Obstacle problem for SPDE with nonlinear Neumann boundary condition via reflected generalized backward doubly SDEs

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Abstract

This paper is intended to give a probabilistic representation for stochastic viscosity solution of semi-linear reflected stochastic partial differential equations with nonlinear Neumann boundary condition. We use its connection with reflected generalized backward doubly stochastic differential equations.

Key Words: Backward doubly SDEs, Stochastic PDEs, Obstacle problem, stochastic viscosity solutions.

MSC: 60H15; 60H20

1 Introduction

Backward stochastic differential equations (BSDEs, for short) were introduced by Pardoux and Peng [10] in 1990, and it was shown in various papers that stochastic differential equations (SDEs) of this type give a probabilistic representation for solution (at least in the viscosity

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sence) of a large class of system of semi-linear parabolic partial differential equations (PDEs). Thereafter a new class of BSDEs, called backward doubly stochastic (BDSDEs), was considered by Pardoux and Peng [11]. The new kind of BSDEs seems suitable for giving a probabilistic representation for a system of parabolic stochastic partial differential equations (SPDEs). We refer to Pardoux and Peng [11] for the link between SPDEs and BDSDEs in the particular case where solutions of SPDEs are regular. The more general situation is much more delicate to treat because of the difficulties of extending the notion of viscosity solutions to SPDEs.

The notion of viscosity solution for PDEs was introduced by Crandall, Ishii and Lions [5] for certain first-order Hamilton-Jacobi equations. Today the theory has become an important tool in many applied fields, especially in optimal control theory and numerous subjects related to it.

The stochastic viscosity solution for semi-linear SPDEs was introduced for the first time in Lions and Souganidis [8]. They use the so-called "stochastic characteristic" to remove the stochastic integrals from a SPDEs. On the other hand, two other ways of defining a stochastic viscosity solution of SPDEs is considered by Buckdahn and Ma respectively in [2, 3] and [4]. In the two first paper, they used the "Doss-Sussman" transformation to connect the stochastic viscosity solution of SPDEs with the BDSDEs. In the second one, they introduced the stochastic viscosity solution by using the notion of stochastic sub and super jets. Next, in order to give a probabilistic representation for viscosity solution of SPDEs with nonlinear Neumann boundary condition, Boufoussi et al. [1] introduced the so-called generalized BDSDEs. They refer the first technique (Doss-Sussman transformation) of Buckdhan and Ma [2, 3].

Based on the work of Boufoussi et al. [1] and employing the penalized method from Ren et al. [13], the aim of this paper, is to establish the existence result for semi-linear reflected SPDEs with nonlinear Neumann boundary condition of the form:

$$\begin{cases} \min\big\{u(t,x)-h(t,x), \frac{\partial}{\partial t}u(t,x) - [Lu(t,x) - f(t,x,u(t,x),\sigma^*(x)\nabla u(t,x))] \\ \qquad\qquad -g(t,x,u(t,x))\Diamond B_s\big\} = 0, \quad (t,x)\in[0,T]\times\Theta \\ u(0,x) = l(x), \qquad x\in\mathbb{R}^d \\ \frac{\partial u}{\partial n}(t,x) + \phi(t,x,u(t,x)) = 0, \qquad x\in\partial\Theta, \end{cases}$$

where $\Diamond$ denotes the Wick product and, thus, indicates that the differential is to understand in Itô's sense. Here B is a standard Brownian motion, L is an infinitesimal generator of a diffusion process X, Θ is a connected bounded domain and f, g, ϕ, l, h are some measurable functions. More precisely, we give some direct links between the stochastic viscosity solution

of the previous reflected SPDE and the solution of the following reflected generalized BDSDE:

$$\begin{aligned} Y_t &= \xi + \int_0^t f(s, Y_s, Z_s)ds + \int_0^t \phi(s, Y_s)dA_s + \int_0^t g(s, Y_s)\, dB_s \\ &\quad - \int_0^t Z_s \downarrow dW_s + K_t, \ \ 0 \leq t \leq T. \end{aligned}$$

ξ is the terminal value, A is a positive real-valued increasing process and $\downarrow dW_s$ denote the classical backward Itô integral with respect the Brownian motion W. Note that our work can be considered as a generalization of two results. First the one given in [13], where the authors treat deterministic reflected PDEs with nonlinear Neumann boundary conditions i.e $g \equiv 0$. The second result appears in [1] where the non reflected SPDE with nonlinear Neumann boundary condition is considered.

The present paper is organized as follows. An existence and uniqueness result for solutions to reflected generalized BDSDEs is shown in Section 2. Section 3 is devoted to give a definition of a reflected stochastic solution to SPDEs and by the same occasion establishes its existence result.

2 Reflected generalized backward doubly stochastic differential equation

2.1 Notation, assumptions and definition.

The scalar product of the space $\mathbb{R}^d (d \geq 2)$ will be denoted by $< . >$ and the associated Euclidian norm by $\|.\|$.

In what follows let us fix a positive real number $T > 0$. First of all $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, \ 0 \leq t \leq T\}$ are two mutually independent standard Brownian motions with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^\ell$, defined respectively on the two probability spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. Let $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{t\geq 0}$ denote the natural filtration generated by B, augmented by the $\mathbb{P}_1$-null sets of $\mathcal{F}_1$; and let $\mathcal{F}^B = \mathcal{F}_\infty^B$. On the other hand we consider the following family of σ-fields:

$$\mathcal{F}_{t,T}^W = \sigma\{W_s - W_T, t \leq s \leq T\} \vee \mathcal{N}_2,$$

where $\mathcal{N}_2$ denotes all the $\mathbb{P}_2-$ null sets in $\mathcal{F}_2$. Denote $\mathbf{F}_T^W = \{\mathcal{F}_{t,T}^W\}_{0\leq t\leq T}$.
Next we consider the product space $(\Omega, \mathcal{F}, \mathbb{P})$ where

$$\Omega = \Omega_1 \times \Omega_2, \ \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \ \text{ and } \mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2.$$

For each $t \in [0, T]$, we define

$$\mathcal{F}_t = \mathcal{F}_t^B \otimes \mathcal{F}_{t,T}^W.$$

Let us remark that the collection $\mathbf{F} = \{\mathcal{F}_t,\ t \in [0,T]\}$ is neither increasing nor decreasing and it does not constitute a filtration.
Further, we assume that, random variables, $\xi(\omega_1),\ \omega_1 \in \Omega_1$ and $\zeta(\omega_2),\ \omega_2 \in \Omega_2$ are considered as random variables on Ω via the following identification:

$$\xi(\omega_1, \omega_2) = \xi(\omega_1); \quad \zeta(\omega_1, \omega_2) = \zeta(\omega_2).$$

In the sequel, let $\{A_t,\ 0 \leq t \leq T\}$ be a continuous, increasing and $\mathbf{F}$-adapted real valued process such that $A_0 = 0$. For any $d \geq 1$, we consider the following spaces of processes:

1. $M^2(0,T,\mathbb{R}^d)$ denote the Banach space of all equivalence classes (with respect to the measure $d\mathbb{P} \times dt$) where each equivalence class contains an d-dimensional jointly measurable stochastic process $\varphi_t; t \in [0,T]$, which satisfies :

   (i) $\|\varphi\|^2_{M^2} = \mathbb{E}\int_0^T |\varphi_t|^2 dt < \infty$;

   (ii) φ_t is $\mathcal{F}_t-$ measurable , for any $t \in [0,T]$.

2. $S^2([0,T],\mathbb{R})$ is the set of one dimensional continuous stochastic processes which verify:

   (iii) $\|\varphi\|^2_{S^2} = \mathbb{E}(\sup_{0\leq t\leq T} |\varphi_t|^2) < \infty$;

   (iv) φ_t is $\mathcal{F}_t-$ measurable , for any $t \in [0,T]$.

Let us give the data (ξ, f, g, ϕ, S) which satisfy:

$(\mathbf{H}_1)$ ξ is a square integrable random variable which is $\mathcal{F}_T-$ mesurable such that for all $\mu > 0$

$$\mathbb{E}\left(e^{\mu A_T}|\xi|^2\right) < \infty.$$

$(\mathbf{H}_2)$ $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^\ell$, and $\phi : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$, are three functions such that:

   (a) There exist $\mathcal{F}_t-$adapted processes $\{f_t,\ \phi_t,\ g_t\ :\ 0 \leq t \leq T\}$ with values in $[1, +\infty)$ and with the property that for any $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, and $\mu > 0$, the following hypotheses are satisfied for some strictly positive finite constant K:

$$\begin{cases} f(t,y,z),\ \phi(t,y), \text{ and } g(t,y,z) \text{ are } \mathcal{F}_t - \text{measurable processes}, \\ |f(t,y,z)| \leq f_t + K(|y| + \|z\|), \\ |\phi(t,y)| \leq \phi_t + K|y|, \\ |g(t,y,z)| \leq g_t + K(|y| + \|z\|), \\ \mathbb{E}\left(\int_0^T e^{\mu A_t} f_t^2 dt + \int_0^T e^{\mu A_t} g_t^2 dt + \int_0^T e^{\mu A_t} \phi_t^2 dA_t\right) < \infty. \end{cases}$$

(*b*) There exist constants $c > 0, \beta < 0$ and $0 < \alpha < 1$ such that for any (y_1, z_1), $(y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{cases} (i)\, |f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2), \\ (ii)\, |g(t, y_1, z_1) - g(t, y_2, z_2)|^2 \leq c|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2, \\ (iii)\, \langle y_1 - y_2, \phi(t, y_1) - \phi(t, y_2)\rangle \leq \beta|y_1 - y_2|^2. \end{cases}$$

(**H**$_3$) The obstacle $\{S_t, 0 \leq t \leq T\}$, is a continuous $\mathcal{F}_t-$progressively measurable real-valued process satisfying

$$E\left(\sup_{0\leq t\leq T} \left|S_t^+\right|^2\right) < \infty.$$

We shall always assume that $S_T \leq \xi$ *a.s.*

One of our main goal in this paper is the study of the general case of reflected generalized BDSDE,

$$\begin{aligned} Y_t &= \xi + \int_0^t f(s, Y_s, Z_s)ds + \int_0^t \phi(s, Y_s)dA_s + \int_0^t g(s, Y_s, Z_s)\, dB_s \\ &\quad - \int_0^t Z_s \downarrow dW_s + K_t, \;\; 0 \leq t \leq T. \end{aligned} \tag{2.1}$$

First of all let us give a definition to the solution of this BDSDE.

