Mácajová and Škoviera Conjecture on Cubic Graphs.

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Abstract

A conjecture of Máčajová and Škoviera [8] asserts that every bridgeless cubic graph has two perfect matchings whose intersection does not contain any odd edge cut. We prove this conjecture for graphs with few vertices and we give a stronger result for traceable graphs.
keywords : Cubic graph; Edge-partition; Traceable graphs

1 Introduction

The following conjecture first appeared in [5] is known as the Fulkerson Conjecture, see [9].

Conjecture 1 *If G is a bridgeless cubic graph, then there exist* 6 *perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$.*

A consequence of the Fulkerson conjecture would be that every bridgeless cubic graph has 3 perfect matchings with empty intersection (take 3 distinct of the 6 perfect matchings given by the conjecture). The following weakening of this (also suggested by Berge) is still open.

Conjecture 2 *There exists a fixed integer k such that every bridgeless cubic graph has a list of k perfect matchings with empty intersection.*

Fan and Raspaud [3] conjectured that any bridgeless cubic graph can be provided with three perfect matchings with empty intersection(we shall say also *non intersecting perfect matchings*).

Conjecture 3 *[3] Every bridgeless cubic graph contains perfect matching M_1, M_2, M_3 such that*

$$M_1 \cap M_2 \cap M_3 = \emptyset$$

The following Conjecture is due to Máčajová and Škoviera in [8]

Conjecture 4 *Every bridgeless cubic graph has two perfect matchings M_1, M_2 such that $M_1 \cap M_2$ does not contain an odd edge cut.*

A *join* in a graph G is a set $J \subseteq E(G)$ such that the degree of every vertex in G has the same parity as its degree in the graph $(V(G), J)$. A perfect matching being a particular join in a cubic graph Kaiser and Raspaud conjectured in [6]

Conjecture 5 *[6] Every bridgeless cubic graph admits two perfect matchings M_1, M_2 and a join J such that*

$$M_1 \cap M_2 \cap J = \emptyset$$

As a matter of fact Conjectures 4 and 5 are equivalent. Equivalence comes from the fact that a set of edges contains a join if and only if this set intersects all odd edge cuts.

If true Conjecture 1 implies Conjecture 3 which itself implies Conjectures 4 and 5. All those conjectures being obviously true for cubic graphs with chromatic index 3, it is useful to consider the following parameter for cubic graphs. The *oddness* of a cubic graph G is the minimum number of odd circuits in a 2-factor of G. In [6] Kaiser and Raspaud proved that Conjecture 5 holds true for bridgeless cubic graph of oddness two.

In this paper, we consider Conjecture 4. We prove that a minimal counterexample to Conjecture 5 must have at least 50 vertices.

Moreover, we prove that Conjectures 4 and 5 hold true while the order of the graph is less than a function of the cyclic edge connectivity. Finally, we give a refining of Kaiser and Raspaud result [6] when considering cubic bridgeless traceable graphs.

When A is a set of edges, $V(A)$ will denote the set of vertices that are an endpoint of some edge in A. If M is a perfect matching of a cubic graph $G = (V, E)$, then denote by G_M the 2-factor $G_M = (V, E - M)$. When X is a set of vertices, δX denotes the set of edges with precisely one end in X. An edge cut is a set of edges whose removal renders the graph disconnected and which is inclusion-wise minimal with this property. The *cyclic edge connectivity* of a cubic graph is the size of a smallest edge cut in a graph such that at least two of the connected components contain cycles. A graph G is said to be traceable whenever G has a *Hamiltonian path* that is a path which visits each vertex exactly once.

Notations $\prec_W$ and $W(z,t)$, concatenation of sub-walks. Let W be a walk of G. Writing $W = x \dots y$ induces a natural order on the vertices of W, let us denote $\prec_W$ this order. When $W = x \dots y$, W will be said to *start* with x and to *end* with y.When z and t are vertices of W such that $z \prec_W t$, the sub-walk $z \dots t$ of W whose endpoints are z and t will be denoted $W(z,t)$. When a walk W ($W = W(x,y)$) and a walk W' ($W' = W'(x',y')$) have a common vertex, say a, we can *concatenate* the sub-walks $W(x,a)$ and $W'(a,y')$ in order

to obtain another walk say W'', also denoted $W(x,a) + W'(a,y')$, such that $W''(x,a) = W(x,a)$ and $W''(a,y') = W'(a,y')$.

