Obstacle problem for SPDE with nonlinear Neumann boundary condition via reflected generalized backward doubly SDEs

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Abstract

This paper is intended to give a probabilistic representation for stochastic viscosity solution of semi-linear reflected stochastic partial differential equations with nonlinear Neumann boundary condition. We use it connection with reflected generalized backward doubly stochastic differential equations.

Key Words: Backward doubly SDEs, Stochastic PDEs, Obstacle problem, stochastic viscosity solutions.

MSC: 60H15; 60H20

1 Introduction

Backward stochastic differential equations (BSDEs, for short) were introduced by Pardoux and Peng [8] in 1990, and it was shown in various paper that stochastic differential equations (SDEs) of this type give a probabilistic representation for solution (at least in the viscosity sence) of a

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large class of system of semi-linear parabolic partial differential equations (PDEs). Thereafter a new class of BSDEs, called backward doubly stochastic (BDSDEs), was considered by Pardoux and Peng [9]. The new kind of BSDEs seems to suitable giving a probabilistic representation for a system of parabolic stochastic partial differential equations (SPDEs). We refer to Pardoux and Peng [9] for the link between SPDEs and BDSDEs in the particular case where solutions of SPDEs are regular. The more general situation is much more delicate to treat because of the difficulties of extending the notion of stochastic viscosity solutions to SPDEs.

The notion of viscosity solution for PDEs was introduced by Crandall and Lions [5] for certain first-order Hamilton-Jacobi equations. Today the theory has become an important tool in many applied fields, especially in optimal control theorey and numerous subject related to it.

The stochastic viscosity solution for semi-linear SPDEs was introduced for the first time in Lions and Souganidis [11]. They use the so-called "stochastic characteristic" to remove the stochastic integrals from a SPDEs. On the other hand, two other way of defining a stochastic viscosity solution of SPDEs is considered by Buckdahn and Ma respectively in [2, 3] and [4]. In the two first paper, they used the "Doss-Sussman" transformation to connect the stochastic viscosity solution of SPDE with the BDSDE. In the second one, they introduced the stochastic viscosity solution by using the notion of stochastic sub and super jets. In this dynamic, and in order to give a probabilistic representation for viscosity solution of SPDE with nonlinear Neumann boundary condition, Boufoussi et al. [1] introduced the so-called generalized BDSDE. They refer the first technic (Doss-Sussman transformation) of Buckdan and Ma [2, 3].

The aim of this paper, is to refer the works of Boufoussi et al and by using the penalized method appear in Ren et al [12] to establish the existence result for semi-linear reflected SPDE with nonlinear Neumann boundary condition of the form:

$$\begin{cases} \min\left\{u(t,x)-h(t,x), du(t,x)+[Lu(t,x)+f(t,x,u(t,x),\sigma^*(x)\nabla u(t,x))]dt \right. \\ \qquad\qquad \left. +g(t,x,u(t,x))\,dB_s\right\}=0, \quad (t,x)\in[0,T]\times\Theta \\ u(0,x)=l(x), \qquad x\in\mathbb{R}^d \\ \dfrac{\partial u}{\partial n}(t,x)+\phi(t,x,u(t,x))=0, \qquad x\in\partial\Theta. \end{cases}$$

Here B is a standard Brownian motion; L is an infinitesimal generator of a diffusion process X that will be defined later; Θ is a connected bounded domain; and f, g, ϕ, l and h are some measurable functions. More precisely, we give some direct links between the stochastic viscosity solution of the previous reflected SPDE and the solution of the following reflected generalized

BDSDE:

$$\begin{aligned} Y_t &= \xi + \int_0^t f(s, Y_s, Z_s)ds + \int_0^t \phi(s, Y_s)dA_s \int_0^t g(s, Y_s, Z_s)\, dB_s \\ &\quad - \int_0^t Z_s \downarrow dW_s + K_t, \;\; 0 \le t \le T, \end{aligned}$$

where ξ is the terminal value, A is a positive real-valued increasing process and $\downarrow dW_s$ denote the classical backward Itô integral with respect the Brownian motion W. Note that our work can be considered as a generalized of two results. First the one give in [12], where the authors treat deterministic reflected PDEs with nonlinear Neumann boundary conditions i.e $g \equiv 0$. The second result appear in [1] in which, they consider non reflected SPDE with nonlinear Neumann boundary condition.

The present paper is organized as follows. An existence and uniqueness result for solutions to reflected generalized BDSDEs is shown in Section 2. Section 3 is devoted to give a definition of a reflected stochastic solution to SPDEs and by the same occasion establish its existence result.

2 Reflected generalized backward doubly stochastic differential equation

2.1 Notation, assumptions and definition.

The scalar product of the space $\mathbb{R}^d, (d \ge 2)$ will be denote by $< . >$ and the Euclidian norm associated by $\|.\|$.

For the remaining of the paper, let us fix a positive real number $T > 0$. First of all $\{W_t, 0 \le t \le T\}$ and $\{B_t,\ 0 \le t \le T\}$ are two mutually independent standard Brownian motion with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^\ell$, defined respectively on the two probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. Let $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{t\ge0}$ denote the natural filtration generated by B, augmented by $\mathbb{P}_1$-null set of $\mathcal{F}_1$; and let $\mathcal{F}^B = \mathcal{F}_\infty^B$. On the other hand we consider the following family of σ-fields:

$$\mathcal{F}_{t,T}^W = \sigma\{W_s - W_T, t \le s \le T\} \vee \mathcal{N}_2,$$

where $\mathcal{N}_2$ denotes all the $\mathbb{P}_2-$ null set in $\mathcal{F}_2$. Denote $F_T^W = \{\mathcal{F}_{t,T}^W\}_{0\le t\le T}$.
Next we consider the product space $(\Omega, \mathcal{F}, \mathbb{P})$ where

$$\Omega = \Omega_1 \times \Omega_2,\;\; \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \;\text{ and } \mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2;$$

For each $t \in [0, T]$, we define

$$\mathcal{F}_t = \mathcal{F}_t^B \otimes \mathcal{F}_{t,T}^W.$$

Let us remark that the collection $F = \{\mathcal{F}_t,\ t \in [0,T]\}$ is neither increasing nor decreasing and it does not constitute a filtration.
Further, we assume that random variables $\xi(\omega_1),\ \omega_1 \in \Omega_1$ and $\zeta(\omega_2),\ \omega_2 \in \Omega_2$ are considered as random variables on Ω via the following identification:

$$\xi(\omega_1,\omega_2) = \xi(\omega_1); \quad \zeta(\omega_1,\omega_2) = \zeta(\omega_2).$$

In the sequel, let $\{A_t,\ 0 \le t \le T\}$ be a continuous, increasing and $\mathcal{F}_t$-adapted real valued process such that $A_0 = 0$. For any $d \ge 1$, we consider the following spaces of processes:

1. $M^2(0,T,\mathbb{R}^d)$ denote the Banach space of all equivalence classes (with respect to the measure $d\mathbb{P} \times dt$) where each equivalence class contains an d-dimensional jointly measurable stochastic processes $\varphi_t; t \in [0,T]$, which satisfy :

   (i) $\|\varphi\|^2_{M^2} = \mathbb{E}\displaystyle\int_0^T |\varphi_t|^2 dt < \infty;$

   (ii) φ is $\mathcal{F}_t-$ measurable , for any $t \in [0,T]$.

2. $S^2([0,T],\mathbb{R})$ is the set of one dimensional continuous stochastic processes, which verify:

   (iii) $\|\varphi\|^2_{S^2} = \mathbb{E}(\sup\limits_{0\le t\le T} |\varphi_t|^2) < \infty;$

   (iv) φ is $\mathcal{F}_t-$ measurable , for any $t \in [0,T]$.