Definition 2.1 *By a solution of the reflected generalized BSDE* (ξ, f, ϕ, g, S) *we mean a triplet of processes* $(Y, Z, K) \in S^2([0, T]; I\!R) \times M^2(0, T; I\!R^d) \times S^2([0, T]; I\!R)$, *which satisfies (2.1) such that the following holds* $I\!P-$ *a.s*

(*i*) *the map* $s \mapsto Y_s$ *is continuous*

(*ii*) $Y_t \geq S_t, \quad 0 \leq t \leq T,$

(*iii*) K *is an increasing process such that* $K_0 = 0$ *and* $\displaystyle\int_0^T (Y_t - S_t)\, dK_t = 0.$

Remark 2.1 *We note that although the equation (*2.1*) looks like a forward SDE, it is indeed a backward one because a terminal condition is given at* $t = 0$ *(*$Y_0 = \xi$*). We use this technique of reversal time due to the set-up of our problem that is, its connection to the obstacle problem for SPDE with nonlinear Neumann boundary condition.*

In the sequel, C denotes a positive constant which may vary from one line the other.

2.2 Comparison theorem

Let us give this comparison theorem related of the generalized BDSDE, which we will need in the proof of our main result. The proof follows with the same computation as in [14], with slight modification due to the presence of the integral with respect the increasing process A. So we just repeat the main step.

Theorem 2.1 *(Comparison theorem for generalized BDSDE) Let (Y,Z) and (Y',Z') be the unique solution of the non reflected generalized BDSDE associated to (ξ,f,ϕ,g) and (ξ',f',ϕ,g) respectively. If $\xi\leq\xi'$, $f(t,Y'_t,Z'_t)\leq f'(t,Y'_t,Z'_t)$ and $\phi(t,Y'_t)\leq\phi'(t,Y'_t)$, then $Y_t\leq Y'_t$, $\forall\, t\in[0,T]$.*

Proof. Let us set $\Delta Y=Y-Y'$, $\Delta Z=Z-Z'$ and $(\Delta Y)^+=(Y-Y')^+$ (with $f^+=\sup\{f,0\}$). Using Itô's formula, we get for all $0\leq t\leq T$

$$\begin{aligned}
&\mathbb{E}((\Delta Y_t)^+)^2+\mathbb{E}\int_0^t\|\Delta Z_s\|^2\mathbf{1}_{\{Y_s>Y'_s\}}ds\\
\leq&(\xi-\xi')^{+^2}+2\mathbb{E}\int_0^t(\Delta Y_s)^+\mathbf{1}_{\{Y_s>Y'_s\}}\left\{f(s,Y_s,Z_s)-f'(s,Y'_s,Z'_s)\right\}ds\\
&+2\mathbb{E}\int_0^t(\Delta Y_s)^+\mathbf{1}_{\{Y_s>Y'_s\}}\left\{\phi(s,Y_s)-\phi'(s,Y'_s)\right\}dA_s\\
&+\mathbb{E}\int_0^t\left\|g(s,Y_s,Z_s)-g(s,Y'_s,Z'_s)\right\|^2\mathbf{1}_{\{Y_s>Y'_s\}}ds, \qquad (2.2)
\end{aligned}$$

where $\mathbf{1}_\Gamma$ denote the characteristic function of a given set $\Gamma\in\mathbf{F}$ defined by

$$\mathbf{1}_\Gamma(\omega)=\begin{cases}1 \text{ if } \omega\in\Gamma\\ 0 \text{ if } \omega\in\Gamma.\end{cases}$$

From $(\mathbf{H}_2)(b)$ we have

$$\begin{aligned}
2(\Delta Y_s)^+\left\{f(s,Y_s,Z_s)-f'(s,Y'_s,Z'_s)\right\} &\leq 2(\Delta Y_s)^+\left\{f(s,Y_s,Z_s)-f(s,Y'_s,Z'_s)\right\}\\
&\leq (\frac{1}{\varepsilon}+\varepsilon c)((\Delta Y_s)^+)^2+\varepsilon c\|\Delta Z_s\|^2,\\
2(\Delta Y_s)^+\left\{\phi(s,Y_s)-\phi'(s,Y'_s)\right\} &\leq 2(\Delta Y_s)^+\left\{\phi(s,Y_s)-\phi(s,Y'_s)\right\}\\
&\leq \beta((\Delta Y_s)^+)^2
\end{aligned}$$

and

$$\left\|g(s,Y_s,Z_s)-g(s,Y'_s,Z'_s)\right\|^2\mathbf{1}_{\{Y_s>Y'_s\}} \leq c((\Delta Y_s)^+)^2\mathbf{1}_{\{Y_s>Y'_s\}}+\alpha\|\Delta Z_s\|^2\mathbf{1}_{\{Y_s>Y'_s\}}.$$

Plugging these inequalities on (2.2) and choosing $\varepsilon=\dfrac{1-\alpha}{2c}$, we conclude that

$$\mathbb{E}((\Delta Y_t)^+)^2\leq 0$$

which leads to $\Delta Y_t^+=0$ a.s. and so $Y'_t\geq Y_t$ a.s. for all $t\leq T$. ■

2.3 Existence and Uniqueness result

Our main goal in this section is to prove the following theorem.

Theorem 2.2 *Under the hypotheses* $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$ *and* $(\boldsymbol{H}_3)$, *there exists a unique solution for the reflected generalized BDSDE* (ξ, f, ϕ, g, S).

Our proof is based on a penalization method but is slightly different from El Karoui et al [7], because of the presence of the two integral with respect the increasing process A and the Brownian motion B, and also because of the time reversal.

For each $n \in \mathbb{N}^*$ we set

$$f_n(s, y, z) = f(s, y, z) + n(y - S_s)^- \tag{2.3}$$

and consider the generalized BDSDE

$$\begin{aligned} Y_t^n &= \xi + \int_0^t f_n(s, Y_s^n, Z_s^n)ds + \int_0^t \phi(s, Y_s^n)dA_s \\ &\quad + \int_0^t g(s, Y_s^n, Z_s^n)\, dB_s - \int_0^t Z_s^n \downarrow dW_s, \end{aligned} \tag{2.4}$$

obtained by the penalized method. We point out that the previous version of generalized BDSDE is, in fact, the time reversal version of that considered in Boufoussi et al [1], due to the set-up of our problem. We nonetheless use the same name because they are similar in nature. Consequently, it is well known (see Boufoussi et al., [1]) that, there exist a unique $(Y^n, Z^n) \in S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d)$ solution of the generalized BDSDE (2.4) such that for each $n \in \mathbb{N}^*$,

$$\mathbb{E}\left(\sup_{0\le t\le T} |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2\, ds\right) < \infty.$$

In order to prove Theorem 2.2, we state the following lemmas that will be useful.

Lemma 2.1 *Let us consider* $(Y^n, Z^n) \in S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d)$ *solution of BDSDE* (2.4). *Then there exists* $C > 0$ *such that,*

$$\sup_{n\in\mathbb{N}^*} \mathbb{E}\left(\sup_{0\le t\le T} |Y_t^n|^2 + \int_0^T |Y_s^n\|^2\, dA_s + \int_0^T \|Z_s^n\|^2\, ds + |K_T^n|^2\right) < C$$

where

$$K_t^n = n\int_0^t (Y_s^n - S_s)^- ds, \;\; 0 \le t \le T. \tag{2.5}$$

Proof. From Itô's formula, it follows that

$$
\begin{aligned}
&|Y_t^n|^2 + \int_0^t \|Z_s^n\|^2 \, ds \\
&\leq |\xi|^2 + 2\int_0^t Y_s^n f(s, Y_s^n, Z_s^n) ds + 2\int_0^t Y_s^n \phi(s, Y_s^n) dA_s + \int_0^t \|g(s, Y_s^n, Z_s^n)\|^2 ds \\
&+ 2\int_0^t S_s dK_s^n + 2\int_0^t \langle Y_s^n, g(s, Y_s^n, Z_s^n) dB_s \rangle - 2\int_0^t \langle Y_s^n, Z_s^n \downarrow dW_s \rangle, \qquad (2.6)
\end{aligned}
$$

where we have used $\int_0^t (Y_s^n - S_s) dK_s^n \leq 0$ and the fact that

$$
\int_0^t Y_s^n dK_s^n \;=\; \int_0^t (Y_s^n - S_s) dK_s^n + \int_0^t S_s dK_s^n \leq \int_0^t S_s dK_s^n.
$$