For basic graph-theoretic terms, we refer the reader to Bondy and Murty [1].

2 Preliminary results

2.1 Fractional perfect matchings

The following result belongs to folklore

Theorem 6 *Let G be a cubic bridgeless graph. G is 3-edge colourable if and only if there is a perfect matching in G that does not contain any odd edge cut.*

Proof Assume that G has a 3-edge colouring using colours α, β and γ. Let M_α be the set of edges coloured with α, if M_α contains an odd edge cut, there must be a partition (V_1, V_2) of $V(G)$ into two odd sets such that the edges of X have one end in V_1 and the other end in V_2. Since all the edges of X are coloured with α, the set of edges in V_1 coloured with β must be a perfect matching in V_1, a contradiction since V_1 has an odd number of vertices.

Conversely, consider a perfect matching M that does not contain any odd edge cut. Suppose that the 2-factor G_M contains an odd cycle C, thus δC is an odd edge cut entirely contained in M, a contradiction. Consequently G_M does not contain any odd cycle, it follows that G is 3-edge colourable. □

In order to prove Conjecture 4 for bridgeless cubic graphs with few vertices, we will consider the notion of *fractional perfect matching* as used in [7].

For a graph $G = (V, E)$, a vector w of $\mathbb{R}^E$ is said to be a fractional perfect matching whenever w satisfies the following properties (the entry of w corresponding to $e \in E$ being denoted $w(e)$ and $w(A) = \Sigma_{e \in A} w(e)$, for $A \subseteq E$) :
• $0 \leq w(e) \leq 1$ for each edge e of G
• $w(\delta\{v\}) = 1$ for each vertex v of G
• $w(\delta X) \geq 1$ for each $X \subseteq V$ of odd cardinality.

The perfect matching polytope is the convex hull of the set of incidence vectors of perfect matchings of G. In [2] Edmonds showed that a vector $w \in \mathbb{R}^E$ belongs to the perfect matching polytope of G if and only if it is a fractional perfect matching

Moreover, when χ^A denotes the characteristic vector of the edge set A we will use the following tool :

Lemma 7 *[7] If w is a fractional perfect matching in a graph $G = (V, E)$ and $c \in \mathbb{R}^E$, then G has a perfect matching M such that $c.\chi^M \geq c.w$ where . denotes the scalar product. Moreover, there exists such a perfect matching M that contains exactly one edge of each edge cut C with $w(C) = 1$.*

It is shown in [7], among other results, that there must exist a perfect matching M_1 that intersects all edge cuts of size 3 into a single edge and a perfect

matching M_2 such that $|M_2 - M_1| \geq \frac{4}{15}|E(G)|$. When the graph has n vertices, since the size of a perfect matching is precisely $\frac{n}{2}$, it must be pointed out that $|M_1 \cap M_2| \leq \frac{n}{10}$.

Observe that there is an alternate proof of Theorem 6 in terms of fractional perfect matchings . Consider indeed a perfect matching M that does not contain any odd edge cut. We define a fractional perfect matching as follows : $w(e) = 0$ when $e \in M$ and $w(e) = \frac{1}{2}$ otherwise. Given an odd set of vertices, say X, δX is an odd edge cut which intersects M in a odd number of edges, since $\delta X \nsubseteq M$, w is a fractional perfect matching. By Lemma 7 there is a perfect matching M' such that

$$c.\chi^{M'} \geqslant c.w = \frac{1}{2} \times \frac{2}{3} \times |E| = \frac{n}{2}.$$

When $c = 1 - \chi^M$, since $c.\chi^{M'} = |M' \backslash M|$ we have $|M' \backslash M| = \frac{n}{2} = |M'|$ and thus $M \cap M' = \emptyset$. It follows that $\chi'(G) = 3$.

2.2 Balanced perfect matchings

Let M be a perfect matching of a cubic graph and let $\mathcal{C} = \{C_1, C_2 \dots C_k\}$ be the 2-factor G_M. $A \subseteq M$ is a *balanced* $M-$matching whenever there is a perfect matching M' such that $M \cap M' = A$. That means that each odd cycle of $\mathcal{C}$ is incident to an odd number of edges in A and the sub-paths determined by the ends of $A \cap M'$ on the cycles of $\mathcal{C}$ incident to A have odd lengths.

