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Galois coverings of weakly shod algebras

Patrick Le Meur

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Abstract

We investigate the Galois coverings of weakly shod algebras. For a weakly shod algebra not quasi-tilted of canonical type, we establish a correspondence between its Galois coverings and the Galois coverings of its connecting component. As a consequence we show that a weakly shod algebra which is not quasi-tilted of canonical type is simply connected if and only if its first Hochschild cohomology group vanishes.

Introduction

Let A be a finite dimensional k -algebra where k is an algebraically closed field. In order to study the category $\text{mod } A$ of finite dimensional (right) A -modules we assume that A is basic and connected. The study of $\text{mod } A$ has risen important classes of algebras. For example: The *representation-finite* algebras, that is, with only finitely many isomorphism classes of indecomposable modules; the *hereditary* algebras, that is, path algebras kQ of finite quivers Q with no oriented cycles; the *tilted* algebras of type Q , that is, endomorphism algebras $\text{End}_{kQ}(T)$ of tilting kQ -modules (see [19]); and the *quasi-tilted* algebras, that is, endomorphism algebras $\text{End}_{\mathcal{H}}(T)$ of tilting objects T in a hereditary abelian category \mathcal{H} (see [18], a quasi-tilted algebra which is not tilted is called of canonical type). For the last three classes, each class is a generalisation of the previous one. More recently, a new class of algebras has arisen (see [2, 27, 31]): That of *laura* algebras. The algebra A is called *laura* if there is an upper bound in the number of isomorphism classes of indecomposable modules which can appear in an oriented path of non-zero morphisms between indecomposable A -modules starting from an injective and ending at a projective. It appears that this class contains the four classes cited above. A *laura* algebra which not quasi-tilted is characterised by the existence of a unique non semi-regular component (that is, containing both a projective and an injective) in its Auslander-Reiten quiver. It is called the connecting component as a generalisation of the connecting components of tilted algebras. Hence a *laura* algebra which is not quasi-tilted of canonical type has at least one, and at most two, connecting components (actually, it has two if and only if A is concealed). Recall that quasi-tilted algebras of finite representation type are tilted ([18, Cor. 2.3.6]) and those of infinite representation type are characterised by the existence of a sincere separating family of semi-regular standard tubes ([24]). *Laura* algebras comprise the *weakly shod* algebras defined by the existence of an upper bound for the length of a path of non-zero non-isomorphisms from an injective to a projective. Actually a *laura* algebra which is not quasi-tilted is weakly shod if and only if the connecting component contains no oriented cycles. Weakly shod algebras were introduced in [14] as a generalisation of shod algebras which were defined in [15, 26] as the class of algebras for which any indecomposable module has injective dimension or projective dimension at most 1. For example, quasi-tilted algebras are shod and therefore weakly shod.

On the other hand, the covering techniques ([11, 28]) have permitted important progress in the study of representation-finite algebras (see [9, 12, 17]). These techniques need to consider algebras as k -categories. If $\mathcal{C} \rightarrow A$ is a Galois covering, then $\text{mod } A$ and $\text{mod } \mathcal{C}$ are related by the so-called push-down functor $F_\lambda: \text{mod } \mathcal{C} \rightarrow \text{mod } A$. When A has no proper Galois covering by a connected and locally bounded k -category (or, equivalently, when the fundamental group of any presentation of A in the sense of [25] is trivial), we say that A is *simply connected* (see [8]). Simply connected algebras are of special interest because of the reduction allowed by the push-down functors. Also they have been object of many investigations (see [8, 10] for instance). For example, Bongartz and Gabriel ([11]) have classified the simply connected representation-finite standard algebras using graded trees. Therefore a nice characterisation of simply connected algebras would be very useful. In [29, Pb. 1], Skowroński asked the following question for a tame and triangular algebra A :

Is A simply connected if and only if $\text{HH}^1(A) = 0$? (Q)

Up to now, there have been partial answers to Q (regardless the tame assumption): For algebras derived equivalent to a hereditary algebra in [22] (and therefore for tilted algebras), for tame quasi-tilted algebras in [3] and for tame weakly shod algebras in [5]. Therefore it is natural to try to answer Q for *laura* algebras. This shall be done in a forthcoming text ([1]). In the present text we study the case of weakly shod algebras not quasi-tilted of canonical type, which will serve for the study made in [1]. For this purpose we prove the following main result.

Theorem A. *Let A be connected, weakly shod and not quasi-tilted of canonical type. Let Γ_A be a connecting component of $\Gamma(\text{mod } A)$. Let G be a group. Then A admits a Galois covering with group G by a connected and locally bounded k -category if and only if Γ_A admits a Galois covering with group G of translation quivers. In particular A admits a Galois covering with group $\pi_1(\Gamma_A)$ by a connected and locally bounded k -category.*

By [11, 4.2], the fundamental group $\pi_1(\Gamma_A)$ of a connecting component Γ_A is free and isomorphic to the fundamental group of its orbit-graph. If A is concealed, then its two connecting components are the unique postprojective and the unique preinjective components, so they have isomorphic fundamental groups. As a consequence of our main result we prove that \mathcal{Q} has a positive answer for weakly shod algebras.

Corollary B. *Let A be connected, weakly shod and not quasi-tilted of canonical type. Let Γ_A be a connecting component of $\Gamma(\text{mod } A)$. The following conditions are equivalent:*

- (a) A is simply connected.
- (b) $\text{HH}^1(A) = 0$.
- (c) The orbit-graph $\mathcal{O}(\Gamma_A)$ of Γ_A is a tree.
- (d) Γ_A is simply connected.

Our proof of Corollary B is independent of the one given in [5] for the tame case. Actually we make no distinction between the different representation types (finite, tame or wild). The proof of Theorem A decomposes in two main steps:

1. If $F: \mathcal{C} \rightarrow A$ is a Galois covering with group G , then every module $X \in \Gamma_A$ is isomorphic to $F_\lambda \tilde{X}$ for some $\tilde{X} \in \text{mod } \mathcal{C}$. The modules \tilde{X} , for X in Γ_A , form an Auslander-Reiten component of \mathcal{C} . This component is a Galois covering with group G of Γ_A .
2. A admits a Galois covering with group $\pi_1(\Gamma_A)$ associated to the universal cover of the orbit-graph $\mathcal{O}(\Gamma_A)$.

As an application of the methods we use, we prove the last main result of the text.

Theorem C. *Let $A' \rightarrow A$ be a Galois covering with finite group G where A' is a basic and connected finite dimensional k -algebra. Then:*

- (a) A is tilted if and only if A' is tilted.
- (b) A is quasi-tilted if and only if A' is quasi-tilted.
- (c) A is weakly shod if and only if A' is weakly shod.

The text is organised as follows. In Section 1 we fix some notations and recall some useful definitions. In Section 2 we give some preliminary results: First, we prove some useful facts on covering techniques; second, we compare the Auslander-Reiten quiver of A and the one of B when $A = B[M]$. Section 3 is the very core of the text and is devoted to the first step of Theorem A. In Section 4 we proceed the second step. In Section 5, we prove Theorem A and Corollary B. Finally, we prove Theorem C in Section 6.

1 Definitions and notations

Notations on k -categories

We refer the reader to [11, 2.1] for notions on k -categories and locally bounded k -categories. All locally bounded k -categories are small and all functors between k -categories are k -linear (of course, our module categories will be skeletally small). For a locally bounded k -category \mathcal{C} , its objects set is denoted by \mathcal{C}_o and the space of morphisms from an object x to an object y is denoted by $\mathcal{C}(x, y)$. If A is a basic finite dimensional k -algebra, it is equivalently a locally bounded k -category as follows. Fix a complete set $\{e_1, \dots, e_n\}$ of pairwise orthogonal primitive idempotents. Then $A_o = \{e_1, \dots, e_n\}$ and $A(e_i, e_j) = e_j A e_i$ for every i, j . In the sequel, A will always denote a basic finite dimensional k -algebra.

Notations on modules

Let \mathcal{C} be a k -category. Following [11, 2.2], a (right) \mathcal{C} -module is a k -linear functor $M: \mathcal{C}^{op} \rightarrow \text{MOD } k$ where $\text{MOD } k$ is the category of k -vector spaces. If \mathcal{C}' is another k -category, a $\mathcal{C} - \mathcal{C}'$ -bimodule is a k -linear functor $\mathcal{C} \times \mathcal{C}'^{op} \rightarrow \text{MOD } k$. We write $\text{MOD } \mathcal{C}$ for the category of \mathcal{C} -modules and $\text{mod } \mathcal{C}$ for the full subcategory of finite dimensional \mathcal{C} -modules, that is, those modules M such that $\sum_{x \in \mathcal{C}_o} \dim_k M(x) < \infty$. The standard

duality $\text{Hom}_k(-, k)$ is denoted by D . We write $\text{ind } \mathcal{C}$ for the full subcategory of $\text{mod } \mathcal{C}$ containing exactly one representative of each isomorphism class of indecomposable modules. A set \mathcal{X} of modules is called *faithful* if $\bigcap_{X \in \mathcal{X}} \text{Ann}(X) = 0$ where $\text{Ann}(X)$ is the annihilator of X , that is, the $\mathcal{C} - \mathcal{C}$ -subbimodule of \mathcal{C} such that $\text{Ann}(X)(x, y) = \{u \in \mathcal{C}(x, y) \mid mu = 0 \text{ for every } m \in X(y)\}$. If S is a set of finite dimensional \mathcal{C} -modules, then $\text{add}(S)$ denotes the smallest full subcategory of $\text{mod } \mathcal{C}$ containing S and stable under direct sums and direct summands.

Assume that \mathcal{C} is locally bounded. We write $\Gamma(\text{mod } \mathcal{C})$ for the Auslander-Reiten quiver and $\tau_{\mathcal{C}} = D\text{Tr}$ for the Auslander-Reiten translation. Let Γ be a component of $\Gamma(\text{mod } \mathcal{C})$. Then Γ is called *generalised standard* if $\text{rad}^\infty(X, Y) = 0$ for every $X, Y \in \Gamma$ (see [30]). Here rad denotes the radical of $\text{mod } \mathcal{C}$, that is, the ideal generated by the non-isomorphisms between indecomposable modules, rad^n denotes the n -th power of the radical and $\text{rad}^\infty = \bigcap_{n \geq 1} \text{rad}^n$. The component Γ is called *non semi-regular* if it contains both an injective module and a projective module. We recall the definition of the *orbit-graph* $\mathcal{O}(\Gamma)$ of Γ in the case Γ has no periodic module (see [11, 4.2] for the general case). First, fix a *polarisation* in Γ , that is, for every arrow $\alpha: x \rightarrow y$ in Γ with y non-projective we fix an arrow $\sigma(\alpha): \tau_{\mathcal{C}} y \rightarrow x$ in such a way that σ induces a bijection from the set of arrows $x \rightarrow y$ to the set of arrows $\tau_{\mathcal{C}} y \rightarrow x$ (see [11, 1.1]). Then $\mathcal{O}(\Gamma)$ is the graph whose vertices are the $\tau_{\mathcal{C}}$ -orbits of

the vertices in Γ and such that there is an edge $(X)^{\tau_C} - (Y)^{\tau_C}$ for every σ -orbit of arrows between a module in $(X)^{\tau_C}$ and a module in $(Y)^{\tau_C}$.

We refer the reader to [7, Chap. VIII, IX] for a background on tilting theory.

Weakly shod algebras ([14])

Let \mathcal{C} be a locally bounded k -category and $X, Y \in \text{ind } \mathcal{C}$. A path $X \rightsquigarrow Y$ in $\text{ind } \mathcal{C}$ (or in $\Gamma(\text{mod } \mathcal{C})$) is a sequence of non-zero morphisms (or of irreducible morphisms, respectively) between indecomposable \mathcal{C} -modules $X = X_0 \xrightarrow{f_1} X_1 \rightarrow \dots \rightarrow X_{n-1} \xrightarrow{f_n} X_n = Y$ (with $n \geq 0$). We then say that X is a predecessor of Y and that Y is a successor of X in $\text{ind } \mathcal{C}$ (or in $\Gamma(\text{mod } \mathcal{C})$, respectively). Hence X is a successor and a predecessor of itself.

The algebra A is called *weakly shod* if and only if the length of paths in $\text{ind } A$ from an injective to a projective is bounded. We write \mathcal{P}_A^f for the set of indecomposable projectives which are successors of indecomposable injectives. When A is weakly shod this set is partially ordered ([5, 4.3]) by the relation: $P \leq Q$ if and only if P is a predecessor of Q in $\text{ind } A$. We need the following properties when A is weakly shod and connected:

- (a) If $\mathcal{P}_A^f = \emptyset$, then A is quasi-tilted ([18, Thm. II.1.14]).
- (b) If $\mathcal{P}_A^f \neq \emptyset$, then $\Gamma(\text{mod } A)$ has a unique non semi-regular component ([14, 1.6, 5.4]). This component is generalised standard, faithful, has no oriented cycle and contains all the modules lying on a path in $\text{ind } A$ form an injective to a projective. In particular, every module lying on it is a brick ([7, IV.1.4]). This component is called *the connecting component of $\Gamma(\text{mod } A)$* (or of A).

Assume that A is connected, weakly shod and that $\mathcal{P}_A^f \neq \emptyset$. Let $P_m \in \mathcal{P}_A^f$ be maximal and e the idempotent such that $P_m = eA$. Then $A = B[M]$ where $M = \text{rad}(P_m)$ and $B = (1 - e)A(1 - e)$. Moreover:

- (a) Any component B' of B is weakly shod. It is moreover tilted if $\mathcal{P}_{B'}^f = \emptyset$ ([14, 4.8]).
- (b) Let $M' \in \text{ind } B$ be a summand of M and B' the component of B such that $M' \in \text{ind } B'$. Then B' is weakly shod and not quasi-tilted of canonical type and M' lies on a connecting component of $\Gamma(\text{mod } B')$ ([5, 5.3]).

