On the Concept of Time-Frequency Distributions Based on Complex-Lag Moments

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Work context: Analysis of Signals with complex Time-Frequency behaviour

Signals issued from diverse physical phenomena:

- Very challenging to analyse in Time-Frequency (TF) domain
- Signals composed of several TF components characterized by various non-linear contents
  $\Rightarrow$ inner interferences due to the TF non-linearity
  $\Rightarrow$ cross terms due to the multi-component structures

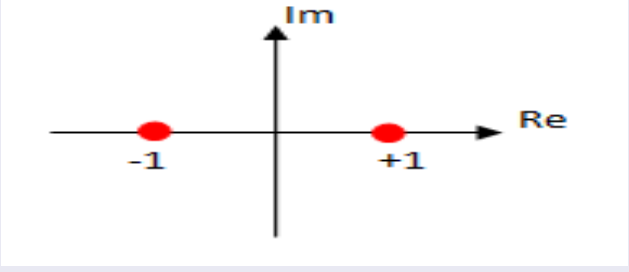
Outline

1. Time-Frequency Distribution Based on Complex-Lag Arguments

Concept of complex-lag distributions

- A way for inner interferences reduction with respect of Wigner distribution.

$$s(t) = Ae^{j\phi(t)}$$

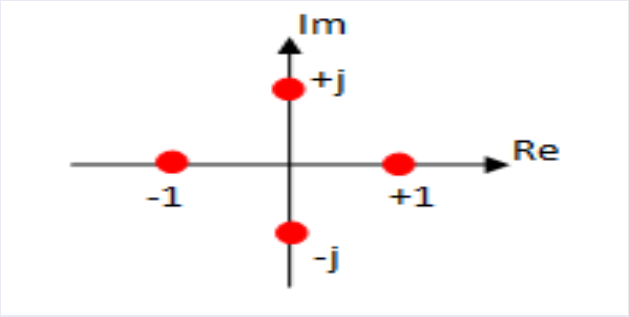


$$WVD(t,\omega) = \mathfrak{F}_\tau \left[s(t+\tfrac{\tau}{2}) s^*(t-\tfrac{\tau}{2}) \right]$$

$$WVD(t,\omega) = \delta\left(\omega - \phi^{'}(t)\right) *_\omega \mathfrak{F}_\tau \left[e^{jQ_{wv}(t,\tau)} \right]$$

with *Spread Factor* $Q_{wv}(t,\tau) = \phi^{(3)}(t)\frac{\tau^3}{2^2\,3!} + \phi^{(5)}(t)\frac{\tau^5}{2^4\,5!} + \dots$

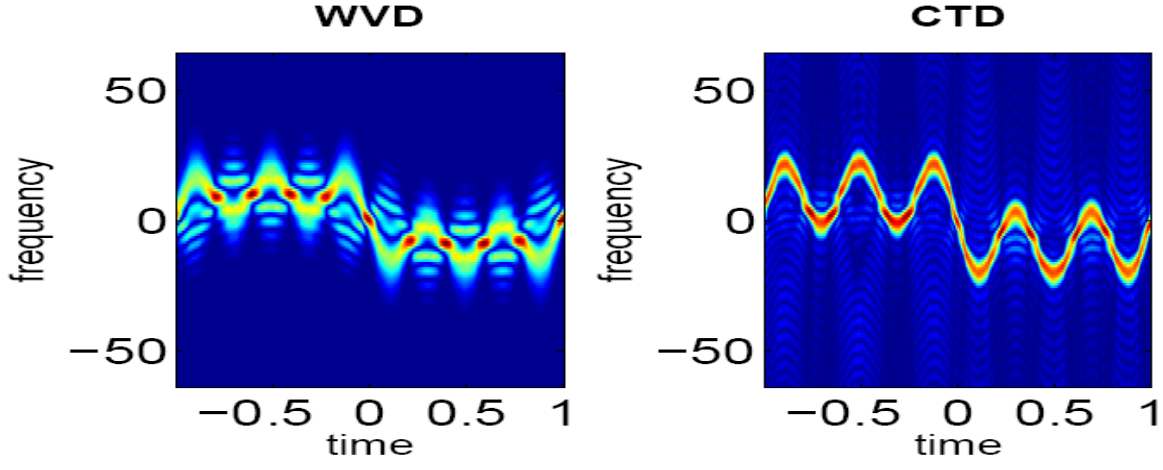
Lj. Stankovic, 2002: Consideration of lags on imaginary axis



