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# Resonance of fractional transfer functions of the second kind

Rachid MALTI, Xavier MOREAU, and Firas KHEMANE

\* Bordeaux University – IMS, 351, cours de la Libération, 33405  
Talence Cedex, France.  
{firstname.lastname}@ims-bordeaux.fr

**Abstract:** Canonical fractional transfer function of the second kind is studied in this paper. Stability and resonance conditions are determined in terms of pseudo-damping factor and commensurable order.

Keywords: Resonance, fractional system, second order transfer function, canonical form.

## 1. INTRODUCTION

Commensurable fractional systems can be represented in a transfer function form as:

$$H(s) = \frac{T(s^\nu)}{R(s^\nu)} = \frac{\sum_{j=0}^{m_B} b_j s^{\nu j}}{1 + \sum_{i=1}^{m_A} a_i s^{\nu i}}, \quad (1)$$

where  $(a_i, b_j) \in \mathbb{R}^2$ ,  $\nu$  is the commensurable differentiation order,  $m_B$  and  $m_A$  are respectively numerator and denominator degrees, with  $m_A > m_B$  for strictly causal systems. Stability of fractional differentiation systems is addressed in the following theorem.

*Theorem 1.1.* (Stability Matignon (1998)). A commensurable transfer function with a commensurable order  $\nu$ , as in (4), with  $T$  and  $R$  two coprime polynomials, is stable if and only if (iff)  $0 < \nu < 2$  and  $\forall p \in \mathbb{C}$  such as  $R(p) = 0$ ,  $|\arg(p)| > \nu \frac{\pi}{2}$ .  $\square$

The commensurable transfer function (1) can always be decomposed in a modal form:

$$H(s) = \sum_{k=1}^N \sum_{q=1}^{v_k} \frac{A_{k,q}}{(s^\nu + p_k)^q}, \quad (2)$$

where  $(-p_k)$ , with  $k = 1, \dots, N$ , represent the  $s^\nu$ -poles of integer multiplicity  $v_k$ .

The representation (2) is constituted of elementary transfer functions of the first kind:

$$\tilde{F}(s) = \frac{\tilde{K}}{s^\nu + b}, \quad (3)$$

studied in Hartley and Lorenzo (1998).

Combining two elementary functions of the first kind (3) yields an elementary function of the second kind written in a canonical form as:

$$F(s) = \frac{K}{1 + 2\zeta \left(\frac{s}{\omega_0}\right)^\nu + \left(\frac{s}{\omega_0}\right)^{2\nu}}. \quad (4)$$

As in rational systems,  $K$  and  $\omega_0$  in (4) represent respectively steady-state gain and cut-off frequency. However, the parameter  $\zeta$  does not have the same meaning as in

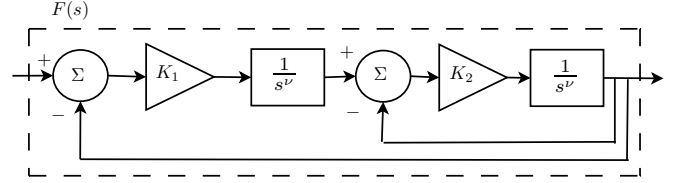


Fig. 1. closed-loop transfer function equivalence

rational systems: it is not a damping factor, unless  $\nu = 1$ . It will be referred to as *pseudo-damping factor* in this paper.

When two complex conjugate  $s^\nu$ -poles are present, the representation (4) is generally preferred to (3), because all parameters in (4) are real-valued.

The transfer function of the second kind (4), can also be viewed as two nested closed-loop transfer functions of Fig. 1 with gains,

$$K_1 = \frac{\omega_0^\nu}{2\zeta}, \quad (5)$$

$$K_2 = 2\zeta\omega_0^\nu, \quad (6)$$

and two integrators of order  $\nu$ .

The closed-loop representation of Fig. 1 will be used in the open-loop transfer function analysis of §4, based on Nichols charts.

Plotting the asymptotic frequency response of a transfer function such as (2) requires usually to decompose (2) into elementary transfer functions of the first and the second kind and the contribution of each elementary function is plotted as it appears depending on the transitional frequencies. The elementary transfer function of the first kind has already been studied in Hartley and Lorenzo (1998). On the other hand, the properties of rational second order systems, with  $(\nu = 1)$ , are well known:

- the system is stable, if the damping factor  $\zeta > 0$ ,
- the system is resonant, if  $0 < \zeta < \frac{\sqrt{2}}{2}$ ,
- the system has two complex conjugate poles, if  $0 < \zeta < 1$ ,
- the system has a real double pole if  $\zeta = 1$ ,
- the system is overdamped, if  $\zeta > 1$ .