Let us give the data (ξ, f, g, ϕ, S) which satisfy:

$(\mathbf{H}_1)$ ξ is a square integrable random variable which is $\mathcal{F}_T-$ mesurable such that for all $\mu > 0$

$$\mathbb{E}\left(e^{\mu A_T}|\xi|^2\right) < \infty.$$

$(\mathbf{H}_2)$ $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^\ell$, and $\phi : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$, are tree functions such that:

   (a) There exist $\mathcal{F}_t-$adapted processes $\{f_t,\ \phi_t,\ g_t :\ 0 \le t \le T\}$ with values in $[1,+\infty)$ and with the property that for any $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, and $\mu > 0$, the following hypotheses are satisfied for some strictly positive finite constant K:

$$\begin{cases} f(t,y,z),\ \phi(t,y), \text{ and } g(t,y,z) \text{ are } \mathcal{F}_t - \text{measurable processes}, \\ |f(t,y,z)| \le f_t + K(|y| + \|z\|), \\ |\phi(t,y)| \le \phi_t + K|y|, \\ |g(t,y,z)| \le g_t + K(|y| + \|z\|), \\ \mathbb{E}\left(\displaystyle\int_0^T e^{\mu A_t} f_t^2 dt + \int_0^T e^{\mu A_t} g_t^2 dt + \int_0^T e^{\mu A_t} \phi_t^2 dA_t\right) < \infty. \end{cases}$$

(b) There exist constants $c>0, \beta<0$ and $0<\alpha<1$ such that for any (y_1,z_1), $(y_2,z_2)\in$ $\mathbb{R}\times\mathbb{R}^d$,

$$\begin{cases} (i)\ |f(t,y_1,z_1)-f(t,y_2,z_2)|^2 \le c(|y_1-y_2|^2+\|z_1-z_2\|^2), \\ (ii)\ |g(t,y_1,z_1)-g(t,y_2,z_2)|^2 \le c|y_1-y_2|^2+\alpha\|z_1-z_2\|^2, \\ (iii)\ \langle y_1-y_2, \phi(t,y_1)-\phi(t,y_2)\rangle \le \beta|y_1-y_2|^2. \end{cases}$$

($\mathbf{H}_3$) The obstacle $\{S_t, 0\le t\le T\}$, is a continuous $\mathcal{F}_t-$progressively measurable real-valued process satisfying

$$E\left(\sup_{0\le t\le T}\left|S_t^+\right|^2\right)<\infty.$$

We shall always assume that $S_T\le\xi$ *a.s.*

As mentioned above one of our main goal in this paper, is the study of the following reflected generalized BDSDE,

$$\begin{aligned} Y_t &= \xi+\int_0^t f(s,Y_s,Z_s)ds+\int_0^t \phi(s,Y_s)dA_s+\int_0^t g(s,Y_s,Z_s)\,dB_s \\ &\quad -\int_0^t Z_s\downarrow dW_s+K_t,\ \ 0\le t\le T. \end{aligned} \tag{2.1}$$

First of all let us give a definition to the solution of this BDSDE.

Definition 2.1 *By solution of the reflected generalized BDSDE (ξ,f,ϕ,g,S), we mean a triplet of processes $(Y,Z,K)\in S^2([0,T];\mathbb{R})\times M^2(0,T;\mathbb{R}^d)\times S^2([0,T];\mathbb{R})$, which satisfies (2.1) such that the following holds $\mathbb{P}-$ a.s*

(i) $Y_t\ge S_t,\quad 0\le t\le T,$

(ii) K is an increasing process such that $K_0=0$ and $\displaystyle\int_0^T (Y_t-S_t)\,dK_t=0;$

Remark 2.1 *We note that although the equation (2.1) looks like a forward SDE, it is indeed a backward one because a terminal condition is given at $t=0$ ($Y_0=\xi$). We use this reversal time technic due to the set-up of our problem, that is it connection to the obstacle problem for SPDE with nonlinear Neumann boundary condition.*

In the sequel, C denotes a positive constant which may vary from one line to another.

2.2 Comparison theorem

Let us give this comparison theorem related of the generalized BDSDE, which we will need in the proof of our main result. The proof follows with same computation in [13], with slight modification due to presence of the integral which respect the increase process A, so we just repeat the main step.

Theorem 2.1 *(Comparison theorem for generalized BDSDE.) Let (Y,Z) and (Y',Z') the unique solution of the non reflected generalized BDSDE associated to (ξ,f,ϕ,g) and (ξ',f',ϕ,g) respectively. If $\xi\leq\xi'$, $f(t,Y_t',Z_t')\leq f'(t,Y_t',Z_t')$ and $\phi(t,Y_t')\leq\phi'(t,Y_t')$, then $Y_t\leq Y_t'$, $\forall\, t\in[0,T]$.*

Proof. Using Itô's formula, we get for all $0\leq t\leq T$

$$\begin{aligned}
&\mathbb{E}(\Delta Y_t^+)^2+\mathbb{E}\int_0^t \mathbb{I}_{(Y_s>Y_s')}\|\Delta Z_s\|^2ds\\
\leq&(\xi-\xi')^{+^2}+2\mathbb{E}\int_0^t \mathbb{I}_{(Y_s>Y_s')}\Delta Y_s^+\left\{f(s,Y_s,Z_s)-f'(s,Y_s',Z_s')\right\}ds\\
&+2\mathbb{E}\int_0^t \mathbb{I}_{(Y_s>Y_s')}\Delta Y_s^+\left\{\phi(s,Y_s)-\phi'(s,Y_s')\right\}dA_s\\
&+\mathbb{E}\int_0^t \mathbb{I}_{(Y_s>Y_s')}\left\|g(s,Y_s,Z_s)-g(s,Y_s',Z_s')\right\|^2ds.
\end{aligned}\tag{2.2}$$

From $(\mathbf{H}_2)(b)$ we have

$$\begin{aligned}
2\Delta Y_s^+\left\{f(s,Y_s,Z_s)-f'(s,Y_s',Z_s')\right\}&\leq 2\Delta Y_s^+\left\{f(s,Y_s,Z_s)-f(s,Y_s',Z_s')\right\}\\
&\leq (\frac{1}{\varepsilon}+\varepsilon c)(\Delta Y_s^+)^2+\varepsilon c\|\Delta Z_s\|^2,
\end{aligned}$$

$$\begin{aligned}
2\Delta Y_s^+\left\{\phi(s,Y_s)-\phi'(s,Y_s')\right\}&\leq 2\Delta Y_s^+\left\{\phi(s,Y_s)-\phi(s,Y_s')\right\}\\
&\leq \beta(\Delta Y_s^+)^2
\end{aligned}$$

and

$$\mathbb{I}_{(Y_s>Y_s')}\left\|g(s,Y_s,Z_s)-g(s,Y_s',Z_s')\right\|^2 \leq \mathbb{I}_{(Y_s>Y_s')}(\Delta Y_s^+)^2c+\mathbb{I}_{(Y_s>Y_s')}\alpha\|\Delta Z_s\|^2.$$

Plugging these inequalities on (2.2) and choosing $\varepsilon=\dfrac{1-\alpha}{2c}$, we conclude that

$$\mathbb{E}(\Delta Y_t^+)^2\leq 0$$

which leads to $\Delta Y_t^+=0$ a.s. and so $Y_t'\geq Y_t$ a.s. for all $\{t\leq T\}$. ■

2.3 Existence and Uniqueness result

Our main goal in this section is to prove the following theorem.

Theorem 2.2 *Under the hypotheses* $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$ *and* $(\boldsymbol{H}_3)$, *there exists a unique solution for the reflected generalized BDSDE* (ξ, f, ϕ, g, S).

Our proof is based on a penalization method, which is slightly different from El Karoui et al [7], because of the presence of two integral which respect the increasing process A and the Brownian motion B and the time reversal.