Lemma 8 *Let M be a perfect matching of a cubic graph G. A matching A is a balanced M-matching if and only if the connected components of G_{M-A} are either odd paths or even cycles.*

Proof Since G_M is a 2-factor of G, the connected components of the subgraph induced by $V(G) - V(A)$ must be cycles or paths. Since A is a balanced M-matching, the connected components of this graph must be even cycles or odd paths.

Conversely, assume that the connected components of $G_M - V(A)$ are odd paths or even cycles. Let A' be a perfect matching of $G_M - V(A)$, we set $M' = A \cup A'$ and we are done. □

Lemma 9 *A bridgeless cubic graph contains 3 non intersecting perfect matching if and only if there is a perfect matching M and two balanced disjoint balanced $M-$matchings.*

Proof Assume that M_1, M_2, M_3 are three perfect matchings of G such that $M_1 \cap M_2 \cap M_3 = \emptyset$. Let $M = M_1$, $A = M_1 \cap M_2$ and $B = M_1 \cap M_3$. Since $A \cap B = M_1 \cap M_2 \cap M_3$, A and B are two balanced $M-$matchings with empty intersection.

Conversely, assume that M is a perfect matching and that A and B are two balanced $M-$matchings with empty intersection. Let $M_1 = M$, M_2 be a

perfect matching such that $M_2 \cap M_1 = A$ and M_3 be a perfect matching such that $M_3 \cap M_1 = B$. We have $M_1 \cap M_2 \cap M_3 = A \cap B$ and the three perfect matchings M_1, M_2 and M_3 have an empty intersection. □

3 On cubic graphs with few vertices

We first prove that Conjecture 4 holds true for bridgeless cubic graphs having less than 50 vertices

Theorem 10 *Let G be a bridgeless cubic graph of order $n < 50$. There are perfect matchings M and M' such that $M \cap M'$ does not contain any edge cut.*

Proof We know from [7] that there must exist a perfect matching M which intersects all edge cuts of size 3 into a single edge and a perfect matching M' such that $|M \cap M'| \leq \frac{n}{10}$. It is assumed $n < 50$, thus $|M \cap M'| < 5$. Hence an odd edge cut, say C in $M \cap M'$ must be of size 3, but a such edge cut cannot exist since M intersects C in precisely one edge. □

Let us now consider cyclic edge connectivity in cubic graphs.

Theorem 11 *Let G be a cubic graph of order n with cyclic edge connectivity $k \geq 3$. One of the following holds.*

1. *There are two perfect matchings M and M' such that $|M \cap M'| \leq \frac{n}{2(2\lfloor \frac{k}{2} \rfloor + 3)}$.*
2. *For all perfect matching M there is an edge cut of size $2\lfloor \frac{k}{2} \rfloor + 1$ entirely contained in M.*

Proof For convenience we set $s = 2\lfloor \frac{k}{2} \rfloor + 3$. Let M be a perfect matching that does not contain any odd edge cut of size $s-2$. The graph being cyclically k-edge connected $s-2$ is the minimum size of an odd edge cut in G. We set $w(e) = \frac{1}{s}$ when $e \in M$ and $w(e) = \frac{s-1}{2s}$ otherwise. If X is an odd set of vertices, δX is an odd edge cut of size at least $s-2$. If $|\delta X| \geq s$ then $w(\delta X) \geq 1$. If $|\delta X| = s-2$ then there are at least 2 edges of δX which are not in M and $w(\delta X) \geq 1$ again. Hence w is a fractional perfect matching.

Applying Lemma 7 with $c = 1 - \chi^M$ we get a perfect matching, say M' such that $c.\chi^{M'} \geq c.w = n \times \frac{s-1}{2s}$. Since $c.\chi^{M'} = |M' - M|$ and $|M'| = \frac{n}{2}$ it follows that $|M \cap M'| \leq \frac{n}{2s}$, as claimed. □

Theorem 12 *Let G be a cubic graph of order n with cyclic edge connectivity $k \geq 4$. If $n < 2(2\lfloor \frac{k}{2} \rfloor + 3)(2\lfloor \frac{k}{2} \rfloor + 1)$ then there are two perfect matchings whose intersection does not contain any odd edge cut.*

Proof Once again we denote $s = 2\lfloor \frac{k}{2} \rfloor + 3$. We can assume that every perfect matching contains an odd edge cut of size $s-2$. Otherwise, from Theorem 11 there are two perfect matchings whose intersection contains less than $\frac{n}{2s} = s-2$ edges and we are done.