The main concern of this paper is to study elementary properties of fractional transfer functions of the second kind written in the canonical form (4). First, stability conditions are established in terms of the pseudo-damping factor  $\zeta$  and the commensurable differentiation order  $\nu$ . Then, resonance conditions are established.

The authors came to this study when they wanted to simulate a resonant and stable transfer function of the second kind as in (4). They noticed then that the above-mentioned properties of second order rational systems do not apply for fractional systems.

## 2. STABILITY OF FRACTIONAL TRANSFER FUNCTIONS OF THE SECOND KIND

The application of theorem 1.1 on (4), requires the computation of both  $s^\nu$ -poles:

$$s_{1,2}^\nu = \omega_0^\nu \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right), \quad (7)$$

which can either be real if  $|\zeta| \geq 1$ , or complex conjugate if  $|\zeta| < 1$ . Hence, two cases are distinguished.

### 2.1 Case $|\zeta| \geq 1$

Two real  $s^\nu$ -poles are present. According to theorem 1.1, the transfer function (4) is stable if both  $s^\nu$ -poles are negative:

$$s_{1,2}^\nu < 0 \Rightarrow \omega_0^\nu \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) < 0 \quad (8)$$

$$\Rightarrow -\zeta \pm \sqrt{\zeta^2 - 1} < 0. \quad (9)$$

In this case, this condition is always true and hence the system is always stable.

### 2.2 Case $|\zeta| < 1$

Two complex conjugate  $s^\nu$ -poles are present:

$$s_{1,2}^\nu = \omega_0^\nu \left( -\zeta \pm j\sqrt{1 - \zeta^2} \right) \quad (10)$$

$$= \omega_0^\nu e^{\pm j\theta}, \quad (11)$$

where  $0 < \theta < \pi$  and  $\theta$  is given by:

$$\theta = \begin{cases} \arctan \left( \frac{\sqrt{1 - \zeta^2}}{-\zeta} \right) & \text{if } -1 < \zeta \leq 0 \\ \arctan \left( \frac{\sqrt{1 - \zeta^2}}{-\zeta} \right) + \pi & \text{if } 0 \leq \zeta < 1. \end{cases} \quad (12)$$

According to theorem 1.1, the system is stable iff:

$$0 < \nu \frac{\pi}{2} < \theta < \pi. \quad (13)$$

Both conditions expressed in (12) are treated below.

*Subcase  $-1 < \zeta \leq 0$ :* In this case,

$$0 < \nu \leq 1 \quad \text{and} \quad (14)$$

$$\left( \frac{\sqrt{1 - \zeta^2}}{-\zeta} \right) > 0. \quad (15)$$

Consequently,  $\theta$  is in the first quadrant:  $\theta \in ]0, \frac{\pi}{2}]$ . According to theorem 1.1, the system is stable iff:

$$0 < \nu \frac{\pi}{2} < \theta \leq \frac{\pi}{2}. \quad (16)$$

Substituting (12) in (16) yields

$$\tan \left( \nu \frac{\pi}{2} \right) < \frac{\sqrt{1 - \zeta^2}}{-\zeta} \leq \tan \left( \frac{\pi}{2} \right) \quad (17)$$

$$\tan^2 \left( \nu \frac{\pi}{2} \right) < \frac{1 - \zeta^2}{\zeta^2} \leq \infty \quad (18)$$

$$\zeta^2 < \cos^2 \left( \nu \frac{\pi}{2} \right). \quad (19)$$

Since  $\zeta$  is negative, it must satisfy the following inequalities:

$$-1 < -\cos \left( \nu \frac{\pi}{2} \right) < \zeta \leq 0. \quad (20)$$

*Subcase  $0 \leq \zeta < 1$ :* In this case,

$$1 \leq \nu < 2 \quad \text{and} \quad (21)$$

$$\left( \frac{\sqrt{1 - \zeta^2}}{-\zeta} \right) < 0. \quad (22)$$

Hence,  $\theta$  is in the second quadrant:  $\theta \in [\frac{\pi}{2}, \pi[$ . According to theorem 1.1, the system is stable iff:

