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# Bayesian Cramer-Rao Bounds for Complex Gain Parameters Estimation of Slowly Varying Rayleigh Channel in OFDM Systems

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## Abstract

This paper deals with on-line Bayesian Cramer-Rao (BCRB) lower bound for complex gains dynamic estimation of time-varying multi-path Rayleigh channels. We propose three novel lower bounds for 4-QAM OFDM systems in case of negligible channel variation within one symbol, and assuming both channel delay and Doppler frequency related information. We derive the true BCRB for data-aided (DA) context and, two closed-form expressions for non-data aided (NDA) context.

## Index Terms

Bayesian Cramer-Rao Bound, OFDM, Rayleigh complex gains.

## I. INTRODUCTION

Dynamic estimation of frequency selective and time-varying channel is a fundamental function [2] for orthogonal frequency division multiplexing (OFDM) mobile communication systems. In radio-frequency transmissions, channel estimation can be generally obtained by estimating only some physical propagation parameters, such as multi-path delays and multi-path complex gains [3] [1] [4]. Moreover, in slowly varying channels, the number of paths and time delays can be easily obtained [3], since delays are quasi-invariant over a large number of symbols. Assuming full availability of delay related information, which is the ultimate accuracy that can be achieved with channel estimation methods ? Tools to face this problem

are available from parameters estimation theory [10] in form of the Cramer-Rao Bounds (CRBs), which give fundamental lower limits of the mean square error (MSE) achievable by any unbiased estimator. A modified CRB (MCRB), easier to evaluate than the Standard CRB (SCRB), has been introduced in [5] [6]. The MCRB is effective when, in addition to the parameter to be estimated, the observed data also depend on other unwanted parameters. More recently, the problem of deriving CRBs, suited to time-varying parameters, has been addressed throughout the Bayesian context. In [8], the authors propose a general framework for deriving analytical expression of on-line CRBs. In [9], the authors introduce a new asymptotic bound, namely the asymptotic bayesian CRB (ABCRB), for non-data-aided (NDA) scenario. This bound is closer to the classical BCRB than the Modified BCRB (MBCRB) and it is easier to be evaluated than BCRB. In this paper, we investigate the BCRB related to the estimation of the complex gains of a rayleigh channel, assuming negligible time variation within one OFDM symbol and, both channel delay and Doppler frequency related information. Explicit expressions of the the BCRB and its variants, MBCRB and ABCRB, are provided for NDA and DA 4-QAM on-line scenarios.

*Notations* :  $[\mathbf{x}]_k$  denotes the  $k$ th entry of the vector  $\mathbf{x}$ , and  $[\mathbf{X}]_{k,m}$  the  $[k, m]$ th entry of the matrix  $\mathbf{X}$ . As in matlab,  $\mathbf{X}_{[k_1:k_2, m_1:m_2]}$  is a submatrix extracted from rows  $k_1$  to  $k_2$  and from columns  $m_1$  to  $m_2$  of  $\mathbf{X}$ .  $\text{diag}\{\mathbf{x}\}$  is a diagonal matrix with  $\mathbf{x}$  on its diagonal,  $\text{diag}\{\mathbf{X}\}$  is a vector whose elements are the elements of the diagonal of  $\mathbf{X}$  and  $\text{blkdiag}\{\mathbf{X}, \mathbf{Y}\}$  is a block diagonal matrix with the matrices  $\mathbf{X}$  and  $\mathbf{Y}$  on its diagonal.  $E_{x,y}[\cdot]$  denotes the expectation over  $x$  and  $y$ .  $J_0(\cdot)$  is the zeroth-order Bessel function of the first kind.  $\nabla_{\mathbf{x}}$  and  $\Delta_{\mathbf{y}}^{\mathbf{x}}$  represent the first and the second-order partial derivatives operator *i.e.*,  $\nabla_{\mathbf{x}} = [\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}]^T$  and  $\Delta_{\mathbf{y}}^{\mathbf{x}} = \nabla_{\mathbf{y}}^* \nabla_{\mathbf{x}}^T$ .

## II. SYSTEM MODEL

Consider an OFDM system with  $N$  sub-carriers, and a cyclic prefix of length  $N_g$ . The duration of an OFDM symbol is  $T = vT_s$ , where  $T_s$  is the sampling time and  $v = N + N_g$ . Let  $\mathbf{x}_{(n)} = [x_{(n)}[-\frac{N}{2}], x_{(n)}[-\frac{N}{2} + 1], \dots, x_{(n)}[\frac{N}{2} - 1]]^T$  be the  $n$ -th transmitted OFDM symbol, where  $\{x_{(n)}[b]\}$  are normalized 4-QAM symbols. It is assumed that the transmission is over a multi-path Rayleigh channel, with negligible variation within one OFDM symbol, characterized by the impulse response:

$$h(nT, \tau) = \sum_{l=1}^L \alpha_l^{(n)} \delta(\tau - \tau_l T_s) \quad (1)$$

where  $L$  is the total number of propagation paths,  $\alpha_l$  is the  $l$ th complex gain of variance  $\sigma_{\alpha_l}^2$  (with  $\sum_{l=1}^L \sigma_{\alpha_l}^2 = 1$ ), and  $\tau_l \times T_s$  is the  $l$ th delay ( $\tau_l$  is not necessarily an integer, but  $\tau_L < N_g$ ). The  $L$  individual elements of  $\{\alpha_l^{(n)}\}$  are uncorrelated with respect to each other. They are wide-sense stationary

narrow-band complex Gaussian processes, with the so-called Jakes' power spectrum [11] with Doppler frequency  $f_d$ . It means that  $\alpha_l^{(n)}$  are correlated complex gaussian variables with zero-means and correlation coefficients given by:

$$\mathbf{R}_{\alpha_l}^{(p)} = \mathbb{E}[\alpha_l^{(n)} \alpha_l^{(n-p)*}] = \sigma_{\alpha_l}^2 J_0(2\pi f_d T p) \quad (2)$$