Let M be a perfect matching of G. We set $w(e) = \frac{1}{s-2}$ when $e \in M$ and $w(e) = \frac{s-3}{2(s-2)}$ otherwise. The weight of an edge being at least $\frac{1}{s-2}$ and an odd edge cut having at least $s-2$ edges, w is a fractional perfect matching. If $c = 1 - \chi^M$, by Lemma 7 there is a perfect matching M' which intersects in a single edge every edge cut C such that $w(C) = 1$.

In addition we know that $c.\chi^{M'} \geq c.w$, in other words $|M \cup M'| \geq \frac{2}{3} \times |E| \times \frac{s-3}{2(s-2)} = \frac{n}{2} \times \frac{s-3}{s-2}$. Consequently $|M \cap M'| \leq \frac{n}{2(s-2)}$. Since $n < 2s(s-2)$ we have that $|M \cap M'| \leq s$.

Assume that $M \cap M'$ contains an odd edge cut C. By the above relation $|C| = s-2$ and then $w(C) = 1$, a contradiction since M' intersects the edge cuts of size $s-2$ in a single edge. □

An example of consequence of Theorem 12 is that Conjecture 4 and therefore Conjecture 5 hold true for cyclically 4-edge-connected graphs having less than 70 vertices.

Remark 13 Kaiser, Král and Norine in [7] showed that every bridgeless cubic graph contains two perfect matchings whose intersection has at most $\frac{n}{10}$ edges. This result strengthen Fulkerson conjecture. Indeed, if we have a set of 6 perfect matchings such that any edge of a bridgeless cubic graph is covered exactly twice by this set, we certainly have two of them whose intersection has at most $\frac{n}{10}$ edges. Observe that this upper bound would be implied by Conjecture 1. A challenging question is thus to characterize the bridgeless cubic graphs for which $\frac{n}{10}$ is optimal. The Petersen graph is obviously such a graph, but no other graph is known with that property and we can conjecture that this is the only graph.

Theorem 11 above says that in a cyclically 4−edge connected cubic graph, with chromatic index 4, either we can find two perfect matchings whose intersection has at most $\frac{n}{14}$ edges or every two perfect matchings has an intersection containing an odd cutset of size 5, a support to the above conjecture.

4 On cubic traceable graphs

In [6] Kaiser and Raspaud proved that Conjecture 5 holds true for bridgeless cubic graph of oddness two. In the following we prove a stronger result for cubic bridgeless traceable graphs.

4.1 An auxiliary graph

Let us consider a Hamiltonian path of G. It will be convenient to denote the vertices of G as integers from 1 to n and the Hamiltonian path will be merely

denoted $1 \dots n$. Hence ij ($i \neq j \in \{1 \dots n\}$) denotes an edge of G while the edge joining i to $i+1 (1 \leq i \leq n-1)$ will be denoted e_i.

Suppose that chromatic index of G is 4, we can colour the edges of G in the following way. The edges e_i $(1 \leq i \leq n-1)$ of the Hamiltonian path are alternately coloured with α and β (the first edge e_1 being coloured with α). The remaining edges are coloured with γ with the exception of one edge incident with 1 and one edge incident with n. These two edges are coloured by δ. The set M_α of edges coloured with α is a perfect matching and the 2−factor $G_{M_\alpha} = \{C_1 \dots C_k\}$ is composed of a set of even cycles whose edges are coloured β or γ and two odd cycles C_1 and C_k. Without loss of generality we suppose that 1 is a vertex of C_1 and n is a vertex of C_k.

The edges $e_{min(C)}$ and $e_{max(C)}$. For $C \in \{C_1, C_2 \dots C_k\}$ we denote $max(C)$ the greatest index i such that e_i is an edge of C, similarly $min(C)$ denotes the smallest index i such that e_i belongs to C. Observe that $max(C)$ and $min(C)$ are even numbers and that the corresponding edges are coloured with β. Moreover the endpoints of $e_{min(C)}$ are $min(C)$ and $min(C)+1$ as well as the endpoints of $e_{max(C)}$ are $max(C)$ and $max(C)+1$

Observe that $min(C)$ and $max(C)$ are always defined and that $min(C) = max(C)$ if and only if C is a triangle.