For each $n \in \mathbb{N}^*$ we set

$$f_n(s, y, z) = f(s, y, z) + n(y - S_s)^- \tag{2.3}$$

and consider the generalized BDSDE

$$\begin{aligned} Y_t^n &= \xi + \int_0^t f_n(s, Y_s^n, Z_s^n) ds + \int_0^t \phi(s, Y_s^n) dA_s \\ &\quad + \int_0^t g(s, Y_s^n, Z_s^n)\, dB_s - \int_0^t Z_s^n \downarrow dW_s, \end{aligned} \tag{2.4}$$

that is the penalized version of reflected generalized BDSDE (ξ, f, g, ϕ, S). We point out that our version of generalized BDSDE is in fact a time reversal of that considered by Boufoussi et al [1], due to the set-up of our problem. We nonetheless use the same name because they have similar nature. Consequently, it is well known (see Boufoussi et al., [1]) that, there exist a unique $(Y^n, Z^n) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$ solution of the generalized BDSDE (2.4) such that for each $n \in \mathbb{N}^*$,

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2\, ds\right) < \infty.$$

In order to prove Theorem 2.2, we state the following lemmas that will be useful

Lemma 2.1 *Let us consider* $(Y^n, Z^n) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$ *solution of BDSDE* (2.4). *Then there exists* $C > 0$ *such that,*

$$\sup_{n\in\mathbb{N}^*} \mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t^n|^2 + \int_0^T |Y_s^n\|^2\, dA_s + \int_0^T \|Z_s^n\|^2\, ds + |K_T^n|^2\right) < C$$

where

$$K_t^n = n\int_0^t (Y_s^n - S_s)^- ds, \;\; 0 \leq t \leq T. \tag{2.5}$$

Proof.
From Itô's formula, it follows that

$$
\begin{aligned}
&|Y_t^n|^2 + \int_0^t \|Z_s^n\|^2\, ds \\
&\leq |\xi|^2 + 2\int_0^t Y_s^n f(s,Y_s^n,Z_s^n)ds + 2\int_0^t Y_s^n \phi(s,Y_s^n)dA_s + \int_0^t \|g(s,Y_s^n,Z_s^n)\|^2 ds \\
&+ 2\int_0^t S_s dK_s^n + 2\int_0^t \langle Y_s^n g(s,Y_s^n,Z_s^n), dB_s\rangle - 2\int_0^t \langle Y_s^n Z_s^n, \downarrow dW_s\rangle,
\end{aligned}
\tag{2.6}
$$

where we have used $\int_0^t (Y_s^n - S_s)dK_s^n \leq 0$ and the fact that

$$
\int_0^t Y_s^n dK_s^n \;=\; \int_0^t (Y_s^n - S_s)dK_s^n + \int_0^t S_s dK_s^n \leq \int_0^t S_s dK_s^n.
$$

Using $(\mathbf{H}_2)$ and the elementary inequality $2ab \leq \gamma a^2 + \frac{1}{\gamma}b^2,\ \forall \gamma > 0$,

$$
\begin{aligned}
2Y_s^n f(s,Y_s^n,Z_s^n) &\leq (c\gamma_1 + \frac{1}{\gamma_1})|Y_s^n|^2 + 2c\gamma_1\|Z_s^n\|^2 + 2\gamma_1 f_s^2, \\
2Y_s^n \phi(s,Y_s^n) &\leq (\gamma_2 - 2|\beta|)|Y_s^n|^2 + \frac{1}{\gamma_2}\phi_s^2, \\
\|g(s,Y_s^n,Z_s^n)\|^2 &\leq K(1+\gamma_3)c|Y_s^n|^2 + \alpha(1+\gamma_3)\|Z_s^n\|^2 + (\frac{1}{\gamma_3}+1)g_s^2.
\end{aligned}
$$

Taking expectation in both sides of the inequality (2.6) and choosing $\gamma_1 = \dfrac{1-\alpha}{6c}$, $\gamma_2 = |\beta|$ and $\gamma_3 = \dfrac{1-\alpha}{2\alpha}$ we obtain for all $\varepsilon > 0$

$$
\begin{aligned}
&\mathbb{E}\,|Y_t^n|^2 + |\beta|\mathbb{E}\int_0^t |Y_s^n|^2\, dA_s + \frac{1-\alpha}{6}\mathbb{E}\int_0^t \|Z_s^n\|^2\, ds \\
&\leq C\mathbb{E}\left\{|\xi|^2 + \int_0^T |Y_s^n|^2 ds + \int_0^T f_s^2 ds + \int_0^T \phi_s^2 dA_s + \int_0^T g_s^2 ds\right\} \\
&+ \frac{1}{\varepsilon}\mathbb{E}\left(\sup_{0\leq s\leq T}(S_s^+)^2\right) + \varepsilon\mathbb{E}\,(K_T^n)^2.
\end{aligned}
\tag{2.7}
$$

On the other hand, we get from (2.4) that for all $0 \leq t \leq T$

$$
K_t^n = Y_t^n - \xi - \int_0^t f_n(s,Y_s^n,Z_s^n)ds - \int_0^t \phi(s,Y_s^n)dA_s - \int_0^t g(s,Y_s^n,Z_s^n)dB_s + \int_0^t Z_s^n d\downarrow W_s, \tag{2.8}
$$

then

$$
E(K_T^n)^2 \leq C\mathbb{E}\left\{\int_0^T f_s^2 ds + \int_0^T \phi_s^2 dA_s + \int_0^T g_s^2 ds\right\} + C\mathbb{E}\left(\int_0^T |Y_s^n|^2\, ds + \int_0^T \|Z_s^n\|^2 ds\right) \tag{2.9}
$$

which yield, in light of (2.7), that for $\varepsilon = \dfrac{1-\alpha}{12c}$

$$\begin{aligned}
&\mathbb{E}\,|Y_t^n|^2 + \mathbb{E}\int_0^t |Y_s^n|^2\, dA_s + \mathbb{E}\int_0^t \|Z_s^n\|^2\, ds \\
&\leq C\mathbb{E}\left\{|\xi|^2 + \int_0^T |Y_s^n|^2 ds + \int_0^T f_s^2 ds + \int_0^T \phi_s^2 dA_s + \int_0^T g_s^2 ds + \mathbb{E}\left(\sup_{0\leq s\leq T}(S_s^+)^2\right)\right\}
\end{aligned}$$

Consequently, it follows from Gronwall's lemma and (2.9) that

$$\begin{aligned}
&\mathbb{E}\left\{|Y_t^n|^2 + \int_0^t |Y_s^n|^2\, dA_s + \int_0^t \|Z_s^n\|^2 ds + |K_T^n|^2\right\} \\
&\leq C\mathbb{E}\left\{|\xi|^2 + \int_0^T f_s^2 ds + \int_0^T \phi_s^2 dA_s + \int_0^T g_s^2 ds + \sup_{0\leq t\leq T}(S_t^+)^2\right\}.
\end{aligned}$$

Finally applied Burkholder-Davis-Gundy inequality we obtain

$$\begin{aligned}
\mathbb{E}\left\{\sup_{0\leq t\leq T}|Y_t^n|^2 + \int_0^T \|Z_s^n\|^2 ds + |K_T^n|^2\right\} \;\leq\; & C\mathbb{E}\left\{|\xi|^2 + \int_0^T f_s^2 ds + \int_0^T \phi_s^2 dA_s\right. \\
& \left. + \int_0^T g_s^2 ds + \sup_{0\leq t\leq T}(S_t^+)^2\right\},
\end{aligned}$$

which end the proof of this Lemma. ■

Now we give a convergence result which is the key point on the proof of our main result. We begin by suppose that g is independent from (Y, Z). Precisely we consider the following equation

$$Y_t \;=\; \xi + \int_0^t f(s, Y_s, Z_s)ds + \int_0^t \phi(s, Y_s)dA_s + \int_0^t g(s)\, dB_s - \int_0^t Z_s \downarrow dW_s + K_t, \tag{2.10}$$

The penalized equation is given by

$$\begin{aligned}
Y_t^n \;=\; & \xi + \int_0^t f(s, Y_s^n, Z_s^n)ds + n\int_0^t (Y_s^n - S_s)^- ds + \int_0^t \phi(s, Y_s^n)dA_s \\
& + \int_0^t g(s)\, dB_s - \int_0^t Z_s^n \downarrow dW_s,
\end{aligned} \tag{2.11}$$