Using $(\mathbf{H}_2)$ and the elementary inequality $2ab \leq \gamma a^2 + \frac{1}{\gamma} b^2, \ \forall \gamma > 0$,

$$
\begin{aligned}
2Y_s^n f(s, Y_s^n, Z_s^n) &\leq (c\gamma_1 + \frac{1}{\gamma_1})|Y_s^n|^2 + 2c\gamma_1 \|Z_s^n\|^2 + 2\gamma_1 f_s^2, \\
2Y_s^n \phi(s, Y_s^n) &\leq (\gamma_2 - 2|\beta|)|Y_s^n|^2 + \frac{1}{\gamma_2}\phi_s^2, \\
\|g(s, Y_s^n, Z_s^n)\|^2 &\leq (1+\gamma_3)c|Y_s^n|^2 + \alpha(1+\gamma_3)\|Z_s^n\|^2 + (\frac{1}{\gamma_3} + 1) g_s^2.
\end{aligned}
$$

Taking expectation in both sides of the inequality (2.6) and choosing $\gamma_1 = \dfrac{1-\alpha}{6c}$, $\gamma_2 = |\beta|$ and $\gamma_3 = \dfrac{1-\alpha}{2\alpha}$ we obtain for all $\varepsilon > 0$

$$
\begin{aligned}
&\mathbb{E}\,|Y_t^n|^2 + |\beta|\mathbb{E}\int_0^t |Y_s^n|^2 \, dA_s + \frac{1-\alpha}{6}\mathbb{E}\int_0^t \|Z_s^n\|^2 \, ds \\
&\leq C\mathbb{E}\left\{ |\xi|^2 + \int_0^t |Y_s^n|^2 ds + \int_0^t f_s^2 ds + \int_0^t \phi_s^2 dA_s + \int_0^t g_s^2 ds \right\} \\
&+ \frac{1}{\varepsilon}\mathbb{E}\left(\sup_{0\leq s\leq t} (S_s^+)^2 \right) + \varepsilon \mathbb{E}\,(K_t^n)^2 . \qquad (2.7)
\end{aligned}
$$

On the other hand, we get from (2.4) that for all $0 \leq t \leq T$,

$$
K_t^n = Y_t^n - \xi - \int_0^t f(s, Y_s^n, Z_s^n) ds - \int_0^t \phi(s, Y_s^n) dA_s - \int_0^t g(s, Y_s^n, Z_s^n) dB_s + \int_0^t Z_s^n \downarrow dW_s. \qquad (2.8)
$$

Then we have

$$
\begin{aligned}
\mathbb{E}(K_t^n)^2 \leq 5\mathbb{E}\Bigg\{ &|\xi|^2 + |Y_t^n|^2 + \left| \int_0^t f(s, Y_s^n, Z_s^n) ds \right|^2 \\
&+ \left| \int_0^t \phi(s, Y_s^n) dA_s \right|^2 + \left| \int_0^t g(s, Y_s^n, Z_s^n) dB_s \right|^2 + \left| \int_0^t Z_s^n \downarrow dW_s \right|^2 \Bigg\}. \qquad (2.9)
\end{aligned}
$$

It follows by Hölder inequality and the isometry equality, together with assumptions $(\mathbf{H_2})(a)$ that

$$\left|\int_0^t f(s,Y_s^n,Z_s^n)ds\right|^2 \leq 3T\int_0^t (f_s^2+K^2|Y_s^n|^2+K^2\|Z_s^n\|^2)ds,$$

$$\mathbb{E}\left|\int_0^t g(s,Y_s^n,Z_s^n)dB_s\right|^2 \leq 3\mathbb{E}\int_0^t [g_s^2+K^2|Y_s^n|^2+K^2\|Z_s^n\|^2]ds.$$

and

$$\mathbb{E}\left|\int_0^t Z_s^n \downarrow dW_s\right|^2 \leq \mathbb{E}\int_0^t |Z_s^n|^2 ds.$$

Next, to estimate $\left|\int_0^t \phi(s,Y_s^n)dA_s\right|^2$, let us assume first that A_T is a bounded real variable. For any $\mu>0$ given in assumptions $(\mathbf{H}_1)$ or $(\mathbf{H}_2)(a)$, we have

$$\begin{aligned}\left|\int_0^t \phi(s,Y_s^n)dA_s\right|^2 &\leq \left(\int_0^t e^{-\mu A_s}dA_s\right)\left(\int_0^t e^{\mu A_s}|\phi(s,Y_s^n)|^2 dAs\right)\\ &\leq \frac{2}{\mu}\int_0^t e^{\mu A_s}(\phi_s^2+K^2|Y_s^n|^2)dA_s\\ &\leq C\int_0^t(\phi_s^2+|Y_s^n|^2)dA_s,\end{aligned}$$

since

$$\left(\int_0^t e^{-\mu A_s}dA_s\right)\leq \frac{1}{\mu}[1-e^{-\mu A_T}]\leq \frac{1}{\mu}.$$

The general case then follows from Fatou's lemma. Therefore, from (2.9) together with the previous inequalities, we get

$$\begin{aligned}\mathbb{E}(K_t^n)^2 \leq C\mathbb{E}\Big\{|\xi|^2+\int_0^t f_s^2 ds+\int_0^t \phi_s^2 dA_s+\int_0^t g_s^2 ds+\int_0^t |Y_s^n|^2\,ds\\ +\mathbb{E}\left(\sup_{0\leq s\leq t}(S_s^+)^2\right)+\int_0^t |Y_s^n|^2\,dA_s+\int_0^t \|Z_s^n\|^2 ds\Big\}.\end{aligned} \tag{2.10}$$

Recalling again (2.7) and taking ε small enough such that $\varepsilon C<\min\{|\beta|,\frac{1-\alpha}{6}\}$, we obtain

$$\begin{aligned}&\mathbb{E}\,|Y_t^n|^2+\mathbb{E}\int_0^t |Y_s^n|^2\,dA_s+\mathbb{E}\int_0^t \|Z_s^n\|^2\,ds\\ &\leq C\mathbb{E}\left\{|\xi|^2+\int_0^t |Y_s^n|^2 ds+\int_0^t f_s^2 ds+\int_0^t \phi_s^2 dA_s+\int_0^t g_s^2 ds+\mathbb{E}\left(\sup_{0\leq s\leq T}(S_s^+)^2\right)\right\}\end{aligned}$$

Consequently, it follows from Gronwall's lemma and (2.10) that

$$\begin{aligned}&\mathbb{E}\left\{|Y_t^n|^2+\int_0^t|Y_s^n|^2\,dA_s+\int_0^t\|Z_s^n\|^2ds+|K_T^n|^2\right\}\\&\leq C\mathbb{E}\left\{|\xi|^2+\int_0^T f_s^2ds+\int_0^T\phi_s^2dA_s+\int_0^T g_s^2ds+\sup_{0\leq t\leq T}(S_t^+)^2\right\}.\end{aligned}$$

Finally, by application of Burkholder-Davis-Gundy inequality we obtain from (2.6)

$$\begin{aligned}\mathbb{E}\left\{\sup_{0\leq t\leq T}|Y_t^n|^2+\int_0^T\|Z_s^n\|^2ds+|K_T^n|^2\right\}\;&\leq\;C\mathbb{E}\left\{|\xi|^2+\int_0^T f_s^2ds+\int_0^T\phi_s^2dA_s\right.\\&\left.+\int_0^T g_s^2ds+\sup_{0\leq t\leq T}(S_t^+)^2\right\},\end{aligned}$$

which end the proof of this Lemma. ■

Now we give a convergence result which is the key point on the proof of our main result. We begin by supposing that g is independent from (Y,Z). More precisely, we consider the following equation

$$Y_t\;=\;\xi+\int_0^t f(s,Y_s,Z_s)ds+\int_0^t\phi(s,Y_s)dA_s+\int_0^t g(s)\,dB_s-\int_0^t Z_s\downarrow dW_s+K_t. \tag{2.11}$$

The penalized equation is given by

$$\begin{aligned}Y_t^n\;&=\;\xi+\int_0^t f(s,Y_s^n,Z_s^n)ds+n\int_0^t(Y_s^n-S_s)^-ds+\int_0^t\phi(s,Y_s^n)dA_s\\&\quad+\int_0^t g(s)\,dB_s-\int_0^t Z_s^n\downarrow dW_s.\end{aligned} \tag{2.12}$$

Since the sequence of functions $(y\mapsto n(y-S_t)^-)_{n\geq1}$ is nondecreasing, then thanks to the comparison theorem 2.1, the sequence $(Y^n)_{n>0}$ is non-decreasing. Hence, Lemma 2.1 implies that there exists a $\mathcal{F}_t$- progressively measurable process Y such that $Y_t^n\nearrow Y_t$ $a.s.$ So the following result holds.