$$CTD(t,\omega) = \mathfrak{F}_\tau \left[s(t+\tfrac{\tau}{4}) s^*(t-\tfrac{\tau}{4}) s^{-j}(t+j\tfrac{\tau}{4}) s^{j}(t-j\tfrac{\tau}{4}) \right]$$

$$CTD(t,\omega) = \delta\left(\omega - \phi^{'}(t)\right) *_\omega \mathfrak{F}_\tau \left[e^{jQ_{ct}(t,\tau)} \right]$$

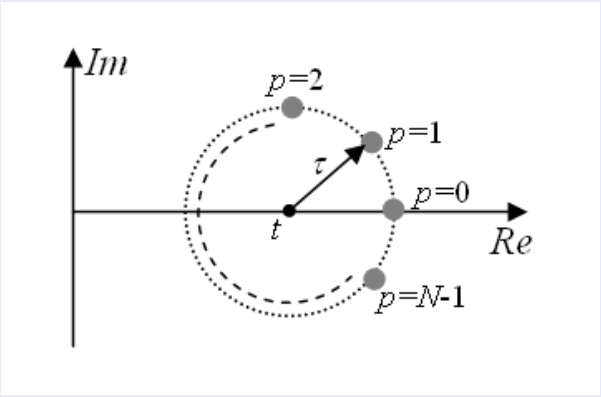
with *Spread Factor* $Q_{ct}(t,\tau) = \phi^{(5)}(t)\frac{\tau^5}{4^4\,5!} + \phi^{(9)}(t)\frac{\tau^9}{4^8\,9!} + \dots$



C. Cornu and all., 2007. *"Time-frequency distributions with generalized complex lag argument"*

- Cauchy's integral formula $\Rightarrow$ computation of the K^{th} order derivative of ϕ at instant t as

$$\phi^{(K)}(t) = \frac{K!}{2\pi j} \oint_{\gamma} \frac{\phi(z)}{(z-t)^{K+1}}\, dz$$



$$\phi^{(K)}(t) = \frac{K!}{N\tau^K} \sum_{p=0}^{N-1} \phi\left(t + \tau e^{j\frac{2\pi p}{N}}\right) e^{-j\frac{2\pi pK}{N}} + \varepsilon$$

$$GCD_N^K[s](t,\omega) = \mathfrak{F}_\tau\left[GCM_N^K[s](t,\tau)\right] = \delta\left(\omega - \phi^{(K)}(t)\right) *_\omega \mathfrak{F}_\tau\left[e^{jQ(t,\tau)}\right]$$

where $\boxed{\mathrm{GCM}_N^K[s](t,\tau) = \prod_{p=0}^{N-1} s^{\omega_{N,p}^{N-K}}\left(t + \omega_{N,p}\sqrt[K]{\tau\frac{K!}{N}}\right)}$

with $\omega_{N,p} = e^{j2\pi p/N}$

and $Q(t,\tau) = N\sum_{p=1}^{+\infty}\phi^{(Np+K)}(t)\frac{\tau^{\frac{Np}{K}+1}}{(Np+K)!}\left(\frac{K!}{N}\right)^{\frac{Np}{K}+1}$