Recall ([27, Thm. 3.1]) that if a connected algebra A admits a non semi-regular component which is faithful, generalised standard and has no oriented cycle, then A is weakly shod.

Galois coverings of translation quivers ([11, 28])

Let Γ and Γ' be translation quivers and assume that Γ is connected. A *covering of translation quivers* $p: \Gamma' \rightarrow \Gamma$ is a morphism of quivers such that: (a) p is a covering of unoriented graphs; (b) $p(x)$ is projective (or injective, respectively) if and only if so is x ; (c) p commutes with the translations. It is called a *Galois covering with group G* if, moreover, the group G acts on Γ' in such a way that: (d) G acts freely on vertices; (e) $pg = p$ for every $g \in G$; (f) the translation quiver morphism $\Gamma'/G \rightarrow \Gamma$ induced by p is an isomorphism; (g) Γ' is connected. Given a connected translation quiver Γ , there exists a group $\pi_1(\Gamma)$ (called the *fundamental group of Γ*) and a Galois covering $\tilde{\Gamma} \rightarrow \Gamma$ with group $\pi_1(\Gamma)$ called the *universal cover of Γ* , which factors through any covering $\Gamma' \rightarrow \Gamma$. If $p: \Gamma' \rightarrow \Gamma$ is a covering (or a Galois covering with group G), then it naturally induces a covering (or a Galois covering with group G , respectively) $\mathcal{O}(\Gamma') \rightarrow \mathcal{O}(\Gamma)$ between the associated orbit-graphs. It is proved in [11, 4.2] that if Γ has only finitely many τ -orbits and if $p: \Gamma' \rightarrow \Gamma$ is the universal cover of translation quiver, then $\mathcal{O}(\Gamma') \rightarrow \mathcal{O}(\Gamma)$ is the universal cover of graphs, that is, $\pi_1(\Gamma)$ is isomorphic to $\pi_1(\mathcal{O}(\Gamma))$ (and therefore is free).

Group actions on module categories ([17])

Let G be a group. A G -category is a k -category \mathcal{C} together with a group morphism $G \rightarrow \text{Aut}(\mathcal{C})$. This defines an action of G on $\text{MOD } \mathcal{C}$: if $M \in \text{MOD } \mathcal{C}$ and $g \in G$, then ${}^gM = M \circ g^{-1}$. We write $G_M := \{g \in G \mid {}^gM \simeq M\}$ for the *stabiliser* of M . We say that G *acts freely on \mathcal{C}* if the induced action on \mathcal{C}_o is free. Assume that \mathcal{C} is locally bounded. Then this G -action preserves Auslander-Reiten sequences and commutes with $\tau_{\mathcal{C}}$. Also it induces an action on $\Gamma(\text{mod } \mathcal{C})$ and on $\mathcal{O}(\Gamma)$ for any G -stable component Γ of $\Gamma(\text{mod } \mathcal{C})$.

Galois coverings of categories ([17])

Let G be a group and $F: \mathcal{E} \rightarrow \mathcal{B}$ a functor between k -categories. We set $\text{Aut}(F) = \{g \in \text{Aut}(\mathcal{E}) \mid F \circ g = F\}$. We say that F is a *Galois covering with group G* if there is a group morphism $G \rightarrow \text{Aut}(F)$ such that G acts freely on \mathcal{E} and the induced functor $\overline{F}: \mathcal{E}/G \rightarrow \mathcal{B}$ is an isomorphism. We need the following characterisation for a functor $F: \mathcal{E} \rightarrow \mathcal{B}$ to be a Galois covering ([17, Sect. 3]). The group morphism $G \rightarrow \text{Aut}(F)$ is such that F is Galois with group G if and only if: (a) the fibres $F^{-1}(x)$ ($x \in \mathcal{B}_o$) are non-empty and G acts on these freely and transitively and (b) F is a *covering functor* in the sense of [11, 3.1], that is, for every $x, y \in \mathcal{E}_o$ the two maps $\bigoplus_{g \in G} \mathcal{E}(x, gy) \rightarrow \mathcal{B}(Fx, Fy)$ and $\bigoplus_{g \in G} \mathcal{E}(gy, x) \rightarrow \mathcal{B}(Fy, Fx)$ induced by F are isomorphisms. A Galois covering $F: \mathcal{E} \rightarrow \mathcal{B}$ with \mathcal{E} and \mathcal{B} locally bounded and connected is called *connected*. In such a case, the morphism $G \rightarrow \text{Aut}(F)$ is an isomorphism ([20, Prop. 6.1.37]). A connected and locally bounded k -category \mathcal{B} is called *simply connected* if and only if there is no connected Galois covering $\mathcal{E} \rightarrow \mathcal{B}$ with non trivial group. This definition is equivalent ([21, Cor. 4.5]) to the original one of [8] and it is more convenient for our purposes.

Covering techniques ([11] and [17])

Let $F: \mathcal{E} \rightarrow \mathcal{B}$ be a Galois covering between locally bounded k -categories. We write $F_\lambda: \text{MOD } \mathcal{E} \rightarrow \text{MOD } \mathcal{B}$ and $F: \text{MOD } \mathcal{B} \rightarrow \text{MOD } \mathcal{E}$ for the *push-down* functor and the *pull-up* functor, respectively. Recall ([11, 3.2]) that $F = X \circ F$ for every $X \in \text{MOD } \mathcal{B}$ and that for $M \in \text{MOD } \mathcal{E}$, the \mathcal{B} -module $F_\lambda M$ is such that $F_\lambda M(x) = \bigoplus_{F x' = x} M(x')$ for every $x \in \mathcal{B}_0$. We list some needed properties on these functors. Both F_λ and F are exact; (F_λ, F) is adjoint; $F_\lambda M$ is projective (or injective) if and only if M is projective (or injective, respectively); $F_\lambda(\text{mod } \mathcal{E}) \subseteq \text{mod } \mathcal{B}$; the functor F_λ is G -invariant, that is, $F_\lambda \circ g = F_\lambda$ for every $g \in G$; for every $M \in \text{mod } \mathcal{E}$ we have $F_\lambda M \simeq \bigoplus_{g \in G} {}^g M$ ([17, 3.2]); and F_λ commutes with the duality, that is, $D \circ F_\lambda \simeq F_\lambda^{op} \circ D$ on $\text{mod } \mathcal{E}$.

Finally, it satisfies a property which will be referred to as the *covering property of F_λ* : For $M, N \in \text{mod } \mathcal{E}$, the two maps $\bigoplus_{g \in G} \text{Hom}_{\mathcal{E}}({}^g M, N) \rightarrow \text{Hom}_{\mathcal{B}}(F_\lambda M, F_\lambda N)$ and $\bigoplus_{g \in G} \text{Hom}_{\mathcal{E}}(M, {}^g N) \rightarrow \text{Hom}_{\mathcal{B}}(F_\lambda M, F_\lambda N)$ induced by F_λ are k -linear isomorphisms. A module $X \in \text{ind } \mathcal{B}$ is called *of the first kind (with respect to F)* if and only if there exists $\tilde{X} \in \text{mod } \mathcal{E}$ (necessarily indecomposable) such that $F_\lambda \tilde{X} \simeq X$ in $\text{mod } \mathcal{B}$. Note that if \tilde{X} exists, then $X = F_\lambda \tilde{X}$ for some $\tilde{X} \in \text{ind } \mathcal{E}$; and, if $\tilde{X}, \hat{X} \in \text{ind } \mathcal{E}$ are such that $F_\lambda \tilde{X} \simeq F_\lambda \hat{X} \simeq X$, then $\tilde{X} \simeq {}^g \hat{X}$ for some $g \in G$ (see [17, 3.5]).

2 Preliminaries

Some results on covering techniques

Let $F: \mathcal{C} \rightarrow \mathcal{A}$ be a Galois covering with group G where \mathcal{C} is locally bounded. We prove some useful comparisons between $\Gamma(\text{mod } \mathcal{A})$ and $\Gamma(\text{mod } \mathcal{C})$. First, we give a necessary condition on a morphism in $\text{mod } \mathcal{C}$ to be mapped by F_λ to a section or a retraction.

Lemma 2.1. *Let $X, Y \in \text{mod } \mathcal{C}$ and $f \in \text{Hom}_{\mathcal{C}}(X, Y)$.*

- (a) $F_\lambda(f)$ is a section (or a retraction) if and only if so is f .
- (b) If $F_\lambda(f)$ is irreducible, then so is f .
- (c) Let $u: E \rightarrow X$ (or $v: X \rightarrow E$) be a right (or left) minimal almost split morphism in $\text{mod } \mathcal{C}$. Assume that $G_X = 1$. Then so is $F_\lambda(u)$ (or $F_\lambda(v)$, respectively).
- (d) $F_\lambda \tau_{\mathcal{C}} X \simeq \tau_{\mathcal{A}} F_\lambda X$.

Proof: (a) Obviously, if f is a section (or a retraction), then so is $F_\lambda(f)$. Assume that $F_\lambda(f)$ is a section. So $\text{Id}_{F_\lambda X} = r \circ F_\lambda(f)$ with $r \in \text{Hom}_{\mathcal{A}}(F_\lambda X, F_\lambda Y)$. Moreover, $r = \sum_g F_\lambda(r_g)$ with $(r_g)_{g \in G} \in \bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}(Y, {}^g X)$, using the covering property of F_λ . Therefore $\text{Id}_{F_\lambda X} = \sum_g F_\lambda(r_g \circ f)$. The covering property of F_λ then implies that $\text{Id}_X = r_1 \circ f$, that is, f is a section. Dually, if $F_\lambda(f)$ is a retraction, then so is f .

(b) is a direct consequence of (a).

(c) is due to the proof of [17, 3.6, (b)].

(d) follows from the fact that F_λ is exact, maps projective modules to projective modules (in particular, F_λ maps a minimal projective resolution in $\text{mod } \mathcal{C}$ to a minimal projective resolution in $\text{mod } \mathcal{A}$) and commutes with the duality. ■

Lemma 2.2. *Let Γ be a component of $\Gamma(\text{mod } \mathcal{A})$ made of modules of the first kind and $\tilde{\Gamma}$ the full subquiver of $\Gamma(\text{mod } \mathcal{C})$ generated by $\{X \in \Gamma(\text{mod } \mathcal{C}) \mid F_\lambda X \in \Gamma\}$. Then:*

- (a) Let $u: M \rightarrow P$ be a right minimal almost split morphism in $\text{mod } \mathcal{C}$ with P indecomposable projective. Then $F_\lambda(u)$ is right minimal almost split.
- (b) Let $X \in \tilde{\Gamma}$ be non projective. Then F_λ transforms any almost split sequence ending at X into an almost split sequence ending at $F_\lambda X$.
- (c) Let $u \in \text{Hom}_{\mathcal{C}}(X, Y)$ with $X, Y \in \tilde{\Gamma}$. Then u is irreducible if and only if so is $F_\lambda(u)$.
- (d) Γ is stable under predecessors and under successors in $\Gamma(\text{mod } \mathcal{C})$ and under the action of G .

Proof: (a) follows from [11, 3.2].

(b) Fix an almost split sequence $0 \rightarrow \tau_{\mathcal{C}} X \xrightarrow{\varphi} E \xrightarrow{\theta} X \rightarrow 0$ in $\text{mod } \mathcal{C}$. By 2.1, (d), we have an exact sequence $0 \rightarrow \tau_{\mathcal{A}} F_\lambda X \xrightarrow{F_\lambda(\varphi)} F_\lambda E \xrightarrow{F_\lambda(\theta)} F_\lambda X \rightarrow 0$ in $\text{mod } \mathcal{A}$. By 2.1, (a), it does not split. Moreover, $F_\lambda X$ is indecomposable and non-projective. Let $v: Z \rightarrow F_\lambda X$ be right minimal almost split. We only need to prove that v factors through $F_\lambda \theta$. Write $v: Z \rightarrow F_\lambda X$ as $v = [v_1 \ \cdots \ v_n]: Z_1 \oplus \cdots \oplus Z_n \rightarrow F_\lambda X$ where $Z_1, \dots, Z_n \in \text{ind } \mathcal{A}$. We prove that each v_i factors through $F_\lambda \theta$. We have $Z_i \in \Gamma$ because v_i is irreducible. Therefore $Z_i = F_\lambda \tilde{Z}_i$ for some $\tilde{Z}_i \in \text{mod } \mathcal{C}$ indecomposable. So $v_i = \sum_g F_\lambda(v_{i,g})$ where $(v_{i,g})_{g \in G} \in$

$\bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}({}^g \tilde{Z}_i, X)$. Note that ${}^g \tilde{Z}_i \not\cong X$ for every $g \in G$ because $Z_i \not\cong F_\lambda X$. Thus $v_{i,g} = \theta \circ w_{i,g}$ for some

$w_{i,g} \in \text{Hom}_{\mathcal{C}}({}^g \tilde{Z}_i, E)$ for every g . We may assume that $w_{i,g} = 0$ if $v_{i,g} = 0$. Then $v_i = F_\lambda(\theta) \circ \left(\sum_{g \in G} F_\lambda(w_{i,g}) \right)$

where $\sum_{g \in G} F_\lambda(w_{i,g}) \in \text{Hom}_{\mathcal{A}}(Z_i, F_\lambda X)$ for every i . Thus v_1, \dots, v_n factor through $F_\lambda \theta$. Therefore so does v .