Hence, the  $n$ th received OFDM symbol  $\mathbf{y}_{(n)} = [y_{(n)}[-\frac{N}{2}], y_{(n)}[-\frac{N}{2} + 1], \dots, y_{(n)}[\frac{N}{2} - 1]]^T$  is given by [3] [1]:

$$\mathbf{y}_{(n)} = \mathbf{H}_{(n)} \mathbf{x}_{(n)} + \mathbf{w}_{(n)} \quad (3)$$

where  $\mathbf{w}_{(n)}$  is a  $N \times 1$  zero-mean complex Gaussian noise vector with covariance matrix  $\sigma^2 \mathbf{I}_N$ , and  $\mathbf{H}_{(n)}$  is a  $N \times N$  diagonal matrix with diagonal elements given by [3] [1]:

$$[\mathbf{H}_{(n)}]_{k,k} = \sum_{l=1}^L \left[ \alpha_l^{(n)} \times e^{-j2\pi(\frac{k-1}{N} - \frac{1}{2})\tau_l} \right] \quad (4)$$

This coefficients are the Fourier Transform of (1) evaluated at the discrete frequency  $f_k = (k-1 - \frac{N}{2}) \frac{1}{NT_s}$  with  $k \in [1, N]$ . Using (4), the observation model in (3) for the  $n$ th OFDM symbol can be re-written as:

$$\mathbf{y}_{(n)} = \text{diag}\{\mathbf{x}_{(n)}\} \mathbf{F} \boldsymbol{\alpha}_{(n)} + \mathbf{w}_{(n)} \quad (5)$$

where  $\boldsymbol{\alpha}_{(n)} = [\alpha_1^{(n)}, \dots, \alpha_L^{(n)}]^T$  is a  $L \times 1$  vector and  $\mathbf{F}$  is the  $N \times L$  Fourier matrix defined by:

$$[\mathbf{F}]_{k,l} = e^{-j2\pi(\frac{k-1}{N} - \frac{1}{2})\tau_l} \quad (6)$$

### III. BAYESIAN CRAMER-RAO BOUNDS (BCRBS)

In this section, we present a general formulation for BCRB which is related to the estimation of the multi-path complex gains. In NDA context, we derive a closed-form expression of a BCRB, *i.e.*, the Asymptotic BCRB or the Modified BCRB. In DA context, the true BCRB is equal to the MBCRB in NDA.  $\hat{\boldsymbol{\alpha}}(\mathbf{y})$  denotes an unbiased estimator of  $\boldsymbol{\alpha} = [\alpha_{(1)}^T, \dots, \alpha_{(K)}^T]^T$  using the set of measurements  $\mathbf{y} = [\mathbf{y}_{(1)}^T, \dots, \mathbf{y}_{(K)}^T]^T$ . In the on-line scenario, the receiver estimates  $\boldsymbol{\alpha}_{(n)}$  based on the current and previous observations only, *i.e.*,  $\mathbf{y} = [\mathbf{y}_{(1)}^T, \dots, \mathbf{y}_{(n)}^T]^T$ .

#### A. Bayesian Cramer-Rao bound

The BCRB is particularly suited when a priori information is available. The BCRB has been proposed in [10] such that:

$$\mathbb{E}_{\mathbf{y}, \boldsymbol{\alpha}} \left[ (\hat{\boldsymbol{\alpha}}(\mathbf{y}) - \boldsymbol{\alpha})(\hat{\boldsymbol{\alpha}}(\mathbf{y}) - \boldsymbol{\alpha})^H \right] \geq \mathbf{BCRB}(\boldsymbol{\alpha}) \quad (7)$$

where  $\mathbf{X} \geq \mathbf{Y}$  is interpreted as meaning that the matrix  $\mathbf{X} - \mathbf{Y}$  is positive semidefinite. The BCRB is the inverse of the Bayesian Information Matrix (BIM), which can be written as:

$$\mathbf{B} = \mathbf{E}_\alpha[\mathcal{F}(\alpha)] + \mathbf{E}_\alpha[-\Delta_\alpha^\alpha \ln(p(\alpha))] \quad (8)$$

where  $p(\alpha)$  is the prior probability density function (pdf) and  $\mathcal{F}(\alpha)$  is the Fisher Information Matrix (FIM) defined as:

$$\mathcal{F}(\alpha) = \mathbf{E}_{\mathbf{y}|\alpha}[-\Delta_\alpha^\alpha \ln(p(\mathbf{y}|\alpha))] \quad (9)$$

where  $p(\mathbf{y}|\alpha)$  is the conditional pdf of  $\mathbf{y}$  given  $\alpha$ . The on-line BCRB associated to observation vector  $\mathbf{y} = [\mathbf{y}_{(1)}^T, \dots, \mathbf{y}_{(K)}^T]^T$  will be obtained [9] by:

$$\text{BCRB}(\alpha_{(K)})_{online} = \text{Tr}\left(\mathbf{BCRB}(\alpha)_{[i(K), i(K)]}\right) \quad (10)$$

where  $i(n)$  is a sequence of indices defined by  $i(n) = 1 + (n-1)L : nL$  with  $n \in [1, K]$ . The definition (10) will stand for the closed form BCRBs.