Since the function $y \mapsto n(y - S_t)^-$ is non-decreasing, then thanks to the comparison theorem 2.1, the sequence $(Y^n)_{n>0}$ is non-decreasing. Hence, Lemma 2.1 implies that there exists a $\mathcal{F}_t$-progressively measurable process Y such that $Y_t^n \nearrow Y_t$ $a.s.$ So the following result hold

Lemma 2.2 *If g does not dependent on (Y,Z), then for each $n \in \mathbb{N}^*$,*

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|(Y_t^n - S_t)^-\right|^2\right) \longrightarrow 0, \quad as\ n \longrightarrow \infty.$$

Proof. Since $Y_t^n \ge Y_t^0$, we can w.l.o.g. replace S_t by $S_t \vee Y_t^0$, i.e. we may assume that $\mathbb{E}(\sup_{0\le t\le T} S_t^2) < \infty$. We want to compare a.s. Y_t and S_t for all $t \in [0,T]$, while we do not know yet if Y is a.s. continuous. Indeed, let us introduce the following processes

$$\begin{cases} \overline{\xi} := \xi + \displaystyle\int_0^T g(s)\, dB_s \\ \overline{S}_t := S_t + \displaystyle\int_t^T g(s)\, dB_s \\ \overline{Y}_t^n := Y_t^n + \displaystyle\int_t^T g(s)\, dB_s \end{cases}$$

Hence,

$$\overline{Y}_t^n = \overline{\xi} + \int_0^t f(s, Y_s^n, Z_s^n)\, ds + n\int_0^t \left(\overline{Y}_s^n - \overline{S}_s\right)^- ds + \int_0^t \phi(s, Y_s^n)\, dA_s - \int_0^t Z_s^n \downarrow dW_s. \quad (2.12)$$

and we define $\quad \overline{Y}_t := \sup_n \overline{Y}_t^n$.

From Theorem 2.1, we have that a.s., $\overline{Y}_t^n \ge \widetilde{Y}_t^n$, $0 \le t \le T$, $n \in \mathbb{N}^*$, where $\left\{(\widetilde{Y}_t^n, \widetilde{Z}_t^n),\ 0 \le t \le T\right\}$ is the unique solution of the BDSDE

$$\widetilde{Y}_t^n = \overline{S}_T + \int_0^t f(s, Y_s^n, Z_s^n)\, ds + n\int_0^t (\overline{S}_s - \widetilde{Y}_s^n) ds + \int_0^t \phi(s, Y_s^n)\, dA_s - \int_0^t \widetilde{Z}_s^n \downarrow dW_s.$$

Recalling that $\mathcal{F}_t$ is not a filtration, let ν be a $(\mathcal{G})_t$−stopping time such that $0 \le \nu \le T$, where $\mathcal{G}_t = \mathcal{F}_{t,T}^W \otimes \mathcal{F}_{0,T}^B$, which is a filtration. So we can write

$$\begin{aligned} \widetilde{Y}_\nu^n &= \mathbb{E}\left\{e^{-n\nu}\overline{S}_T + \int_0^\nu e^{-n(\nu-s)} f(s, Y_s^n, Z_s^n) ds + n\int_0^\nu e^{-n(\nu-s)}\overline{S}_s ds \right. \\ &\quad \left. + \int_0^\nu e^{-n(\nu-s)}\phi(s, Y_s^n) dA_s \mid \mathcal{G}_\nu \right\}. \end{aligned}$$

Moreover

$$\int_0^\nu e^{-n(\nu-s)} f(s, Y_s^n, Z_s^n) ds \le \frac{1}{\sqrt{2n}}\left(\int_0^\nu (f(s, Y_s^n, Z_s^n))^2\, ds\right)^{\frac{1}{2}}$$

and

$$\int_0^\nu e^{-n(\nu-s)}\phi(s, Y_s^n) dA_s \le \left(\int_0^\nu e^{-2n(\nu-s)} dA_s\right)^{\frac{1}{2}}\left(\int_0^\nu |\phi(s, Y_s^n)|^2 dA_s\right)^{\frac{1}{2}},$$

which imply that

$$\int_0^{\nu} e^{-n(\nu-s)} f(s, Y_s^n, Z_s^n)ds + \int_0^{\nu} e^{-n(\nu-s)} \phi(s, Y_s^n)dA_s \to 0, \text{ in } L^2 \text{ as } n \to \infty$$

Consequently, we get that $\widetilde{Y}_\nu^n \longrightarrow \overline{S}_\nu$ *a.s.* and in $L^2(\Omega)$, as $n \to \infty$, and $\overline{Y}_\nu \geq \overline{S}_\nu$ a.s. which yields that $Y_\nu \geq S_\nu$ a.s. From this and section Theorem in Dellacherie and Meyer [6], the last inequality hold for all $t \in [0,T]$. Further $(Y_t^n - S_t)^- \downarrow 0$, a.s. and from Dini's theorem, the convergence is uniform in t. Finally, as $(Y_t^n - S_t)^- \leq (S_t - Y_t^0)^+ \leq |S_t| + \left|Y_t^0\right|$, the dominated convergence theorem ensures that

$$\lim_{n \longrightarrow +\infty} \mathbb{E}(\sup_{0 \leq t \leq T} | (Y_t^n - S_t)^- |^2) = 0.$$

■

Proof of Theorem 2.2. Existence The proof of existence will be divided in two steps.

Step 1. g does not dependent on (Y, Z).

Recalling that $Y_t^n \nearrow Y_t$ *a.s.* Then, Fatou's lemma and Lemma 2.1 ensure

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t|^2 \right) < +\infty,$$

It then follows from Lemma 2.1 and Lebegue's dominated convergence theorem that

$$\mathbb{E}\left(\int_0^T |Y_s^n - Y_s|^2 \, ds \right) \longrightarrow 0, \quad \text{as } n \to \infty. \tag{2.13}$$

Next, we will prove that the sequence of processes Z^n converges in $M^2(0,T;\mathbb{R}^d)$. For this end, for $n \geq p \geq 1$, Itô's formula gives,

$$\begin{aligned}
&|Y_t^n - Y_t^p|^2 + \int_0^t |Z_s^n - Z_s^p|^2 \, ds \\
= \;& 2\int_0^t (Y_s^n - Y_s^p)[f(s, Y_s^n, Z_s^n) - f(s, Y_s^p, Z_s^p)]ds + 2\int_0^t (Y_s^n - Y_s^p)[\phi(s, Y_s^n) - \phi(s, Y_s^p)]dA_s \\
& -2\int_0^t (Y_s^n - Y_s^p)\langle (Z_s^n - Z_s^p), \downarrow dW_s \rangle + 2\int_0^t (Y_s^n - Y_s^p)(dK_s^n - dK_s^p).
\end{aligned}$$

From the same step as previous, by use again assumptions (**H**2), there exists a constant $C > 0$, such that

$$\begin{aligned}
& \mathbb{E}\left\{ |Y_t^n - Y_t^p|^2 + \int_0^t |Y_s^n - Y_s^p|^2 \, dA_s + \int_0^t \|Z_s^n - Z_s^p\|^2 \, ds \right\} \\
\leq \;& C\mathbb{E}\left\{ \int_0^t |Y_s^n - Y_s^p|^2 ds + \sup_{0 \leq s \leq T} (Y_s^n - S_s)^- K_T^p + \sup_{0 \leq s \leq T} (Y_s^p - S_s)^- K_T^n \right\},
\end{aligned}$$

which, by Gronwall lemma, Hölder inequality and Lemma 2.1 respectively, imply

$$\begin{aligned} E\left\{|Y_t^n - Y_t^p|^2 + \int_0^t \|Z_s^n - Z_s^p\|^2\,ds\right\} &\le C\left\{\mathbb{E}\left(\sup_{0\le s\le T} |\,(Y_s^n - S_s)^-\,|^2\right)\right\}^{1/2} \\ &\quad + C\left\{\mathbb{E}\left(\sup_{0\le s\le T} |\,(Y_s^p - S_s)^-\,|^2\right)\right\}^{1/2}. \quad (2.14) \end{aligned}$$