This proves (b).

(c) is a direct consequence of (a), (b) and 2.1.

(d) Clearly, $\tilde{\Gamma}$ is stable under the action of G . We prove the stability under predecessors (the proof for successors is dual). Let $u \in \text{Hom}_{\mathcal{C}}(X, Y)$ be irreducible with $X \in \text{ind } \mathcal{C}$ and $Y \in \tilde{\Gamma}$. We claim that $F_{\lambda}X \in \text{add}(\Gamma)$. If Y is projective, then X is a direct summand of $\text{rad}(Y)$ and $u: X \rightarrow Y$ is the inclusion. So $F_{\lambda}Y$ is indecomposable projective, $F_{\lambda}X$ is a direct summand of $F_{\lambda}(\text{rad}(Y)) = \text{rad}(F_{\lambda}Y)$ ([11, 3.2]) and $F_{\lambda}(u): F_{\lambda}X \rightarrow F_{\lambda}Y$ is injective. Since $F_{\lambda}Y \in \Gamma$ we have $\text{rad}(F_{\lambda}Y) \in \text{add}(\Gamma)$ and therefore $F_{\lambda}X \in \text{add}(\Gamma)$. Assume that Y is not projective. So there is an almost split sequence in $\text{mod } \mathcal{C}$:

$$0 \rightarrow \tau_{\mathcal{C}}Y \rightarrow E \oplus X \xrightarrow{\begin{bmatrix} ? \\ u \end{bmatrix}} Y \rightarrow 0.$$

By (a), there is an almost split sequence in $\text{mod } A$:

$$0 \rightarrow \tau_A F_{\lambda}Y \rightarrow F_{\lambda}E \oplus F_{\lambda}X \xrightarrow{\begin{bmatrix} ? \\ F_{\lambda}u \end{bmatrix}} F_{\lambda}Y \rightarrow 0.$$

Since $F_{\lambda}Y \in \Gamma$, we have $F_{\lambda}X \in \text{add}(\Gamma)$. This proves the claim. Now we prove that $F_{\lambda}X$ is indecomposable. Since $F_{\lambda}X \in \text{add}(\Gamma)$, there exist $\tilde{E}_1, \dots, \tilde{E}_n \in \tilde{\Gamma}$ and an isomorphism $\varphi: F_{\lambda}X \xrightarrow{\sim} \tilde{E}_1 \oplus \dots \oplus \tilde{E}_n$. From the covering property of F_{λ} , we have $\varphi = \sum_{g \in G} F_{\lambda}(\varphi_g)$ where $(\varphi_g)_{g \in G} \in \bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}({}^g X, \tilde{E}_1 \oplus \dots \oplus \tilde{E}_n)$. Since φ is an isomorphism, there exists $g \in G$ such that $F_{\lambda}(\varphi_g) \notin \text{rad}(F_{\lambda}X, F_{\lambda}\tilde{E}_1 \oplus \dots \oplus F_{\lambda}\tilde{E}_n)$. So there exists i such that the restriction $F_{\lambda}({}^g X \rightarrow F_{\lambda}\tilde{E}_i)$ of $F_{\lambda}(\varphi_g)$ is an isomorphism so that ${}^g X \simeq \tilde{E}_i \in \Gamma$. ■

The following proposition describes the action of F_{λ} on almost split sequences in $\text{mod } \mathcal{C}$ under suitable conditions. Note that if we assume that G acts freely on indecomposable \mathcal{C} -modules (that is, $G_X = 1$ for any $X \in \text{ind } \mathcal{C}$), then the last three points follow at once from [17, 3.6].

Proposition 2.3. *Keep the hypotheses and notations of 2.2.*

- (a) Γ is faithful if and only if $\tilde{\Gamma}$ is.
- (b) Γ is generalised standard if and only if $\text{rad}^{\infty}(X, Y) = 0$ for every $X, Y \in \tilde{\Gamma}$.
- (c) $\tilde{\Gamma}$ is a (disjoint) union of components of $\Gamma(\text{mod } \mathcal{C})$. In particular, $\tilde{\Gamma}$ is a translation subquiver of $\Gamma(\text{mod } \mathcal{C})$.
- (d) The map $X \mapsto F_{\lambda}X$ extends to a covering of translation quivers $\tilde{\Gamma} \rightarrow \Gamma$. If $\tilde{\Gamma}$ is connected and $G_X = 1$ for every $X \in \tilde{\Gamma}$, then this is a Galois covering with group G .
- (e) Γ has an oriented cycle if and only if $\tilde{\Gamma}$ has a non trivial path of the form $X \rightsquigarrow {}^g X$.

Proof: (a) Assume that Γ is faithful. Let $u \in \text{Ann}(\tilde{\Gamma})(x, y)$, that is, $u \in \mathcal{C}(x, y)$ and $mu = 0$ for every $m \in X(y)$, $X \in \tilde{\Gamma}$. We claim that $F(u) \in \text{Ann}(\Gamma)(F_x, F_y)$. Let $X \in \Gamma$ and $m \in X(F_y)$. We may assume that $X = F_{\lambda}\tilde{X}$ with $\tilde{X} \in \tilde{\Gamma}$. So $m = (m_g)_{g \in G} \in \bigoplus_{g \in G} \tilde{X}(gy)$ and, therefore, $mF(u) = (m_g g(u))_{g \in G}$. On the other hand, $g(u) \in \text{Ann}(\tilde{\Gamma})(gx, gy)$ because $\tilde{\Gamma}$ is G -stable. So $m_g g(u) = 0$ for every $g \in G$ so that $mF(u) = 0$. Thus $F(u) \in \text{Ann}(\Gamma)(F_x, F_y) = 0$ and, therefore, $u = 0$. So $\tilde{\Gamma}$ is faithful.

Conversely, assume that $\tilde{\Gamma}$ is faithful and let $u \in \text{Ann}(\Gamma)(F_x, F_y)$. So $u = \sum_g F(u_g)$ where $(u_g)_{g \in G} \in \bigoplus_{g \in G} \mathcal{C}(gx, y)$. We claim that $u_g \in \text{Ann}(\tilde{\Gamma})(gx, y)$ for every $g \in G$. Indeed, let $X \in \tilde{\Gamma}$ and $m \in X(y)$. Then $m \in F_{\lambda}X(F_y)$ and $0 = mu = (mu_g)_{g \in G} \in \bigoplus_{g \in G} F_{\lambda}X(gx)$. So $mu_g = 0$ for every g . Thus $u_g \in \text{Ann}(\tilde{\Gamma})(gx, y)$ for every $g \in G$ and, therefore, $u = 0$ because $\tilde{\Gamma}$ is faithful. So $\tilde{\Gamma}$ is faithful.

(b) Assume that $\text{rad}^{\infty}(X, Y) = 0$ for every $X, Y \in \tilde{\Gamma}$. Let $X, Y \in \tilde{\Gamma}$. We prove that $\text{rad}^{\infty}(F_{\lambda}X, F_{\lambda}Y) = 0$. Since $\text{Hom}_A(F_{\lambda}X, F_{\lambda}Y)$ is finite dimensional and isomorphic to $\bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}(X, {}^g Y)$, there exists $n \geq 1$ such that

$\text{rad}^n(X, {}^g Y) = 0$ for every $g \in G$. Let $f \in \text{rad}^l(F_{\lambda}X, F_{\lambda}Y)$ with $l \geq 1$. Let $[u_1 \dots u_t]^t: X \rightarrow E_1 \oplus \dots \oplus E_t$ be left minimal almost split in $\text{mod } \mathcal{C}$. By 2.2, (a) and (b), there exist $f_i \in \text{Hom}_A(F_{\lambda}E_i, F_{\lambda}Y)$ for every i , such that $f = \sum_i f_i \circ F_{\lambda}(u_i)$. More generally an induction on l shows that there exist morphisms $\delta_1: X \rightarrow X_1, \dots, \delta_s: X \rightarrow X_s$ in $\text{mod } \mathcal{C}$ all equal to compositions of l irreducible morphisms between indecomposable modules and there exist $h_i \in \text{Hom}_A(F_{\lambda}X_i, F_{\lambda}Y)$ for every i , such that $f = \sum_i h_i \circ F_{\lambda}(\delta_i)$. On the other hand, $h_i = \sum_g F_{\lambda}(h_{i,g})$ with $(h_{i,g})_{g \in G} \in \bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}(X_i, {}^g Y)$ by the covering property of F_{λ} . Therefore:

$$f = \sum_g F_{\lambda} \left(\sum_i h_{i,g} \circ \delta_i \right)$$

where $\sum_i h_{i,g} \circ \delta_i \in \text{rad}^l(X, {}^g Y)$ for every g . In the particular case where $l = n$, we have $f = 0$. Thus $\text{rad}^n(F_{\lambda}X, F_{\lambda}Y) = 0$. This proves that Γ is generalised standard.

Conversely, assume that Γ is generalised standard. Let $f \in \text{rad}^l(X, Y)$ with $X, Y \in \tilde{\Gamma}$ and $l \geq 1$. The arguments used above show that there exist morphisms $\delta_1: X \rightarrow X_1, \dots, \delta_s: X \rightarrow X_s$ in $\text{mod } \mathcal{C}$ all equal to compositions of l irreducible morphisms between indecomposable modules and there exist morphisms $h_1: X_1 \rightarrow Y, \dots, h_s: X_s \rightarrow Y$ such that $f = \sum_i h_i \circ \delta_i$. By 2.2, (c), we therefore have $F_\lambda(f) \in \text{rad}^l(F_\lambda X, F_\lambda Y)$. Hence $F_\lambda(\text{rad}^\infty(X, Y)) \subseteq \text{rad}^\infty(F_\lambda X, F_\lambda Y) = 0$. Since F_λ is faithful, we have $\text{rad}^\infty(X, Y) = 0$ for every $X, Y \in \tilde{\Gamma}$.

(c) This is a direct consequence of 2.2, (d).

(d) By assumption and 2.2, F_λ preserves indecomposability, irreducibility and almost split sequences in $\tilde{\Gamma}$. Consequently, for each $X \in \tilde{\Gamma}$ there is a bijection between the set of arrows in $\tilde{\Gamma}$ which leave (or arrive at) X and the set of arrows in Γ which leave (or arrive at, respectively) $F_\lambda X$. Whence the covering $\tilde{\Gamma} \rightarrow \Gamma$ extending the map $X \mapsto F_\lambda X$. The rest of the assertion is a consequence of the arguments presented in the proof of [17, 3.6].

(e) follows from (d). \blacksquare

Remark 2.4. Assume, in 2.2, that $\tilde{\Gamma}$ is connected and $G_X = 1$ for every $X \in \tilde{\Gamma}$. By 2.3, (d), there is a Galois covering with group G of graphs $p: \mathcal{O}(\tilde{\Gamma}) \rightarrow \mathcal{O}(\Gamma)$ such that $p((X)^{\tau c}) = (F_\lambda X)^{\tau_A}$ for every vertex $X \in \tilde{\Gamma}$. The G -action on $\mathcal{O}(\tilde{\Gamma})$ is given by ${}^g((X)^{\tau c}) = ({}^g X)^{\tau c}$ for every $g \in G, X \in \tilde{\Gamma}$. In particular, if $g: \mathcal{O}(\tilde{\Gamma}) \rightarrow \mathcal{O}(\tilde{\Gamma})$ is an automorphism of graphs such that $p \circ g = p$, then there exists $g' \in G$ such that g is induced by g' .

Remark 2.5. In view of the proof of 2.3, (a), if $X \in \text{mod } \mathcal{C}$ is faithful, then so is $F_\lambda X$. However, one can easily find examples where $F_\lambda X$ is faithful and X is not.

Comparisons between the Auslander-Rieten quivers of A and B when $A = B[M]$

In this paragraph we assume that A is connected and weakly shod and $\mathcal{P}_A^f \neq \emptyset$. Let $P_m \in \mathcal{P}_A^f$ be maximal and $A = B[M]$ the associated one-point extension. We give a useful relationship between the connecting component Γ_A of $\Gamma(\text{mod } A)$ and the connecting components associated to the connected components of B . It follows from the work made in [14] (see also [5, Lem. 4.1] who treated the case where the extension point is separating). For convenience, we give the details below. Note that:

- (a) Any strict predecessor of P_m in $\text{ind } A$ is a B -module.
- (b) If $P \in \text{ind } B$ is projective, then any predecessor of P in $\text{ind } A$ is a B -module.

We begin with the following lemma.

Lemma 2.6. *Let \mathcal{X} be the full subcategory of $\text{ind } A$ generated by*

$$\{X \in \text{ind } A \mid X \not\cong P_m \text{ and } X \text{ is a predecessor in } \text{ind } A \text{ of an indecomposable projective } A\text{-module}\}.$$

Then:

- (a) \mathcal{X} is made of B -modules.
- (b) \mathcal{X} is stable under predecessors in $\text{ind } A$ and contains no successor of P_m in $\text{ind } A$.
- (c) τ_A and τ_B coincide on \mathcal{X} .
- (d) The full subquivers of $\Gamma(\text{mod } A)$ and $\Gamma(\text{mod } B)$ generated by \mathcal{X} coincide.