"Inner interferences reduction" property of GCD

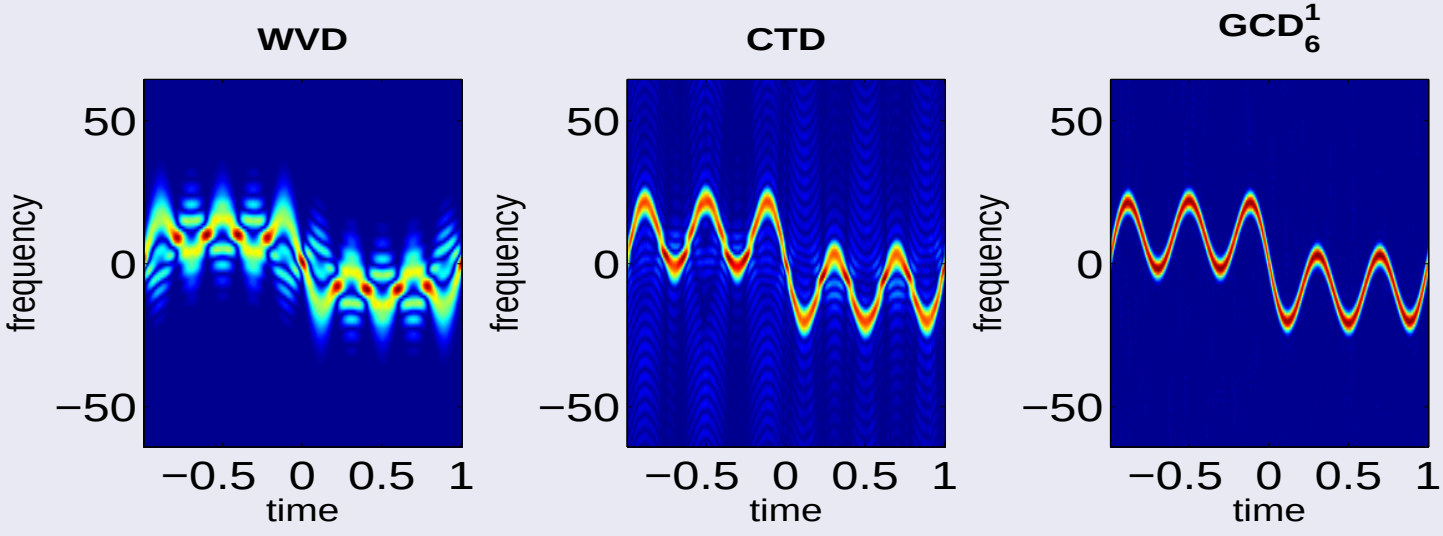
Increasing **parameter N** ($N \Leftrightarrow$ number of complex lags) leads to a reduction of Spread Factor and so attenuation of inner interferences.

Illustration

- Spread Factors in some Time-Frequency Distributions

Distribution	**Spread factor**
$STFT \Leftrightarrow \mathrm{GCD}_1^1$	$Q(t,\tau) = \phi^{(2)}(t)\frac{\tau^2}{2!} + \phi^{(3)}(t)\frac{\tau^3}{3!} + \dots$
$WVD \Leftrightarrow \mathrm{GCD}_2^1$	$Q(t,\tau) = \phi^{(3)}(t)\frac{\tau^3}{2^2\,3!} + \phi^{(5)}(t)\frac{\tau^5}{2^4\,5!} + \dots$
$CTD \Leftrightarrow \mathrm{GCD}_4^1$	$Q(t,\tau) = \phi^{(5)}(t)\frac{\tau^5}{4^4\,5!} + \phi^{(9)}(t)\frac{\tau^9}{4^8\,9!} + \dots$
GCD_6^1	$Q(t,\tau) = \phi^{(7)}(t)\frac{\tau^7}{6^6\,7!} + \phi^{(13)}(t)\frac{\tau^{13}}{6^{12}\,13!} + \dots$