$$\frac{\pi}{2} \leq \nu \frac{\pi}{2} < \theta < \pi. \quad (23)$$

Consequently, substituting (12) in (23) yields:

$$\tan \left( \nu \frac{\pi}{2} - \pi \right) < \frac{\sqrt{1 - \zeta^2}}{-\zeta} < \tan(0) \quad (24)$$

$$\tan^2 \left( \nu \frac{\pi}{2} \right) > \frac{1 - \zeta^2}{\zeta^2} > 0 \quad (25)$$

$$\zeta^2 > \cos^2 \left( \nu \frac{\pi}{2} \right). \quad (26)$$

Since  $\zeta$  is positive, it must satisfy the following inequalities:

$$0 < -\cos \left( \nu \frac{\pi}{2} \right) < \zeta < 1. \quad (27)$$

### 2.3 Summary

The transfer function (4) is stable iff:

$$-\cos \left( \nu \frac{\pi}{2} \right) < \zeta < \infty \quad \text{and} \quad 0 < \nu < 2. \quad (28)$$

The stable transfer function (4) has two complex conjugate  $s^\nu$ -poles iff:

$$-\cos \left( \nu \frac{\pi}{2} \right) < \zeta < 1. \quad (29)$$

Condition (29) is not necessary for the transfer function of the second kind (4) to be resonant.

## 3. RESONANCE OF FRACTIONAL TRANSFER FUNCTIONS OF THE SECOND KIND

The frequency response of (4) is given by:

$$F(j\omega) = \frac{K}{1 + 2\zeta \left( \frac{j\omega}{\omega_0} \right)^\nu + \left( \frac{j\omega}{\omega_0} \right)^{2\nu}}. \quad (30)$$

Define  $\Omega = \frac{\omega}{\omega_0}$  as the normalized frequency. Since the resonance does not depend on the gain  $K$ , it will be set to one in the following. Hence, define:

$$\mathcal{F}(j\Omega) = \frac{1}{1 + 2\zeta (j\Omega)^\nu + (j\Omega)^{2\nu}}. \quad (31)$$

The gain of  $\mathcal{F}(j\Omega)$ <sup>1</sup>,

$$|\mathcal{F}(j\Omega)| = \frac{1}{|1 + 2\zeta e^{j\nu\frac{\pi}{2}}\Omega^\nu + e^{j\nu\pi}\Omega^{2\nu}|}, \quad (32)$$

is further detailed in (33)-(36).

In case  $F(s)$  is resonant,  $|\mathcal{F}(j\Omega)|$  has one or multiple maxima at positive frequencies. Since log is a strictly increasing function, finding the maximum of  $|\mathcal{F}(j\Omega)|$  corresponds to finding the maximum of  $|\mathcal{F}(j\Omega)|_{\text{dB}}$ . Hence,  $F(s)$  is resonant if  $\frac{d|\mathcal{F}(j\Omega)|_{\text{dB}}}{d\Omega} = 0$  has at least one real and strictly positive solution. Based on (36), all real and strictly positive solutions of the following equation need to be evaluated:

$$\begin{aligned} \frac{d|\mathcal{F}(j\Omega)|_{\text{dB}}}{d\Omega} = 0 \Rightarrow \Omega^{4\nu-1} + 3\zeta \cos\left(\nu\frac{\pi}{2}\right)\Omega^{3\nu-1} + \\ (2\zeta^2 + \cos(\nu\pi))\Omega^{2\nu-1} + \zeta \cos\left(\nu\frac{\pi}{2}\right)\Omega^{\nu-1} = 0. \end{aligned} \quad (37)$$

Since  $\Omega = 0$ , is not an acceptable solution, the common factor  $\Omega^{\nu-1}$  in (37) can be simplified, so as to obtain:

$$\begin{aligned} \frac{d|\mathcal{F}(j\Omega)|_{\text{dB}}}{d\Omega} = 0 \Rightarrow \Omega^{3\nu} + 3\zeta \cos\left(\nu\frac{\pi}{2}\right)\Omega^{2\nu} + \\ (2\zeta^2 + \cos(\nu\pi))\Omega^\nu + \zeta \cos\left(\nu\frac{\pi}{2}\right) = 0. \end{aligned} \quad (38)$$