Proof: (a) and (b) follow from the definition of \mathcal{X} . For $L \in \text{mod } A$ let \bar{L} be the B -module obtained by restriction of scalars, that is, $\bar{L} = L.(1 - e)$ if $e \in A$ is the idempotent such that $P_m = eA$. Assume that $0 \rightarrow \tau_A X \xrightarrow{u} E \xrightarrow{v} X \rightarrow 0$ is an almost split sequence in $\text{mod } A$ with $X \in \text{ind } B$. Then it is easily verified that $\overline{\tau_A X} = \tau_B X$ and $0 \rightarrow \tau_B X \xrightarrow{\bar{u}} \bar{E} \xrightarrow{\bar{v}} \bar{X} \rightarrow 0$ is almost split in $\text{mod } B$. Also, if X is not a successor of P_m , then the two exact sequences coincide. Then (c) and (d) follow from these facts. \blacksquare

The category \mathcal{X} of the preceding lemma serves to compare connecting components as follows.

Lemma 2.7. *Let \mathcal{X} be as in the preceding lemma, $M' \in \text{ind } B$ a direct summand of M and B' the component of B such that $M' \in \text{ind } B'$. If Γ' is the component of $\Gamma(\text{mod } B')$ containing M' , then:*

- (a) The connecting component Γ_A of $\Gamma(\text{mod } A)$ contains every module lying on both Γ' and \mathcal{X} .
- (b) The full subquivers of Γ_A and Γ' generated by the modules lying on both Γ' and \mathcal{X} coincide.
- (c) Every $\tau_{B'}$ -orbit of Γ' contains a module lying on \mathcal{X} .

Proof: (a) Let X lie on both Γ' and \mathcal{X} . By [5, 1.1], $\tau_{B'}^m X$ is a predecessor in $\Gamma(\text{mod } B')$ (and therefore in $\Gamma(\text{mod } A)$), by 2.6) of M' or of a projective $P \in \text{ind } B'$ for some $m \geq 0$. By [14, Lem. 5.3], $P \in \Gamma_A$. So $\tau_{B'}^m X \in \Gamma_A$. On the other hand, 2.6, (c), implies that $\tau_{B'}^m X = \tau_A^m X$. So $X \in \Gamma_A$.

(b) Let \mathcal{X}_1 and \mathcal{X}_2 be the full subquivers of Γ_A and Γ' , respectively, generated by the modules lying on both \mathcal{X} and Γ' . By (a), \mathcal{X}_1 and \mathcal{X}_2 have the same vertices. Then 2.6, (d), implies that $\mathcal{X}_1 = \mathcal{X}_2$.

(c) is obtained using similar arguments as those used to prove (a). \blacksquare

Remark 2.8. Using 2.7 we get the following description of the orbit-graph $\mathcal{O}(\Gamma_A)$. For simplicity, we write $\mathcal{O}(\Gamma_A) \setminus \{(P_m)^{\tau_A}\}$ for the full subgraph of $\mathcal{O}(\Gamma_A)$ generated by the vertices different from $(P_m)^{\tau_A}$.

- (a) Let B' be a component of B and $\Gamma_{B'}$ the (unique) connecting component of B' containing a direct summand of M . Then $\mathcal{O}(\Gamma_{B'})$ is a component of $\mathcal{O}(\Gamma_A) \setminus \{(P_m)^{\tau_A}\}$ and all the components of $\mathcal{O}(\Gamma_A) \setminus \{(P_m)^{\tau_A}\}$ have this form.
- (b) If X is an indecomposable direct summand of M with multiplicity d , then $(X)^{\tau_B}$ lies on exactly one of the connected components of $\mathcal{O}(\Gamma_A) \setminus \{(P_m)^{\tau_A}\}$ and $\mathcal{O}(\Gamma_A)$ contains exactly d edges $(X)^{\tau_A} - (P_m)^{\tau_A}$. Moreover all the arrows connected to $(P_m)^{\tau_A}$ have this form.

3 Components of the first kind for weakly shod algebras

Let A be weakly shod. We examine when a component of $\Gamma(\text{mod } A)$ is made of modules of the first kind with respect to any Galois covering of A . We study two cases: When the component is connecting and when it is semi-regular and not regular.

Connecting components of the first kind

The aim of this paragraph is to prove the following proposition.

Proposition 3.1. *Let A be connected, weakly shod and not quasi-tilted of canonical type, Γ_A a connecting component of A and $F: \mathcal{C} \rightarrow A$ a connected Galois covering with group G . Then Γ_A is made of modules of the first kind. Moreover the full subquiver $\Gamma_{\mathcal{C}}$ of $\Gamma(\text{mod } \mathcal{C})$ generated by the modules $X \in \text{ind } \mathcal{C}$ such that $F_{\lambda}X \in \Gamma_A$ is a G -stable faithful and generalised standard component of $\Gamma(\text{mod } \mathcal{C})$ with no non-trivial path of the form $X \rightsquigarrow {}^g X$. Finally, the map $X \mapsto F_{\lambda}X$ on the vertices of $\Gamma_{\mathcal{C}}$ extends to a Galois covering of translation quivers $\Gamma_{\mathcal{C}} \rightarrow \Gamma_A$ with group G .*

In order to prove this result, we proceed along the following steps:

- (a) Any $X \in \Gamma_A$ satisfies $X \simeq F_{\lambda}\tilde{X}$ for some $\tilde{X} \in \text{ind } \mathcal{C}$ such that $G_{\tilde{X}} = 1$.
- (b) $\text{rad}^{\infty}(X, Y) = 0$ for every $X, Y \in \Gamma_{\mathcal{C}}$.
- (c) $\Gamma_{\mathcal{C}}$ is a component of $\Gamma(\text{mod } \mathcal{C})$.

We prove each step in a separate lemma.

Lemma 3.2. *Let A be connected, weakly shod and not quasi-tilted of canonical type, Γ_A a connecting component of A and $F: \mathcal{C} \rightarrow A$ a Galois covering with group G where \mathcal{C} is locally bounded. Then for every $X \in \Gamma_A$ there exists $\tilde{X} \in \text{ind } \mathcal{C}$ such that $F_{\lambda}\tilde{X} \simeq X$ and $G_{\tilde{X}} = 1$.*

Proof: Note that if $Y = \tau_A^m X$ for some $m \in \mathbb{Z}$, then the conclusion holds true for X if and only if it holds true for Y . We prove the lemma by induction on $\text{rk}(K_0(A))$ and begin with the case where A is tilted. If A is tilted then Γ_A has a complete slice $\{T_1, \dots, T_n\}$. By [22, Cor. 4.5, Prop. 4.6] and the above remark, the lemma holds true for A . Now assume that A is not tilted and that the lemma holds true for algebras whose rank of the Grothendieck group is smaller than $\text{rk}(K_0(A))$. So $\mathcal{P}_A^f \neq \emptyset$. Let $P_m \in \mathcal{P}_A^f$ be maximal and $A = B[M]$ the associated one-point extension. Recall ([20, Prop. 6.1.40, Prop. 6.1.41]) that for any component B' of B the Galois covering $F: \mathcal{C} \rightarrow A$ restricts to a Galois covering $F^{-1}(B') \rightarrow B'$ with group G . The conclusion of the lemma clearly holds true for $X = P_m$. Let B' be a component of B and X lie in a connecting component of B' . By the induction hypothesis, we have $X \simeq F_{\lambda}'\tilde{X}$ where $\tilde{X} \in \text{ind } F^{-1}(B')$ is such that $G_{\tilde{X}} = 1$ and $F': F^{-1}(B') \rightarrow B'$ is the restriction of F . In particular $X \simeq F_{\lambda}\tilde{X}$. By the above remark and 2.8, the proposition therefore holds true for A . ■

Lemma 3.3. *Keep the notations and hypotheses 3.2. Let $\Gamma_{\mathcal{C}}$ be the full subquiver of $\Gamma(\text{mod } \mathcal{C})$ generated by the modules $X \in \text{ind } \mathcal{C}$ such that $F_{\lambda}X \in \Gamma_A$. Then:*

- (a) $\Gamma_{\mathcal{C}}$ is a (disjoint) union of components of $\Gamma(\text{mod } \mathcal{C})$.
- (b) $\Gamma_{\mathcal{C}}$ is faithful, has no non trivial path of the form $X \rightsquigarrow {}^g X$ and $\text{rad}^{\infty}(X, Y) = 0$ for every $X, Y \in \Gamma_{\mathcal{C}}$.

Proof: This follows from 2.3 and the fact that Γ_A is faithful, generalised standard, and has no oriented cycle. ■

Lemma 3.4. *Keep the notations and hypotheses of 3.1. Then $\Gamma_{\mathcal{C}}$ is a component of $\Gamma(\text{mod } \mathcal{C})$.*

Proof: Following [18], we define the left part \mathcal{L}_A of $\text{mod } A$ as the full subcategory of $\text{ind } A$ generated by:

$$\{M \in \text{ind } A \mid \text{pd}_A L \leq 1 \text{ for every predecessor } L \text{ of } M \text{ in } \text{ind } A\}$$

where pd_A is the projective dimension. Let T be the direct sum of the indecomposable A -modules which are either Ext-injective in \mathcal{L}_A or not in \mathcal{L}_A and projective. Then T is a basic tilting A -module ([3, 4.2.4.4]) and for every $X \in \Gamma_A$ there exists $m \in \mathbb{Z}$ such that $\tau_A^m X$ is a direct summand of T . Fix an indecomposable decomposition $T = T_1 \oplus \dots \oplus T_n$ in $\text{mod } A$. By 3.2, there exist $\tilde{T}_1, \dots, \tilde{T}_n \in \Gamma_{\mathcal{C}}$ such that $F_{\lambda}\tilde{T}_i \simeq T_i$ and $G_{\tilde{T}_i} = 1$, for every i . Let \mathcal{E} be the full subcategory of $\text{ind } \mathcal{C}$ generated by $\{{}^g \tilde{T}_i \mid i \in \{1, \dots, n\} \text{ and } g \in G\}$. So \mathcal{C} and \mathcal{E} have equivalent derived categories (see the proof of [22, Lem. 4.8]). In particular \mathcal{E} is connected. So, by 3.3, (b), there is a component Γ of $\Gamma_{\mathcal{C}}$ which contains $\{{}^g \tilde{T}_i\}_{i,g}$. We claim that $\Gamma = \Gamma_{\mathcal{C}}$. If $X \in \Gamma_{\mathcal{C}}$, then $F_{\lambda}X \in \Gamma_A$ so that $\tau_A^m F_{\lambda}X \simeq T_i$ for some $i \in \{1, \dots, n\}$ and $m \in \mathbb{Z}$. Consequently $\tau_{\mathcal{C}}^m X \simeq {}^g \tilde{T}_i$ for some g and therefore

$X \in \Gamma$. Thus $\Gamma_C = \Gamma$ is connected. ■

Now we prove 3.1.

Proof of 3.1: The proposition is a direct consequence of 3.2, 3.3 and 3.4. ■

Remark 3.5. Assume that $\mathcal{P}_A^f = \emptyset$ and A admits two connecting components: Its unique postprojective component and its unique preinjective one. With the hypotheses and notations of 3.1, assume that Γ_A is the postprojective component (or the preinjective component) of A . Then it is not difficult to check that Γ_C is the unique postprojective component (or the unique preinjective component, respectively) of $\Gamma(\text{mod } \mathcal{C})$.

Semi-regular components of the first kind

Now we treat the case of semi-regular components containing a projective or an injective. Most of the work in this paragraph is based on the following lemma which does not assume A to be weakly shod.

Lemma 3.6. *Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G where \mathcal{C} is locally bounded. Let Γ be a component of $\Gamma(\text{mod } A)$ such that:*

- (a) Γ has no multiple arrows and every vertex in Γ is the source of at most two arrows and the target of at most two arrows.
- (b) There exists $M_0 \in \Gamma$ which is either the source of exactly one arrow or the target of exactly one arrow, and which is isomorphic to $F_\lambda \tilde{M}_0$ where $\tilde{M}_0 \in \text{ind } \mathcal{C}$ is such that $G_{\tilde{M}_0} = 1$.

Then every $X \in \Gamma$ is isomorphic to $F_\lambda \tilde{X}$ for some $\tilde{X} \in \text{ind } \mathcal{C}$ such that $G_{\tilde{X}} = 1$.

Proof: Let \mathcal{X} be the set of those modules $X \in \Gamma$ for which the conclusion of the lemma holds. Therefore \mathcal{X} contains M_0 and \mathcal{X} is stable under τ_A and τ_A^{-1} because of 2.1, (d). Assume by absurd that $\mathcal{X} \subsetneq \Gamma$. Then by considering an unoriented path in Γ starting from a module $X \in \Gamma \setminus \mathcal{X}$, ending at M_0 and of minimal length, we have the following (or its dual treated dually): There exists an irreducible morphism $u: Y \rightarrow X$ with $X \in \mathcal{X}$, $Y \in \Gamma \setminus \mathcal{X}$ and such that if $E \rightarrow X$ is right minimal almost split, then either $E = Y$, or $E = Y \oplus Y'$ for some $Y' \in \mathcal{X}$. We prove that $Y \simeq F_\lambda \tilde{Y}$ for some $\tilde{Y} \in \text{ind } \mathcal{C}$. For this purpose, we distinguish two cases according to whether E is indecomposable or not. We fix $\tilde{X} \in \text{ind } \mathcal{C}$ such that $F_\lambda \tilde{X} \simeq X$ and $G_{\tilde{X}} = 1$. Assume first that $E = Y$ is indecomposable. Let $\tilde{u}: \tilde{Y} \rightarrow \tilde{X}$ be a right minimal almost split morphism in $\text{mod } \mathcal{C}$. Thus 2.1, (c), implies that so is $F_\lambda(\tilde{u}): F_\lambda \tilde{Y} \rightarrow F_\lambda \tilde{X}$. Therefore $F_\lambda \tilde{Y} \simeq Y$. Now assume that $E = Y \oplus Y'$ with $Y' \in \mathcal{X}$. In particular, $Y' \simeq F_\lambda Y''$ for some $Y'' \in \text{ind } \mathcal{C}$. We thus have a right minimal almost split morphism $[u, u']: Y \oplus Y' \rightarrow X$ in $\text{mod } A$. Let $f: \tilde{E} \rightarrow \tilde{X}$ be a right minimal almost split morphism in $\text{mod } \mathcal{C}$. As above, we deduce that so is $F_\lambda(f): F_\lambda \tilde{E} \rightarrow F_\lambda \tilde{X}$ in $\text{mod } A$. Therefore $F_\lambda \tilde{E} \simeq Y \oplus F_\lambda \tilde{Y}'$. Applying F yields $\bigoplus_{g \in G} {}^g \tilde{E} \simeq F(Y \oplus \bigoplus_{g \in G} {}^g \tilde{Y}')$. Since $\tilde{Y} \in \text{ind } \mathcal{C}$, we deduce that ${}^g \tilde{E} = \tilde{Y}' \oplus \tilde{Y}$ for some $g \in G$ and some $\tilde{Y} \in \text{mod } \mathcal{C}$.