Consequently, it follows from Lemma 2.2 that,

$$\mathbb{E}\left\{|Y_s^n - Y_s^p|^2 + \int_0^t \|Z_s^n - Z_s^p\|^2\,ds\right\} \longrightarrow 0, \quad \text{as } n,p \longrightarrow \infty.$$

Finally, from Bürkhölder-Davis-Gundy's inequality, we obtain

$$\mathbb{E}\left(\sup_{0\le s\le T} |Y_s^n - Y_s^p|^2 + \int_0^T \|Z_s^n - Z_s^p\|^2\,ds\right) \longrightarrow 0, \quad \text{as } n,p \longrightarrow \infty,$$

and from (2.8) we can deduce

$$\mathbb{E}\left\{\sup_{0\le s\le T} |K_s^n - K_s^p|^2\right) \longrightarrow 0, \text{ as } n,p \to \infty,$$

which provide that the sequence of processes (Y^n, Z^n, K^n) is a Cauchy in the Banach space $S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d) \times S^2([0,T];\mathbb{R})$. Consequently there exists a triplet $(Y,Z,K) \in S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d) \times S^2([0,T];\mathbb{R})$ such that

$$\mathbb{E}\left\{\sup_{0\le s\le T} |Y_s^n - Y_s|^2 + \int_0^T \|Z_s^n - Z_s\|^2\,ds + \sup_{0\le s\le T} |K_s^n - K_s|^2\right) \to 0, \text{ as } n \to \infty.$$

It remains to show that (Y,Z,K) solve reflected BDSDE (ξ, f, ϕ, g, S). In this fact, in view of Lemma 2.2, $Y_t \ge S_t$ a.s., and $\int_0^T (Y_s - S_s)dK_s \ge 0$. Moreover $\int_0^T (Y_s^n - S_s)dK_s^n = -n\int_0^T |(Y_s^n - S_s)^-|^2 ds \le 0$ and passing to the limit we get $\int_0^T (Y_s - S_s)dK_s \le 0$, which together with the above proved (ii) of the definition. Finally passing to the limit in (2.11) we proved that (Y,Z,K) verify (2.10).

Step 2. The general case. In light of the above step, the BDSDE

$$Y_t = \xi + \int_0^t f(s,Y_s,Z_s)ds + \int_0^t \phi(s,Y_s)dA_s + \int_0^t g(s,\bar{Y}_s,\bar{Z}_s)\,dB_s - \int_0^t Z_s \downarrow dW_s + K_t$$

has a unique solution (Y,Z,K). So, we can define the mapping

$$\begin{aligned} \Psi:\ \ M^2(0,T;\mathbb{R}^d) &\longrightarrow M^2(0,T;\mathbb{R}^d) \\ (\bar{Y},\bar{Z}) &\longmapsto (Y,Z) = \Psi(\bar{Y},\bar{Z}). \end{aligned}$$

Now, let (Y,Z), (Y',Z'), $(\bar{Y},\bar{Z})$ and $(\bar{Y'},\bar{Z'})\in M^2(0,T;\mathbb{R}^d)$ such that $(Y,Z)=\Psi(\bar{Y},\bar{Z})$ and $(Y',Z')=\Psi(\bar{Y'},\bar{Z'})$. Put $\Delta\eta=\eta-\eta'$ for $\eta=Y,\bar{Y},Z,\bar{Z}$. By virtue of Itô's formula, we have

$$
\begin{aligned}
&\mathbb{E}e^{-\mu t}|\Delta Y_t|^2+\mathbb{E}\int_0^t e^{-\mu s}\|\Delta Z_s\|^2ds\\
&=2\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s\left\{f(s,Y_s,Z_s)-f(s,Y'_s,Z'_s)\right\}ds+2\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s\left\{\phi(s,Y_s)-\phi(s,Y'_s)\right\}dA_s\\
&+2\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s d(K_s-K'_s)+\int_0^t e^{-\mu s}\left\|g(s,\bar{Y}_s,\bar{Z}_s)-g(s,\bar{Y'}_s,\bar{Z'}_s)\right\|^2ds-\mu\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta Y_s\right|^2ds.
\end{aligned}
$$

But $\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s d(K_s-K'_s)\le 0$, then we obtain from (**H**2)

$$
\begin{aligned}
&\mathbb{E}e^{-\mu t}|\Delta Y_t|^2+|\beta|\mathbb{E}\int_0^t e^{-\mu s}|\Delta Y_s|^2dA_s+\mu\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta Y_s\right|^2ds+\mathbb{E}\int_0^t e^{-\mu s}\|\Delta Z_s\|^2ds\\
&\le 2\mathbb{E}\int_0^t e^{-\mu s}\Delta Y_s\left\{f(s,Y_s,Z_s)-f(s,Y'_s,Z'_s)\right\}ds+\int_0^t e^{-\mu s}\left\|g(s,\bar{Y}_s,\bar{Z}_s)-g(s,\bar{Y'}_s,\bar{Z'}_s)\right\|^2ds
\end{aligned}
$$

By use again assumptions (**H**2), there exists constants $c(\alpha)>0$, such that,

$$
\begin{aligned}
&(\mu-c(\alpha))\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta Y_s\right|^2ds+\mathbb{E}\int_0^t e^{-\mu s}\|\Delta Z_s\|^2ds\\
&\le\frac{1+\alpha}{2}\left(\bar{c}\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta\bar{Y}_s\right|^2ds+\mathbb{E}\int_0^t e^{-\mu s}\left\|\Delta\bar{Z}_s)\right\|^2ds\right)
\end{aligned}
$$

Choosing μ such that $\mu-c(\alpha)=\bar{c}>0$, we obtain

$$
\begin{aligned}
&\bar{c}\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta Y_s\right|^2ds+\mathbb{E}\int_0^t e^{-\mu s}\|\Delta Z_s\|^2ds\\
&\le\frac{1+\alpha}{2}\left(\bar{c}\mathbb{E}\int_0^t e^{-\mu s}\left|\Delta\bar{Y}_s\right|^2ds+\mathbb{E}\int_0^t e^{-\mu s}\left\|\Delta\bar{Z}_s)\right\|^2ds\right)
\end{aligned}
$$

Since $\dfrac{1+\alpha}{2}<1$, then it follows that Ψ is a contraction and its unique fixed point solves our BDSDE.

Uniqueness
Let us define

$$\{(\Delta Y_t,\Delta Z_t,\Delta K_t)\,,\ 0\le t\le T\}=\{(Y_t-Y'_t,Z_t-Z'_t,K_t-K'_t),\ 0\le t\le T\}$$

where $\{(Y_t,Z_t,K_t)\,,\ 0\le t\le T\}$ and $\{(Y'_t,Z'_t,K'_t),\ 0\le t\le T\}$ denote two solutions of the reflected BDSDE associated to the data (ξ,f,g,ϕ,S). Let us first note that

$$\int_0^T\Delta Y_s\Delta dK_s\le 0. \tag{2.15}$$

Moreover, Itô formula yields that for every $0 \leq t \leq T$

$$
\begin{aligned}
& |\Delta Y_t|^2 + \int_0^t \|\triangle Z_s\|^2 ds \\
= \;& 2\int_0^t \triangle Y_s(f(s,Y_s,Z_s) - f(s,Y'_s,Z'_s))ds + \int_0^t \|g(s,Y_s,Z_s) - g(s,Y'_s,Z'_s)\|^2 ds \\
& 2\int_0^t \triangle Y_s(\phi(s,Y_s) - \phi(s,Y'_s))dA_s + \int_0^t \langle \triangle Y_s, (g(s,Y_s,Z_s) - g(s,Y'_s,Z'_s))\rangle dB_s \\
& -2\int_0^t \langle \triangle Y_s, \triangle Z_s dW_s\rangle + 2\int_0^t \triangle Y_s \triangle dK_s.
\end{aligned}
$$

Then, by using similar computation as the proof of existence and (2.15) one have

$$
\mathbb{E}\left\{|\Delta Y_t|^2 + \int_0^T |\triangle Y_s| dA_s + \int_0^T \|\triangle Z_s\|^2 ds\right\} \;\leq\; C\mathbb{E}\int_0^T |\triangle Y_s| ds,
$$

from which, we deduce that $\Delta Y_t = 0$ and further $\Delta Z_t = 0$. On the other hand since

$$
\begin{aligned}
\Delta K_t \;=\; & \Delta Y_t - \int_0^t \left(f(s,Y_s,Z_s) - f(s,Y'_s,Z'_s)\right) ds - \int_0^t \left(\phi(s,Y_s) - \phi(s,Y'_s)\right) dA_s \\
& - \int_0^t \left(g(s,Y_s,Z_s) - g(s,Y'_s,Z'_s)\right) dB_s + \int_0^t \triangle Z_s \downarrow dW_s,
\end{aligned}
$$

we have $\Delta K_t = 0$ which end the proof of the theorem. ■

3 Connection to stochastic viscosity solution for reflected SPDEs with nonlinear Neumann boundary condition

In this section we will investigate the reflected generalized BDSDEs studied in the previous section in order to give a probabilistic interpretation for the stochastic viscosity solution of a class of nonlinear reflected SPDEs with nonlinear Neumann boundary condition.

