# INVERSION OF SOME SERIES OF FREE QUASI-SYMMETRIC FUNCTIONS 

FLORENT HIVERT, JEAN-CHRISTOPHE NOVELLI, AND JEAN-YVES THIBON


#### Abstract

We give a combinatorial formula for the inverses of the alternating sums of free quasi-symmetric functions of the form $\mathbf{F}_{\omega(I)}$ where $I$ runs over compositions with parts in a prescribed set $C$. This proves in particular three special cases (no restriction, even parts, and all parts equal to 2 ) which were conjectured by B. C. V. Ung in [Proc. FPSAC'98, Toronto].


## 1. Introduction

The algebra of Free Quasi-Symmetric Functions FQSym [5] is a graded algebra of noncommutative polynomials whose bases are parametrized by permutations. Under commutative image, it is mapped onto Gessel's algebra of quasi-symmetric functions, whence its name.

Quasi-symmetric functions generalize symmetric functions in a natural way, and many classical results admit quasi-symmetric extensions or analogs. However, very few results resembling symmetric series identities, like those of Schur or Littlewood (see, e.g., [11]) are known. In [15], B. C. V. Ung proves a quasi-symmetric analog of Schur's identity, and conjectures three further combinatorial inversions of quasisymmetric series, which are even stated at the level of FQSym.

In this note, we prove a master identity, which consists in a combinatorial formula for the inverses of the alternating sums of free quasi-symmetric functions of the form $\mathbf{F}_{\omega(I)}$ where $I$ runs over compositions with parts in a prescribed set $C$. Here $\mathbf{F}_{\sigma}$ denotes the standard basis of FQSym (mapped onto Gessel's fundamental basis), and $\omega(I)$ is the longest permutation with descent composition $I$. Ung's conjectures boil down to the following special cases : no restriction on the parts, even parts, and all parts equal to 2 .

Acknowledgements. This project has been partially supported by the grant ANR-06-BLAN-0380. The authors would also like to thank the contributors of the MuPAD project, and especially of the combinat part, for providing the development environment for their research (see [9] for an introduction to MuPAD-Combinat).

## 2. Background and notations

2.1. Free quasi-symmetric functions. Let $A$ be a totally ordered alphabet. Recall that the standardized $\operatorname{Std}(w)$ of a word $w \in A^{*}$ is the permutation obtained by iteratively scanning $w$ from left to right, and labelling $1,2, \ldots$ the occurrences of its
smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\sigma=\operatorname{Std}(w)^{-1}$ can be characterized as the unique permutation of minimal length such that $w \sigma$ is a nondecreasing word. For example, $\operatorname{Std}(b b a c a b)=341625$.

An elementary observation, which is at the basis of the constructions of [5] , is that the noncommutative polynomials

$$
\begin{equation*}
\mathbf{G}_{\sigma}(A)=\sum_{w \in A^{*} ; \operatorname{Std}(w)=\sigma} w \tag{1}
\end{equation*}
$$

span a subalgebra of $\mathbb{Q}\langle A\rangle$. When $A$ is infinite, this subalgebra admits a natural Hopf algebra structure, but this fact will not be needed here. This is FQSym, the algebra of Free Quasi-Symmetric Functions.

Let $\mathbf{F}_{\sigma}=\mathbf{G}_{\sigma^{-1}}$. The scalar product is defined by

$$
\begin{equation*}
\left\langle\mathbf{F}_{\sigma}, \mathbf{G}_{\tau}\right\rangle=\delta_{\sigma, \tau} . \tag{2}
\end{equation*}
$$

For a word $w$ on the alphabet $\{1,2, \ldots\}$, denote by $w[k]$ the word obtained by replacing each letter $i$ by the integer $i+k$. If $u$ and $v$ are two words, with $u$ of length $k$, one defines the shifted concatenation

$$
\begin{equation*}
u \bullet v=u \cdot(v[k]) \tag{3}
\end{equation*}
$$

and the shifted shuffle

$$
\begin{equation*}
u \uplus v=u Ш(v[k]) . \tag{4}
\end{equation*}
$$

where $u ш u^{\prime}$ is the usual shuffle product on words.
The product formula in the $\mathbf{F}$ basis is

$$
\begin{equation*}
\mathbf{F}_{\alpha} \mathbf{F}_{\beta}=\sum_{\gamma \in \alpha \uplus \beta} \mathbf{F}_{\gamma} . \tag{5}
\end{equation*}
$$