**1) Computation of  $\mathbf{E}_\alpha[-\Delta_\alpha^\alpha \ln(p(\alpha))]$ :**  $\alpha$  is a complex Gaussian vector with zero mean and covariance matrix  $\mathbf{R}_\alpha = E\{\alpha\alpha^H\}$  of size  $KL \times KL$  defined as:

$$[\mathbf{R}_\alpha]_{i(l,p), i(l',p')} = \begin{cases} \mathbf{R}_{\alpha_i}^{(p-p')} & \text{for } l=l' \in [1, L] \quad p, p' \in [0, K-1] \\ 0 & \text{for } l \neq l' \quad " \end{cases} \quad (11)$$

where  $i(l, p) = 1 + (l-1) + pL$  and  $\mathbf{R}_{\alpha_i}^{(p)}$  is defined in (2). For example, if  $K = L = 2$  then,  $\mathbf{R}_\alpha$  is given by:

$$\mathbf{R}_\alpha = \begin{bmatrix} \mathbf{R}_{\alpha_1}^{(0)} & 0 & \mathbf{R}_{\alpha_1}^{(-1)} & 0 \\ 0 & \mathbf{R}_{\alpha_2}^{(0)} & 0 & \mathbf{R}_{\alpha_2}^{(-1)} \\ \mathbf{R}_{\alpha_1}^{(1)} & 0 & \mathbf{R}_{\alpha_1}^{(0)} & 0 \\ 0 & \mathbf{R}_{\alpha_2}^{(1)} & 0 & \mathbf{R}_{\alpha_2}^{(0)} \end{bmatrix} \quad (12)$$

Thus, the pdf  $p(\alpha)$  is defined as:

$$p(\alpha) = \frac{1}{|\pi \mathbf{R}_\alpha|} e^{-\alpha^H \mathbf{R}_\alpha^{-1} \alpha} \quad (13)$$

Taking the second derivative of the natural logarithm of (13) with respect to  $\alpha$  and making the expectation over  $\alpha$ , hence:

$$\mathbf{E}_\alpha[-\Delta_\alpha^\alpha \ln(p(\alpha))] = \mathbf{R}_\alpha^{-1} \quad (14)$$

**2) Computation of  $E_{\alpha}[\mathcal{F}(\alpha)]$ :** Using the whiteness of the noise and the independence of the transmitted OFDM symbols, one obtains from the observation model in (5) that:

$$\Delta_{\alpha}^{\alpha} \ln(p(\mathbf{y}|\alpha)) = \sum_{n=1}^K \Delta_{\alpha}^{\alpha} \ln(p(\mathbf{y}_{(n)}|\alpha_{(n)})) \quad (15)$$

Each term of the sum (15) is a  $KL \times KL$  block diagonal matrix with only one nonzero  $L \times L$  block matrix, namely:

$$\Delta_{\alpha}^{\alpha} \ln(p(\mathbf{y}_{(n)}|\alpha_{(n)}))_{[i(n),i(n)]} = \Delta_{\alpha_{(n)}}^{\alpha_{(n)}} \ln(p(\mathbf{y}_{(n)}|\alpha_{(n)})) \quad (16)$$

As a direct consequence,  $\Delta_{\alpha}^{\alpha} \ln(p(\mathbf{y}|\alpha))$  is a block diagonal matrix with the  $n$ th diagonal block given by (16). Moreover, because of the circularity of the observation noise, the expectation of (16) with respect to  $\mathbf{y}_{(n)}$  and  $\alpha_{(n)}$  does not depend on  $\alpha_{(n)}$ . One then obtains:

$$E_{\alpha}[\mathcal{F}(\alpha)] = \text{blkdiag}\{\mathbf{J}, \mathbf{J}, \dots, \mathbf{J}\} \quad (17)$$

where  $\mathbf{J}$  is a  $L \times L$  matrix defined as:

$$\mathbf{J} = E_{\mathbf{y},\alpha}[-\Delta_{\alpha_{(n)}}^{\alpha_{(n)}} \ln(p(\mathbf{y}_{(n)}|\alpha_{(n)}))] \quad (18)$$

The log-likelihood function in (18) can be expanded as:

$$\ln(p(\mathbf{y}_{(n)}|\alpha_{(n)})) = \ln\left(\sum_{\mathbf{x}_{(n)}} p(\mathbf{y}_{(n)}|\mathbf{x}_{(n)}, \alpha_{(n)})p(\mathbf{x}_{(n)})\right) \quad (19)$$