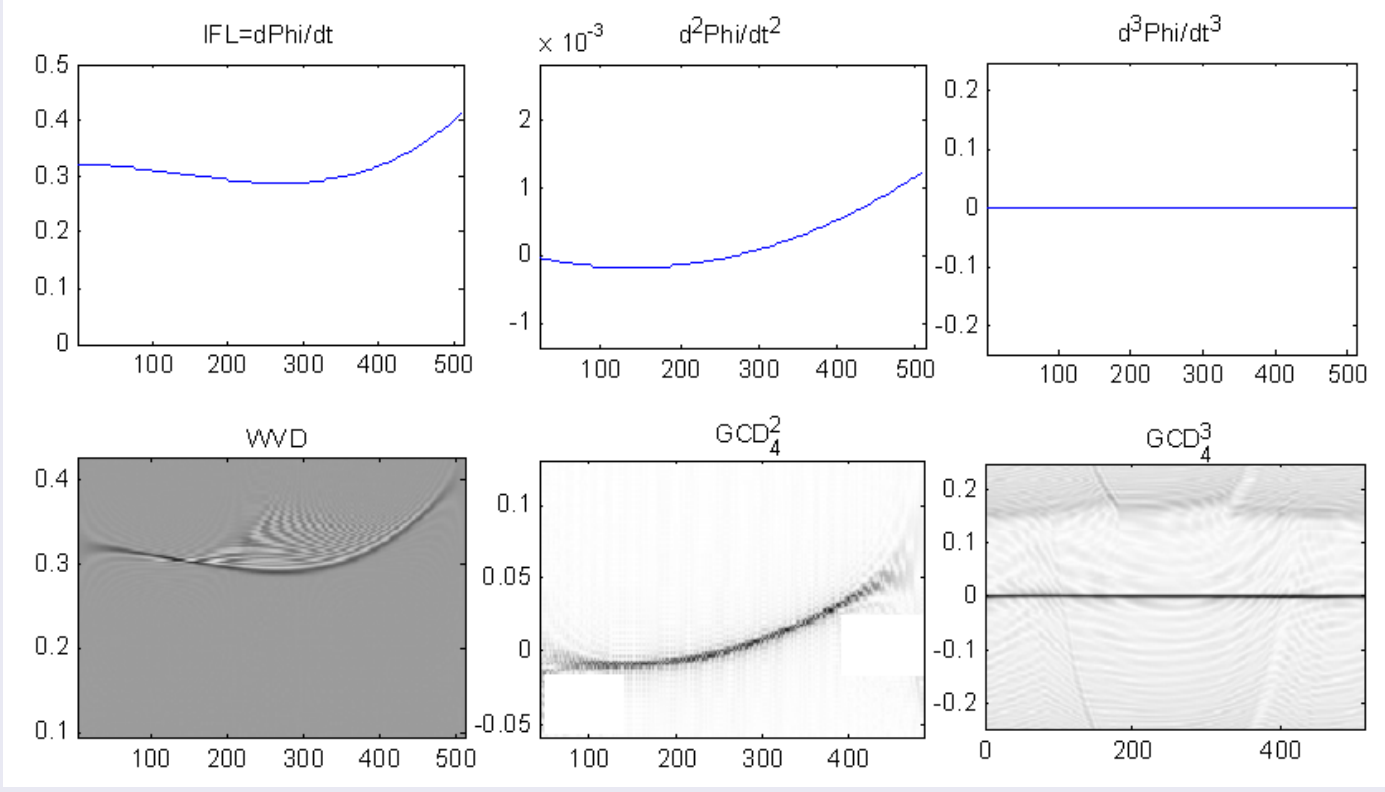
- Example for a test signal $s(t) = e^{j(6cos(\pi t)+\frac{2}{3}cos(3\pi t)+\frac{4}{3}cos(5\pi t))}$



Inner interferences for WVD, CTD and GCD_6^1

GCD ⇔ Time-Phase Derivative Representation

- GCD offers representations for derivatives of any order **K** of the instantaneous phase law of the signal.
- Example for a 4^{th} order polynomial phase signal



The theoretical derivatives (top) are correctly represented by GCDs of corresponding order (bottom)