The sum of the inverses of the permutations occuring in $\alpha^{-1} \mathbb{\Psi} \beta^{-1}$ is called convolution and denoted by $\alpha \star \beta$. Thus, FQSym provides a realization of the convolution algebra of permutations studied in [13, (12].
2.2. Descent classes and compositions. Recall that the descent set of a permutation $\sigma$ is $D=\{i \mid \sigma(i)>\sigma(i+1)\}$. If $\sigma \in \mathfrak{S}_{n}$ has descent set $D=\left\{d_{1}<\ldots<\right.$ $\left.d_{k}\right\} \subseteq[n-1]$, the descent composition $I=C(\sigma)$ is the composition $I=\left(i_{1}, \ldots, i_{k+1}\right)$ of $n$ defined by $i_{s}=d_{s}-d_{s-1}$, where $d_{0}=0$ and $d_{k+1}=n$. The symbol $I \vDash n$ means that $I$ is a composition of $n$, and $l(I)$ denotes the length of $I$.

The descent class $D_{I}=\left\{\sigma \in \mathfrak{S}_{n} \mid C(\sigma)=I\right\}$ has a unique element of minimal (resp. maximal) length denoted by $\alpha(I)$ (resp. $\omega(I)$ ). Actually, descent classes are intervals $D_{I}=[\alpha(I), \omega(I)]$ for the left weak order on $\mathfrak{S}_{n}$ (see, e.g., [3] ).

The mirror image of a word $w=a_{1} a_{2} \cdots a_{m}$ is $\bar{w}=a_{m} a_{m-1} \cdots a_{1}$. We shall use this notation for compositions and permutations as well.

Finally, the diameter of a descent class is the permutation

$$
\begin{equation*}
\operatorname{diam}(I):=\alpha(I) \omega(I)^{-1}=\alpha(I) \omega(\bar{I}) \tag{6}
\end{equation*}
$$

2.3. A multiplicative basis of FQSym. The left-shifted concatenation of words is

$$
\begin{equation*}
u \triangleright v=u[l] \cdot v \quad \text { if } u \in A^{k}, v \in A^{l} \tag{7}
\end{equation*}
$$

similar to the usual shifted concatenation •, but with the shift on the first factor. The following basis is introduced in (4]:

$$
\begin{equation*}
\mathbf{S}^{\sigma}:=\sum_{\tau \leq \sigma} \mathbf{G}_{\tau} \tag{8}
\end{equation*}
$$

where $\leq$ is the left weak order. It has the property

$$
\begin{equation*}
\mathbf{S}^{\sigma}=\mathbf{S}^{\sigma_{1}} \mathbf{S}^{\sigma_{2}} \cdots \mathbf{S}^{\sigma_{r}} \tag{9}
\end{equation*}
$$

whenever $\sigma=\sigma_{1} \triangleright \sigma_{2}>\sigma_{r}$.
The Moebius function of the left weak order is explicitely known [6, 园, 3], and gives in particular

$$
\begin{equation*}
\mathbf{G}_{\sigma}=\sum_{I \preceq C\left(\sigma^{-1}\right)}(-1)^{l(I)-1} \mathbf{S}^{\alpha(I) \sigma} \tag{10}
\end{equation*}
$$

## 3. The main result

3.1. Ung's conjectures. In [15], Ung made the following conjectures. The inverses of the series

$$
\begin{aligned}
& H_{1}=\sum_{I}(-1)^{\ell(I)} \mathbf{F}_{\omega(I)} \\
& H_{2}=\sum_{n \geq 0}(-1)^{n} \mathbf{F}_{\omega\left(2^{n}\right)} \\
& H_{3}=\sum_{I}(-1)^{\ell(I)} \mathbf{F}_{\omega(2 I)}
\end{aligned}
$$

are as follows. For a permutation $\sigma$ of shape $I$, let $\hat{\sigma}=\sigma \omega(I)^{-1}$. Then,

$$
\begin{aligned}
H_{1}^{-1} & =\sum_{\alpha} \mathbf{G}_{\hat{\alpha}} \\
H_{2}^{-1} & =\sum_{\beta} \mathbf{G}_{\hat{\beta}} \\
H_{3}^{-1} & =\sum_{\gamma} \mathbf{G}_{\hat{\gamma}}
\end{aligned}
$$

where $\alpha$ runs over all permutations, $\beta \in \mathfrak{S}_{2 p}$ runs over permutations of shape $2^{2 p}$, and $\gamma \in \mathfrak{S}_{2 p}$ runs over permutations with descent set contained in $\{2,4, \ldots, 2 p-2\}$.