The vector  $\mathbf{y}_{(n)}$  for given  $\mathbf{x}_{(n)}$  and  $\alpha_{(n)}$  is complex Gaussian with mean vector  $\mathbf{m}_{(n)} = \text{diag}\{\mathbf{x}_{(n)}\}\mathbf{F}\alpha_{(n)}$  and covariance matrix  $\sigma^2\mathbf{I}_N$ . Thus, the conditional pdf is :

$$p(\mathbf{y}_{(n)}|\mathbf{x}_{(n)}, \alpha_{(n)}) = \frac{1}{|\pi\sigma^2\mathbf{I}_N|} e^{-\frac{1}{\sigma^2}(\mathbf{y}_{(n)}-\mathbf{m}_{(n)})^H(\mathbf{y}_{(n)}-\mathbf{m}_{(n)})} \quad (20)$$

Since each element of the vector  $\mathbf{m}_{(n)}$  depends on only one element of  $\mathbf{x}_{(n)}$  then, using the Gaussian nature of the noise and the equiprobability of the normalized QAM symbols, one finds (see Appendix A) that:

$$\ln(p(\mathbf{y}_{(n)}|\alpha_{(n)})) = \ln\left[\frac{1}{|\pi\sigma^2\mathbf{I}_N|} e^{-\frac{1}{\sigma^2}(\mathbf{y}_{(n)}^H\mathbf{y}_{(n)}+\alpha_{(n)}^H\mathbf{F}^H\mathbf{F}\alpha_{(n)})} \prod_{k=1}^N \cosh\left(\frac{\sqrt{2}}{\sigma^2}\text{Re}(a_n(k))\right) \cosh\left(\frac{\sqrt{2}}{\sigma^2}\text{Im}(a_n(k))\right)\right] \quad (21)$$

where  $a_n(k) = [\mathbf{y}_{(n)}]_k^* \mathbf{g}_k^T \alpha_{(n)}$  and  $\mathbf{g}_k^T$  is the  $k$ th row of the matrix  $\mathbf{F}$ . The result of the second derivative of (21) with respect to  $\alpha_{(n)}$  is given by:

$$\Delta_{\alpha_{(n)}}^{\alpha_{(n)}} \ln(p(\mathbf{y}_{(n)}|\alpha_{(n)})) = -\frac{1}{\sigma^2}\mathbf{F}^H\mathbf{F} + \sum_{k=1}^N \left[ \frac{1}{2\sigma^4} [\mathbf{y}_{(n)}]_k [\mathbf{y}_{(n)}]_k^* \mathbf{g}_k^* \mathbf{g}_k^T \left( 2 - \tanh^2\left(\frac{\sqrt{2}}{\sigma^2}\text{Re}(a_n(k))\right) - \tanh^2\left(\frac{\sqrt{2}}{\sigma^2}\text{Im}(a_n(k))\right) \right) \right] \quad (22)$$

The expectation of (22) with respect to  $\mathbf{y}_{(n)}|\boldsymbol{\alpha}_{(n)}$  does not have any simple analytical solution. Hence, we have to resort to either numerical integration methods or some approximations. In the following, we present both the high-SNR and the low-SNR approximations of the BCRB, as defined in [9].

### B. Asymptotic BCRB

**1) High-SNR BCRB Asymptote:** From the definition of BIM (8), only the first term (*i.e.*,  $E_{\boldsymbol{\alpha}}[\mathcal{F}(\boldsymbol{\alpha})]$ ) depends on the SNR, which is fully characterized by  $\mathbf{J}$ . Hence, we focus on the behavior of  $\mathbf{J}$ . At high SNR (*i.e.*,  $\sigma^2 \rightarrow 0$ ), the  $\tanh$ -function in (22) can be approximated as:  $\tanh\left(\frac{\sqrt{2}}{\sigma^2}x\right) \approx \text{sgn}(x)$ . Hence, we obtain the high-SNR asymptote of  $\mathbf{J}$ , which is:

$$\mathbf{J}_h = \frac{1}{\sigma^2} \mathbf{F}^H \mathbf{F} \quad (23)$$

**2) Low-SNR BCRB Asymptote:** Following the same reasoning as before, at low SNR (*i.e.*,  $\sigma^2 \rightarrow +\infty$ ), we have  $\tanh(x) \approx x$  around  $x = 0$ . Hence, we obtain:

$$\Delta_{\boldsymbol{\alpha}_{(n)}}^{\boldsymbol{\alpha}_{(n)}} \ln(p(\mathbf{y}_{(n)}|\boldsymbol{\alpha}_{(n)})) \approx -\frac{1}{\sigma^2} \mathbf{F}^H \mathbf{F} + \sum_{k=1}^N \left[ \frac{1}{\sigma^8} [\mathbf{y}_{(n)}]_k [\mathbf{y}_{(n)}]_k^* \mathbf{g}_k^* \mathbf{g}_k^T \left( \sigma^4 - a_n(k) a_n^*(k) \right) \right] \quad (24)$$

Plugging (24) into (18), we obtain the low-SNR asymptote of  $\mathbf{J}$ , which is (see Appendix B):

$$\mathbf{J}_l = \left( \frac{\beta}{\sigma^4} + \frac{8\beta^2}{\sigma^6} + \frac{6\beta^3}{\sigma^8} \right) \mathbf{F}^H \mathbf{F} \quad (25)$$

where  $\beta = \sum_{l=1}^L \sigma_{\alpha_l}^2$  is the total channel energy.