Lemma 2.2 *If g does not dependent on (Y,Z), then for each $n\in\mathbb{N}^*$,*

$$I\!E\left(\sup_{0\leq t\leq T}\left|(Y_t^n-S_t)^-\right|^2\right)\longrightarrow0,\quad as\;\;n\longrightarrow\infty.$$

Proof. Since $Y_t^n\geq Y_t^0$, we can w.l.o.g. replace S_t by $S_t\vee Y_t^0$, i.e. we may assume that $\mathbb{E}(\sup_{0\leq t\leq T}S_t^2)<\infty$. We want to compare a.s. Y_t and S_t for all $t\in[0,T]$, while we do not

know yet if Y is a.s. continuous. Indeed, let us introduce the following processes

$$\begin{cases} \overline{\xi} := \xi + \int_0^T g(s)\, dB_s \\ \overline{S}_t := S_t + \int_t^T g(s)\, dB_s \\ \overline{Y}_t^n := Y_t^n + \int_t^T g(s)\, dB_s \end{cases}$$

Hence,

$$\overline{Y}_t^n = \overline{\xi} + \int_0^t f(s, Y_s^n, Z_s^n)\, ds + n\int_0^t \left(\overline{Y}_s^n - \overline{S}_s\right)^- ds + \int_0^t \phi(s, Y_s^n)\, dA_s - \int_0^t Z_s^n \downarrow dW_s. \quad (2.13)$$

and we define $\overline{Y}_t := \sup_n \overline{Y}_t^n$.

From Theorem 2.1, we have that a.s., $\overline{Y}_t^n \geq \widetilde{Y}_t^n$, $0 \leq t \leq T$, $n \in \mathbb{N}^*$, where $\left\{(\widetilde{Y}_t^n, \widetilde{Z}_t^n),\ 0 \leq t \leq T\right\}$ is the unique solution of the BSDE

$$\widetilde{Y}_t^n = \overline{S}_T + \int_0^t f(s, Y_s^n, Z_s^n)\, ds + n\int_0^t (\overline{S}_s - \widetilde{Y}_s^n) ds + \int_0^t \phi(s, Y_s^n)\, dA_s - \int_0^t \widetilde{Z}_s^n \downarrow dW_s.$$

Let $\mathbf{G} = (\mathcal{G}_t)_{0\leq t\leq T}$ be a filtration defined by $\mathcal{G}_t = \mathcal{F}_{t,T}^W \otimes \mathcal{F}_{0,T}^B$. We consider ν a $\mathbf{G}$-stopping time such that $0 \leq \nu \leq T$. So we can write

$$\begin{aligned} \widetilde{Y}_\nu^n &= \mathbb{E}\left\{ e^{-n\nu}\overline{S}_T + \int_0^\nu e^{-n(\nu-s)} f(s, Y_s^n, Z_s^n) ds + n\int_0^\nu e^{-n(\nu-s)}\overline{S}_s ds \right. \\ &\quad \left. + \int_0^\nu e^{-n(\nu-s)}\phi(s, Y_s^n) dA_s \mid \mathcal{G}_\nu \right\}. \end{aligned} \quad (2.14)$$

First, with the help of Hölder inequality and assumptions $(\mathbf{H}_2)(a)$, we have

$$\begin{aligned} \mathbb{E}\left(\int_0^\nu e^{-n(\nu-s)} f(s, Y_s^n, Z_s^n) ds\right)^2 &\leq \frac{1}{2n}\mathbb{E}\left(\int_0^\nu |f(s, Y_s^n, Z_s^n|^2 ds\right) \\ &\leq \frac{C}{2n}\mathbb{E}\left(\int_0^T (f_s^2 + |Y_s^n|^2 + \|Z_s^n\|^2) ds\right), \end{aligned}$$

which provide

$$\mathbb{E}\left(\int_0^\nu e^{-n(\nu-s)} f(s, Y_s^n, Z_s^n) ds\right)^2 \longrightarrow 0 \,\text{as}\, n \to \infty, \quad (2.15)$$

since $\mathbb{E}\left(\int_0^T (f_s^2 + |Y_s^n|^2 + \|Z_s^n\|^2) ds\right) < C$ (see Lemma 2.1 and $(\mathbf{H}_2)(a)$).

Next, to prove that

$$\mathbb{E}\left(\int_0^\nu e^{-n(\nu-s)}\phi(s, Y_s^n) dA_s\right)^2 \longrightarrow 0 \,\text{as}\, n \to \infty, \quad (2.16)$$

let us suppose that A_T is a bounded random variable. Using again Hölder inequality and assumption $(\mathbf{H}_2)(a)$, we get

$$\begin{aligned}\mathbb{E}\left(\int_0^\nu e^{-n(\nu-s)}\phi(s,Y_s^n)dA_s\right)^2 &\leq \mathbb{E}\left[\left(\int_0^T e^{-2n(\nu-s)}dA_s\right)\left(\int_0^T|\phi(s,Y_s^n)|^2dA_s\right)\right]\\ &\leq \mathbb{E}\left[|A_T|\left(\int_0^T(\phi_s^2+K^2|Y_s^n|^2)dA_s\right)\right]<C.\end{aligned}$$

The result follows by Lebesgue dominated Theorem, since $\int_0^\nu e^{-n(\nu-s)}\phi(s,Y_s^n)dA_s\to 0$ $a.s.$ as $n\to\infty$. The general case follows also by the approximation argument and Fatou's lemma. On the other hand it is easily seen that

$$e^{-n\nu}\overline{S}_T+n\int_0^\nu e^{-n(\nu-s)}\overline{S}_s ds\to\overline{S}_\nu\mathbf{1}_{\{\nu>0\}}+\overline{S}_T\mathbf{1}_{\{\nu=0\}}\ a.s.\text{ as }n\to\infty. \tag{2.17}$$

According to (2.15)-(2.17), the equality (2.14) provides

$$\widetilde{Y}_\nu^n\longrightarrow\overline{S}_\nu\mathbf{1}_{\{\nu>0\}}+\overline{S}_T\mathbf{1}_{\{\nu=0\}}\ a.s.$$

and in $L^2(\Omega)$, as $n\to\infty$, and $\overline{Y}_\nu\geq\overline{S}_\nu$ a.s. which yields that $Y_\nu\geq S_\nu$ a.s. From this and the Section Theorem in Dellacherie and Meyer [6], it follows that the last inequality holds for all $t\in[0,T]$. Further $(Y_t^n-S_t)^-\downarrow 0$, a.s. and from Dini's theorem, the convergence is uniform in t. Finally, as $(Y_t^n-S_t)^-\leq(S_t-Y_t^0)^+\leq|S_t|+\left|Y_t^0\right|$, the dominated convergence theorem ensures that

$$\lim_{n\longrightarrow+\infty}\mathbb{E}(\sup_{0\leq t\leq T}|\,(Y_t^n-S_t)^-\,|^2)=0.$$

■

Proof of Theorem 2.2. Existence The proof of existence will be divided in two steps.

Step 1. g does not dependent on (Y,Z).

Recall that $Y_t^n\nearrow Y_t$ $a.s.$ Then, Fatou's lemma and Lemma 2.1 ensure

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t|^2\right)<+\infty,$$

It then follows from Lemma 2.1 and Lebegue's dominated convergence theorem that

$$\mathbb{E}\left(\int_0^T|Y_s^n-Y_s|^2\,ds\right)\longrightarrow 0,\quad\text{as }\ n\to\infty. \tag{2.18}$$

Next, we will prove that the sequence of processes Z^n converges in $M^2(0,T;\mathbb{R}^d)$. To this end, for $n \geq p \geq 1$, Itô's formula gives,

$$\begin{aligned}
&|Y_t^n - Y_t^p|^2 + \int_0^t \|Z_s^n - Z_s^p\|^2 ds \\
= \;& 2\int_0^t (Y_s^n - Y_s^p)[f(s,Y_s^n,Z_s^n) - f(s,Y_s^p,Z_s^p)]ds + 2\int_0^t (Y_s^n - Y_s^p)[\phi(s,Y_s^n) - \phi(s,Y_s^p)]dA_s \\
&+2\int_0^t \langle (Y_s^n - Y_s^p), [g(s,Y_s^n,Z_s^n) - g(s,Y_s^p,Z_s^p)]dB_s\rangle + \int_0^t \|g(s,Y_s^n,Z_s^n) - g(s,Y_s^p,Z_s^p)\|^2 ds\rangle \\
&-2\int_0^t \langle Y_s^n - Y_s^p, [Z_s^n - Z_s^p] \downarrow dW_s\rangle + 2\int_0^t (Y_s^n - Y_s^p)(dK_s^n - dK_s^p).
\end{aligned}$$

From the same step as before, by using again assumptions $(\mathbf{H}_2)$, we see that there exists a constant $C > 0$, such that

$$\begin{aligned}
&\mathbb{E}\left\{|Y_t^n - Y_t^p|^2 + \int_0^t |Y_s^n - Y_s^p|^2 dA_s + \int_0^t \|Z_s^n - Z_s^p\|^2 ds\right\} \\
\leq \;& C\mathbb{E}\left\{\int_0^t |Y_s^n - Y_s^p|^2 ds + \sup_{0\leq s\leq T}(Y_s^n - S_s)^- K_T^p + \sup_{0\leq s\leq T}(Y_s^p - S_s)^- K_T^n\right\},
\end{aligned}$$

which, by Gronwall lemma, Hölder inequality and Lemma 2.1 respectively, implies

$$\begin{aligned}
\mathbb{E}\left\{|Y_t^n - Y_t^p|^2 + \int_0^t \|Z_s^n - Z_s^p\|^2 ds\right\} \;\leq\; & C\left\{\mathbb{E}\left(\sup_{0\leq s\leq T} |(Y_s^n - S_s)^-|^2\right)\right\}^{1/2} \\
& + C\left\{\mathbb{E}\left(\sup_{0\leq s\leq T} |(Y_s^p - S_s)^-|^2\right)\right\}^{1/2}. \quad (2.19)
\end{aligned}$$