Consequently $F_\lambda \tilde{E} \simeq F_\lambda \tilde{Y}' \oplus F_\lambda \tilde{Y}$ and finally $Y \simeq F_\lambda \tilde{Y}$. Hence, in any case, we have $Y \simeq F_\lambda \tilde{Y}$ and an irreducible morphism $\tilde{Y} \rightarrow \tilde{X}$ for some $\tilde{Y} \in \text{ind } \mathcal{C}$. Since $Y \notin \mathcal{X}$, there exists $g \in G \setminus \{1\}$ such that ${}^g \tilde{Y} \simeq \tilde{Y}$. Therefore the morphism $\tilde{Y} \rightarrow \tilde{X}$ defines two irreducible morphisms $\alpha: \tilde{Y} \rightarrow \tilde{X}$ and $\beta: \tilde{Y} \rightarrow {}^g \tilde{X}$. Since $G_{\tilde{X}} = 1$, and by 2.1, (c), both $F_\lambda(\alpha): F_\lambda \tilde{Y} \rightarrow F_\lambda \tilde{X}$ and $F_\lambda(\beta): F_\lambda \tilde{Y} \rightarrow F_\lambda {}^g \tilde{X} = F_\lambda \tilde{X}$ are irreducible. On the other hand, Γ has no multiple arrows so there exists an isomorphism $\varphi: F_\lambda \tilde{X} \xrightarrow{\sim} F_\lambda \tilde{X}$ such that $F_\lambda(\beta) = \varphi \circ F_\lambda(\alpha)$. By the covering property of F_λ we have $\varphi = \sum_{h \in G} F_\lambda(\varphi_h)$ with $(\varphi_h)_h \in \bigoplus_{h \in G} \text{Hom}_{\mathcal{C}}(\tilde{Y}, {}^h \tilde{X})$. So $F_\lambda(\beta) = \sum_{g \in G} F_\lambda(\varphi_h \circ \alpha)$ and therefore $\beta = \varphi_g \circ \alpha$ because of the covering property of F_λ . This implies that $\varphi_g: \tilde{X} \rightarrow {}^g \tilde{X}$ is a retraction and therefore an isomorphism. We get a contradiction because $G_{\tilde{X}} = 1$. ■

We apply this lemma to our situation where A is weakly shod and not quasi-tilted of canonical type.

Proposition 3.7. *Let A be connected, weakly shod and not quasi-tilted of canonical type, $F: \mathcal{C} \rightarrow A$ a Galois covering with group G where \mathcal{C} is locally bounded and Γ a semi-regular component of $\Gamma(\text{mod } A)$ containing a projective or an injective. Then for every $X \in \Gamma$ there exists $\tilde{X} \in \text{ind } \mathcal{C}$ such that $F_\lambda \tilde{X} \simeq X$ and $G_{\tilde{X}} = 1$.*

Proof: It follows from [14, 6.2] that at least one of the following cases is satisfied:

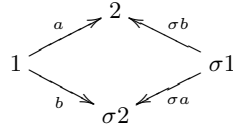
- (a) Γ is a postprojective or a preinjective component.
- (b) Γ is obtained from a tube or from $\mathbb{Z}A_\infty$ by ray or coray insertions.

In case (a), the proposition follows from: 2.1, (d); the fact that the G -action on $\text{mod } \mathcal{C}$ commutes with $\tau_{\mathcal{C}}$; and, the fact that the conclusion of the proposition holds true for indecomposable projective or injective modules. In case (b), there exists a projective $M_0 \in \Gamma$ such that Γ and M_0 satisfy the conditions of 3.6. Whence the proposition. ■

Remark 3.8. Keep the notations and hypotheses of the 3.7. Let $\tilde{\Gamma}$ be the full subquiver of $\Gamma(\text{mod } \mathcal{C})$ generated by the vertices $X \in \text{ind } \mathcal{C}$ such that $F_\lambda X \in \Gamma$. Then $\tilde{\Gamma}$ is a union of semi-regular components and contains a projective or an injective.

The following example shows that 3.7 does not necessarily hold for regular components, even for tilted algebras.

Example 3.9. Let A be the path algebra of the Kronecker quiver $1 \begin{matrix} \xrightarrow{a} \\ \xrightarrow{b} \end{matrix} 2$. It admits a Galois covering $F: A' \rightarrow A$ with group $\mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$ where A' is the path algebra of the following quiver of type $\widetilde{\mathbb{A}}_3$:



with $F(x) = F(\sigma x) = x$ for every $x \in \{1, 2, a, b\}$. Then the indecomposable A -module $k \begin{matrix} \xrightarrow{\text{Id}} \\ \xrightarrow{\text{id}} \end{matrix} k$ lying on a homogeneous tube is not of the first kind with respect to F and, in general, with respect to any non-trivial connected Galois covering of A .

4 The Galois covering of A associated to the universal cover of the connecting component

Let A be weakly shod and not quasi-tilted of canonical type and Γ_A a connecting component. Recall that given a connected Galois covering $F: A' \rightarrow A$ with group G there is a component $\Gamma_{A'}$ of $\Gamma(\text{mod } A')$ and a Galois covering of graphs $\mathcal{O}(\Gamma_{A'}) \rightarrow \mathcal{O}(\Gamma_A)$ with group G (see 3.1 and 2.4). This Galois covering of graphs is called *associated to F* . In this section, we prove the following result which is a counter-part of the work made in 3.1.

Proposition 4.1. *Let A be connected, weakly shod and not quasi-tilted of canonical type, and Γ_A a connecting component. Then there exists a connected Galois covering $F: \tilde{A} \rightarrow A$ with group the fundamental group $\pi_1(\Gamma_A)$ such that the associated Galois covering of graphs $\mathcal{O}(\tilde{\Gamma}_A) \rightarrow \mathcal{O}(\Gamma_A)$ is the universal cover.*

Remark 4.2. Recall that if A has more than one connecting component, then it has two of them: The unique preinjective component and the unique postprojective component. In particular the isomorphism class of $\pi_1(\Gamma_A)$ does not depend on the connecting component.

Until the end of the section we adopt the hypotheses and notations of the above proposition. Here is the strategy of its proof. We use an induction on $\text{rk}(K_0(A))$. If A is tilted of type Q , then $\mathcal{O}(\Gamma_A)$ is the underlying graph of Q . So 4.1 follows from [22, Thm. 1] in that case. If A is not tilted, there exists $P_m \in \mathcal{P}_A^f$ maximal and defining the one-point extension $A = B[M]$. Then we use 2.8 and the Galois covering of B given by the inductive step to construct the desired Galois covering of A .

From now on we assume that A is not tilted, $P_m \in \mathcal{P}_A^f$ is maximal and $A = B[M]$ is the associated one-point extension. The extending object is denoted by $x_0 \in A_0$. Also we assume that 4.1 holds true for the components B_1, \dots, B_t of B ($B = B_1 \times \dots \times B_t$). Thus for every $i \in \{1, \dots, t\}$ there is a connected Galois covering $F^{(i)}: \tilde{B}_i \rightarrow B_i$ with group $\pi_1(\Gamma_i)$ equal to the fundamental group of the (unique) connecting component Γ_i of B_i containing a direct summand of M . We write $\tilde{\Gamma}_i \rightarrow \Gamma_i$ for the universal cover of translation quivers. The construction of a connected Galois covering $F: \tilde{C} \rightarrow A$ with group $\pi_1(\Gamma_A)$ is decomposed into the following steps.

- A reminder on the universal cover of $\mathcal{O}(\Gamma_A)$.
- The construction of a Galois covering $F: \tilde{B} \rightarrow B$ with group $\pi_1(\Gamma_A)$ using $F^{(1)}, \dots, F^{(t)}$.
- The construction of the locally bounded k -category \tilde{A} and the Galois covering $F: \tilde{A} \rightarrow A$.
- The proof that \tilde{A} is connected.

Reminder: the universal cover $\mathcal{O}(\Gamma_A)$

For simplicity, we still denote by x_0 the vertex $(P_m)^{\tau_A}$ of $\mathcal{O}(\Gamma_A)$ and use it as the base-point for the computation of the universal cover of $\mathcal{O}(\Gamma_A)$. Recall that the universal cover $p: \tilde{\mathcal{O}} \rightarrow \mathcal{O}(\Gamma_A)$ is such that:

- $\tilde{\mathcal{O}}$ is the graph with vertices the homotopy classes $[\Gamma]$ of paths $\Gamma: x_0 \rightsquigarrow x$ in $\mathcal{O}(\Gamma_A)$ (where x is any vertex) and such that for every edge $\alpha: x - y$ in $\mathcal{O}(\Gamma_A)$ and every vertex $[\Gamma]$ in $\tilde{\mathcal{O}}$ with end-point x , there is an edge $\alpha: [\Gamma] - [\alpha\Gamma]$ in $\tilde{\mathcal{O}}$.
- With the notations of (a), p maps the vertex $[\Gamma]$ to x and the edge $\alpha: [\Gamma] - [\alpha\Gamma]$ to $\alpha: x - y$.

The Galois covering of $F: \tilde{B} \rightarrow B$ with group $\pi_1(\Gamma_A)$

We construct a Galois covering $F: \tilde{B} \rightarrow B$ with group $\pi_1(\Gamma_A)$ using $F^{(1)}, \dots, F^{(t)}$. We define \tilde{B} as a disjoint union $\coprod_{i=1}^t \prod_{\tilde{?}} \tilde{B}_i$ of (infinitely many) copies of \tilde{B}_i ($i \in \{1, \dots, t\}$). More precisely, let $i \in \{1, \dots, t\}$. Every component \mathcal{U} of $p^{-1}(\mathcal{O}(\Gamma_i))$ is simply connected so the restriction $\mathcal{U} \rightarrow \mathcal{O}(\Gamma_i)$ of p fits into a commutative diagram of graphs:

$$\begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\sim} & \mathcal{O}(\tilde{\Gamma}_i) \\
 & \searrow & \swarrow \\
 & & \mathcal{O}(\Gamma_i)
 \end{array} \tag{D\mathcal{U}}$$

where the horizontal arrow is an isomorphism and the oblique arrow on the right is induced by $\tilde{\Gamma}_i \rightarrow \Gamma_i$. We then attach to \tilde{B} one copy of \tilde{B}_i for each component \mathcal{U} of $p^{-1}(\mathcal{O}(\Gamma_i))$. The Galois coverings $F^{(1)}, \dots, F^{(t)}$ then clearly define a functor $F: \tilde{B} \rightarrow B$ such that F and $F^{(i)}$ coincide on each copy of \tilde{B}_i .

Now we endow \tilde{B} with a $\pi_1(\Gamma_A)$ -action such that $F \circ g = g$ for every $g \in \pi_1(\Gamma_A)$. Let $g \in \pi_1(\Gamma_A)$ and \hat{B}_i be a copy of \tilde{B}_i in \tilde{B} . We define the action of g on \hat{B}_i . Let \mathcal{U} be the component of $p^{-1}(\mathcal{O}(\Gamma_i))$ associated to \hat{B}_i . Then $g(\mathcal{U})$ is also a component of $p^{-1}(\mathcal{O}(\Gamma_i))$ to which corresponds a copy \bar{B}_i of \tilde{B}_i in \tilde{B} . Moreover, the graph morphism $g: \mathcal{U} \rightarrow g(\mathcal{U})$ and the diagrams $(D_{\mathcal{U}})$ and $(D_{g(\mathcal{U})})$ determine an automorphism $\mathcal{O}(\tilde{\Gamma}_i) \xrightarrow{\sim} \mathcal{O}(\tilde{\Gamma}_i)$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{O}(\tilde{\Gamma}_i) & \xrightarrow{\sim} & \mathcal{O}(\tilde{\Gamma}_i) \\ & \searrow & \swarrow \\ & \mathcal{O}(\Gamma_i) & \end{array}$$

Therefore, the automorphism $\mathcal{O}(\tilde{\Gamma}_i) \xrightarrow{\sim} \mathcal{O}(\tilde{\Gamma}_i)$ extends the map $(X)^{\tau_{\tilde{B}_i}} \mapsto (\bar{g}X)^{\tau_{\tilde{B}_i}}$ associated to some $\bar{g} \in \pi_1(\Gamma_i)$ (see 2.4). The action of g on \hat{B}_i is therefore defined as follows: g maps the component \hat{B}_i of \tilde{B} to the component \bar{B}_i and, as a functor, it acts like $\bar{g}: \tilde{B}_i = \hat{B}_i \xrightarrow{\sim} \bar{B}_i = \tilde{B}_i$. This way, we get a $\pi_1(\Gamma_A)$ -action on \tilde{B} such that $F \circ g = F$ for every $g \in G$.