2. Complex-Lag Moment for Multi-Component Signals

Consideration of Cross Terms

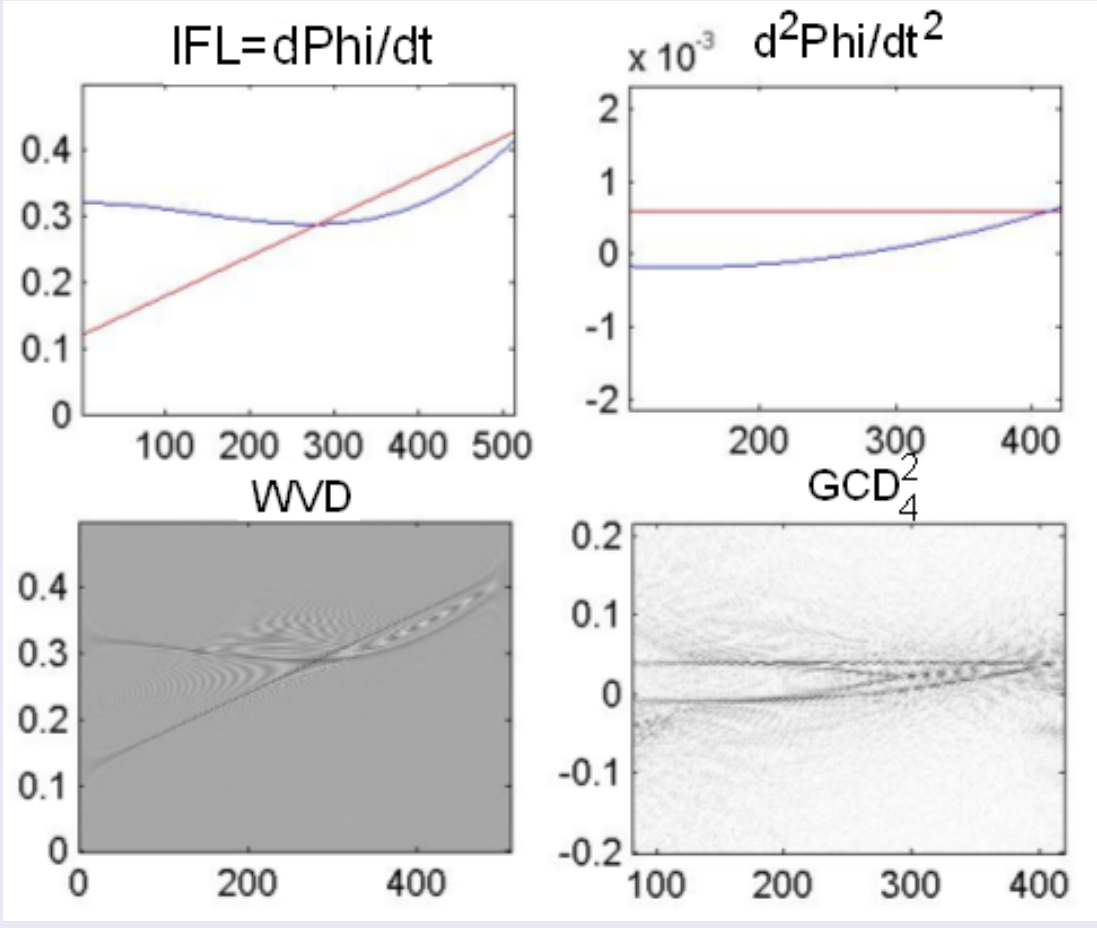
- The GCM has a complicate non-linear form
- Multi-component case $\Rightarrow$ high level of cross terms very difficult to calculate, as follows:

for $s(t) = s_1(t) + s_2(t) = e^{j\phi_1(t)} + e^{j\phi_2(t)}$

$$GCM_N^K\left[s_1 + s_2\right](t,\tau)$$

$$= \prod_{p=0}^{N-1}\left[s_1\left(t + \omega_{N,p}\sqrt[K]{\tau\frac{K!}{N}}\right) + s_2\left(t + \omega_{N,p}\sqrt[K]{\tau\frac{K!}{N}}\right)\right]^{\omega_{N,p}^{N-K}}$$