The Asymptotic BCRB (ABCRB) defined in [9] leads to a lower bound on the MSE. This ABCRB is given by:

$$\mathbf{ABCRB}(\boldsymbol{\alpha}) = \left( \text{blkdiag} \{ \mathbf{J}_{min}, \dots, \mathbf{J}_{min} \} + \mathbf{R}_{\boldsymbol{\alpha}}^{-1} \right)^{-1} \quad (26)$$

where  $\mathbf{J}_{min} = \min(v_l, v_h) \mathbf{F}^H \mathbf{F}$ , with  $v_l = \frac{\beta}{\sigma^4} + \frac{8\beta^2}{\sigma^6} + \frac{6\beta^3}{\sigma^8}$  and  $v_h = \frac{1}{\sigma^2}$ .

### C. Modified BCRB

The analytical computation of  $\mathcal{F}(\boldsymbol{\alpha})$  is quite tedious in case of NDA context because of the OFDM symbols  $\mathbf{x} = [\mathbf{x}_{(1)}^T, \dots, \mathbf{x}_{(K)}^T]^T$ , which are “nuisance parameters”. In order to circumvent this kind of problem, a Modified BCRB (MBCRB) has been proposed in [5]. This MBCRB is the inverse of the following information matrix:

$$\mathbf{C} = E_{\boldsymbol{\alpha}}[\mathbf{G}(\boldsymbol{\alpha})] + E_{\boldsymbol{\alpha}}[-\Delta_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}} \ln(p(\boldsymbol{\alpha}))] \quad (27)$$

where  $\mathbf{G}(\boldsymbol{\alpha})$  is the modified FIM defined as:

$$\mathbf{G}(\boldsymbol{\alpha}) = \mathbf{E}_{\mathbf{x}}\mathbf{E}_{\mathbf{y}|\mathbf{x},\boldsymbol{\alpha}}[-\Delta_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}} \ln(p(\mathbf{y}|\mathbf{x}, \boldsymbol{\alpha}))] \quad (28)$$

It should be noted that the MBCRB in NDA context is equal to the BRCB in DA context (*i.e.* symbols  $\mathbf{x}$  are a priori known). Hence, following the same reasoning as before, we have:

$$\mathbf{E}_{\boldsymbol{\alpha}}[\mathbf{G}(\boldsymbol{\alpha})] = \text{blkdiag}\{\mathbf{J}_m, \mathbf{J}_m, \dots, \mathbf{J}_m\} \quad (29)$$

where  $\mathbf{J}_m$  is a  $L \times L$  matrix defined as:

$$\mathbf{J}_m = \mathbf{E}_{\mathbf{y},\mathbf{x},\boldsymbol{\alpha}}[-\Delta_{\boldsymbol{\alpha}_{(n)}}^{\boldsymbol{\alpha}_{(n)}} \ln(p(\mathbf{y}_{(n)}|\mathbf{x}_{(n)}, \boldsymbol{\alpha}_{(n)})))] \quad (30)$$

By taking the second derivative of the natural logarithm ( $\ln$ ) of (20) with respect to  $\boldsymbol{\alpha}_{(n)}$ , one easily obtains that:

$$\mathbf{J}_m = \mathbf{E}_{\mathbf{x}}\left[\frac{1}{\sigma^2}\mathbf{F}^H \text{diag}\{\mathbf{x}_{(n)}^H\} \text{diag}\{\mathbf{x}_{(n)}\}\mathbf{F}\right] = \frac{1}{\sigma^2}\mathbf{F}^H\mathbf{F} \quad (31)$$

since the QAM-symbols are normalized and uncorrelated with respect to each other. The MBCRB for the estimation of  $\boldsymbol{\alpha}$  is given by:

$$\mathbf{MBCRB}(\boldsymbol{\alpha}) = \left(\text{blkdiag}\{\mathbf{J}_m, \mathbf{J}_m, \dots, \mathbf{J}_m\} + \mathbf{R}_{\boldsymbol{\alpha}}^{-1}\right)^{-1} \quad (32)$$

We see that  $\mathbf{J}_h = \mathbf{J}_m$  hence, the high-SNR asymptote of the BCRB is equal to the MBCRB. This corroborates the result of [7] for a scalar parameter in non-Bayesian case.

#### IV. DISCUSSION AND CONCLUSION

In this section, we illustrate the behavior of the previous bounds for the complex gains estimation. A 4-QAM OFDM system with normalized symbols,  $N = 128$  subcarriers and  $N_g = \frac{N}{8}$  is used. The normalized Rayleigh channel contains  $L = 6$  paths and others parameters given in [1].

Fig. 1 presents the on-line BCRB (evaluated by Monte-Carlo trials), ABCRB and MBCRB versus  $\text{SNR} = \frac{1}{\sigma^2}$ , for a block-observation length  $K = 20$  and a normalized Doppler frequency  $f_d T = 10^{-3}$ . We plot also as reference the SCRB (*i.e.*, the prior information is not used). We observe that both ABCRB and the MBCRB are lower than SCRB since the prior information of the complex gains is considered. We also verify that  $\text{MBCRB} \leq \text{ABCRB} \leq \text{BCRB}$ , as in [9]. At high SNR, the MBCRB and the ABCRB are very close, as predicted by our theoretical analysis.