3.1 Notion of stochastic viscosity solution for reflected SPDEs with nonlinear Neumann boundary condition

With the same notations as in Section 2, let recall $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{0\leq t\leq T}$, where B is a one dimensional Brownian motion. By $\mathcal{M}_{0,T}^B$, we denote all the $\mathbf{F}^B-$ stopping times τ such $0 \leq \tau \leq T$, a.s. $\mathcal{M}_\infty^B$ will be all $\mathbf{F}^B-$ stopping times that are almost surely finite. For generic Euclidean spaces E and E_1 we introduce the following vector spaces of functions:

1. The symbol $\mathcal{C}^{k,n}([0,T]\times E;E_1)$ stands for the space of all E_1-valued functions defined on $[0,T]\times E$ which are k-times continuously differentiable in t and n-times continuously differentiable in x, and $\mathcal{C}_b^{k,n}([0,T]\times E;E_1)$ denotes the subspace of $\mathcal{C}^{k,n}([0,T]\times E;E_1)$ in which all functions have uniformly bounded partial derivatives.

2. For any sub-σ-field $\mathcal{G}\subseteq\mathcal{F}_T^B$, $\mathcal{C}^{k,n}(\mathcal{G},[0,T]\times E;E_1)$ (resp. $\mathcal{C}_b^{k,n}(\mathcal{G},[0,T]\times E;E_1)$) denote the space of all $\mathcal{C}^{k,n}([0,T]\times E;E_1)$ (resp. $\mathcal{C}_b^{k,n}([0,T]\times E;E_1)$-valued random variable that are $\mathcal{G}\otimes\mathcal{B}([0,T]\times E)$-measurable;

3. $\mathcal{C}^{k,n}(\mathbf{F}^B,[0,T]\times E;E_1)$ (resp.$\mathcal{C}_b^{k,n}(\mathbf{F}^B,[0,T]\times E;E_1)$) is the space of all random fields $\phi\in\mathcal{C}^{k,n}(\mathcal{F}_T,[0,T]\times E;E_1$ (resp. $\mathcal{C}^{k,n}(\mathcal{F}_T,[0,T]\times E;E_1)$, such that for fixed $x\in E$, the mapping $(t,\omega_1)\rightarrow\alpha(t,\omega_1,x)$ is $\mathbf{F}^B$-progressively measurable.

4. For any sub-σ-field $\mathcal{G}\subseteq\mathcal{F}^B$ and a real number $p\geq 0$, $L^p(\mathcal{G};E)$ to be all E-valued $\mathcal{G}$-measurable random variable ξ such that $\mathbb{E}|\xi|^p<\infty$.

Furthermore, regardless their dimensions we denote by $\langle .,\rangle$ and $|.|$ the inner product and norm in E and E_1, respectively. For $(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}$, we denote $D_x=(\frac{\partial}{\partial x_1},....,\frac{\partial}{\partial x_d})$, $D_{xx}=(\partial^2_{x_ix_j})_{i,j=1}^d$, $D_y=\frac{\partial}{\partial y}$, $D_t=\frac{\partial}{\partial t}$. The meaning of D_{xy} and D_{yy} is then self-explanatory.

Let Θ be an open connected bounded domain of $\mathbb{R}^d$ $(d\geq 1)$. We suppose that Θ is smooth domain, which is such that for a function $\psi\in\mathcal{C}_b^2(\mathbb{R}^d)$, Θ and its boundary $\partial\Theta$ are characterized by $\Theta=\{\psi>0\}$, $\partial\Theta=\{\psi=0\}$ and, for any $x\in\partial\Theta$, $\nabla\psi(x)$ is the unit normal vector pointing towards the interior of Θ.

In this section, we consider the continuous coefficients f and ϕ,

$$\begin{aligned} f &: \Omega_1\times[0,T]\times\overline{\Theta}\times\mathbb{R}\times\mathbb{R}^d\longrightarrow\mathbb{R} \\ \phi &: \Omega_1\times[0,T]\times\overline{\Theta}\times\mathbb{R}\longrightarrow\mathbb{R} \end{aligned}$$

with the property that for all $x\in\overline{\Theta}$, $f(.,x,.,.)$ and $g(.,x,.)$ are Lipschitz continuous in x and satisfy the conditions (H_1') and (H_2), uniformly in x, where, for some constant $K>0$, the condition (H_1') is:

$$(H_1')\begin{cases} |f(t,x,y,z)|\leq K(1+|x|+|y|+\|z\|), \\ |\phi(t,x,y)|\leq K(1+|x|+|y|). \end{cases}$$

Furthermore, we shall make use the following assumptions:

(H_3) The function $\sigma:\mathbb{R}^d\longrightarrow\mathbb{R}^{d\times d}$ and $b:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ are uniformly Lipschitz continuous , with common Lipschitz constant $K>0$.

(H_4) The functions $l : \overline{\Theta} \longrightarrow \mathrm{I\!R}$ and $h : [0,T] \times \overline{\Theta} \longrightarrow \mathrm{I\!R}$ are continuous such that, for some $K > 0$,

$$
\begin{aligned}
|l(x)| &\leq K(1+|x|) \\
|h(t,x)| &\leq K(1+|x|) \\
h(0,x) &\leq l(x), \qquad x \in \overline{\Theta}.
\end{aligned}
$$

(H_5) The function $g \in \mathcal{C}_b^{0,2,3}([0,T] \times \overline{\Theta} \times \mathrm{I\!R}; \mathrm{I\!R})$.

Let us consider the related obstacle problem for SPDEs with nonlinear Neumann boundary condition:

$$
\mathcal{OP}^{(f,\phi,g,h,l)} \begin{cases} \min \{u(t,x) - h(t,x),\ du(t,x) + [Lu(t,x) + f(t,x,u(t,x),\sigma^*(x)D_x u(t,x))]dt \\ \qquad +g(t,x,u(t,x))\, dB_s\} = 0, \quad (t,x) \in [0,T] \times \Theta \\ u(0,x) = l(x), \qquad x \in \Theta \\ \dfrac{\partial u}{\partial n}(t,x) + \phi(t,x,u(t,x)) = 0, \quad (t,x) \in [0,T] \times \partial\Theta, \end{cases}
$$

where

$$
L = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma(x)\sigma^*(x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}, \quad \forall\, x \in \Theta,
$$

and

$$
\frac{\partial}{\partial n} = \sum_{i=1}^{d} \frac{\partial \psi}{\partial x_i}(x) \frac{\partial}{\partial x_i}, \quad \forall\, x \in \partial\Theta.
$$

As the work of Buckdahn-Ma [2, 3], our next goal is to define the notion of stochastic viscosity to $\mathcal{OP}^{(f,\phi,g,h)}$. So, we shall recall some of their notation. Let $\eta \in \mathcal{C}(\mathbf{F}^B, [0,T] \times \mathrm{I\!R}^d \times \mathrm{I\!R})$ be the solution to the equation

$$
\eta(t,x,y) = y + \int_0^t \langle g(s,x,\eta(s,x,y)), \circ dB_s \rangle. \tag{3.1}
$$

then the mapping $y \mapsto \eta(s,x,y)$ defines a diffeomorphism for all t, x $a.s.$ Hence if we denote by $\varepsilon(s,x,y)$ its y-inverse, we obtain

$$
\varepsilon(t,x,y) = y - \int_0^t \langle D_y \varepsilon(s,x,y) g(s,x,y), \circ dB_s \rangle. \tag{3.2}
$$

To simplify the notation in the sequel we denote

$$A_{f,g}(\varphi(t,x)) = \mathbf{L}\varphi(t,x) + f(t,x,\varphi(t,x),\sigma^* D_x\varphi(t,x)) - \frac{1}{2}(g, D_y g)(t,x,\varphi(t,x))$$
and $\Psi(t,x) = \eta(t,x,\varphi(t,x))$.