One can check easily that for rational systems,  $\nu = 1$ , (38) reduces to:

$$\Omega^3 + (2\zeta^2 - 1)\Omega = 0, \quad (39)$$

which strictly positive solution is given, as expected, by

$$\Omega_r = \sqrt{1 - 2\zeta^2}, \quad (40)$$

provided that the following known condition is satisfied:

$$1 - 2\zeta^2 > 0 \Rightarrow \zeta < \frac{\sqrt{2}}{2}. \quad (41)$$

The third order equation in  $\Omega^\nu$  (38), can have positive real-valued, negative real-valued or complex-valued solutions. The number of resonant frequencies of the studied system, zero one or two, depends on the number of positive real-valued solutions corresponding to maxima of  $\mathcal{F}(j\Omega)$ . Care must be taken, because some of the strictly positive solutions correspond to minima (anti-resonance), especially when a double resonance is present, then the gain presents a minimum between the two maxima.

Solving (38) analytically is not easy. A numerical solution is obtained for all combinations of  $\nu$  and  $\zeta$  and plotted in Fig.2 (yellow and green regions represent combinations of  $\nu$  and  $\zeta$  which produce resonant systems). The main result of this paper is plotted in Fig.2 and summarized below.

- if  $0 < \nu \leq 0.5$  and  $-\cos\nu\frac{\pi}{2} < \zeta < 0 \Rightarrow$  the stable system is always resonant,
- if  $0 < \nu \leq 0.5$  and  $0 < \zeta \Rightarrow$  the stable system is never resonant,
- if  $0.5 < \nu \leq 1$  and  $-\cos\nu\frac{\pi}{2} < \zeta < \infty \Rightarrow$  the stable system is resonant if an additional condition is satisfied  $\zeta < \zeta_0$ , where  $\zeta_0$  is computed numerically and plotted in Fig.2 as the upper bound of the resonant region in the interval  $\zeta \in ]0.5, 1]$ . In the particular case when  $\nu = 1$ ,  $\zeta_0 = \frac{\sqrt{2}}{2}$ ,

<sup>1</sup> The multi-valued function  $s^\nu$  becomes holomorphic in the complement of its branch cut line as soon as a branch cut line, i.e.  $\mathbb{R}_-$  is specified. The following restrictions on arguments of  $s$  are imposed:  $-\pi < \arg(s) < \pi$ . Hence, the only possible argument of  $j$  is  $\frac{\pi}{2}$ .

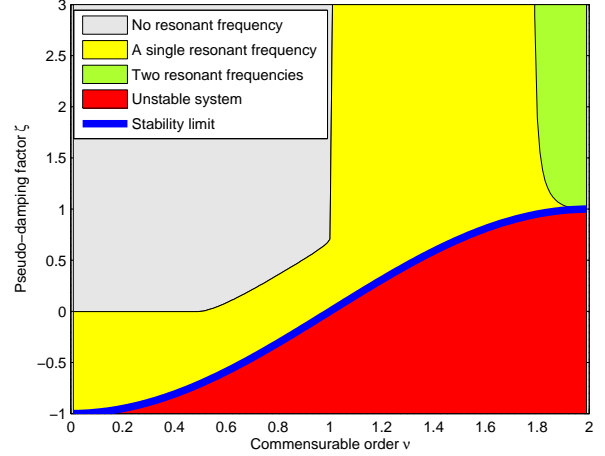


Fig. 2. Resonance and stability regions of the fractional system (4) in the  $\zeta$  versus  $\nu$  plane

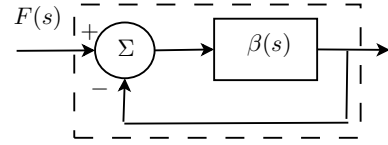


Fig. 3. closed-loop transfer function equivalence

- if  $1 < \nu \leq 2$  and  $-\cos\nu\frac{\pi}{2} < \zeta < \infty \Rightarrow$  the stable system is always resonant,
- if  $\nu_0 < \nu \leq 2$  and  $\zeta_0 < \zeta < \infty$  where  $\nu_0$  and  $\zeta_0$  are computed numerically and plotted as the lower-left limits of the green region of Fig2  $\Rightarrow$  the stable system has two resonant frequencies.