Fig. 2 presents the on-line ABCRB versus time index  $K$  for different normalized Doppler frequencies  $10^{-5} \leq f_d T \leq 5 \cdot 10^{-3}$  and  $\text{SNR} = 10\text{dB}$ . When the number of observations increases, the estimation can



be significantly improved when the estimator takes also into account the previous information; the bound thus decreases and converges to an asymptote. The estimation gain is larger using previous symbols with slow channel variations (low  $f_d T$ ). In brief, our contribution permit to measure the benefit of using additional previous OFDM symbols for channel estimation process of the current symbol, whereas most methods use only the current symbol [2].

## APPENDIX A

### DERIVATION OF EXPRESSION (21) AND (22)

Plugging (20) into (19), we obtain:

$$\ln(p(\mathbf{y}_{(n)}|\boldsymbol{\alpha}_{(n)})) = -\frac{1}{\sigma^2} \left( \mathbf{y}_{(n)}^H \mathbf{y}_{(n)} + \mathbf{m}_{(n)}^H \mathbf{m}_{(n)} \right) + \ln \left( \frac{p(\mathbf{x}_{(n)})}{|\pi\sigma^2 \mathbf{I}_N|} \sum_{\mathbf{x}_{(n)}} e^{\frac{2}{\sigma^2} \text{Re}(\mathbf{y}_{(n)}^H \mathbf{m}_{(n)})} \right) \quad (33)$$

since the 4QAM-symbols are equiprobable (*i.e.*,  $p(\mathbf{x}_{(n)}) = \frac{1}{4^N}$ ). However,  $\mathbf{m}_{(n)} = \text{diag}\{\mathbf{x}_{(n)}\} \mathbf{F} \boldsymbol{\alpha}_{(n)}$  then,  $\mathbf{y}_{(n)}^H \mathbf{m}_{(n)} = \sum_{k=1}^N a_n(k) [\mathbf{x}_{(n)}]_k$  where  $a_n(k)$  is defined in section III part A. Hence, one obtains:

$$\sum_{\mathbf{x}_{(n)}} e^{\frac{2}{\sigma^2} \text{Re}(\mathbf{y}_{(n)}^H \mathbf{m}_{(n)})} = \prod_{k=1}^N \left( \sum_{[\mathbf{x}_{(n)}]_k} e^{\frac{2}{\sigma^2} \text{Re}(a_n(k) [\mathbf{x}_{(n)}]_k)} \right) \quad (34)$$

Since  $[\mathbf{x}_{(n)}]_k = \frac{1}{\sqrt{2}}(\pm 1 \pm j)$  (4QAM-symbol), we obtain:

$$\sum_{[\mathbf{x}_{(n)}]_k} e^{\frac{2}{\sigma^2} \text{Re}(a_n(k) [\mathbf{x}_{(n)}]_k)} = 4 \cosh\left(\frac{\sqrt{2}}{\sigma^2} \text{Re}(a_n(k))\right) \cosh\left(\frac{\sqrt{2}}{\sigma^2} \text{Im}(a_n(k))\right) \quad (35)$$

Inserting this result into (33), we obtain the expression in (21). Taking the second derivative of (21) with respect to  $\boldsymbol{\alpha}_{(n)}$  and using  $\nabla_{\boldsymbol{\alpha}_{(n)}} \text{Re}(a_n(k)) = \frac{1}{2} [\mathbf{y}_{(n)}]_k^* \mathbf{g}_k$  and  $\nabla_{\boldsymbol{\alpha}_{(n)}} \text{Im}(a_n(k)) = \frac{1}{2j} [\mathbf{y}_{(n)}]_k^* \mathbf{g}_k$ , we obtain finally the expression in (22).

## APPENDIX B

### EVALUATION OF $\mathbf{J}_l$ IN (25)

Inserting the definition of  $a_n(k)$  into (24) and plugging the result into (18), one obtains:

$$\mathbf{J}_l = \frac{1}{\sigma^2} \mathbf{F}^H \mathbf{F} - \frac{1}{\sigma^4} \sum_{k=1}^N \mathbf{g}_k^* \mathbf{E}_{\boldsymbol{\alpha}} \mathbf{E}_{\mathbf{y}} | \boldsymbol{\alpha} \left[ [\mathbf{y}_{(n)}]_k [\mathbf{y}_{(n)}]_k^* \right] \mathbf{g}_k^T + \frac{1}{\sigma^8} \sum_{k=1}^N \mathbf{g}_k^* \mathbf{g}_k^T \mathbf{E}_{\boldsymbol{\alpha}} \left[ \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H \mathbf{g}_k^* \mathbf{E}_{\mathbf{y}} | \boldsymbol{\alpha} \left[ ([\mathbf{y}_{(n)}]_k [\mathbf{y}_{(n)}]_k^*)^2 \right] \right] \mathbf{g}_k^T \quad (36)$$