Consequently, it follows from Lemma 2.2 that,

$$\mathbb{E}\left\{|Y_s^n - Y_s^p|^2 + \int_0^t \|Z_s^n - Z_s^p\|^2 ds\right\} \longrightarrow 0, \quad \text{as } n,p \longrightarrow \infty.$$

Finally, from Burkhölder-Davis-Gundy's inequality, we obtain

$$\mathbb{E}\left(\sup_{0\leq s\leq T} |Y_s^n - Y_s^p|^2 + \int_0^T \|Z_s^n - Z_s^p\|^2 ds\right) \longrightarrow 0, \quad \text{as } n,p \longrightarrow \infty,$$

and from (2.8) we can deduce

$$\mathbb{E}\left\{\sup_{0\leq s\leq T} |K_s^n - K_s^p|^2\right) \longrightarrow 0, \text{ as } n,p \to \infty,$$

which provides that the sequence of processes (Y^n, Z^n, K^n) is Cauchy in the Banach space $S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d) \times S^2([0,T];\mathbb{R})$. Consequently, there exists a triplet $(Y,Z,K) \in S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d) \times S^2([0,T];\mathbb{R})$ such that

$$\mathbb{E}\left\{ \sup_{0\leq s\leq T} |Y_s^n - Y_s|^2 + \int_0^T \|Z_s^n - Z_s\|^2 \, ds + \sup_{0\leq s\leq T} |K_s^n - K_s|^2 \right) \to 0, \text{ as } n \to \infty.$$

It remains to show that (Y,Z,K) solves the reflected BSDE (ξ, f, ϕ, g, S). In this fact, since $(Y_t^n, K_t^n)_{0\leq t\leq T}$ tends to $(Y_t, K_t)_{0\leq t\leq T}$ uniformly in t in probability, the measure dK_n converges to dK weakly in probability, so that $\int_0^T (Y_s^n - S_s) dK_s^n \to \int_0^T (Y_s - S_s) dK_s$ in probability as $n \to \infty$. On the other hand, in view of Lemma 2.2, $Y_t \geq S_t$ a.s., and thus $\int_0^T (Y_s - S_s) dK_s \geq 0$. Moreover, $\int_0^T (Y_s^n - S_s) dK_s^n = -n \int_0^T |(Y_s^n - S_s)^-|^2 ds \leq 0$ and passing to the limit we get $\int_0^T (Y_s - S_s) dK_s \leq 0$, which together with the above proved (ii) of the definition. Finally, passing to the limit in (2.12) we proved that (Y,Z,K) verifies (2.11).

Step 2. The general case. In light of the above step, the BDSDE

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s) ds + \int_0^t \phi(s, Y_s) dA_s + \int_0^t g(s, \bar{Y}_s, \bar{Z}_s) \, dB_s - \int_0^t Z_s \downarrow dW_s + K_t$$

has a unique solution (Y,Z,K). So, we can define the mapping

$$\begin{array}{rccc} \Psi: & S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d) & \longrightarrow & S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d) \\ & (\bar{Y}, \bar{Z}) & \longmapsto & (Y,Z) = \Psi(\bar{Y}, \bar{Z}). \end{array}$$

Now, let (Y,Z), (Y',Z'), $(\bar{Y}, \bar{Z})$ and $(\bar{Y}', \bar{Z}') \in S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d)$ such that $(Y,Z) = \Psi(\bar{Y}, \bar{Z})$ and $(Y',Z') = \Psi(\bar{Y}', \bar{Z}')$. Put $\Delta\eta = \eta - \eta'$ for $\eta = Y, \bar{Y}, Z, \bar{Z}$. By virtue of Itô's formula, we have

$$\begin{aligned} &\mathbb{E}e^{-\mu t}|\Delta Y_t|^2 + \mathbb{E}\int_0^t e^{-\mu s}\|\Delta Z_s\|^2 ds \\ &= 2\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s \left\{ f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s) \right\} ds + 2\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s \left\{ \phi(s, Y_s) - \phi(s, Y'_s) \right\} dA_s \\ &+2\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s d(\Delta K_s) + \int_0^t e^{-\mu s} \left\| g(s, \bar{Y}_s, \bar{Z}_s) - g(s, \bar{Y}'_s, \bar{Z}'_s) \right\|^2 ds - \mu\mathbb{E}\int_0^t e^{-\mu s} |\Delta Y_s|^2 \, ds. \end{aligned}$$

But since $\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s d(K_s - K'_s) \leq 0$, then we obtain from $(\mathbf{H}_2)$

$$\begin{aligned} &\mathbb{E}e^{-\mu t}|\Delta Y_t|^2 + |\beta|\mathbb{E}\int_0^t e^{-\mu s}|\Delta Y_s|^2 dA_s + \mu\mathbb{E}\int_0^t e^{-\mu s} |\Delta Y_s|^2 \, ds + \mathbb{E}\int_0^t e^{-\mu s}\|\Delta Z_s\|^2 ds \\ &\leq 2\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s \left\{ f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s) \right\} ds + \int_0^t e^{-\mu s} \left\| g(s, \bar{Y}_s, \bar{Z}_s) - g(s, \bar{Y}'_s, \bar{Z}'_s) \right\|^2 ds \end{aligned}$$

By use again assumptions $(\mathbf{H}_2)$, there exists constants $c(\alpha) > 0$, such that,

$$\begin{aligned}
&(\mu - c(\alpha))\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta Y_s\right|^2 ds + \mathbb{E}\int_0^t e^{-\mu s}\|\Delta Z_s\|^2 ds \\
&\leq \frac{1+\alpha}{2}\left(\bar{c}\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta \bar{Y}_s\right|^2 ds + \mathbb{E}\int_0^t e^{-\mu s}\left\|\Delta \bar{Z}_s)\right\|^2 ds\right)
\end{aligned}$$

Choosing μ such that $\mu - c(\alpha) = \bar{c} > 0$, we obtain

$$\begin{aligned}
&\bar{c}\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta Y_s\right|^2 ds + \mathbb{E}\int_0^t e^{-\mu s}\|\Delta Z_s\|^2 ds \\
&\leq \frac{1+\alpha}{2}\left(\bar{c}\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta \bar{Y}_s\right|^2 ds + \mathbb{E}\int_0^t e^{-\mu s}\left\|\Delta \bar{Z}_s)\right\|^2 ds\right)
\end{aligned}$$

Since $\dfrac{1+\alpha}{2} < 1$, then it follows that Ψ is a contraction and its unique fixed point solves our BDSDE.

Uniqueness Let us define

$$\left\{(\Delta Y_t, \Delta Z_t, \Delta K_t)\,,\ 0 \leq t \leq T\right\} = \left\{(Y_t - Y'_t, Z_t - Z'_t, K_t - K'_t),\ 0 \leq t \leq T\right\}$$

where $\left\{(Y_t, Z_t, K_t)\,,\ 0 \leq t \leq T\right\}$ and $\left\{(Y'_t, Z'_t, K'_t),\ 0 \leq t \leq T\right\}$ denote two solutions of the reflected BDSDE associated to the data (ξ, f, g, ϕ, S). Let us first note that

$$\int_0^T \Delta Y_s d(\Delta K_s) \leq 0. \tag{2.20}$$

Moreover, Itô formula yields that for every $0 \leq t \leq T$

$$\begin{aligned}
&|\Delta Y_t|^2 + \int_0^t \|\Delta Z_s\|^2 ds \\
=\ & 2\int_0^t \Delta Y_s(f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s))ds + \int_0^t \|g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)\|^2 ds \\
&+2\int_0^t \Delta Y_s(\phi(s, Y_s) - \phi(s, Y'_s))dA_s + \int_0^t \langle \Delta Y_s, (g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s))dB_s\rangle \\
&-2\int_0^t \langle \Delta Y_s, \Delta Z_s dW_s\rangle + 2\int_0^t \Delta Y_s d(\Delta K_s).
\end{aligned}$$

Then, by using similar computation as in the proof of existence and (2.20) we have

$$\mathbb{E}\left\{|\Delta Y_t|^2 + \int_0^T |\Delta Y_s| dA_s + \int_0^T \|\Delta Z_s\|^2 ds\right\} \ \leq\ C\mathbb{E}\int_0^T |\Delta Y_s|^2 ds,$$

from which, we deduce that $\Delta Y_t = 0$ and further $\Delta Z_t = 0$. On the other hand since

$$\begin{aligned}\Delta K_t &= \Delta Y_t - \int_0^t (f(s,Y_s,Z_s) - f(s,Y'_s,Z'_s))\,ds - \int_0^t (\phi(s,Y_s) - \phi(s,Y'_s))\,dA_s \\ &\quad - \int_0^t (g(s,Y_s,Z_s) - g(s,Y'_s,Z'_s))\,dB_s + \int_0^t \Delta Z_s \downarrow dW_s,\end{aligned}$$

we have $\Delta K_t = 0$. The proof is complete now. ■

3 Connection to stochastic viscosity solution for reflected SPDEs with nonlinear Neumann boundary condition

In this section we will investigate the reflected generalized BDSDEs studied in the previous section in order to give a probabilistic interpretation for the stochastic viscosity solution of a class of nonlinear reflected SPDEs with nonlinear Neumann boundary condition.