The sequence $(\Gamma_j)_{j=1\dots h}$. We define a sequence $(\Gamma_j)_{j=1\dots h}$, $2 \leq h \leq k$ of members of $\{C_1, \dots C_k\}$ as follows :

- We set $\Gamma_1 = C_1$.
- If $max(\Gamma_j) < min(C_k)$, since the edge $e_{max(\Gamma_j)+1}$ is not a bridge there is a cycle C in G_{M_α} with $min(C) < max(\Gamma_j) < max(C)$. Among all such cycles, let us denote by Γ_{j+1} the cycle C for which $max(C)$ is maximum.
- If $max(\Gamma_j) > min(C_k)$, we set $h = j+1$ and $\Gamma_h = C_k$.

Observe that by construction, when $h = 2$, we have

$$1 < min(\Gamma_2) = min(C_k) < max(C_1) = max(\Gamma_1) < n$$

and when $h > 2$, we have

$$min(\Gamma_j) < max(\Gamma_{j-1}) < min(\Gamma_{j+1}) < max(\Gamma_j) \qquad 1 < j < h$$

and

$$min(\Gamma_h) = min(C_k) < n$$

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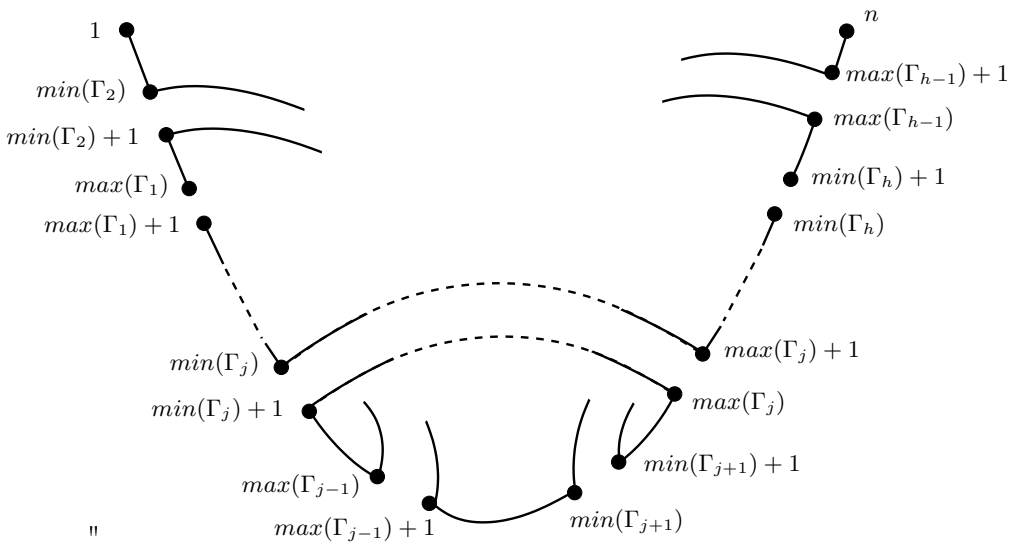

Figure 1: An auxiliary graph (with $h > 2$)

An auxiliary graph H. We consider an auxiliary graph H, where $V(H) = V(G)$ and $E(H)$ is obtained from $E(G)$ as follows (see Figure 1) :

- We delete the edges $e_{max(\Gamma_1)}$ and $e_{min(\Gamma_h)}$ and the edges $e_{min(\Gamma_j)}$ and $e_{max(\Gamma_j)}$ of each cycle Γ_j $(1 < j < h)$.
- Since Γ_j $(1 < j < h)$ is an even cycle, when deleting the edges $e_{min(\Gamma_j)}$ and $e_{max(\Gamma_j)}$, we get two odd paths with one end in $\{min(\Gamma_j), min(\Gamma_j)+1\}$ and the other end in $\{max(\Gamma_j), max(\Gamma_j)+1\}$, namely P_j and P'_j. We put in $E(H)$ two new edges (denoted in the following as *additional* edges) one edge connecting the endpoints of P_j while the endpoints of the other edge are the endpoints of P'_j. We will say in the following that the first edge *represents* the path P_j while the other one *represents* the path P'_j
- Finally, we delete all the edges of G being coloured with γ and δ (that is the chords of the Hamiltonian path).

