$dV = \frac{1}{4\pi} dR dR' \sinh^2 r dr. \tag{B3}$$

We complete this Appendix with some explicit formulas for the matrices $d^{\rho,N}_{kk'm}(r)$, referred to in the main text. In particular, we show that the asymptotic behavior (35) holds. We closely follow Secs. 4 and 5 of [18]. We start with the following useful expression:

$$\begin{aligned} d^{\rho,N}_{kk'm}(r) &= \{\cdots\}^{(1/2)}(2\sinh r)^{-k-k'} \\ &\quad \times \sum_{\nu,\mu} c_{\nu\mu} e^{\mu r} \frac{\sinh(r(i\rho/2+\nu))}{(i\rho/2+\nu)\sinh r}, \end{aligned} \tag{B4}$$

where $\nu + k'$ and $\mu + k'$ are integers, and

$$\begin{aligned} \{\cdots\} = \Big\{&(2k+1)(2k'+1) \\ &\times \frac{(k+\frac{N}{2})!(k-\frac{N}{2})!(k'+\frac{N}{2})!(k'-\frac{N}{2})!}{(k+m)!(k-m)!(k'+m)!(k'-m)!}\Big\}. \end{aligned} \tag{B5}$$

To define the coefficients $c_{\nu\mu}$, it is useful to redefine the summation labels $(\nu, \mu) \to (a, b)$, while introducing a new sum over integers (n_1, n_2):

$$2b = \mu + \nu + k' - k + \frac{N}{2} + 2n_1 - m, \tag{B6}$$

$$2a = \nu - \mu + k' + k + m - \frac{N}{2} - 2n_1. \tag{B7}$$

The sum over (ν, μ) can then be traded by a sum over (n_1, n_2, a, b):

$$\begin{aligned} \sum_{\nu\mu} c_{\nu\mu}(\cdots) = &\sum_{n_1,n_2} \binom{k+m}{n_1} \binom{k-m}{n_1 - m + \frac{N}{2}} \binom{k'+m}{n_2} \\ &\times \binom{k'-m}{n_2 - m + \frac{N}{2}} \sum_{a,b} (-1)^{a+b+m-(N/2)} \\ &\times \binom{k+k'-n_1-n_2+m-\frac{N}{2}}{a} \\ &\times \binom{n_1+n_2-m+\frac{N}{2}}{b} (\cdots), \end{aligned} \tag{B8}$$

where all summations extend over the domain where the binomial coefficients do not vanish. From Eq. (B4), one sees that the asymptotic behavior for $r \to \infty$ is of the form:

$$d^{\rho,N}_{kk'm}(r) \sim e^{-r(k+k'+1-(\mu+\nu)_{\max})}, \tag{B9}$$

for $(\mu + \nu)$ taking its maximal value. One can check that this maximal value is given by

$$(\mu + \nu)_{\max} = k + k' - \left| m + \frac{N}{2} \right| \tag{B10}$$

which then gives the asymptotic behavior

$$d^{\rho,N}_{kk'm}(r) \sim e^{-r[1+|m+(N/2)|]} \tag{B11}$$

as advertised in the main text. A last step, which is not necessary for the proof of finiteness but should be very useful for the finiteness analysis of the state-sum model, is the evaluation of the maximum of $e^{(1-\epsilon)r}d^{\rho,N}_{kk'm}(r)$, as a function of r. This would allow for the explicit expression of the coefficients $C^{\rho,N,\epsilon}_{kk'}$ in (42) in terms of the representation labels.

APPENDIX C: CONSIDERATION OF FULL TRIANGULATION AND THE GAUGE-FIXING INTERPRETATION OF THE REGULARIZATION

In the main text, for brevity, we did not derive the spinfoam sum from a discrete path integral on the full triangulation. Because of the new nature of the derivation—specifically the use of nongauge invariant tensors—the derivation of (15) using the full triangulation has a small difference from the standard derivation. We review this difference. With the derivation based on the full triangulation in mind, we then review the internal gauge-fixing procedure in [15], which is the standard procedure in lattice gauge theories [19]. We will see that this gauge-fixing procedure cannot be used in our case, but must be modified; the modified procedure will be equivalent to the regularization proposed in the main text.

Given the parallel transports V_{tv} around the link of a face f, let $U_f(t) := U_f(t,t)$ denote their composition in clockwise order starting at t. The discrete action [5,6,11] is

$$\begin{aligned} S_{\rm disc} &= \frac{1}{16\pi G}\sum_{f\in\Delta}\mathrm{tr}\Big(\big(B_f(t)+\frac{1}{\gamma}{}^\star B_f(t)\big)\Lambda[U_f(t)]\Big) \\ &= \sum_{f\in\Delta}\mathrm{tr}(J_f(t)\Lambda[U_f(t)]), \end{aligned} \tag{C1}$$

where Λ: $SL(2,\mathbb{C}) \to SO(3,1)$ denotes the standard 2-1 homomorphism. Next, for any four $SL(2,\mathbb{C})$ irreps $(\rho_4, N_4), \ldots, (\rho_4, N_4)$, we have the following resolution of the identity on $\mathcal{H}_{N_1,\rho_1}\otimes\cdots\otimes\mathcal{H}_{N_4,\rho_4}$:

$$\begin{aligned} \mathbb{1}^{(k_1,m_1)\cdots(k_4,m_4)}{}_{(k'_1,m'_1)\cdots(k'_4,m'_4)} &= \delta^{k_1}_{k'_1}\cdots\delta^{k_4}_{k'_4}\sum_{w\in\mathcal{B}_{\tilde k_1,\ldots,\tilde k_4}} w^{m_1\cdots m_4}(w^\dagger)_{m'_1\cdots m'_4} \\ &= \sum_{\tilde k_1,\ldots,\tilde k_4}\sum_{w\in\mathcal{B}_{\tilde k_1,\ldots,\tilde k_4}}[(\otimes^4_{i=1}P_{\tilde k_i})\otimes w]^{m_1\cdots m_4}[(\otimes^4_{i=1}P_{\tilde k_i})\otimes w]^\dagger{}_{m'_1\cdots m'_4}, \end{aligned} \tag{C2}$$

where $\mathcal{B}_{\tilde k_1,\ldots,\tilde k_4}$ is a fixed orthonormal basis of $\mathcal{H}_{\tilde k_1}\otimes\cdots\otimes\mathcal{H}_{\tilde k_4}$ for each 4-tuple of $SU(2)$ spins $\tilde k_1,\ldots,\tilde k_4$, and the P_k are as in Eq. (10). We then compute the partition function for (C1) using the same strategy as in [3], except using the resolution of the identity on the full tensor space, instead of just on the intertwiner space. This yields

$$\begin{aligned} \mathcal{Z} &:= \int\prod_f dJ_f(t)\prod_{(t,v)}dV_{tv}e^{iS_{\rm disc}[J,V]} \\ &= \sum_{k_{ft},w_t}\Big[\prod_f\sum_{n=0}^\infty\int_{-\infty}^\infty(N_f^2+\rho_f^2)d\rho\Big]\prod_v A_v[\rho_f,N_f;k_{tf},w_t], \end{aligned} \tag{C3}$$

where for each vertex v, the integrations over the five connection variables $\{V_{vt}\}_{t\in v}$ are absorbed into the expression for A_v in Eq. (14).

Now, a discrete gauge transformation is specified by a group element $G_v \in SL(2,\mathbb{C})$ at each vertex and a group element $G_t \in SL(2,\mathbb{C})$ at each tetrahedron, with action $V_{vt} \mapsto G_v V_{vt} G_t^{-1}$, $B_f(t) \mapsto G_t B_f(t) G_t^{-1}$. Let us review the gauge-fixing procedure of [15]: we will then see why the gauge-fixing procedure of [15] does not work in our case, and then we will present a different procedure.

First, if one does not impose the simplicity constraints, (C3) is a partition function for the BF theory, so that the gauge-fixing strategy of [15] applies. One first chooses a maximal tree T of the 1-skeleton of the cell complex Δ^* dual to the triangulation Δ. Each 1-cell of T is an edge e dual to a tetrahedron t, with parallel transport $V_{vt}V_{tv'}$. As T contains no closed loops, one can use the aforementioned gauge freedom to fix to the identity all V_{vt} in T. This is the gauge-fixing procedure of [15] (adapted to the present variables).

In the new models, however, we must impose the simplicity constraints (12), $k_{ft} = \frac{N_f}{2} = \frac{\rho_f}{2\gamma}$. As noted in the main text, because k_{ft} is the quantum number of a non-Lorentz invariant quantity, these constraints break $SL(2,\mathbb{C})$ gauge symmetry at the tetrahedra, reducing the gauge there to $SU(2) \ni G_t \equiv g_t$ (see Fig. 1). As a consequence one

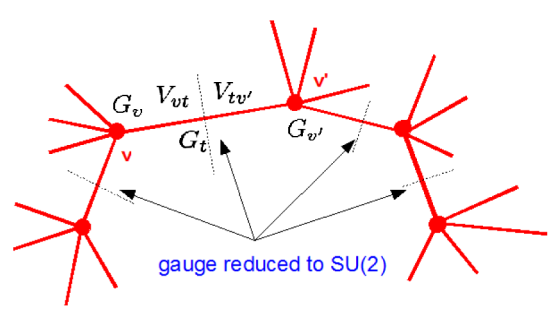

FIG. 1 (color online). The figure stands for a general dual triangulation. The gauge invariance is reduced at the tetrahedra.

will no longer be able to completely fix the group elements on a maximal tree (one may fix the rotation part of the group leaving it a pure boost, but this does not help us in the proof of finiteness). At the end of the day, we are able to fix to the identity only one V_{tv} per 4-simplex, which is equivalent to the regularization procedure presented earlier in this paper. Let $V_{t_v v}$ denote the group element in v that we gauge fix to the identity. This gauge-fixing condition implies $G_v g_t^{-1} = \mathbb{1}$, i.e. $G_v = g_t$. Thus the G_v gauge freedom is precisely the gauge that has been fixed by $V_{t_v v} = \mathbb{1}$, leaving the $SU(2)$ gauge transformations at the tetrahedra free.

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