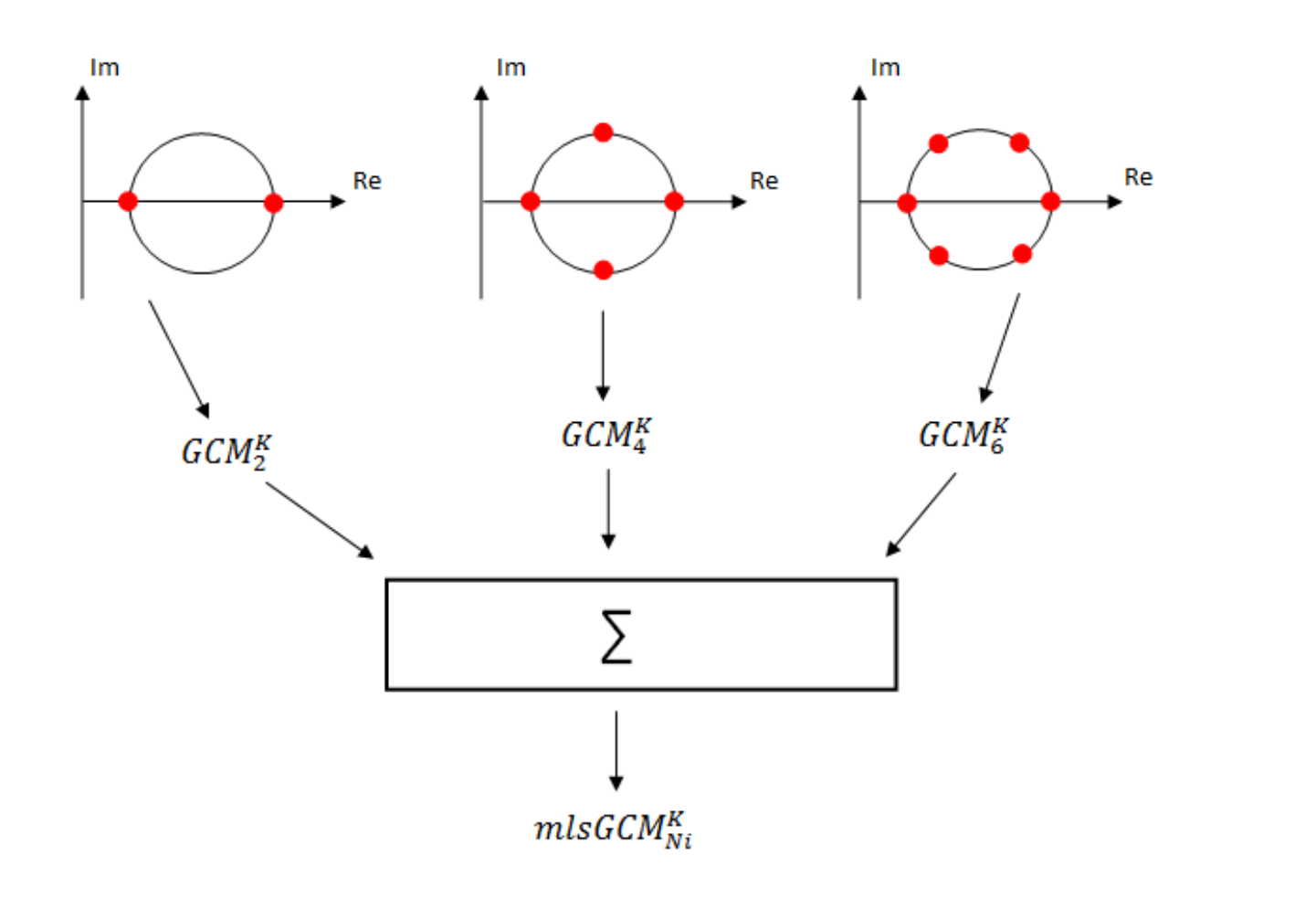
$$= e^{j\phi_1^{(K)}(t)\tau + jQ_1(t,\tau)} + e^{j\phi_2^{(K)}(t)\tau + jQ_2(t,\tau)} + CT_{\phi_1,\phi_2}(N,K)$$



Case of a two-component signal: cross terms have strong energetic level and corrupt the visibility of auto-terms