Definition 3.1 *A random field $u \in \mathcal{C}\left(\mathbf{F}^B, [0,T] \times \overline{\Theta}\right)$ is called a stochastic viscosity subsolution of the stochastic obstacle problem $\mathcal{OP}^{(f,\phi,g,h,l)}$ if $u(0,x) \le l(x)$, for all $x \in \overline{\Theta}$, and if for any stopping time $\tau \in \mathcal{M}_{0,T}^B$, any state variable $\xi \in L^0\left(\mathcal{F}_\tau^B, \Theta\right)$, and any random field $\varphi \in \mathcal{C}^{1,2}\left(\mathcal{F}_\tau^B,\ [0,T] \times \mathbb{R}^d\right)$, with the property that for $\mathbb{P}$-almost all $\omega \in \{0 < \tau < T\}$ the inequality*

$$u(t,\omega,x) - \Psi(t,\omega,x) \le 0 = u(\tau(\omega),\xi(\omega)) - \Psi(\tau(\omega),\xi(\omega))$$

is fulfilled for all (t,x) in some neighborhood $\mathcal{V}(\omega,\tau(\omega),\xi(\omega))$ of $(\tau(\omega),\xi(\omega))$, the following conditions are satisfied:

(a) on the event $\{0 < \tau < T\}$ the inequality

$$\min\left\{u(\tau,\xi) - h(\tau,\xi), \mathrm{A}_{f,g}(\Psi(\tau,\xi)) - D_y\Psi(\tau,\xi)\, D_t\varphi(\tau,\xi)\right\} \le 0 \tag{3.3}$$

holds $\mathbb{P}$-almost surely;

(b) on the event $\{0 < \tau < T\} \cap \{\xi \in \partial\Theta\}$ the inequality

$$\begin{aligned}&\min\Big[\min\left\{u(\tau,\xi) - h(\tau,\xi), \mathrm{A}_{f,g}(\Psi(\tau,\xi)) - D_y\Psi(\tau,\xi)\, D_t\varphi(\tau,\xi)\right\},\\ &\qquad -\frac{\partial\Psi}{\partial n}(\tau,\xi) - \phi(\tau,\xi,\psi(\tau,\xi))\Big] \le 0\end{aligned} \tag{3.4}$$

holds $\mathbb{P}$-almost surely.

A random field $u \in \mathcal{C}\left(\mathbf{F}^B, [0,T] \times \overline{\Theta}\right)$ is called a stochastic viscosity supersolution of the stochastic obstacle problem $\mathcal{OP}^{(f,\phi,g,h,l)}$ if $u(0,x) \ge l(x)$, for all $x \in \overline{\Theta}$, and if for any stopping time $\tau \in \mathcal{M}_{0,T}^B$, any state variable $\xi \in L^0\left(\mathcal{F}_\tau^B, \Theta\right)$, and any random field $\varphi \in \mathcal{C}^{1,2}\left(\mathcal{F}_\tau^B,\ [0,T] \times \mathbb{R}^d\right)$, with the property that for $\mathbb{P}$-almost all $\omega \in \{0 < \tau < T\}$ the inequality

$$u(t,\omega,x) - \Psi(t,\omega,x) \ge 0 = u(\tau(\omega),\xi(\omega)) - \Psi(\tau(\omega),\xi(\omega))$$

is fulfilled for all (t,x) in some neighborhood $\mathcal{V}(\omega,\tau(\omega),\xi(\omega))$ of $(\tau(\omega),\xi(\omega))$, the following conditions are satisfied:

(a) on the event $\{0 < \tau < T\}$ the inequality

$$\min\left\{u(\tau,\xi) - h(\tau,\xi), \mathrm{A}_{f,g}(\Psi(\tau,\xi)) - D_y\Psi(\tau,\xi)\, D_t\varphi(\tau,\xi)\right\} \ge 0 \tag{3.5}$$

holds $\mathbb{P}$-almost surely;

(b) on the event $\{0 < \tau < T\} \cap \{\xi \in \partial\Theta\}$ the inequality

$$\max\left[\min\left\{u(\tau,\xi) - h(\tau,\xi), \mathrm{A}_{f,g}\left(\Psi\left(\tau,\xi\right)\right) - D_y\Psi\left(\tau,\xi\right)D_t\varphi\left(\tau,\xi\right)\right\}\right., \\ \left. -\frac{\partial\Psi}{\partial n}\left(\tau,\xi\right) - \phi\left(\tau,\xi,\psi\left(\tau,\xi\right)\right)\right] \geq 0 \tag{3.6}$$

holds $\mathbb{P}$-almost surely.

Finally, a random field $u \in \mathcal{C}\left(\mathbf{F}^B, [0,T] \times \overline{\Theta}\right)$ is called a stochastic viscosity solution of the stochastic obstacle problem $\mathcal{OP}^{(f,\phi,g,h,l)}$, if it is both a stochastic viscosity subsolution and a supersolution.

Remark 3.1 *Observe that if f, ϕ are deterministic and $g \equiv 0$ the definition* 3.1 *coincide with the definition of (deterministic) viscosity solution of PDE $\mathcal{OP}^{(f,\phi,0,h,l)}$ giving by Ren et al [12].*

3.2 Existence of stochastic viscosity solutions for SPDE with nonlinear Neumann boundary condition

The main objective of this subsection is to show how the stochastic obstacle problem $\mathcal{OP}^{(f,\phi,g,h,l)}$ is related to refected generalized BDSDE (2.1) introduced in Section 1. In this fact, we recall some known results on reflected diffusions. We consider

$$\begin{aligned} &s \mapsto A_s^{t,x} \text{ is increasing} \\ &X_s^{t,x} = x + \int_s^t b\left(X_r^{t,x}\right)dr + \int_s^t \sigma\left(X_r^{t,x}\right)d\downarrow W_r + \int_0^t \nabla\psi\left(X_r^{t,x}\right)dA_r^{t,x}, \quad \forall\, s \in [0,t]\,, \\ &A_s^{t,x} = \int_0^s I_{\left\{X_r^{t,x} \in \partial\Theta\right\}}\, dA_r^{t,x}. \end{aligned} \tag{3.7}$$

We note here that due to the direction of the Itô integral, (3.7) should be viewed as going from t to 0 (i.e., $X_0^{t,x}$ should be understood as the terminal value of the solution $X^{t,x}$). It is then clear that under conditions (H_3) on the coefficients b and σ, 3.7 will have a strong solution that is $\mathcal{F}_s^W$-adapted, for $0 \leq s \leq t$.