#### 4. OPEN-LOOP – CLOSED-LOOP ANALYSIS

The transfer function (4) can be viewed as the closed-loop system of Fig.1, or by considering only the outer loop, as the closed-loop system of Fig.3:

$$F(s) = \frac{\beta(s)}{1 + \beta(s)}, \quad (42)$$

where the open-loop transfer function  $\beta(s)$  is given by:

$$\beta(s) = \frac{K_1 K_2}{s^{2\nu}(1 + \frac{K_2}{s^\nu})}, \quad (43)$$

$K_1$  and  $K_2$  being defined in (5) and (6).

The open-loop transfer function  $\beta(s)$  is studied for different values of  $\nu$  and  $\zeta$  and its frequency response plotted for  $\zeta = -0.7$ ,  $\zeta = +0.7$ , and  $\zeta = 2$  in the Nichols charts of Figs 4, 5, and 6.

For negative  $\zeta$ , see for instance Fig.4 with  $\zeta = -0.7$ , the steady state gain of the open-loop transfer function,  $\beta(s)$ , is negative. Hence, in low frequencies, the Nichols chart of  $\beta(s)$  is inside the Nichols magnitude contours. When stability condition (28) is satisfied, here  $\zeta > -\cos(0.7\frac{\pi}{2}) = -0.45$ ,  $\beta(s)$  passes on the right of the critical point; otherwise it passes on its left. Consequently, for negative  $\zeta$ , the system can either be stable and resonant or unstable.

For  $0 < \zeta \leq 1$ , see for instance Fig.5 with  $\zeta = +0.7$ , the steady state gain of  $\beta(s)$  is positive. When  $\nu < \nu_0$ ,

$$|\mathcal{F}(j\Omega)| = \frac{1}{\left| (1 + 2\zeta \cos(\nu\frac{\pi}{2}) \Omega^\nu + \cos(\nu\pi) \Omega^{2\nu}) + j(2\zeta \sin(\nu\frac{\pi}{2}) \Omega^\nu + \sin(\nu\pi) \Omega^{2\nu}) \right|} \quad (33)$$

The gain in dB is now given by:

$$|\mathcal{F}(j\Omega)|_{\text{dB}} = -10 \log \left[ \left( 1 + 2\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^\nu + \cos(\nu\pi) \Omega^{2\nu} \right)^2 + \left( 2\zeta \sin\left(\nu\frac{\pi}{2}\right) \Omega^\nu + \sin(\nu\pi) \Omega^{2\nu} \right)^2 \right] \quad (34)$$

$$|\mathcal{F}(j\Omega)|_{\text{dB}} = -10 \log \left[ 1 + 4\zeta^2 \cos^2\left(\nu\frac{\pi}{2}\right) \Omega^{2\nu} + \cos^2(\nu\pi) \Omega^{4\nu} + 4\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^\nu + 2 \cos(\nu\pi) \Omega^{2\nu} + 4\zeta \cos\left(\nu\frac{\pi}{2}\right) \cos(\nu\pi) \Omega^{3\nu} + 4\zeta^2 \sin^2\left(\nu\frac{\pi}{2}\right) \Omega^{2\nu} + \sin^2(\nu\pi) \Omega^{4\nu} + 4\zeta \sin\left(\nu\frac{\pi}{2}\right) \sin(\nu\pi) \Omega^{3\nu} \right] \quad (35)$$

$$|\mathcal{F}(j\Omega)|_{\text{dB}} = -10 \log \left[ 1 + 4\zeta^2 \Omega^{2\nu} + \Omega^{4\nu} + 4\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^\nu + 2 \cos(\nu\pi) \Omega^{2\nu} + 4\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^{3\nu} \right] \quad (36)$$

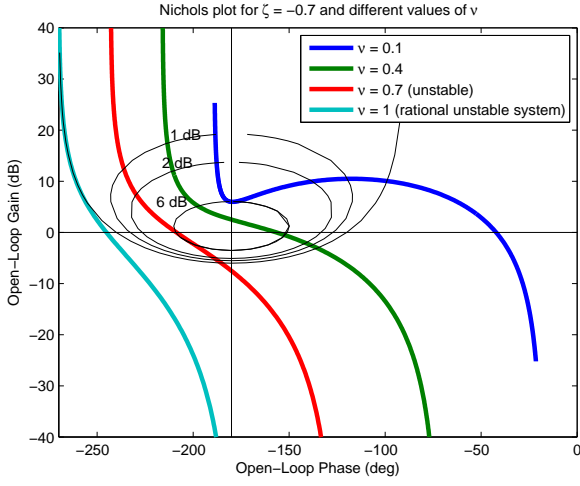


Fig. 4. Nichols charts for  $\zeta = -0.7$  and different values of  $\nu$ .