Reduction of Cross Terms: "Multi-lag sets" principle

- Auto-terms **don't** depend of number of lags N



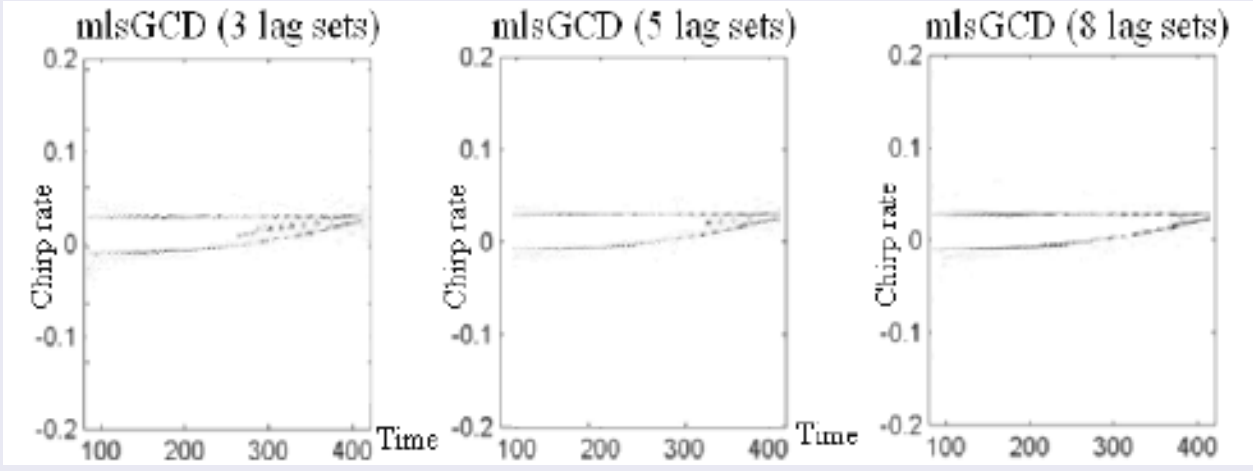
- First step: Compute GCMs for several lags sets N_i
  $\Rightarrow$ auto terms keep the same structures
  $\Rightarrow$ cross terms depend of N and are consequently differently located
- Second step: Sum the different GCMs

$$mlsGCM^{K}_{\{N_i\}}[s](t,\tau)$$

$$= \sum_{i=1}^{P} GCM^{K}_{N_i}[s](t,\tau)$$

$$= P\sum_{l} e^{j\phi_l^{(K)}(t)\tau + jQ_l(t,\tau)} + \sum_{i=1}^{P} CT_{\phi_l}(N_i,K)$$

Highlighting of auto-terms / Decreasing of cross-terms

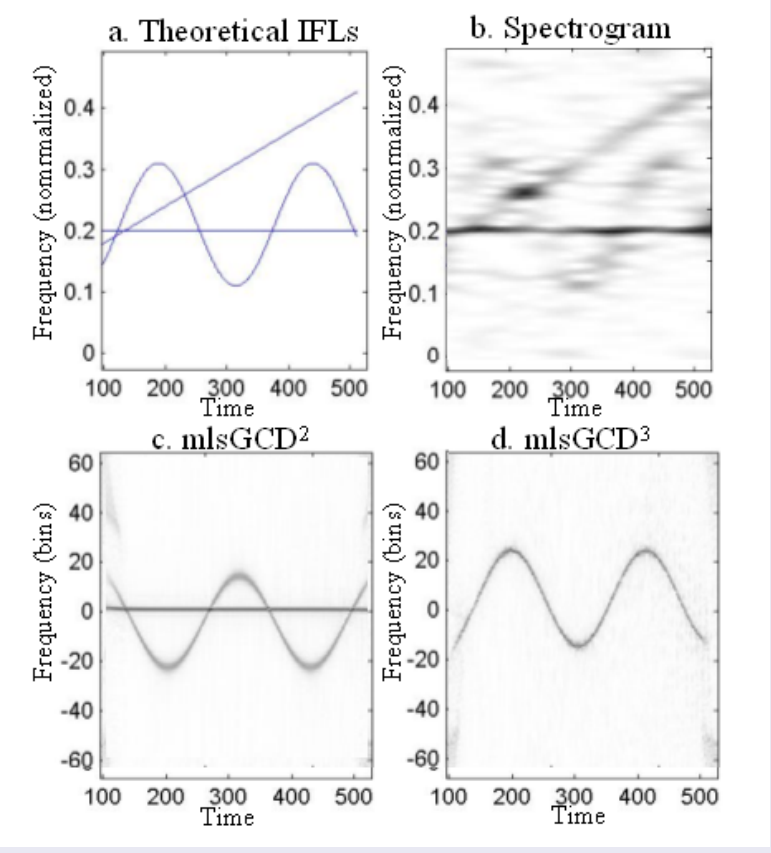


Case of a two-component signal: cross terms level reduction using *mlsGCD*

3. Results

GCD and mlsGCD can perform in many applications dealing with complex TF modulations.