3.1 Notion of stochastic viscosity solution for reflected SPDEs with nonlinear Neumann boundary condition

With the same notations as in Section 2, let $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{0\le t\le T}$ be the filtartion generated by B, where B is a one dimensional Brownian motion. By $\mathcal{M}_{0,T}^B$ we denote all the $\mathbf{F}^B$-stopping times τ such $0 \le \tau \le T$, a.s. $\mathcal{M}_\infty^B$ is the set of all $\mathbf{F}^B$-stopping times that are almost surely finite. For generic Euclidean spaces E and E_1 we introduce the following vector spaces of functions:

1. The symbol $\mathcal{C}^{k,n}([0,T]\times E;E_1)$ stands for the space of all E_1-valued functions defined on $[0,T]\times E$ which are k-times continuously differentiable in t and n-times continuously differentiable in x, and $\mathcal{C}_b^{k,n}([0,T]\times E;E_1)$ denotes the subspace of $\mathcal{C}^{k,n}([0,T]\times E;E_1)$ in which all functions have uniformly bounded partial derivatives.

2. For any sub-σ-field $\mathcal{G}\subseteq\mathcal{F}_T^B$, $\mathcal{C}^{k,n}(\mathcal{G},[0,T]\times E;E_1)$ (resp. $\mathcal{C}_b^{k,n}(\mathcal{G},[0,T]\times E;E_1)$) denotes the space of all $\mathcal{C}^{k,n}([0,T]\times E;E_1)$ (resp. $\mathcal{C}_b^{k,n}([0,T]\times E;E_1)$-valued random variable that are $\mathcal{G}\otimes\mathcal{B}([0,T]\times E)$-measurable;

3. $\mathcal{C}^{k,n}(\mathbf{F}^B,[0,T]\times E;E_1)$ (resp.$\mathcal{C}_b^{k,n}(\mathbf{F}^B,[0,T]\times E;E_1)$) is the space of all random fields $\phi\in\mathcal{C}^{k,n}(\mathcal{F}_T,[0,T]\times E;E_1$ (resp. $\mathcal{C}^{k,n}(\mathcal{F}_T,[0,T]\times E;E_1)$, such that for fixed $x\in E$, the mapping $(t,\omega_1)\to\alpha(t,\omega_1,x)$ is $\mathbf{F}^B$-progressively measurable.

4. For any sub-σ-field $\mathcal{G}\subseteq\mathcal{F}^B$ and a real number $p\ge 0$, $L^p(\mathcal{G};E)$ to be all E-valued $\mathcal{G}$-measurable random variable ξ such that $\mathbb{E}|\xi|^p<\infty$.

Furthermore, regardless their dimensions we denote by $< .,. >$ and $|.|$ the inner product and norm in E and E_1, respectively. For $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$, we denote $D_x = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d})$, $D_{xx} = (\partial^2_{x_i x_j})_{i,j=1}^d$, $D_y = \frac{\partial}{\partial y}$, $D_t = \frac{\partial}{\partial t}$. The meaning of D_{xy} and D_{yy} is then self-explanatory.

Let Θ be an open connected bounded domain of $\mathbb{R}^d\,(d \geq 1)$. We suppose that Θ is smooth domain, which is such that for a function $\psi \in \mathcal{C}_b^2(\mathbb{R}^d)$, Θ and its boundary $\partial\Theta$ are characterized by $\Theta = \{\psi > 0\}$, $\partial\Theta = \{\psi = 0\}$ and, for any $x \in \partial\Theta$, $\nabla\psi(x)$ is the unit normal vector pointing towards the interior of Θ.

In this section, we consider the continuous coefficients f and ϕ,

$$\begin{array}{rcl} f & : & \Omega_1 \times [0,T] \times \overline{\Theta} \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R} \\ \phi & : & \Omega_1 \times [0,T] \times \overline{\Theta} \times \mathbb{R} \longrightarrow \mathbb{R} \end{array}$$

with the property that for all $x \in \overline{\Theta}$, $f(.,x,.,.)$ and $g(.,x,.)$ are Lipschitz continuous in x and satisfy the conditions $(\mathbf{H'}_1)$ and $(\mathbf{H}_2)$, uniformly in x, where, for some constant $K > 0$, the condition $(\mathbf{H'}_1)$ is:

$$(H'_1) \left\{ \begin{array}{l} |f(t,x,y,z)| \leq K(1 + |x| + |y| + \|z\|), \\ |\phi(t,x,y)| \leq K(1 + |x| + |y|). \end{array} \right.$$

Furthermore, we shall make use of the following assumptions:

- $(\mathbf{H}_3)$ The function $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ and $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are uniformly Lipschitz continuous , with common Lipschitz constant $K > 0$.

- $(\mathbf{H}_4)$ The functions $l : \overline{\Theta} \longrightarrow \mathbb{R}$ and $h : [0,T] \times \overline{\Theta} \longrightarrow \mathbb{R}$ are continuous such that, for some $K > 0$,

$$\begin{array}{rcll} |l(x)| & \leq & K(1 + |x|) & \\ |h(t,x)| & \leq & K(1 + |x|) & \\ h(0,x) & \leq & l(x), & x \in \overline{\Theta}. \end{array}$$

- $(\mathbf{H}_5)$ The function $g \in \mathcal{C}_b^{0,2,3}([0,T] \times \overline{\Theta} \times \mathbb{R}; \mathbb{R})$.

Let us consider the related obstacle problem for SPDEs with nonlinear Neumann boundary condition:

$$\mathcal{OP}^{(f,\phi,g,h,l)} \begin{cases} \min\Big\{u(t,x)-h(t,x),\ -\dfrac{\partial u(t,x)}{\partial t}-[Lu(t,x)+f(t,x,u(t,x),\sigma^*(x)D_xu(t,x))]dt \\ \qquad\qquad -g(t,x,u(t,x))\diamondsuit B_s\Big\}=0, \quad (t,x)\in[0,T]\times\Theta \\ u(0,x)=l(x), \qquad x\in\Theta \\ \dfrac{\partial u}{\partial n}(t,x)+\phi(t,x,u(t,x))=0, \quad (t,x)\in[0,T]\times\partial\Theta, \end{cases}$$

where

$$L=\frac{1}{2}\sum_{i,j=1}^{d}(\sigma(x)\sigma^*(x))_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i(x)\frac{\partial}{\partial x_i}, \quad \forall\, x\in\Theta,$$

and

$$\frac{\partial}{\partial n}=\sum_{i=1}^{d}\frac{\partial\psi}{\partial x_i}(x)\frac{\partial}{\partial x_i}, \quad \forall\, x\in\partial\Theta.$$

As in the work of Buckdahn-Ma [2, 3], our next goal is to define the notion of stochastic viscosity to $\mathcal{OP}^{(f,\phi,g,h)}$. So, we shall recall some of their notation. Let $\eta\in\mathcal{C}(\mathbf{F}^B,[0,T]\times\mathbb{R}^d\times\mathbb{R})$ be the solution to the equation

$$\eta(t,x,y) \;=\; y+\int_0^t\langle g(s,x,\eta(s,x,y)),\circ dB_s\rangle,$$

where the stochastic integrals have to be interpreted in Stratonowich sense. We have the following relation with the standard Itô integral:

$$\int_0^t\langle g(s,x,\eta(s,x,y)),\circ dB_s\rangle \;=\; \frac{1}{2}\int_0^t\langle g,D_yg\rangle(s,x,\eta(s,x,y)ds+\int_0^t\langle g(s,x,\eta(s,x,y)dB_s.$$

Under the assumption $(\mathbf{H}_5)$ the mapping $y\mapsto\eta(s,x,y)$ defines a diffeomorphism for all $t,x,\ a.s.$ Hence if we denote by $\varepsilon(s,x,y)$ its y-inverse, one can show that (cf. Buckdahn and Ma [2])

$$\varepsilon(t,x,y)=y-\int_0^t\langle D_y\varepsilon(s,x,y)g(s,x,y),\circ dB_s\rangle. \tag{3.1}$$

To simplify the notation in the sequel we denote

$$A_{f,g}(\varphi(t,x))=\mathbf{L}\varphi(t,x)+f(t,x,\varphi(t,x),\sigma^*D_x\varphi(t,x))-\frac{1}{2}(g,D_yg)(t,x,\varphi(t,x))$$
$$\text{and } \Psi(t,x)=\eta(t,x,\varphi(t,x)).$$