in this example  $\nu_0 \approx 1$ , the Nichols chart remains outside the 0dB Nichols magnitude contour. For  $\nu_0 < \nu < 2$ , the Nichols chart crosses the 0dB Nichols magnitude contour which makes the closed-loop system resonant. The system is unstable if the stability condition (28), expressed in terms of  $\nu$ , is not satisfied, here  $\nu > \arccos(-\zeta)\frac{2}{\pi} = 1.50$ .

For  $\zeta > 1$ , see for instance Fig.6 with  $\zeta = 2$ , the steady state gain of  $\beta(s)$  is positive. When  $\nu \leq 1$ , the Nichols chart remains outside the 0dB Nichols magnitude contour. For  $1 < \nu < 2$ , the Nichols chart crosses the 0dB Nichols magnitude contour which makes the closed-loop system resonant. The system is unstable if  $\nu \geq 2$  as specified by theorem 1.1.

## 5. TIME-DOMAIN SIMULATIONS OF FRACTIONAL MODELS

Due to the consideration that real physical systems generally have bandlimited fractional behavior and due to the practical limitations of input and output signals (Shannon's cut-off frequency for the upper band and the spectrum of the input signal for the lower band), fractional operators are usually approximated by high order rational models. As a result, a fractional model and its rational approximation have the same dynamics within a limited frequency band. The most commonly used approximation of  $s^\nu$  in the frequency band  $[\omega_A, \omega_B]$  is the recursive distribution of zeros and poles proposed by Oustaloup

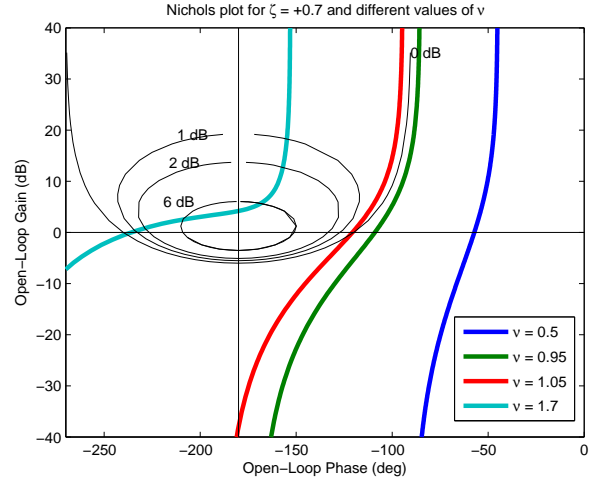


Fig. 5. Nichols charts for  $\zeta = +0.7$  and different values of  $\nu$ .

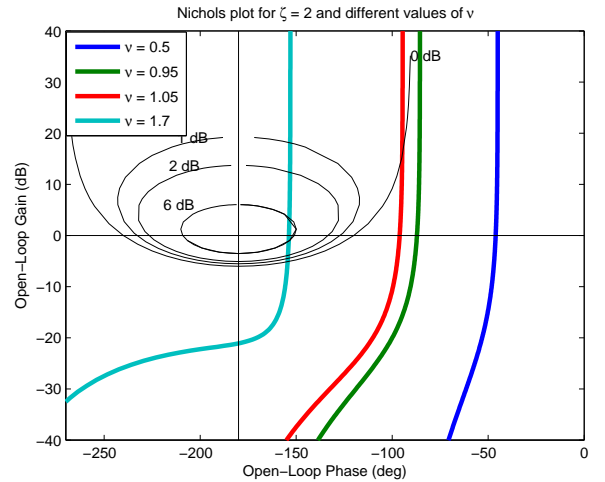


Fig. 6. Nichols charts for  $\zeta = 2$  and different values of  $\nu$ .