Lemma 4.3. *The $\pi_1(\Gamma_A)$ -action on \tilde{B} is free, \tilde{B} is locally bounded and $F: \tilde{B} \rightarrow B$ is a Galois covering with group $\pi_1(\Gamma_A)$.*

Proof: Let $x \in \tilde{B}_o$ and $g \in \pi_1(\Gamma_A)$ be such that $gx = x$. We write \hat{B}_i for the copy of \tilde{B}_i in \tilde{B} containing x and \mathcal{U} for the corresponding component of $p^{-1}(\mathcal{O}(\Gamma_i))$. In particular, $g(\mathcal{U}) = \mathcal{U}$ and there exists $g' \in \pi_1(\Gamma_i)$ such that the action of g on \hat{B}_i is given by $g': \tilde{B}_i = \hat{B}_i \xrightarrow{\sim} \hat{B}_i = \tilde{B}_i$. Since $gx = x$, this means that $g'x = x$. So $g' = \text{Id}_{\tilde{B}_i}$ and g is the identity map on \mathcal{U} . Thus, g is the identity on the universal cover $\mathcal{O}(\tilde{\Gamma}_i)$ and therefore on \tilde{B} . This proves that the $\pi_1(\Gamma_A)$ -action on \tilde{B} is free.

By construction, \tilde{B} is locally bounded.

Now we prove that $\pi_1(\Gamma_A)$ acts transitively on $F^{-1}(x)$ for every $x \in \tilde{B}_o$. Let $x, y \in \tilde{B}_o$ be such that $Fx = Fy$. By construction of F , there exists i such that x and y lie on copies \hat{B}_i and \bar{B}_i of \tilde{B}_i in \tilde{B} , respectively. We write \mathcal{U} and \mathcal{V} for the components of $p^{-1}(\mathcal{O}(\Gamma_i))$ corresponding to \hat{B}_i and \bar{B}_i , respectively. So there exists $g \in \pi_1(\Gamma_A)$ such that $g(\mathcal{U}) = \mathcal{V}$. Therefore gx lies on \bar{B}_i and $F(gx) = Fy$. So we may assume that $\bar{B}_i = \hat{B}_i$. Using $(D_{\mathcal{U}})$, we identify the map $\mathcal{U} \rightarrow \mathcal{O}(\Gamma_i)$ induced by p with the universal cover $\mathcal{O}(\tilde{\Gamma}_i) \rightarrow \mathcal{O}(\Gamma_i)$. Since F coincides with $F^{(i)}: \tilde{B}_i \rightarrow B_i$ on \hat{B}_i , there exists $g' \in \pi_1(\Gamma_i)$ such that $g'(x) = y$. Moreover, there exists $g'' \in \pi_1(\Gamma_A)$ such that g'' and g' coincide on some vertex of \mathcal{U} (because $p: \tilde{\mathcal{O}} \rightarrow \mathcal{O}(\Gamma_A)$ is a Galois covering with group $\pi_1(\Gamma_A)$) and therefore on \mathcal{U} (because $\mathcal{U} \rightarrow \mathcal{O}(\Gamma_i)$ is a Galois covering). We thus have $g''x = y$ with $g'' \in \pi_1(\Gamma_A)$. This shows the transitivity of $\pi_1(\Gamma_A)$ on the fibres of $F: \tilde{B}_o \rightarrow B_o$.

Therefore $F: \tilde{B} \rightarrow B$ is, by construction, a covering functor, $\pi_1(\Gamma_A)$ is a group acting freely on \tilde{B} such that $F \circ g = g$ for every $g \in \pi_1(\Gamma_A)$ and $\pi_1(\Gamma_A)$ acts transitively on the fibres of $F: \tilde{B}_o \rightarrow B_o$. So F is a Galois covering with group $\pi_1(\Gamma_A)$. \blacksquare

The Galois covering $F: \tilde{A} \rightarrow A$ with group $\pi_1(\Gamma_A)$

Now we extend $F: \tilde{B} \rightarrow B$ to a Galois covering $F: \tilde{A} \rightarrow A$ with group $\pi_1(\Gamma_A)$. Recall that $A = B[M]$. Accordingly let \tilde{A} be the category:

$$\tilde{A} = \begin{bmatrix} S & \tilde{M} \\ 0 & \tilde{B} \end{bmatrix} \quad (*)$$

where S is the category with objects set $S_o = \pi_1(\Gamma_A) \times \{x_0\}$ and no non-zero morphism except the scalar multiples of the identity morphisms and \tilde{M} is an $S - \tilde{B}$ -bimodule defined as follows. Fix an indecomposable decomposition $M = \bigoplus_{i=1}^t \bigoplus_{j=1}^{n_i} M_{i,j}$ such that $M_{i,j} \in \text{ind } B_i$ for every i, j . Let i, j be such indices. Then the

homotopy class of the edge $x_0 - (M_{i,j})^{\tau_A}$ associated to the inclusion morphism $M_{i,j} \hookrightarrow P_m$ is a vertex in $\tilde{\mathcal{O}}$ (see 2.8). Also it lies on some component \mathcal{U} of $p^{-1}(\mathcal{O}(\Gamma_i))$ to which corresponds a copy \hat{B}_i of \tilde{B}_i in \tilde{B} . By 3.2, there exists $\tilde{M}_{i,j} \in \text{ind } \tilde{B}_i$ such that $F_{\lambda}^{(i)} \tilde{M}_{i,j} = M_{i,j}$. We thus consider $\tilde{M}_{i,j}$ as an indecomposable \tilde{B} -module such that $\tilde{M}_{i,j} \in \text{ind } \tilde{B}_i$. In particular we have $F_{\lambda} \tilde{M}_{i,j} = M_{i,j}$. The $S - \tilde{B}$ -bimodule \tilde{M} is then defined as follows:

$$\begin{aligned} \tilde{M}: \quad S \times \tilde{B}^{op} &\rightarrow \text{mod } k \\ ((g, x_0), x) &\mapsto \bigoplus_{i=1}^t \bigoplus_{j=1}^{n_i} {}^g \tilde{M}_{i,j}(x) . \end{aligned}$$

The k -category \tilde{A} is thus completely defined. Now we extend the $\pi_1(\Gamma_A)$ -action on \tilde{B} to an action on \tilde{A} . We let $\pi_1(\Gamma_A)$ act on $\pi_1(\Gamma_A) \times \{x_0\}$ in the obvious way. Let $g \in \pi_1(\Gamma_A)$ and $u \in \tilde{M}_{i,j}(h^{-1}x) \subseteq \tilde{M}((h, x_0), x)$. We define $g.u$ to be the morphism u viewed as an element of $\tilde{M}_{i,j}(h^{-1}x) \subseteq \tilde{M}(g.(h, x_0), g.x)$.

Lemma 4.4. *The above construction defines a locally bounded k -category \tilde{A} endowed with a free $\pi_1(\Gamma_A)$ -action.*

Proof: We clearly have defined a k -category and the $\pi_1(\Gamma_A)$ -action is well-defined and free because $\pi_1(\Gamma_A)$ acts freely on $\pi_1(\Gamma_A) \times \{x_0\}$ and on \tilde{B} . We prove that \tilde{A} is locally bounded. Recall that \tilde{B} is locally bounded. Moreover for every $g \in \pi_1(\Gamma_A)$ we have $\bigoplus_{x \in \tilde{B}_o} \tilde{A}((g, x_0), x) = \bigoplus_{x \in \tilde{B}_o, i, j} \tilde{M}_{i, j}(g^{-1}x) = \bigoplus_{x \in \tilde{B}_o} M(x)$ because $F_\lambda \tilde{M}_{i, j} = M_{i, j}$ for every i, j . Thus $\bigoplus_{x \in \tilde{B}_o} \tilde{A}((g, x_0), x)$ is finite dimensional for every $g \in \pi_1(\Gamma_A)$. Finally, for every $x \in \tilde{B}_o$ we have

$\bigoplus_{g \in \pi_1(\Gamma_A)} \tilde{A}((g, x_0), x) = \bigoplus_{g \in \pi_1(\Gamma_A), i, j} \tilde{M}_{i, j}(g^{-1}x) = M(F(x))$. So $\bigoplus_{g \in \pi_1(\Gamma_A)} \tilde{A}((g, x_0), x)$ is finite dimensional for every $x \in \tilde{B}_o$. This proves that \tilde{A} is locally bounded. \blacksquare

We extend the Galois covering $F: \tilde{B} \rightarrow B$ to a functor $F: \tilde{A} \rightarrow A$ as follows:

- (a) $F((g, x_0)) = x_0$ for every $g \in \pi_1(\Gamma_A)$.
- (b) Let $u \in \tilde{M}_{i, j}(g^{-1}x) \subseteq \tilde{M}((g, x_0), x)$. Then $\tilde{M}_{i, j}(g^{-1}x) \subseteq \bigoplus_{h \in \pi_1(\Gamma_A)} \tilde{M}_{i, j}(h^{-1}x) = M_{i, j}(F(x)) \subseteq M(F(x))$ (recall that $F_\lambda \tilde{M}_{i, j} = M_{i, j}$). So we set $F(u) = u \in M(F(x))$.

Lemma 4.5. *The above construction defines a Galois covering $F: \tilde{A} \rightarrow A$ with group $\pi_1(\Gamma_A)$.*

Proof: $F: \tilde{A} \rightarrow A$ is a k -linear functor such that $F \circ g = g$ for every $g \in \pi_1(\Gamma_A)$. Moreover it is a covering functor because so is $F: \tilde{B} \rightarrow B$ and $F_\lambda \tilde{M}_{i, j} = M_{i, j}$ for every i, j . Finally, the group $\pi_1(\Gamma_A)$ acts transitively on $F^{-1}(x)$ for every $x \in \tilde{A}_o$. Indeed, this is the case if $x \in \tilde{B}_o$ because $F: \tilde{B} \rightarrow B$ is a Galois covering with group $\pi_1(\Gamma_A)$ and it is clearly the case if $x = x_0$. So F is a Galois covering with group $\pi_1(\Gamma_A)$. \blacksquare

The category \tilde{A} is connected

We denote by P_m the indecomposable projective \tilde{A} -module associated to the object $(1, x_0)$ of \tilde{A} . Therefore $\text{rad}(P_m) = \bigoplus_{i, j} \tilde{M}_{i, j}$. We need the following lemma whose proof follows from the definitions and where $\tilde{x}_0 = (x_0, 1)$.

Lemma 4.6. *Let $g \in \pi_1(\Gamma_A)$ and $g\tilde{x}_0 - x_1$ be an edge in \tilde{O} . Then there exist i, j such that x_1 is the homotopy class of the edge $\alpha: x_0 - (M_{i, j})^{\tau_A}$ in $\mathcal{O}(\Gamma_A)$ associated to the inclusion $M_{i, j} \hookrightarrow P_m$. Let \mathcal{U} be the component of $p^{-1}(\mathcal{O}(\Gamma_i))$ containing x_1 and \tilde{B}_i the associated copy of \tilde{B}_i in \tilde{B} . Then ${}^g\tilde{M}_{i, j} \in \text{ind } \tilde{B}_i$ (and $\tilde{M}_{i, j}$ is a direct summand of $\text{rad}({}^g\tilde{P}_m)$).*

We use 4.6 to prove that \tilde{A} is connected.

Lemma 4.7. *\tilde{A} is connected*

Proof: It suffices to prove that two indecomposable projective \tilde{A} -modules lie on the same component of $\text{mod } \tilde{A}$. Let $g \in \pi_1(\Gamma_A)$. Since \tilde{O} is connected, there is a sequence of edges in \tilde{O} :

$$\begin{array}{cccccccccccc} \tilde{x}_0 & & g_1\tilde{x}_0 & & g_2\tilde{x}_0 & & \cdots & & g_{n-1}\tilde{x}_0 & & g_n\tilde{x}_0 \\ & \searrow & / \quad \backslash & & / \quad \backslash & & \cdots & & / \quad \backslash & & / \quad \backslash \\ & x_1 & x'_1 & x_2 & x'_2 & x_3 & \cdots & & x'_{n-1} & x_n & x'_n \end{array}$$

where $g = g_n$ and, for every j , the vertices x_j and x'_j lie on the same component of $p^{-1}(\mathcal{O}(\Gamma_{i_j}))$ for some i_j . By 4.6 and because $\tilde{B}_1, \dots, \tilde{B}_t$ are connected, the modules \tilde{P}_m and ${}^g\tilde{P}_m$ lie on the same connected component of $\text{mod } \tilde{A}$.

Now let \tilde{P} be an indecomposable projective \tilde{A} -module associated to an object $x \in \tilde{B}_o$. So $F_\lambda \tilde{P}$ is the indecomposable projective B -module associated to Fx . Let $i \in \{1, \dots, t\}$ be such that Fx is an object of B_i . So x is an object of some copy \tilde{B}_i of \tilde{B}_i in \tilde{B} and we let \mathcal{U} be the associated component of $p^{-1}(\mathcal{O}(\Gamma_i))$. On the other hand, we let \tilde{B}_i be the copy of \tilde{B}_i in \tilde{B} such that $\tilde{M}_{i, 1} \in \text{ind } \tilde{B}_i$ and \mathcal{V} the associated component of $p^{-1}(\mathcal{O}(\Gamma_i))$. In particular there exists $g \in \pi_1(\Gamma_A)$ such that $g(\mathcal{V}) = \mathcal{U}$ so that ${}^g\tilde{M}_{i, 1} \in \text{ind } \tilde{B}_i$. Therefore: \tilde{P} and ${}^g\tilde{M}_{i, 1}$ lie on the same component of $\text{mod } \tilde{A}$ because they are indecomposable \tilde{B}_i -modules and \tilde{B}_i is connected; ${}^g\tilde{M}_{i, 1}$ and ${}^g\tilde{P}_m$ lie on the same component of $\text{mod } \tilde{A}$ because of the inclusion $\tilde{M}_{i, j} \hookrightarrow \tilde{P}_m$; and we already proved that so do \tilde{P}_m and ${}^g\tilde{P}_m$. This shows that \tilde{P} and \tilde{P}_m lie on the same component of $\text{mod } \tilde{A}$. So \tilde{A} is connected. \blacksquare

Now we are in position to prove the main result of the section.