Using that  $[\mathbf{y}_{(n)}]_k = [\mathbf{x}_{(n)}]_k \mathbf{g}_k^T \boldsymbol{\alpha}_{(n)} + [\mathbf{w}_{(n)}]_k$ , the independance between the QAM-symbols and the noise, and these results below:

$$\mathbf{E}_{[\mathbf{x}_{(n)}]_k} \left[ [\mathbf{x}_{(n)}]_k^2 \right] = \mathbf{E}_{[\mathbf{w}_{(n)}]_k} \left[ [\mathbf{w}_{(n)}]_k^2 \right] = 0 \quad \text{and} \quad \mathbf{E}_{[\mathbf{w}_{(n)}]_k} \left[ [\mathbf{w}_{(n)}]_k^2 [\mathbf{w}_{(n)}]_k^{*2} \right] = 2\sigma^4 \quad (37)$$

we obtain:

$$\begin{aligned} \mathbb{E}_{\mathbf{y}|\boldsymbol{\alpha}} \left[ [\mathbf{y}_{(n)}]_k [\mathbf{y}_{(n)}]_k^* \right] &= \mathbf{g}_k^T \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H \mathbf{g}_k^* + \sigma^2 \\ \mathbb{E}_{\mathbf{y}|\boldsymbol{\alpha}} \left[ \left( [\mathbf{y}_{(n)}]_k [\mathbf{y}_{(n)}]_k^* \right)^2 \right] &= 2\sigma^4 + 4\sigma^2 \mathbf{g}_k^T \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H \mathbf{g}_k^* + \mathbf{g}_k^T \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H \mathbf{g}_k^* \mathbf{g}_k^{*T} \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H \mathbf{g}_k^* \end{aligned} \quad (38)$$

Hence,  $\mathbf{J}_l$  becomes:

$$\mathbf{J}_l = \frac{1}{\sigma^4} \sum_{k=1}^N \mathbf{V}_k \mathbf{D} \mathbf{V}_k + \frac{4}{\sigma^6} \sum_{k=1}^N \mathbf{V}_k \mathbb{E}_{\boldsymbol{\alpha}} [\mathbf{T}_1] \mathbf{V}_k + \frac{1}{\sigma^8} \sum_{k=1}^N \mathbf{V}_k \mathbb{E}_{\boldsymbol{\alpha}} [\mathbf{T}_2] \mathbf{V}_k \quad (39)$$

where  $\mathbf{V}_k = \mathbf{g}_k^* \mathbf{g}_k^T$ ,  $\mathbf{T}_1 = \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H \mathbf{V}_k \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H$ ,  $\mathbf{T}_2 = \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H \mathbf{V}_k \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H \mathbf{V}_k \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H$  and  $\mathbf{D} = \mathbb{E}_{\boldsymbol{\alpha}} \left[ \boldsymbol{\alpha}_{(n)} \boldsymbol{\alpha}_{(n)}^H \right] = \text{diag} \{ \sigma_{\alpha_1}^2, \dots, \sigma_{\alpha_L}^2 \}$ . The elements of  $\mathbf{T}_1$  and  $\mathbf{T}_2$  are given by:

$$\begin{aligned} [\mathbf{T}_1]_{l,l'} &= \sum_{l_1=1}^L \sum_{l_2=1}^L [\mathbf{V}_k]_{l_1,l_2} [\boldsymbol{\alpha}_{(n)}]_{l_1} [\boldsymbol{\alpha}_{(n)}]_{l_2} [\boldsymbol{\alpha}_{(n)}]_{l'}^* [\boldsymbol{\alpha}_{(n)}]_{l_1}^* \\ [\mathbf{T}_2]_{l,l'} &= \sum_{l_1=1}^L \sum_{l_2=1}^L \sum_{l_3=1}^L \sum_{l_4=1}^L [\mathbf{V}_k]_{l_1,l_2} [\mathbf{V}_k]_{l_3,l_4} [\boldsymbol{\alpha}_{(n)}]_{l_1} [\boldsymbol{\alpha}_{(n)}]_{l_2} [\boldsymbol{\alpha}_{(n)}]_{l_4} [\boldsymbol{\alpha}_{(n)}]_{l'}^* [\boldsymbol{\alpha}_{(n)}]_{l_1}^* [\boldsymbol{\alpha}_{(n)}]_{l_3}^* \end{aligned} \quad (40)$$

Using that  $\mathbb{E}_{[\mathbf{c}_{(n)}]_l} \left[ [\mathbf{c}_{(n)}]_l^2 \right] = 0$  and the definitions of 4th and 6th order moments for complex gaussian variables, we obtain:

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\alpha}} [\mathbf{T}_1] &= \mathbf{D} \mathbf{V}_k \mathbf{D} + \text{Tr}(\mathbf{V}_k \mathbf{D}) \mathbf{D} \\ \mathbb{E}_{\boldsymbol{\alpha}} [\mathbf{T}_2] &= 2\mathbf{D} \mathbf{V}_k \mathbf{D} \mathbf{V}_k \mathbf{D} + 2\text{Tr}(\mathbf{V}_k \mathbf{D}) \mathbf{D} \mathbf{V}_k \mathbf{D} + \text{Tr}(\mathbf{V}_k \mathbf{D} \mathbf{V}_k \mathbf{D}) \mathbf{D} + \left( \text{Tr}(\mathbf{V}_k \mathbf{D}) \right)^2 \mathbf{D} \end{aligned} \quad (41)$$

Using that  $\mathbf{g}_k^T \mathbf{D} \mathbf{g}_k^* = \text{Tr}(\mathbf{V}_k \mathbf{D}) = \sum_{l=1}^L \sigma_{\alpha_l}^2 = \beta$ ,  $\text{Tr}(\mathbf{V}_k \mathbf{D} \mathbf{V}_k \mathbf{D}) = \beta^2$ ,  $\mathbf{D} \mathbf{V}_k \mathbf{D} \mathbf{V}_k \mathbf{D} = \beta \mathbf{D} \mathbf{V}_k \mathbf{D}$ , and inserting these results into (39), we obtain the expression of  $\mathbf{J}_l$  in (25).