Taking into account (8) and (6), we see that all three identities are of the form (11) below, with $E=\mathbb{N}^{*},\{2\}$ and $2 \mathbb{N}^{*}$, respectively.

### 3.2. Generalization.

Theorem 3.1. Let $E$ be any subset of $\mathbb{N}^{*}$. And let $C(E)$ be the set of all compositions with parts in this subset. Then

$$
\begin{equation*}
\left(\sum_{I \in C(E)}(-1)^{l(I)} \mathbf{G}_{\omega(I)}\right)^{-1}=\sum_{K \in C(E)} \mathbf{S}^{\operatorname{diam}(K)} . \tag{11}
\end{equation*}
$$

Proof - Thanks to (10), the statement to be proved is equivalent to

$$
\begin{equation*}
\left(\sum_{I \in C(E)}(-1)^{l(I)} \sum_{J \preceq \bar{I}}(-1)^{l(J)-1} \mathbf{S}^{\alpha(J) \omega(I)}\right)\left(\sum_{K \in C(E)} \mathbf{S}^{\alpha(K) \omega(\bar{K})}\right)=1, \tag{12}
\end{equation*}
$$

or, opening the parentheses,

$$
\begin{equation*}
\sum_{I, K \in C(E)} \sum_{J \preceq \bar{I}}(-1)^{l(I)+l(J)-1} \mathbf{S}^{\alpha(J) \omega(I)} \mathbf{S}^{\alpha(K) \omega(\bar{K})}=1 . \tag{13}
\end{equation*}
$$

Now,

$$
\begin{equation*}
\mathbf{S}^{\alpha(J) \omega(I)} \mathbf{S}^{\alpha(K) \omega(\bar{K})}=\mathbf{S}^{\alpha\left(J^{\prime}\right) \omega\left(I^{\prime}\right)} \tag{14}
\end{equation*}
$$

where $I^{\prime}=I \bullet \bar{K}$ and $J^{\prime}=K \triangleright J$. Note that $J^{\prime} \preceq \overline{I^{\prime}}$ and that

$$
\begin{equation*}
(-1)^{l(I)+l(J)-1}=-(-1)^{\left(l\left(I^{\prime}\right)+l\left(J^{\prime}\right)-1\right.} . \tag{15}
\end{equation*}
$$

Now, given any non-empty permutation $\sigma$ obtained as a product $\alpha(J) \omega(I)$ with $J \preceq \bar{I}$, it can be decomposed in exactly two ways as a product $\alpha(J) \omega(I)>\alpha(K) \omega(\bar{K})$ : either with $K=\emptyset$ or with $\alpha(K) \omega(\bar{K})$ corresponding to the last anticonnected permutation associated with the decomposition of $\sigma$ into anticonnected permutations. This comes from the fact that $\alpha(J) \omega(I)$ (with $J \preceq \bar{I}$ ) is anticonnected iff $J=\bar{I}$.

Since the coefficients associated with these two decompositions are opposite, such a permutation does not occur in the final result. Hence the result reduces to the contribution of the empty permutation.

## 4. Comments on Ung's other identities

In [15], Ung proves quasi-symmetric analogs of Schur's identity (for the sum of all Schur functions) and of Littlewood's identity (for its inverse). In fact, these analogs may be formulated without further work at the level of FQSym.

The first identity is

$$
\begin{equation*}
\sum_{I} F_{I}=\frac{1}{2}\left[\prod_{i} \frac{1+x_{i}}{1-x_{i}}-1\right]=\frac{1}{2}\left[\lambda_{1}(X) \sigma_{1}(X)-1\right] \tag{16}
\end{equation*}
$$

where $\lambda_{1}$ (resp. $\sigma_{1}$ ) is the sum of the elementary (resp. complete) symmetric functions. Interpreting the right-hand side in the algebra of noncommutative symmetric
functions, we have

$$
\begin{equation*}
\frac{1}{2}\left[\lambda_{1}(A) \sigma_{1}(A)-1\right]=\frac{1}{2}\left[\prod_{i}^{\leftarrow}\left(1+a_{i}\right) \prod_{i}\left(1-a_{i}\right)^{-1}-1\right]=\sum_{n \geq 0} H_{n} \tag{17}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{n}=\sum_{k=0}^{n-1} R_{1^{k}, n-k} \tag{18}
\end{equation*}
$$