Oustaloup (1995). Trigeassou et al. Trigeassou et al. (1999) suggested to use an integrator outside the frequency range  $[\omega_A, \omega_B]$  instead of a gain:

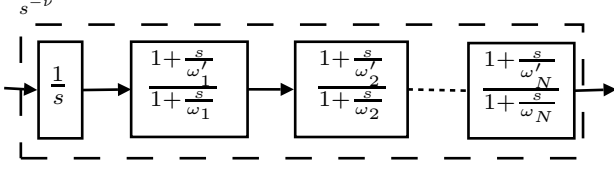


Fig. 7. Approximation of a fractional integrator using a rational model

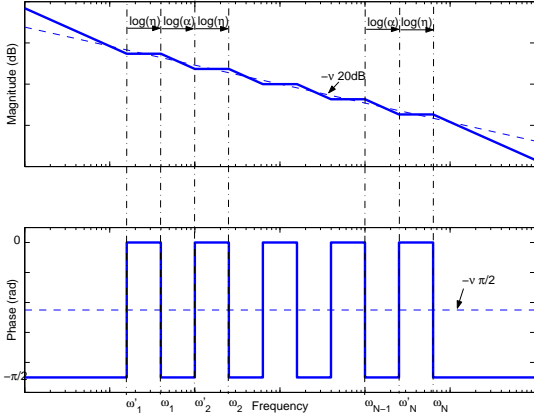


Fig. 8. Bode diagrams of a fractional integrator and the asymptotic behavior of its rational approximation

$$s^{-\nu} \rightarrow s_{[\omega_A, \omega_B]}^{-\nu} = \frac{C_0}{s} \left( \frac{1 + \frac{s}{\omega_A}}{1 + \frac{s}{\omega_B}} \right)^{1-\nu} \approx \frac{C_0}{s} \prod_{k=1}^N \frac{1 + \frac{s}{\omega'_k}}{1 + \frac{s}{\omega_k}}, \quad (44)$$

where  $\omega_k = \alpha\omega'_k$ ,  $\omega'_{k+1} = \eta\omega'_k$  and

$$\nu = 1 - \frac{\log \alpha}{\log \alpha \eta}, \quad (45)$$

$\alpha$  and  $\eta$  are real parameters which depend on the differentiation order  $\nu$  (see Fig 8). The bigger  $N$  the better the approximation of the integrator  $s^{-\nu}$ . The approximation (44) will be used while plotting the impulse and the step responses of systems studied in the following section.

## 6. EXAMPLE

### 6.1 Example 1

In this example, the particular case  $\nu = 0.5$  and  $\zeta = -0.5$  is studied. Eq. (38), which reduces to:

$$\Omega^{1.5} - \frac{3\sqrt{2}}{4}\Omega + \frac{\Omega^{0.5}}{2} - \frac{\sqrt{2}}{4} = 0, \quad (46)$$

has a single strictly positive real-valued solution and two complex conjugate ones:

$$\Omega_{r1} = 0.87, \quad \Omega_{r1,r2} = -0.37 \pm 0.08j. \quad (47)$$

Only the positive solution is acceptable and corresponds to the resonant frequency, as shown in the Bode diagram of Fig. 9. Moreover, the step and impulse responses, plotted in Fig.10, are underdamped as expected due to the resonant frequency.