Proposition 3.1 *There exist a constant $C > 0$ such that for all x, $x' \in \overline{\Theta}$ the following inequality holds:*

$$I\!\!E\left[\sup_{0\leq s\leq t}\left|X_s^{t,x} - X_s^{t,x'}\right|^2\right] \leq C\,|x - x'|^2\,,$$

$$\mathbb{E}\left(\sup_{0\leq s\leq t}\left|A_s^{t,x} - A_t^{t,x'}\right|^2\right) \leq C|x - x'|^2.$$

Moreover, for all $p \geq 1$, there exists a constant C_p such that for all $(t,x) \in \mathbb{R}_+ \times \overline{\Theta}$,

$$\mathbb{E}\left(\left|A_s^{t,x}\right|^p\right) \leq C_p(1+s^p)$$

and for each μ, $0 < s < t$, there exists a constant $C(\mu,t)$ such that for all $x \in \overline{\Theta}$,

$$\mathbb{E}\left(e^{\mu A_t^x}\right) \leq C(\mu,t).$$

On the other hand, let us consider the following reflected generalized BDSDE:

$$\begin{aligned} Y_s^{t,x} &= l\left(X_0^{t,x}\right) + \int_0^s f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr + \int_0^s g\left(r, X_r^{t,x}, Y_r^{t,x}\right) dB_r \\ &+ \int_0^s \phi\left(r, X_r^{t,x}, Y_r^{t.x}\right) dA_r^{t,x} + K_s^{t,x} - \int_0^s \left\langle Z_r^{t,x}, \downarrow dW_r\right\rangle, \quad 0 \leq s \leq t, \end{aligned} \tag{3.8}$$

where the coefficients l, f, g, ϕ and h satisfy the hypotheses (H_1'), (H_2), (H_4) and (H_5).

Proposition 3.2 *Let the ordered triplet $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ be a solution of the BDSDE (3.8). Then the random field $(s,t,x) \mapsto Y_s^{t,x}$, $(s,t,x) \in [0,T] \times [0,T] \times \Theta$ is almost surely continuous.*

Let now define

$$u(t,x) = Y_t^{t,x}, \quad (t,x) \in [0,T] \times \overline{\Theta}. \tag{3.9}$$

Theorem 3.1 *$u \in C(\mathbf{F}^B, [0,T] \times \Theta)$ and is a stochastic viscosity solution of obstacle problem $\mathcal{OP}^{(f,\phi,g,h,l)}$.*

Proof. For each $(t,x) \in [0,T] \times \Theta, n \geq 1$, let $\{{}^nY_s^{t,x}, {}^nZ_s^{t,x},\ \ 0 \leq s \leq t\}$ denote the solution of the generalized BDSDE

$$\begin{aligned} {}^nY_s^{t,x} &= l(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, {}^nY_r^{t,x}, {}^nZ_r^{t,x})dr + n\int_0^s ({}^nY_r^{t,x} - h(r, X_r^{t,x}))^- dr \\ &+ \int_0^s \phi(r, X_r^{t,x}, {}^nY_r^{t,x})dA_r^{t,x} \int_0^s g(r, X_r^{t,x}, {}^nY_r^{t,x})dB_r - \int_0^s {}^nZ_r^{t,x} \downarrow dW_r. \end{aligned}$$

It is know from Boufoussi et al [1] that

$$u_n(t,x) = {}^nY_t^{t,x}, \quad (t,x) \in [0,T] \times \overline{\Theta},$$

is the stochastic viscosity solution of the parabolic SPDE:

$$\begin{cases} du_n(t,x) + [Lu_n(t,x) + f_n(t,x,u_n(t,x),\sigma^* D_x u_n(t,x))]dt \\ +g(t,x,u_n(t,x))\,dB_t = 0, \quad (t,x) \in [0,T]\times\Theta, \\ \\ u_n(0,x) = l(x), \quad x\in\Theta, \\ \\ \dfrac{\partial u_n}{\partial n}(t,x) + \phi(t,x,u_n(t,x)) = 0, \quad (t,x)\in[0,T]\times\partial\Theta. \end{cases} \tag{3.10}$$

where $f_n(t,x,y,z) = f(t,x,y,z) + n(y-h(t,x))^-$. But from the proof of Theorem 2.2, for each $(t,x)\in[0,T]\times\mathbb{R}^d$ we have

$$u_n(t,x)\uparrow u(t,x) \;\; a.s. \quad \text{as } n\to\infty.$$

Since u_n and u are continuous, it follows from Dini's theorem that the above convergence is uniform on any compacts. We now show that u is a stochastic viscosity subsolution of obstacle problem of $\mathcal{OP}^{(f,\phi,g,h,l)}$. Let $(\tau,\xi)\in\mathcal{M}^B_{0,T}\times L^0(\mathcal{F}^B_\tau;\Theta)$ and $\varphi\in\mathcal{C}^{1,2}\left(\mathcal{F}^B_\tau,\ [0,T]\times\Theta\right)$ such that $u(\tau,\xi)>h(\tau,\xi)$, $\mathbb{P}$-a.s. and for $\mathbb{P}-$ almost all $\omega\in\{0<\tau<T\}$, we have

$$u(\omega,t,x)-\eta(\omega,t,x,\varphi(t,x))\le 0 = u(\omega,\tau(\omega),\xi(\omega))-\Psi(\omega,\tau(\omega),\xi(\omega))$$

for all (t,x) in some neighborhood $\mathcal{V}(\omega,\tau(\omega),\xi(\omega))$ of $(\tau(\omega),\xi(\omega))$. On the other hand, recalling that u_n is viscosity solution of SPDE (3.10), there is (τ_k,ξ_k) an $(\tau,\xi)-$ approximation sequence such that

$$u_{n_k}(\omega,t,x)-\Psi(\omega,t,x)\le 0 = u_{n_k}(\omega,\tau_k(\omega),\xi_k(\omega))-\Psi(\omega,\tau_k(\omega),\xi_k(\omega))$$

for all (t,x) in some neighborhood $\mathcal{V}(\omega,\tau_k(\omega),\xi_k(\omega))\subset\mathcal{V}(\omega,\tau(\omega),\xi(\omega))$ for k large enough, and we get the following:

(a) On the event $\{0<\tau_k<T\}$ the inequality

$$\mathrm{A}_{f_{n_k},g}(\Psi(\tau_k,\xi_k)) - D_y\Psi(\tau_k,\xi_k)\,D_t\varphi(\tau_k,\xi_k)\le 0$$

holds $\mathbb{P}$-a.s.

(b) On the event$\{0<\tau_k<T\}\cap\{\xi_k\in\partial\Theta\}$ the inequality

$$\begin{aligned}\min\Big[\mathrm{A}_{f_{n_k},g}(\Psi(\tau_k,\xi_k)) - D_y\Psi(\tau_k,\xi_k)\,D_t\varphi(\tau_k,\xi_k)\,,\\ -\frac{\partial\Psi}{\partial n}(\tau_k,\xi_k)-\phi_{n_k}(\tau_k,\xi_k,\Psi(\tau_k,\xi_k))\Big]\le 0\end{aligned}$$

holds $\mathbb{P}$-a.s.

From the assumption that $u(\tau,\xi) > h(\tau,\xi)$, $\mathbb{P}$-a.s. and the uniform convergence of u_n, it follows that for k enough large, $u_{n_k}(\tau_k,\xi_k) > h(\tau_k,\xi_k)$, $\mathbb{P}$-a.s., hence, taking the limit as $k\to\infty$ in the above inequality yields:

(a) On the event $\{0<\tau<T\}$ the inequality

$$\mathrm{A}_{f,g}\left(\Psi\left(\tau,\xi\right)\right)-D_y\Psi\left(\tau,\xi\right)D_t\varphi\left(\tau,\xi\right)\leq 0$$

holds $\mathbb{P}$-a.s.

(b) On the event$\{0<\tau<T\}\cap\{\xi\in\partial\Theta\}$ the inequality

$$\min\Big[\mathrm{A}_{f,g}\left(\Psi\left(\tau,\xi\right)\right)-D_y\Psi\left(\tau,\xi\right)D_t\varphi\left(\tau,\xi\right), \\ -\frac{\partial\Psi}{\partial n}(\tau,\xi)-\phi(\tau,\xi,\Psi(\tau,\xi))\Big]\leq 0$$

holds $\mathbb{P}$-a.s.

This proved that u is a stochastic viscosity subsolution of $\mathcal{OP}^{f,\phi,g,h,l}$. By the same argument as above one can show that u given by (3.9) is also a stochastic viscosity supersolution of $\mathcal{OP}^{f,\phi,g,h,l}$. So we conclude that u is a stochastic viscosity of $\mathcal{OP}^{f,\phi,g,h,l}$, that end the proof. ■

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