All the vertices of H have degree 2 except 6 vertices, namely 1, $max(\Gamma_1)$, $max(\Gamma_1)+1$, $min(\Gamma_h)$, $min(\Gamma_h)+1$, n which have degree 1. Thus the connected components of H are precisely 3 paths whose endpoints are members of $\{1, max(\Gamma_1), max(\Gamma_1)+1, min(\Gamma_h), min(\Gamma_h)+1, n\}$

Notation.

Two walks of G will be said α-disjoint whenever those walks do not share any edge coloured with α.

In the following, when a cubic brigeless traceable G graph is given we will consider the graph G together with the edge-colouring defined above, the sequence $(\Gamma_j)_{j=1...h}$, the auxiliary graph H and all related notations.

4.2 Technical lemmas

Lemma 14 *Let G be a cubic bridgeless traceable graph such that $\chi'(G) = 4$. There are in G three pairwise α-disjoint odd walks, W_1, W_2, W_3 such that, for $i \in \{1, 2, 3\}$:*

- *W_i has one endpoint say q_i in C_1 and the other endpoint q'_i in C_k*
- *W_i does not share any edge with C_1 nor C_k*

Proof Those walks will be derived from the connected components of the auxiliary graph H. As a matter of fact, when $h = 2$, H is reduced to 3 sub-paths of P, namely: Q_1: $1...min(\Gamma_2)$,Q_2: $min(\Gamma_2) + 1....max(\Gamma_1)$ and Q_3:$max(\Gamma_1) + 1....n$. We can thus suppose that $h > 2$.

Let $Q = x_1 \dots x_r$ be a connected component of H with its endpoint x_1 in $\{1, max(\Gamma_1), max(\Gamma_1) + 1\}$, we will prove that the other endpoint x_r of Q belongs to $\{min(\Gamma_h), min(\Gamma_h)+1, n\}$. Let x be the maximum index in $\{1, \dots n\}$ of a vertex of Q.

Claim 1 $x > max(\Gamma_{h-2})$.

Proof It is easy to check that $x > max(\Gamma_1)$.

If $x > max(\Gamma_{j-1})$ and $x \leq max(\Gamma_j)$ for $1 < j < h - 1$, then the vertex x must be a vertex of one of the 2 sub-paths $max(\Gamma_{j-1}) + 1 \dots min(\Gamma_{j+1})$ or $min(\Gamma_{j+1}) + 1 \dots max(\Gamma_j)$ of P, thus $x = min(\Gamma_{j+1})$ or $x = max(\Gamma_j)$. In both cases, since $j < h-1$ there must be in Q one vertex of $\{max(\Gamma_{j+1}), max(\Gamma_{j+1})+1\}$. But those vertices have an index greater than x, a contradiction. Thus $x > max(\Gamma_{h-2})$. □

Claim 2 *The connected components of H are odd paths with one end in $\{1, max(\Gamma_1), max(\Gamma_1) + 1\}$ and the other end in $\{min(\Gamma_h), min(\Gamma_h) + 1, n\}$.*

Proof From Claim 1, $x > max(\Gamma_{h-2})$ and either $x = min(\Gamma_h) = x_r$ or $x = max(\Gamma_{h-1})$, in that case $x_r = min(\Gamma_h) + 1$, or $x = n = x_r$. Consequently no connected component of H can be a path with both ends in $\{1, max(\Gamma_1), max(\Gamma_1) + 1\}$, the Claim follows. □

To a path, say Q_s ($s \in \{1, 2, 3\}$), of H we can associate an odd walk W_s of G as follows :

- Let q_s be the last vertex of Q_s that belongs to C_1 when running on Q_s from its endpoint in $\{1, max(C_1), max(C_1)+1\}$. Similarly let q'_s be the last vertex of Q_s that belongs to C_k when running on Q_s from its endpoint in