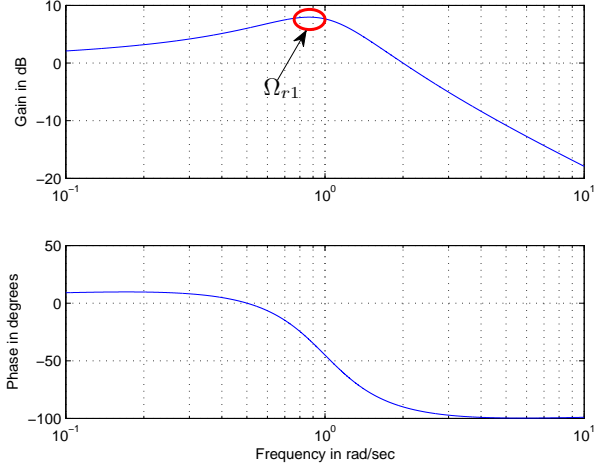


Fig. 9. Bode plot for  $\zeta = -0.5$  and  $\nu = 0.5$

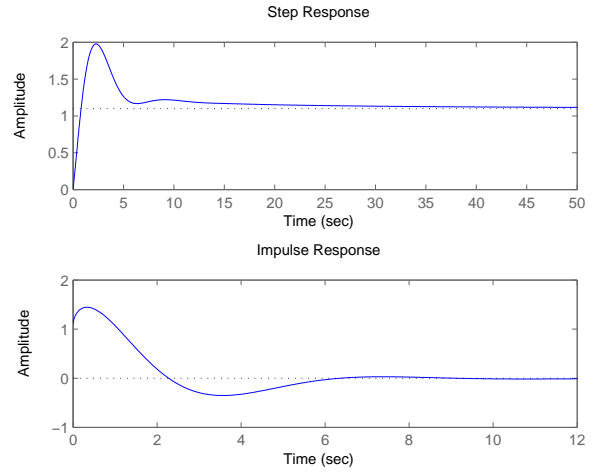


Fig. 10. Step and Impulse responses for  $\zeta = -0.5$  and  $\nu = 0.5$

### 6.2 Example 2

Consider now the case  $\nu = 1.9$ , and  $\zeta = 2$ , which presents the particularity of having a double resonance as established in Fig.2. Eq. (38), now written as:

$$\Omega^{5.7} - 5.93\Omega^{3.8} + 8.95\Omega^{1.9} - 1.98 = 0, \quad (48)$$

has three real-valued solutions:

$$\Omega_{r1} = 0.50, \quad \Omega_{r2} = 1.47, \quad \Omega_{r3} = 1.96. \quad (49)$$

As shown in Fig.11,  $\Omega_{r1}$  corresponds to a resonance,  $\Omega_{r2}$  to an anti-resonance, and  $\Omega_{r3}$  to a second resonance. When a system presents two resonant frequencies, it always has a minimum between these two maxima. Moreover, the step and impulse responses, plotted in Fig 12, are oscillatory as expected due to the resonant frequencies.

## 7. CONCLUSION

The fractional transfer functions of the second kind (4) is studied in this paper. First, stability conditions are determined in terms of the pseudo-damping factor  $\zeta$  (it is known that the commensurable order,  $\nu$ , must satisfy  $0 < \nu < 2$ ) and is summarized in (28). Then, conditions

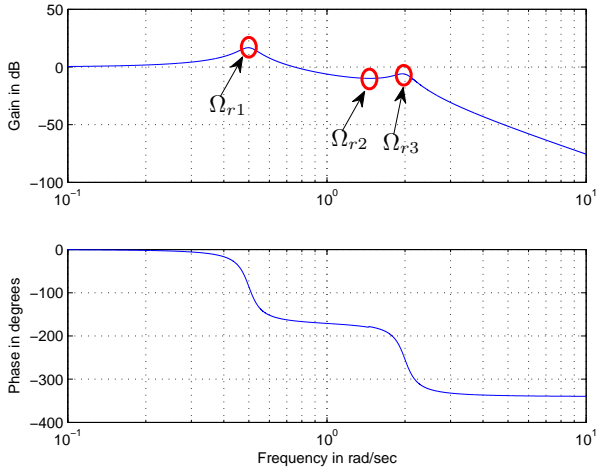


Fig. 11. Bode plot for  $\zeta = 2$  and  $\nu = 1.9$

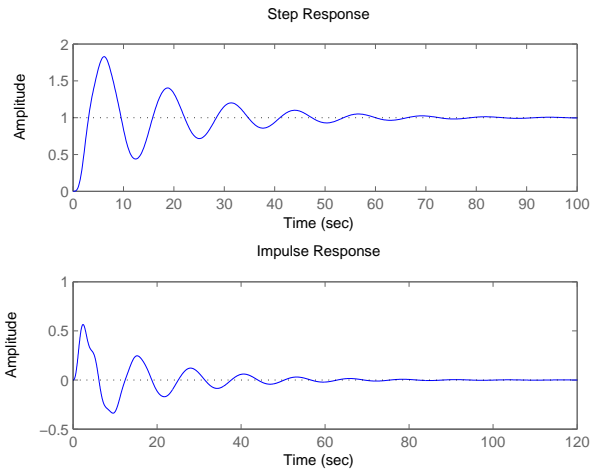


Fig. 12. Step and Impulse responses for  $\zeta = 2$  and  $\nu = 1.9$

on  $\zeta$  and  $\nu$  are determined so that the system is resonant. The latter conditions are difficult to express analytically. They are computed numerically and plotted in Fig.2. It is also shown, in Fig.2, that some combinations of  $\zeta$  and  $\nu$  yield two resonant frequencies.

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