Definition 3.1 *A random field* $u \in \mathcal{C}\left(\mathbf{F}^B, [0,T] \times \overline{\Theta}\right)$ *is called a stochastic viscosity subsolution of the stochastic obstacle problem* $\mathcal{OP}^{(f,\phi,g,h,l)}$ *if* $u(0,x) \leq l(x)$*, for all* $x \in \overline{\Theta}$*, and if for any stopping time* $\tau \in \mathcal{M}_{0,T}^B$*, any state variable* $\xi \in L^0\left(\mathcal{F}_\tau^B, \Theta\right)$*, and any random field* $\varphi \in \mathcal{C}^{1,2}\left(\mathcal{F}_\tau^B,\ [0,T] \times \mathbb{R}^d\right)$*, with the property that for* $\mathbb{P}$*-almost all* $\omega \in \{0 < \tau < T\}$ *the inequality*

$$u(t,\omega,x) - \Psi(t,\omega,x) \leq 0 = u(\tau(\omega),\xi(\omega)) - \Psi(\tau(\omega),\xi(\omega))$$

is fulfilled for all (t,x) *in some neighborhood* $\mathcal{V}(\omega,\tau(\omega),\xi(\omega))$ *of* $(\tau(\omega),\xi(\omega))$*, the following conditions are satisfied:*

(a) on the event $\{0 < \tau < T\} \cap \{\xi \in \Theta\}$ *the inequality*

$$\min\left\{u(\tau,\xi) - h(\tau,\xi), \mathrm{A}_{f,g}(\Psi(\tau,\xi)) - D_y\Psi(\tau,\xi) D_t\varphi(\tau,\xi)\right\} \leq 0 \tag{3.2}$$

holds, $\mathbb{P}$*-almost surely;*

(b) on the event $\{0 < \tau < T\} \cap \{\xi \in \partial\Theta\}$ *the inequality*

$$\begin{aligned}\min\Big[\min\big\{u(\tau,\xi) - h(\tau,\xi), \mathrm{A}_{f,g}(\Psi(\tau,\xi)) - D_y\Psi(\tau,\xi) D_t\varphi(\tau,\xi)\big\}\ ,\\ -\frac{\partial\Psi}{\partial n}(\tau,\xi) - \phi(\tau,\xi,\psi(\tau,\xi))\Big] \leq 0\end{aligned} \tag{3.3}$$

holds, $\mathbb{P}$*-almost surely.*

A random field $u \in \mathcal{C}\left(\mathbf{F}^B, [0,T] \times \overline{\Theta}\right)$ *is called a stochastic viscosity supersolution of the stochastic obstacle problem* $\mathcal{OP}^{(f,\phi,g,h,l)}$ *if* $u(0,x) \geq l(x)$*, for all* $x \in \overline{\Theta}$*, and if for any stopping time* $\tau \in \mathcal{M}_{0,T}^B$*, any state variable* $\xi \in L^0\left(\mathcal{F}_\tau^B, \Theta\right)$*, and any random field* $\varphi \in \mathcal{C}^{1,2}\left(\mathcal{F}_\tau^B,\ [0,T] \times \mathbb{R}^d\right)$*, with the property that for* $\mathbb{P}$*-almost all* $\omega \in \{0 < \tau < T\}$ *the inequality*

$$u(t,\omega,x) - \Psi(t,\omega,x) \geq 0 = u(\tau(\omega),\xi(\omega)) - \Psi(\tau(\omega),\xi(\omega))$$

is fulfilled for all (t,x) *in some neighborhood* $\mathcal{V}(\omega,\tau(\omega),\xi(\omega))$ *of* $(\tau(\omega),\xi(\omega))$*, the following conditions are satisfied:*

(a) on the event $\{0 < \tau < T\} \cap \{\xi \in \Theta\}$ *the inequality*

$$\min\left\{u(\tau,\xi) - h(\tau,\xi), \mathrm{A}_{f,g}(\Psi(\tau,\xi)) - D_y\Psi(\tau,\xi) D_t\varphi(\tau,\xi)\right\} \geq 0 \tag{3.4}$$

holds, $\mathbb{P}$*-almost surely;*

(b) on the event $\{0 < \tau < T\} \cap \{\xi \in \partial\Theta\}$ *the inequality*

$$\begin{aligned}\max\Big[\min\big\{u(\tau,\xi) - h(\tau,\xi), \mathrm{A}_{f,g}(\Psi(\tau,\xi)) - D_y\Psi(\tau,\xi) D_t\varphi(\tau,\xi)\big\}\ ,\\ -\frac{\partial\Psi}{\partial n}(\tau,\xi) - \phi(\tau,\xi,\psi(\tau,\xi))\Big] \geq 0\end{aligned} \tag{3.5}$$

holds, $\mathbb{P}$*-almost surely.*

Finally, a random field $u \in \mathcal{C}\left(\mathbf{F}^B, [0,T] \times \overline{\Theta}\right)$ *is called a stochastic viscosity solution of the stochastic obstacle problem* $\mathcal{OP}^{(f,\phi,g,h,l)}$ *, if it is both a stochastic viscosity subsolution and a supersolution.*

Remark 3.1 *Observe that if* f, ϕ *are deterministic and* $g \equiv 0$ *the definition* 3.1 *coincides with the definition of (deterministic) viscosity solution of PDE* $\mathcal{OP}^{(f,\phi,0,h,l)}$ *given by Ren et al [13].*

3.2 Existence of stochastic viscosity solutions for SPDE with nonlinear Neumann boundary condition

The main objective of this subsection is to show how the stochastic obstacle problem $\mathcal{OP}^{(f,\phi,g,h,l)}$ is related to reflected generalized BDSDE (2.1) introduced in Section 1. For this end we recall some known results on reflected diffusions. We consider

$$
\begin{aligned}
&s \mapsto A_s^{t,x} \text{ is increasing} \\
&X_s^{t,x} = x + \int_s^t b\left(X_r^{t,x}\right) dr + \int_s^t \sigma\left(X_r^{t,x}\right) d \downarrow W_r + \int_s^t \nabla\psi\left(X_r^{t,x}\right) dA_r^{t,x}, \quad \forall\, s \in [0,t]\,, \\
&A_s^{t,x} = \int_s^t I_{\left\{X_r^{t,x} \in \partial\Theta\right\}}\, dA_r^{t,x}. \qquad (3.6)
\end{aligned}
$$

We note here that due to the direction of the Itô integral, (3.6) should be viewed as going from t to 0 (i.e., $X_0^{t,x}$ should be understood as the terminal value of the solution $X^{t,x}$). It is then clear (see [9]) that under conditions $(\mathbf{H}_3)$ on the coefficients b and σ, (3.6) has a unique strong $\mathbf{F}^W$-adapted solution. We refer to Pardoux and Zhang [12]: Propositions 3.1 and 3.2, for the following regularity results.

Proposition 3.1 *There exists a constant* $C > 0$ *such that for all* x, $x' \in \overline{\Theta}$ *the following inequality holds:*

$$
\begin{aligned}
I\!E\left[\sup_{0\le s\le t}\left|X_s^{t,x} - X_s^{t,x'}\right|^2\right] &\le C\,|x-x'|^2\,, \\
\mathbb{E}\left(\sup_{0\le s\le t}\left|A_s^{t,x} - A_s^{t,x'}\right|^2\right) &\le C|x-x'|^2.
\end{aligned}
$$

Moreover, for all $p \ge 1$, *there exists a constant* C_p *such that for all* $(t,x) \in \mathbb{R}_+ \times \overline{\Theta}$,

$$
\mathbb{E}\left(\left|A_s^{t,x}\right|^p\right) \le C_p(1+t^p)
$$

and for each μ, $0 < s < t$, *there exists a constant* $C(\mu,t)$ *such that for all* $x \in \overline{\Theta}$,

$$
\mathbb{E}\left(e^{\mu A_s^{t,x}}\right) \le C(\mu,t).
$$

On the other hand, let us consider the following reflected generalized BDSDE:

$$\begin{cases} Y_s^{t,x} = l\left(X_0^{t,x}\right) + \displaystyle\int_0^s f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr + \int_0^s g\left(r, X_r^{t,x}, Y_r^{t,x}\right) dB_r \\ \qquad\qquad + \displaystyle\int_0^s \phi\left(r, X_r^{t,x}, Y_r^{t.x}\right) dA_r^{t,x} + K_s^{t,x} - \int_0^s \left\langle Z_r^{t,x}, \downarrow dW_r \right\rangle, \quad 0 \le t \le s \le T, \\ h(s, X_s^{t,x}) \le Y_s^{t,x}, \quad \forall\, t \le s \le T \text{ and } \displaystyle\int_0^T \left(Y_r^{t,x} - h(r, X_r^{t,x})\right) dK_r^{t,x} = 0, \end{cases} \tag{3.7}$$

where the coefficients l, f, g, ϕ and h satisfy the hypotheses $(\mathbf{H}'_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_4)$ and $(\mathbf{H}_5)$.