Proof of 4.1: We use an induction on $\text{rk}(K_0(A))$. If A is tilted, then the result follows from [22, Thm. 1]. Assume that A is not tilted and that the conclusion of the proposition holds for algebras B such that $\text{rk}(K_0(B)) < \text{rk}(K_0(A))$. Hence there exists a maximal element $P_m \in \mathcal{P}_A^f$. Let $A = B[M]$ be the associated one-point extension. Let $B = B_1 \times \dots \times B_t$ be an indecomposable decomposition. Then B_1, \dots, B_t are connected, weakly shod and not quasi-tilted of canonical type. Let $\Gamma_1, \dots, \Gamma_t$ be the connecting components of B_1, \dots, B_t , respectively, containing a summand of M . The induction hypothesis implies that, for every i , there exists a connected Galois covering $F^{(i)}: \tilde{B}_i \rightarrow B_i$ with group $\pi_1(\Gamma_i)$ whose associated Galois covering of $\mathcal{O}(\Gamma_i)$

is the universal cover of graph. By 4.5 and 4.7, there exists a connected Galois covering $F: \tilde{A} \rightarrow A$ with group $\pi_1(\Gamma_A)$. Let $\mathcal{O}(\Gamma_{\tilde{A}}) \rightarrow \mathcal{O}(\Gamma_A)$ be the associated Galois covering with group $\pi_1(\Gamma_A)$. Since $\pi_1(\Gamma_A)$ is free, this Galois covering is necessarily the universal covering of graphs. ■

We give some examples to illustrate 4.1. In these examples we write P_x , I_x or S_x for the corresponding indecomposable projective, indecomposable injective or simple, respectively.

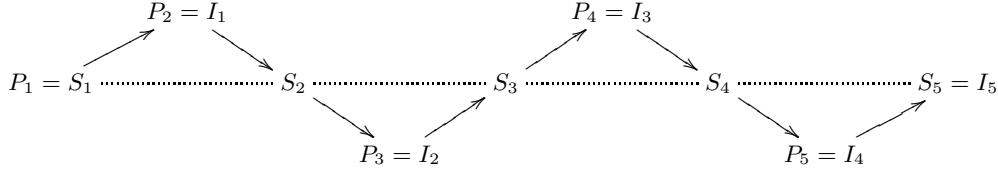
Example 4.8. Let A be the radical square zero algebra with ordinary quiver Q as follows:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \equiv\equiv\equiv 6$$

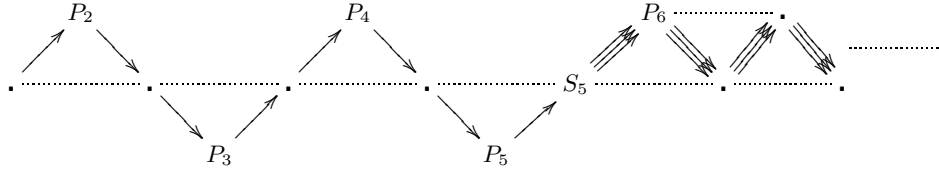
Let $M = \text{rad}(P_6)$. Then $A = B[M]$ where B is the radical square zero algebra with ordinary quiver:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

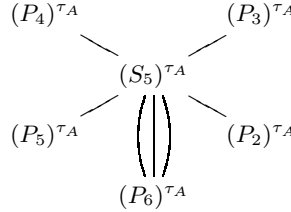
Note that B is of finite representation type and $\Gamma(\text{mod } B)$ is equal to:



The algebra A is wild and weakly shod, it has a unique connecting component of the following shape:



Note that A is not quasi-tilted because the projective dimension of S_5 is equal to 4. The orbit-graph of the connecting component of A is equal to:



The fundamental group of this graph is free of rank 2. So 4.1 implies that A admits a connected Galois covering with group a free group with rank 2. Actually this Galois covering is given by the fundamental group of the monomial presentation of A (see [25]).

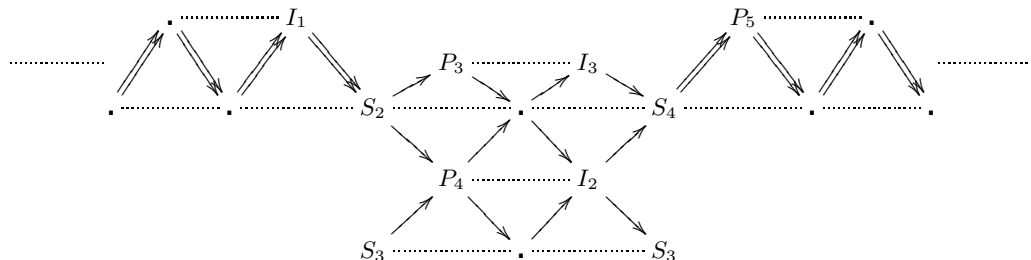
Recall that weakly shod algebras are particular cases of Laura algebras. The following example from [14] shows that 4.1 holds for some Laura algebras which are not weakly shod.

Example 4.9. (see [14, 2.6]) Let A be the radical square zero algebra with ordinary quiver Q as follows:

$$1 \equiv\equiv 2 \longrightarrow 3 \longrightarrow 4 \equiv\equiv 5$$

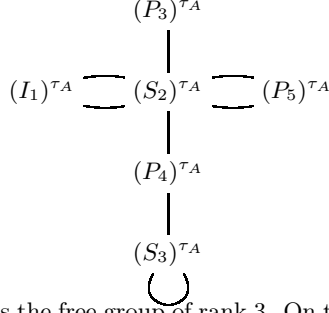
Then A is a Laura algebra. The component of $\Gamma(\text{mod } A)$ consist of:

1. The postprojective components and the homogeneous tubes of the Kronecker algebra with quiver $1 \equiv\equiv 2$.
2. The preinjective component and the homogeneous tubes of the Kronecker algebra with quiver $4 \equiv\equiv 5$.
3. A unique non semi-regular component of the following shape:



where the two copies of the S_3 are identified.

In this example, the orbit-graph of the unique non semi-regular component is the following:



The fundamental group of this graph is the free group of rank 3. On the other hand, if one denotes by (kQ^+) for the ideal of kQ generated by the set of arrows, then the fundamental group of the natural presentation $kQ/(kQ^+)^2 \simeq A$ (in the sense of [25]) is also isomorphic to the free group of rank 3. Hence A admits a connected Galois covering with group isomorphic to the orbit-graph of the connecting component.

5 Proof of Theorem A and of Corollary B

Throughout the section we assume that A is connected and weakly shod. We prove the first two main results of the text presented in the introduction.

Proof of Theorem A: We assume that A is not quasi-tilted of canonical type. Let G be a group and Γ_A a connecting component of $\Gamma(\text{mod } A)$. If $F: \mathcal{C} \rightarrow A$ is a connected Galois covering then 3.1 yields a Galois covering of translation quivers with group G of Γ_A . Conversely, let $\Gamma' \rightarrow \Gamma_A$ be a Galois covering of translation quivers with group G . Therefore $G \simeq \pi_1(\Gamma_A)/N$ for some normal subgroup $N \triangleleft \pi_1(\Gamma_A)$ ([11, 1.4]). On the other hand, 4.1 yields a connected Galois covering $\tilde{A} \rightarrow A$ with group $\pi_1(\Gamma_A)$. Factoring out by N yields a connected Galois covering $\tilde{A}/N \rightarrow A$ with group G . ■

Now we turn to the proof of Corollary B. We need the three following lemmas. The first one follows directly from Theorem A so we omit the proof.

Lemma 5.1. *Assume that A is not quasi-tilted of canonical type. Let Γ_A be a connecting component of A . Then the following conditions are equivalent:*

- (a) A is simply connected,
- (b) The orbit-graph $\mathcal{O}(\Gamma_A)$ is a tree.
- (c) Γ_A is simply connected.

The following lemma expresses the simple connectedness of $A = B[M]$ in terms of the simple connectedness of the components of B . In the case where A is tame weakly shod, the necessity was proved in [5, Lem. 5.1]. We recall that if A is connected and $x_0 \in A_o$ is the extension object in $A = B[M]$, then x_0 is called *separating* if M has exactly as many indecomposable summands as the number of components of B (that is, M restricts to an indecomposable module on each component of B).

Lemma 5.2. *Assume that $\mathcal{P}_A^f \neq \emptyset$. Let $P_m \in \mathcal{P}_A^f$ be maximal, $A = B[M]$ the associated one-point extension and $x_0 \in A_o$ the extending object. Then A is simply connected if and only if the two following conditions are satisfied:*

- (a) B is a product of simply connected algebras,
- (b) x_0 is separating (that is, M is multiplicity-free).

Proof: By [14, 4.5, 4.8], B is a product of connected, weakly shod and not quasi-tilted of canonical type algebras. Assume that A is simply connected. By [4, 2.6], the object x_0 is separating. Let B' be a connected component of B . Since A is connected, M admits an indecomposable summand lying on $\text{ind } B'$. By 2.8 and because the orbit-graph of any connecting component of A is simply connected, the orbit-graph of any connecting component of B' is simply connected. So B' is simply connected by Theorem A. Conversely assume that x_0 is separating and B is a product of simply connected algebras. By Theorem A, for every component B' of B , the orbit-graph of any connecting component of B' is a tree. By 2.8 and because x_0 is separating, we deduce that the orbit-graph of any connecting component of A is a tree. By Theorem A, this implies that A is simply connected. ■

Finally, we recall the following lemma which was proved in [5, 2.5].

Lemma 5.3. *Under the hypothesis and notations of 5.2, the following conditions are equivalent:*

- (a) $\text{HH}^1(A) = 0$.
- (b) $\text{HH}^1(B) = 0$ and x_0 is separating.

Now we can prove Corollary B.

Proof of Corollary B: We use an induction on $\text{rk}(K_0(A))$. By [22, Thm. 1], the corollary holds true if A is tilted. So we assume that A is not quasi-tilted and the corollary holds true for algebras B such that $\text{rk}(K_0(B)) < \text{rk}(K_0(A))$. Since $\mathcal{P}_A^f \neq \emptyset$, there exists $P_m \in \mathcal{P}_A^f$ maximal. Let $A = B[M]$ be the associated one-point extension. Using the induction hypothesis applied to the components of B and using 5.1, 5.2 and 5.3, we deduce that A is simply connected if and only if $\text{HH}^1(A) = 0$. On the other hand, Theorem A shows that A is simply connected if and only if $\mathcal{O}(\Gamma_A)$ is a tree. ■

We finish this section with an example to illustrate Corollary B

Example 5.4. Let A be as in 4.8. Then A is not simply connected and neither is the orbit-graph of its connecting component. On the other hand, a straightforward computation shows that $\dim \text{HH}^0(A) = 1$, $\dim \text{HH}^1(A) = 3$ and $\dim \text{HH}^i(A) = 0$ if $i \geq 2$.

6 The class of weakly shod algebras is stable under finite Galois coverings and under quotients

In this section we prove Theorem C. At first we study the implications of Theorem C in the more general setting of Galois coverings with non necessarily finite groups.

Lemma 6.1. *Let $F: \mathcal{C} \rightarrow A$ be a connected Galois covering with group G . If A is weakly shod and not quasi-tilted, then $\Gamma(\text{mod } \mathcal{C})$ has a unique non semi-regular component $\Gamma_{\mathcal{C}}$. Moreover it is faithful, generalised standard and has no non trivial path of the form $X \rightsquigarrow {}^g X$ with $X \in \Gamma_{\mathcal{C}}$ and $g \in G$.*

Proof: Let Γ_A be the connecting component of A . Let $\Gamma_{\mathcal{C}}$ be as in 3.1. We only need to prove that $\Gamma_{\mathcal{C}}$ is the unique non semi-regular component of $\Gamma(\text{mod } \mathcal{C})$. Note that $\Gamma_{\mathcal{C}}$ contains both a projective and an injective because so does Γ_A . Let $P \in \text{ind } \mathcal{C} \setminus \Gamma_{\mathcal{C}}$ be projective. Then $F_{\lambda} P \in \text{ind } A \setminus \Gamma_A$ is projective and therefore lies on a semi-regular component of $\Gamma(\text{mod } A)$. By 3.8, so does P . Whence the lemma. ■

The preceding lemma has a converse under the additional assumption that the group G acts freely on the indecomposable modules lying on $\Gamma_{\mathcal{C}}$. This last condition is always verified when G is torsion-free.

Lemma 6.2. *Let $F: \mathcal{C} \rightarrow A$ be a connected Galois covering with group G . Assume that $\Gamma(\text{mod } \mathcal{C})$ has a unique non semi-regular component $\Gamma_{\mathcal{C}}$ and that the following conditions are satisfied:*

- (a) $\Gamma_{\mathcal{C}}$ is faithful and generalised standard.
- (b) $\Gamma_{\mathcal{C}}$ has no non trivial path of the form $X \rightsquigarrow {}^g X$.
- (c) $G_X = 1$ for every $X \in \Gamma_{\mathcal{C}}$.