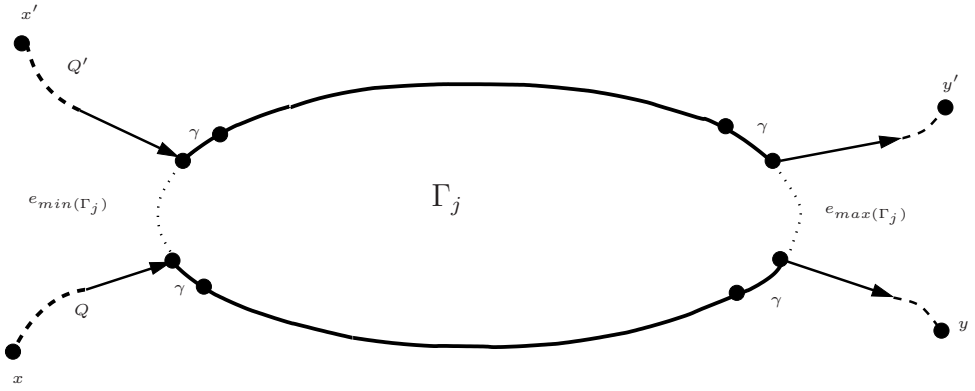

Figure 2: The walks Q and Q' that intersect Γ_j

$\{n, min(C_k), min(C_k)+1\}$. Let Q'_s be the sub-path of Q_s whose endpoints are q_s and q'_s.

- Each *additional* edge of Q'_s represents some odd sub-path (P_j or P'_j) of some cycle Γ_j. The walk W_s is obtained from the path Q'_s by replacement of each additional edge with the sub-path that it represents.

The walks W_s ($s \in \{1,2,3\}$) defined above are α-disjoint walks. Moreover those walks have one end which belongs to C_1 and the other end which belongs to C_k, have no edge of C_1 nor of C_k while their end-edges are coloured with α. □

We shall deal in the next subsection with the particular case where the sequence $(\Gamma_j)_{j=1\ldots h}$ contains only the two odd cycles C_1 and C_k, see Proposition 19. Hence, we assume in the sequel of this subsection that $h > 2$, we give below some notations in order to describe the construction from W_1, W_2 and W_3 of new walks which intersect the even cycles of the sequence.

We intend to derive Q into a walk which set of α-edges can be extended into a perfect matching on Γ_j.

An odd subpath of C_i whose end edges are coloured with γ is a γ-*chain* and we define analogously a β-*chain*. A walk W will be said to *well-intersect* a cycle C of G_{M_α} when either $W \cap C = \emptyset$ or the set of endpoints of the α-edges of W which belong to C, say $\{a_1 \ldots a_p\}$ in that order around C, are such that the consecutive subpaths $\{[a_i \ldots a_{i+1}]\}_{0<i<p}$ are odd..

Lemma 15 *Let W_1, W_2, W_3 be the three walks described in Lemma 14 then, each even cycle of G_{M_α} not involved in the sequence $(\Gamma_j)_{j=1\ldots h}$ is well-intersected by W_i ($i \in \{1,2,3\}$).*

Proof Let C be an even cycle not involved in the sequence $(\Gamma_j)_{j=1\ldots h}$. The walks W_i $i \in \{1,2,3\}$ possibly intersect this cycle in using only edges coloured

β. The α edges of W_i with an end in C determine thus a set of γ-chains or β-chains. The result follows. □

Lemma 16 *Let W_1, W_2, W_3 be the three walks described in Lemma 14 then, for each even cycle of the sequence $(\Gamma_j)_{j=1...h}$ the walk which do not use the vertices $max(\Gamma_j)$ and $max(\Gamma_j)+1$ well-intersects Γ_j.*

Proof Assume without loss of generality that W_1 does not use neither $max(\Gamma_j)$ nor $max(\Gamma_j)+1$, then, by construction, W_1 has not been obtained by replacement of the additional edges representing the two paths P_i or P'_i of $C_i = \Gamma_j$. Hence W_1 possibly intersects Γ_j by using only edges coloured β. The α edges of W_1 with an end in Γ_j determine thus a set of γ-chains or β-chains. The result follows. □

Given an even cycle of the sequence $(\Gamma_j)_{j=1...h}$ say Γ_j, there are precisely two walks in $\{W_1, W_2, W_3\}$ say $Q = Q(x, y)$ and $Q'(x', y')$, $x \in \Gamma_1$, $y \in \Gamma_h$, $x' \in \Gamma_1$, $y' \in \Gamma_h$, both of them containing a subpath of Γ_j with end-edges coloured with γ (see Figure 2). Morover both Q and Q' contain precisely one vertex of $\{max(\Gamma_j), max(\Gamma_j)+1\}$.

The first vertex of Q (resp. Q') following the order given by $\prec_Q$ (resp. $\prec_{Q'}$) that belongs to Γ_j is denoted x_j (resp. x'_j).