Proposition 3.2 *Let the ordered triplet $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ be a solution of the BDSDE (3.7). Then the random field $(s,t,x) \mapsto Y_s^{t,x}$, $(s,t,x) \in [0,T] \times [0,T] \times \Theta$ is almost surely continuous.*

Proof. If we denote by $\mathbb{E}^{\mathcal{F}_s}$ the conditional expectation with respect to $\mathcal{F}_s$, then we can show that there exists a constant $C > 0$ such that for all (t,x), $(t',x') \in [0,T] \times \overline{\Theta}$ the following inequality holds

$$\begin{aligned} &\left|Y_s^{t,x} - Y_s^{t',x'}\right|^2 \\ &\le C\mathbb{E}^{\mathcal{F}_s}\left[e^{\mu k_T}\left|l(X_0^{t,x}) - l(X_0^{t',x'})\right|^2 + \int_0^T e^{\mu k_r}\left|f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) - f\left(r, X_r^{t',x'}, Y_r^{t,x}, Z_r^{t,x}\right)\right|^2 dr\right. \\ &+ \int_0^T e^{\mu k_r}\left|\phi\left(r, X_r^{t,x}, Y_r^{t,x}\right)\right|^2 d\overline{A}_r + \int_0^T e^{\mu k_r}\left|\phi\left(r, X_r^{t,x}, Y_r^{t,x}\right) - \phi\left(r, X_r^{t',x'}, Y_r^{t,x}\right)\right|^2 dA_r^{t',x'} \\ &\left.+ \int_0^T e^{\mu k_r}\left(h(r, X_r^{t',x'}) - h(r, X_r^{t,x})\right) d\Delta K_r\right], \end{aligned}$$

where $\Delta K := K^{t,x} - K^{t',x'}$, $\overline{A} = A^{t,x} - A^{t',x'}$ and $k \triangleq \left|\overline{A}\right| + A^{t',x'}$ where $\left|\overline{A}\right|$ is the total variation of the process $\overline{A}$. Using the assumptions $(\mathbf{H}'_1)$ and $(\mathbf{H}_2)$, we get

$$\begin{aligned} &\left|Y_s^{t,x} - Y_s^{t',x'}\right|^2 \\ &\le C\mathbb{E}^{\mathcal{F}_s}\left[e^{\mu k_T}\left|l(X_0^{t,x}) - l(X_0^{t',x'})\right|^2 + \int_0^T e^{\mu k_r}\left|X_r^{t,x} - X_r^{t',x'}\right|^2 dr\right. \\ &+ \int_0^T e^{\mu k_r}\left|X_r^{t,x} - X_r^{t',x'}\right|^2 dA_r^{t',x'} + \int_0^T e^{\mu k_r}\left(h(r, X_r^{t',x'}) - h(r, X_r^{t,x})\right) d\Delta K_r \\ &\left.+ \sup_{0\le s\le T} e^{\mu k_T}\left(1 + \left|X_s^{t,x}\right|^2 + \left|Y_s^{t,x}\right|^2\right)\left|A^{t,x} - A^{t',x'}\right|_T\right] \end{aligned}$$

The continuity follows from standard arguments using Proposition 3.1 and the continuity of the functions l and h. ■

Let now define

$$u(t,x) = Y_t^{t,x}, \quad (t,x) \in [0,T] \times \overline{\Theta}. \tag{3.8}$$

Theorem 3.1 *$u \in C(\mathbf{F}^B, [0,T] \times \Theta)$ is a stochastic viscosity solution of obstacle problem $\mathcal{OP}^{(f,\phi,g,h,l)}$.*

Proof. For each $(t,x) \in [0,T] \times \Theta, n \geq 1$, let $\{{}^nY_s^{t,x}, {}^nZ_s^{t,x}, \ 0 \leq s \leq t\}$ denote the solution of the generalized BDSDE

$$\begin{aligned} {}^nY_s^{t,x} &= l(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, {}^nY_r^{t,x}, {}^nZ_r^{t,x})dr + n\int_0^s ({}^nY_r^{t,x} - h(r, X_r^{t,x}))^- dr \\ &\quad + \int_0^s \phi(r, X_r^{t,x}, {}^n Y_r^{t,x})dA_r^{t,x} \int_0^s g(r, X_r^{t,x}, {}^n Y_r^{t,x})dB_r - \int_0^s {}^n Z_r^{t,x} \downarrow dW_r. \end{aligned}$$

It is know from Boufoussi et al [1] that

$$u_n(t,x) = {}^nY_t^{t,x}, \quad (t,x) \in [0,T] \times \overline{\Theta},$$

is the stochastic viscosity solution of the parabolic SPDE:

$$\begin{cases} \dfrac{\partial u_n(t,x)}{\partial t} + [Lu_n(t,x) + f_n(t,x,u_n(t,x), \sigma^* D_x u_n(t,x))] + g(t,x,u_n(t,x)) \diamondsuit B_t = 0, \\ (t,x) \in [0,T] \times \Theta, \\ u_n(0,x) = l(x), \quad x \in \Theta, \\ \dfrac{\partial u_n}{\partial n}(t,x) + \phi(t,x,u_n(t,x)) = 0, \quad (t,x) \in [0,T] \times \partial\Theta. \end{cases} \tag{3.9}$$

where $f_n(t,x,y,z) = f(t,x,y,z) + n(y - h(t,x))^-$. However, from the results of the previous section, for each $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$u_n(t,x) \uparrow u(t,x) \ \ a.s. \quad \text{as} \ \ n \to \infty.$$

Since u_n and u are continuous, it follows from Dini's theorem that the above convergence is uniform on any compacts. We now show that u is a stochastic viscosity subsolution of obstacle problem of $\mathcal{OP}^{(f,\phi,g,h,l)}$. Let $(\tau,\xi) \in \mathcal{M}_{0,T}^B \times L^0(\mathcal{F}_\tau^B; \Theta)$ and $\varphi \in \mathcal{C}^{1,2}\left(\mathcal{F}_\tau^B, \ [0,T] \times \Theta\right)$ be such that, for $u(\tau,\xi) > h(\tau,\xi), \ \mathbb{P}$-a.s. and for $\mathbb{P}-$ almost all $\omega \in \{0 < \tau < T\}$, we have

$$u(\omega,t,x) - \Psi(\omega,t,x) < 0 = u(\omega,\tau(\omega),\xi(\omega)) - \Psi(\omega,\tau(\omega),\xi(\omega)) \tag{3.10}$$

for all (t,x) in some neighborhood $\mathcal{V}(\omega,\tau(\omega),\xi(\omega))$ of $(\tau(\omega),\xi(\omega))$. From the assumption that $u(t,x) > h(t,x)$ and the uniformly convergence of u_n, there exists $(\tau_k,\xi_k)_{k\geq 0} \subset \mathcal{M}_{0,T}^B \times$

$L^0(\mathcal{F}_\tau^B;\Theta)$ an (τ,ξ)-approximation sequence (e.g. Definition 2.1 [4]) such that $u_{n_k}(\tau_k,\xi_k) > h(\tau_k,\xi_k)$, $\mathbb{P}$-a.s. Moreover, in virtue of (3.10), we get

$$u_{n_k}(\omega,t,x) - \Psi(\omega,t,x) < 0 = u_{n_k}(\omega,\tau_k(\omega),\xi_k(\omega)) - \Psi(\omega,\tau_k(\omega),\xi_k(\omega)) \tag{3.11}$$

for all (t,x) in some neighborhood $\mathcal{V}(\omega,\tau_k(\omega),\xi_k(\omega)) \subset \mathcal{V}(\omega,\tau(\omega),\xi(\omega))$ for k large enough. To have ξ_k in $L^0(\mathcal{F}_{\tau_k}^B;\Theta)$, we must and do take $\tau_k > \tau$ on $\{\tau < T\}$ and $\tau_k \downarrow \tau$; what is possible from the definition of (τ,ξ)-approximation sequence (e.g. [4]). On the other hand, since u_n is a viscosity solution of SPDE (3.9) and according to (3.11), we get the following:

(a) On the event $\{0<\tau_k<T\}\cap\{\xi_k\in\Theta\}$ the inequality

$$\mathrm{A}_{f_{n_k},g}(\Psi(\tau_k,\xi_k)) - D_y\Psi(\tau_k,\xi_k)\,D_t\varphi(\tau_k,\xi_k) \le 0$$

holds, $\mathbb{P}$-a.s.

(b) On the event$\{0<\tau_k<T\}\cap\{\xi_k\in\partial\Theta\}$ the inequality

$$\min\Big[\mathrm{A}_{f_{n_k},g}(\Psi(\tau_k,\xi_k)) - D_y\Psi(\tau_k,\xi_k)\,D_t\varphi(\tau_k,\xi_k), -\frac{\partial\Psi}{\partial n}(\tau_k,\xi_k) - \phi_{n_k}(\tau_k,\xi_k,\Psi(\tau_k,\xi_k))\Big] \le 0$$

holds, $\mathbb{P}$-a.s.

Therefore, taking the limit as $k\to\infty$ in the above inequality yields:

(a) On the event $\{0<\tau<T\}\cap\{\xi\in\Theta\}$ the inequality

$$\mathrm{A}_{f,g}(\Psi(\tau,\xi)) - D_y\Psi(\tau,\xi)\,D_t\varphi(\tau,\xi) \le 0$$

holds, $\mathbb{P}$-a.s.

(b) On the event$\{0<\tau<T\}\cap\{\xi\in\partial\Theta\}$ the inequality

$$\min\Big[\mathrm{A}_{f,g}(\Psi(\tau,\xi)) - D_y\Psi(\tau,\xi)\,D_t\varphi(\tau,\xi), -\frac{\partial\Psi}{\partial n}(\tau,\xi) - \phi(\tau,\xi,\Psi(\tau,\xi))\Big] \le 0$$

holds, $\mathbb{P}$-a.s.

This proved that u is a stochastic viscosity subsolution of $\mathcal{OP}^{f,\phi,g,h,l}$. By the same argument as above one can show that u given by (3.8) is also a stochastic viscosity supersolution of $\mathcal{OP}^{f,\phi,g,h,l}$. So we conclude that u is a stochastic viscosity of $\mathcal{OP}^{f,\phi,g,h,l}$, that end the proof. ■

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