Then A is weakly shod.

Proof: Note that $\Gamma_{\mathcal{C}}$ is G -stable because of its uniqueness. It follows from the arguments presented in the proof of [17, 3.6] that there is a component Γ of $\Gamma(\text{mod } A)$ such that $\Gamma = \{F_{\lambda} X \mid X \in \Gamma_{\mathcal{C}}\}$. Also the map $X \mapsto F_{\lambda} X$ extends to a Galois covering of translation quivers $\Gamma_{\mathcal{C}} \rightarrow \Gamma$ with group G . In particular Γ is non semi-regular. Moreover 2.3 implies that Γ is faithful, generalised standard and has no oriented cycles. Therefore A is weakly shod. ■

Now we prove the equivalences of Theorem C. Part of the tilted case was treated in [22, Rem. 4.10]. We recall it for convenience.

Proposition 6.3. *Let $F: A' \rightarrow A$ be a connected Galois covering with finite group G . Then A' is tilted if A is tilted.*

Now we prove the equivalence of Theorem C in the quasi-tilted case.

Proposition 6.4. *Let $F: A' \rightarrow A$ be a connected Galois covering with finite group G . Then A' is quasi-tilted if and only if A is quasi-tilted.*

Proof: Recall that \mathcal{L}_A denotes the left part of A . We use the following description of \mathcal{L}_A ([6, Thm. 1.1]):

$$\mathcal{L}_A = \{M \in \text{ind } A \mid \text{pd}_A(L) \leq 1 \text{ for every } L \in \text{ind } A \text{ such that } \text{Hom}_A(L, M) \neq 0\} .$$

Also, by ([18, II Thm. 1.14, II Thm. 2.3]), the following conditions are equivalent for any algebra A :

- (a) A is quasi-tilted.
- (b) A has global dimension at most 2 and $\text{id}_A(X) \leq 1$ or $\text{pd}_A(X) \leq 1$ for every $X \in \text{ind } A$.
- (c) \mathcal{L}_A contains all the indecomposable projective A -modules.

Assume that A is quasi-tilted. Let $u: X \rightarrow P$ be a non-zero morphism of A' -modules with $X, P \in \text{ind } A'$ and P projective. So $F_{\lambda}(u): F_{\lambda} X \rightarrow F_{\lambda} P$ is non zero and $F_{\lambda} P$ is indecomposable projective. Fix an indecomposable decomposition $F_{\lambda} X = X_1 \oplus \dots \oplus X_r$ in $\text{mod } A$. So the restriction $X_i \rightarrow F_{\lambda} P$ of $F_{\lambda}(u)$ is non-zero for some i . Since A is quasi-tilted, we have $F_{\lambda} P \in \mathcal{L}_A$ and therefore $\text{pd}_A(X_i) \leq 1$. On the other hand, $F_{\lambda} X = \bigoplus_{g \in G} {}^g X$,

$F_*F_\lambda X = F_*X_1 \oplus \dots \oplus F_*X_r$ and the projective dimension is unchanged under F_* , F_λ and under the action of G . Consequently $\text{pd}_{A'}(X) = \text{pd}_{A'}(F_*X_i) = \text{pd}_A(X_i) \leq 1$. So $P \in \mathcal{L}_{A'}$. Thus, A' is quasi-tilted.

Conversely, assume that A' is quasi-tilted. In particular, A and A' have the same global dimension, that is, at most 2. Let $X \in \text{ind } A$. Since G is finite, $F_*X \in \text{mod } A'$. Fix an indecomposable decomposition $F_*X = X_1 \oplus \dots \oplus X_r$ in $\text{mod } A'$. We claim that X_1, \dots, X_r have the same projective dimension. Indeed, let $d = \text{pd}_{A'}(X_1)$ and $I = \{i \in \{1, \dots, r\} \mid \text{pd}_{A'}(X_i) = d\}$. Then $F_*X = L \oplus M$ where $L = \bigoplus_{i \in I} X_i$ and $M = \bigoplus_{i \in I^c} X_i$. Since the G -action on $\text{mod } A'$ preserves the projective dimension, we have ${}^g L = L$ and ${}^g M = M$ for every $g \in G$. By [16, 1.2], we deduce that there exist $Y, Z \in \text{mod } A$ such that $X = Y \oplus Z$, $L = F_*Y$ and $M = F_*Z$. Since X is indecomposable and $I \neq \emptyset$, we have $Z = 0$ and, therefore, $I = \{1, \dots, r\}$. Thus $\text{pd}_{A'}(X_i) = \text{pd}_{A'}(X_j) = \text{pd}_A(X)$ (and, dually, $\text{id}_{A'}(X_i) = \text{id}_{A'}(X_j) = \text{id}_A(X)$) for every i, j . Since A' is quasi-tilted, we infer that $\text{pd}_A(X) \leq 1$ of $\text{id}_A(X) \leq 1$. This proves that A is quasi-tilted. \blacksquare

Now we end the proof of Theorem C.

Proof of Theorem C: The necessity in (a) follows from 6.3 and (b) was proved in 6.4. We prove (c) and may assume that neither A nor A' is quasi-tilted. Assume that A is weakly shod and not quasi-tilted. Then 6.1 implies that $\Gamma(\text{mod } A')$ has a unique non semi-regular component which is moreover faithful, generalised standard and has no oriented cycle. Therefore A' is weakly shod. This proves the necessity in (c). From now on, we assume that A' is weakly shod and not quasi-tilted of canonical type. We prove that A is weakly shod. In view of 6.2, we need the following result.

Lemma 6.5. *Assume that A' is weakly shod and not quasi-tilted of canonical type. We have $G_X = 1$ for every indecomposable A' -module X lying on a connecting component of $\Gamma(\text{mod } A')$.*

Proof of 6.5: The conclusion of the lemma holds true for any indecomposable projective or injective A' -module. So does it for non-stable modules because $\tau_{A'}$ commutes with the G -action. Let $\Gamma_{A'}$ be a connecting component of A' and $X \in \Gamma_{A'}$ be stable. We still write $\mathcal{L}_{A'}$ for the left part of $\text{mod } A'$ and we write $\mathcal{R}_{A'}$ for the right part of $\text{mod } A'$, defined dually. Since A' is weakly shod, the set $\text{ind } A' \setminus (\mathcal{L}_{A'} \cup \mathcal{R}_{A'})$ is finite, contained in $\Gamma_{A'}$, and has no periodic module. Therefore there exists $n \in \mathbb{Z}$ such that $\tau_{A'}^n X \in \Gamma_{A'} \cap (\mathcal{L}_{A'} \cup \mathcal{R}_{A'})$. Assume for example that $X' = \tau_{A'}^n X \in \Gamma_{A'} \cap \mathcal{L}_{A'}$ (the remaining case is dealt with dually). Let e be the sum of the primitive idempotents e' of A' such that $e'A' \in \mathcal{L}_{A'}$ and let $B' = eA'e$. Therefore B' is a full convex subcategory of A' , it is a product of tilted algebras, $X' \in \text{ind } B'$ (see [3]) and B' is stable under G because so is $\mathcal{L}_{A'}$. In particular, F restricts to a Galois covering $F': B' \rightarrow B$ with group G , where $B := F(B')$. In order to prove that $G_X = 1$ we prove that $G_{X'} = 1$. By absurd assume that there is $g \in G \setminus \{1\}$ such that ${}^g X' \not\cong X'$. After replacing g by some adequate power, we assume that g is of prime order p . The quotient $\pi: B \rightarrow B/\langle g \rangle$ is a Galois covering with group $\langle g \rangle \simeq \mathbb{Z}/p\mathbb{Z}$. Therefore $\text{Ext}_{B/\langle g \rangle}^1(\pi_\lambda X', \pi_\lambda X') \simeq \bigoplus_{j=0}^{p-1} \text{Ext}_B^1(X', {}^g X') = 0$ because

of [22, 2.1], the isomorphism ${}^g X' \simeq X'$ and the equality $\text{Ext}_B^1(X', X') = 0$. In order to get a contradiction we first prove that $\pi_\lambda X'$ is indecomposable. Fix an indecomposable decomposition $\pi_\lambda X' = M_1 \oplus \dots \oplus M_t$ in $\text{mod}(B/\langle g \rangle)$. Hence $\text{Ext}_{B/\langle g \rangle}^1(M_i, M_i) = 0$ for all i . We claim that M_i lies in the image of π_λ for all i . Indeed, we distinguish two cases according to whether $\text{car}(k) = p$ or $\text{car}(k) \neq p$. If $\text{car}(k) = p$ then the claim follows from [23, Lem. 6.1]. If $\text{car}(k) \neq p$, then $B/\langle g \rangle$ is Morita equivalent to the skew-group algebra $B[\langle g \rangle]$ ([13, Thm. 2.8]) and $B[\langle g \rangle]$ is tilted ([6, Thm. 1.2, (g)]). Therefore $B/\langle g \rangle$ is tilted and the claim follows from [22, Prop. 4.6]. Thus, in all cases, $M_i \simeq \pi_\lambda M'_i$ for some $M'_i \in \text{mod } B$ (necessarily indecomposable). So $\bigoplus_{j=0}^{p-1} {}^g M'_i \simeq \pi_\lambda M_i$ is a summand of $\pi_\lambda X' \simeq \bigoplus_{j=0}^{p-1} {}^g X' \simeq \bigoplus_{j=0}^{p-1} X'$. We thus have $M'_i \simeq X'$ for all i , whereas $\pi_\lambda X' = \pi_\lambda M'_1 \oplus \dots \oplus \pi_\lambda M'_t$. this proves that $\pi_\lambda X'$ is indecomposable. The contradiction is therefore the following. On the one hand, $\text{Ext}_{B/\langle g \rangle}^1(\pi_\lambda X', \pi_\lambda X') = 0$, $\pi_\lambda X' \in \text{ind}(B/\langle g \rangle)$ and $B/\langle g \rangle$ is a product of quasi-tilted algebras (because B is a product of tilted algebras and by 6.4), which imply that $\text{End}_{B/\langle g \rangle}(\pi_\lambda X') \simeq k$. On the other hand, $\text{End}_{B/\langle g \rangle}(\pi_\lambda X') \simeq \bigoplus_{j=0}^{p-1} \text{Hom}_B(X', {}^g X') \simeq \bigoplus_{j=0}^{p-1} \text{End}_B(X')$ as k -vector spaces. This is absurd. So $G_{X'} = 1$ and therefore $G_X = 1$. \blacksquare

Now we can prove that A is weakly shod by applying 6.2. As remarked in the proof of 6.5, a non trivial path in $\text{ind } A$ of the form $X \rightsquigarrow {}^g X$ with $X \in \Gamma_{A'}$ gives rise to a non trivial path $X \rightsquigarrow X$ in $\text{ind } A'$ which is impossible because A' is weakly shod. Therefore all the hypotheses of 6.2 are satisfied and A is weakly shod. This proves (c).

It only remains to prove the necessity in (a). We assume that A' is tilted and prove that so is A . Let $\Gamma_{A'}$ be a connecting component of $\Gamma(\text{mod } A')$. It admits a complete slice Σ' . Clearly, $\Gamma_{A'}$ is G -stable whatever the number of connecting components of A' is (one or two). By 6.2, 6.5 and [17, 3.6], there exists a component Γ of $\Gamma(\text{mod } A)$ such that $\Gamma = \{F_\lambda X \mid X \in \Gamma_{A'}\}$. Moreover there is a Galois covering of translation quivers $\Gamma_{A'} \rightarrow \Gamma$ with group G extending the map $X \mapsto F_\lambda X$. We prove that Γ has a complete slice. For this purpose we use the following lemma.

Lemma 6.6. *${}^g X \in \Sigma'$ for every $g \in G, X \in \Sigma'$.*

Proof of 6.6: Let $g \in G$ and write $\Sigma' = \{X_1, \dots, X_n\}$. So there exist a permutation $i \mapsto g.i$ of $\{1, \dots, n\}$ and integers l_1, \dots, l_n such that ${}^g X_i = \tau_{A'}^{l_i} X_{g.i}$ for every i . Clearly, the modules ${}^g X_1, \dots, {}^g X_n$ form a complete slice

$g(\Sigma')$ in $\Gamma_{A'}$. This implies that $l_1 = l_2 = \dots = l_n$. We write $l = l_1$. Therefore $g(\Sigma') = \tau_{A'}^l(\Sigma')$. On the other hand, g has finite order and $\Gamma_{A'}$ has no oriented cycles. So $l = 0$ and $g(\Sigma') = \Sigma'$. ■

Let Σ be the full subquiver of Γ generated by $\{F_\lambda X \mid X \in \Sigma'\}$. Hence Σ is convex in Γ , has no oriented cycle and intersects each τ_A -orbit of Γ exactly once because Σ' is a G -stable complete slice in $\Gamma_{A'}$. Moreover, the arguments used in the proof of 2.3 show that Σ is faithful because so is Σ' . Finally, given $X, Y \in \Sigma'$, we have $\text{Hom}_A(F_\lambda X, \tau_A F_\lambda Y) \simeq \bigoplus_{g \in G} \text{Hom}_{A'}(X, \tau_{A'}^g Y) = 0$ because of the covering property of F_λ , 2.1 (d) and the fact that Σ' is a G -stable slice in $\Gamma_{A'}$. Thus Σ is a complete slice and A is tilted with Γ as a connecting component. This proves the sufficiency (a) and finishes the proof of Theorem C. ■

Remark 6.7. The reader may find similar equivalences to those of Theorem C about skew-group algebras (instead of Galois coverings) under the additional assumption that $\text{car}(k)$ does not divide the order of the group G (see [6]).

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