## REFERENCES

- [1] H. Hijazi and L. Ros, "Polynomial estimation of time-varying multi-path gains with intercarrier interference mitigation in OFDM systems" in *IEEE Trans. Vehic. Techno.*, To be appeared in January 2009.
- [2] M. Hsieh and C. Wei, "Channel estimation for OFDM systems based on comb-type pilot arrangement in frequency selective fading channels" in *IEEE Trans. Consumer Electron.*, vol.44, no. 1, Feb. 1998.
- [3] B. Yang, K. B. Letaief, R. S. Cheng and Z. Cao, "Channel estimation for OFDM transmission in multipath fading channels based on parametric channel modeling" in *IEEE Trans. Commun.*, vol. 49, no. 3, pp. 467-479, March 2001.
- [4] E. Simon, L. Ros and K. Raoof, "Synchronization over rapidly time-varying multipath channel for CDMA downlink RAKE receivers in Time-Division mode", in *IEEE Trans. Vehicular Tech.*, vol. 56. no. 4, Jul. 2007
- [5] B. Z. Bobrovosky, E. Mayer-Wolf and M. Zakai, "Some classes of global Cramer-Rao bounds" in *Ann. Statistics*, vol. 15, pp. 1421-1438, 1987.
- [6] A. D'Andrea, U. Mengali and R. Reggiannini, "The modified Cramer-Rao bound and its application to synchronization problems" in *IEEE Trans. Commun.*, vol. 42, pp. 1391-1399, Apr. 1994.

- [7] M. Moeneclaey, "On the true and the modified Cramer-Rao bounds for the estimation of a scalar parameter in the presence of nuisance parameters" in *IEEE Trans. Com.*, vol. 46, no. 11, pp. 1536-1544, Nov. 1998.
- [8] P. Tichavsky, C. H. Murachvik and A. Nehorai, "Posterior Cramer-Rao bound for discrete-time nonlinear filtering" in *IEEE Trans. Signal Processing*, vol. 46, pp. 1386-1396, May. 1998.
- [9] S. Bay, C. Herzet, J. M. Brossier, J. P. Barbot and B. Geller, "Analytic and asymptotic analysis of Bayesian Cramer-Rao bound for dynamical phase offset estimation" in *IEEE Trans. Signal Processing*, vol. 56, pp. 61-70, Jan. 2008.
- [10] H. L. Van Trees, *Detection, estimation, and modulation theory: Part I*, Wiley, New York, 1968.
- [11] W. C. Jakes, *Microwave Mobile Communications*. IEEE Press, 1983.

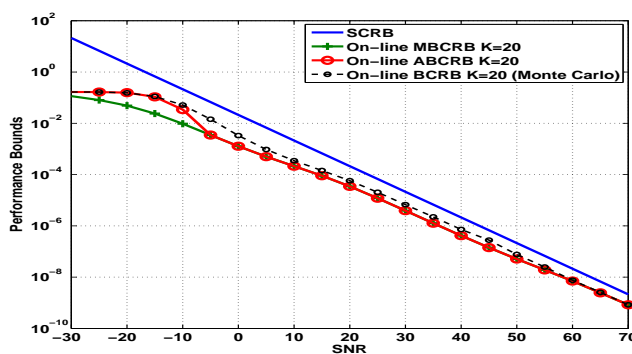


Fig. 1. SCR and BCRBs vs SNR for  $f_d T = 0.001$

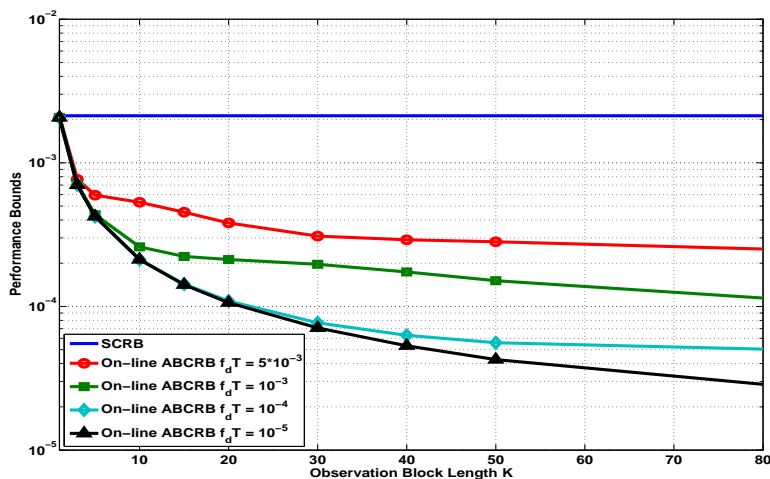


Fig. 2. BCRBs vs number of observations, for SNR=10dB