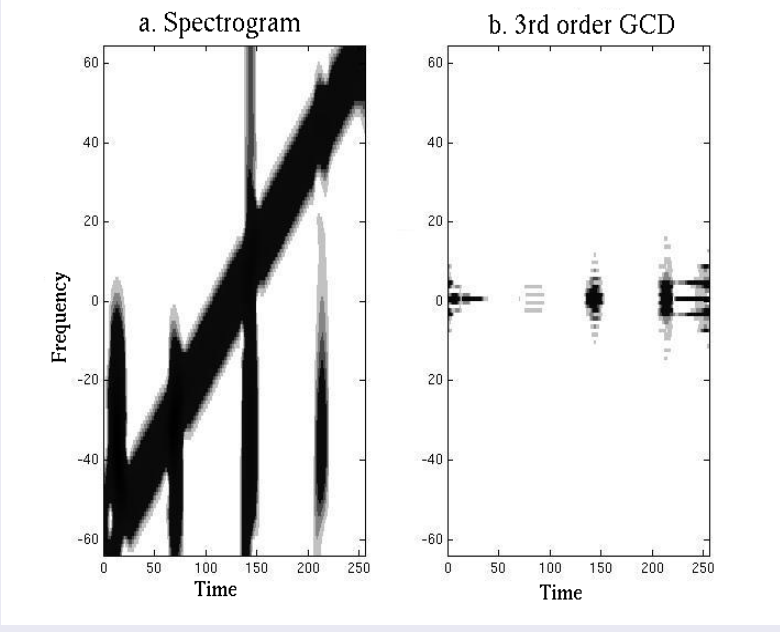
Ex.1: Removal of quasi-stationary corrupting components



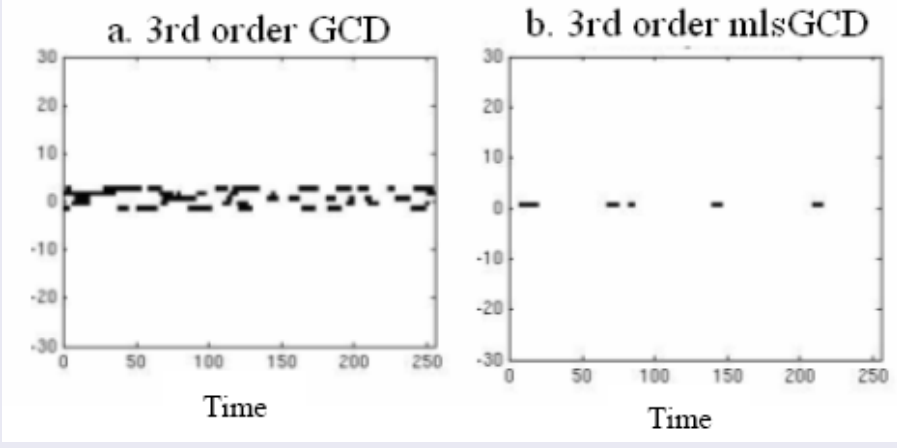
Separation of a Sinusoidal Frequency Modulation corrupted by chirp, tone and noise.

Ex.2: Application to Transient signals

- Infinite derivability of transient signals phase law
- Extraction, via derivability property, of transient impulses corrupted by both noise and coherent perturbations



Transient impulses extraction



Noise reduction by *mlsGCD*

4. Conclusion

We achieved...

Distribution able to:

1. focus on arbitrary derivate of phase laws
2. in multi-component context
3. with highly concentrated and cross-terms reduced representations

Prospects

- Signal-dependent choice of lag sets
- 2D extension of complex lag concept

End

Thank you for your Attention!!!