The commutative image of $R_{1^{k}, n-k}$ is the Schur function $s_{n-k, 1^{k}}$, whose quasi-symmetric expansion is easily found to be

$$
\begin{equation*}
s_{n-k, 1^{k}}=\sum_{I \vdash n, l(I)=k+1} F_{I} . \tag{19}
\end{equation*}
$$

But $R_{1^{k}, n-k}$ can also be interpreted as an element of FQSym,

$$
\begin{equation*}
R_{1^{k}, n-k}=\sum_{C\left(\sigma^{-1}\right)=\left(1^{k}, n-k\right)} \mathbf{F}_{\sigma} \tag{20}
\end{equation*}
$$

so that (16) means that each descent class contains exactly one permutation whose inverse has a hook shape ( $1^{k}, n-k$ ).

The second identity is

$$
\begin{equation*}
\left(\sum_{I} F_{I}\right)^{-1}=1+\sum_{I \vDash 2 n+1}(-1)^{n+1} c_{I} F_{I} \tag{21}
\end{equation*}
$$

where $c_{I}$ is the number of permutations of shape $I$ whose inverse has shape $\left(12^{n}\right)$. This formula is obtained by observing that the inverse of $H=\sum_{n} H_{n}$ is the noncommutative hyperbolic tangent of [7], that is

$$
\begin{equation*}
H^{-1}=1-\sum_{n \geq 0}(-1)^{n} R_{12^{n}} \tag{22}
\end{equation*}
$$

which can again be interpreted as an identity in FQSym

$$
\begin{equation*}
H^{-1}=1+\sum_{n \geq 0}(-1)^{n} \sum_{C\left(\sigma^{-1}\right)=\left(12^{n}\right)} \mathbf{F}_{\sigma} . \tag{23}
\end{equation*}
$$

## References

[1] M. Aguiar and F. Sottile, Structure of the Malvenuto-Reutenauer Hopf algebra of permutations, Adv. in Maths, 191 (2005), 225-275.
[2] A. BJörner, Orderings of Coxeter groups, Combinatorics and algebra (1983), Amer. Math. Soc., Providence, RI, 1984, 175-195.
[3] A. Björner and F. Brenti, Combinatorics of Coxeter groups, Graduate Texts in Mathematics, 231. Springer, New York, 2005. xiv+363 pp.
[4] G. Duchamp, F. Hivert, J.-C. Novelli, and J.-Y. Thibon, Noncommutative symmetric functions VII: free quasi-symmetric functions revisited, preprint.
[5] G. Duchamp, F. Hivert, and J.-Y. Thibon, Noncommutative symmetric functions VI: free quasi-symmetric functions and related algebras, Internat. J. Alg. Comput. 12 (2002), 671-717.
[6] P. Edelman, Geometry and the Möbius function of the weak Bruhat order of the symmetric group, 1983.
[7] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon, Noncommutative symmetric functions, Adv. in Math. 112 (1995), 218-348.
[8] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, [in "Combinatorics and algebra", C. Greene, Ed.], Contemporary Mathematics, 34 (1984), 289-301.
[9] F. Hivert and N. Thiéry, MuPAD-Combinat, an open-source package for research in algebraic combinatorics, Sém. Lothar. Combin. 51 (2004), 70p. (electronic).
[10] D. E. Knuth, The art of computer programming, vol.3: Sorting and searching, (AddisonWesley, 1973).
[11] I.G. Macdonald, Symmetric functions and Hall polynomials, 2nd ed., Oxford University Press, 1995.
[12] C. Malvenuto and C. Reutenauer, Duality between quasi-symmetric functions and Solomon descent algebra, J. Algebra 177 (1995), 967-892.
[13] C. Reutenauer, Free Lie algebras, Oxford University Press, 1993.
[14] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/
[15] B. C. V. Ung, Combinatorial identities for series of quasi-symmetric functions, Proc. FPSAC'08, Toronto (2008).
(Hivert) LITIS, Université de Rouen ; Avenue de l'université ; 76801 Saint Étienne du Rouvray, France,
(Novelli and Thibon) Université Paris-Est, Institut Gaspard Monge, 5 Boulevard Descartes, Champs-Sur-Marne, 77454 Marne-la-Vallée cedex 2, France

E-mail address, Florent Hivert: hivert@univ-rouen.fr
E-mail address, Jean-Christophe Novelli: novelli@univ-mlv.fr
E-mail address, Jean-Yves Thibon: jyt@univ-mlv.fr