Lemma 17 *Let G be a cubic bridgeless traceable graph such that $\chi'(G) = 4$. Let Γ_j be an even cycle of the sequence $(\Gamma_j)_{j=1...h}$. Then there are two α-disjoint walks say R and R' such that*

1. *$R(x, x_j) = Q(x, x_j)$ and $R'(x', x'_j) = Q'(x', x'_j)$.*
2. *R contains one end vertex of $e_{max(\Gamma_j)}$, say y_j, while R' contains the other, say y'_j.*
3. *Either $Q(y_j, y)$ is a subwalk of R and $Q'(y'_j, y')$ a subwalk of R' or $Q'(y'_j, y')$ is a subwalk of R and $Q(y_j, y)$ a subwalk of R'.*
4. *$R(x_j, y_j)$ and $R'(x'_j, y'_j)$ are subpaths of Γ_j.*
5. *$R(x_j, y_j)$ is a γ-chain.*

Proof One of the two paths of Γ_j joining x_j to the endpoints of $e_{max(\Gamma_j)}$ is certainly a γ-chain. Let P be this path. Let y_j be the endpoint of $e_{max(\Gamma_j)}$ which belongs to P while y'_j denotes the other. If Q contains the path P, we set $R = Q$ and $R' = Q'$ otherwise we set $R = Q(x, x_j) + P + Q'(y_j, y')$ and $R' = Q'(x', x'_j) + P' + Q(y'_j, y)$ where P' is a subpath of Γ_j joining x'_j to y'_j (see Figure 3). □

In the following, up to a renaming of the vertices y and y', we assume that y is an endpoint of R while y' is an endpoint of R'.

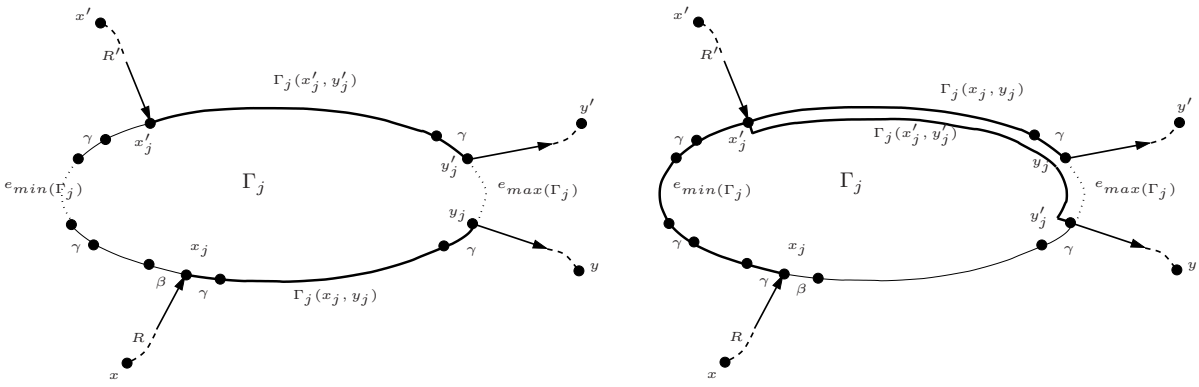

Figure 3: The walks R and R' that intersect Γ_j

Since Γ_j is an even cycle we have $j < h$, thus there certainly exists one cycle in $\{\Gamma_{j+1}, \Gamma_{j+2}\}$ say Γ which have an endpoint of $e_{max(\Gamma)}$ on R. The index Γ in the sequence $(\Gamma_j)_{j=1...h}$ will be denoted $\sigma_R(\Gamma_j)$. The index $\sigma_{R'}(\Gamma_j)$ is defined similarly from the walk R'. By construction we have $\{\sigma_R(\Gamma_j), \sigma_{R'}(\Gamma_j)\} = \{j+1, j+2\}$.

Lemma 18 *Let G be a cubic bridgeless traceable graph such that $\chi'(G) = 4$. Let Γ_j be an even cycle of the sequence $(\Gamma_j)_{j=1...h}$. There are two α-disjoint walks say S and S' such that*

1. *The vertex x_j (resp. x'_j) is an endpoint of S (resp. S').*
2. *S and S' have distinct endpoints in $\{x_{\sigma_R(\Gamma_j)}, x'_{\sigma_{R'}(\Gamma_j)}\}$.*
3. *The vertices of S and S' are vertices of $R(x_j, x_{\sigma_R(\Gamma_j)})$ or vertices of $R'(x'_j, x_{\sigma_{R'}(\Gamma_j)})$ or of Γ_j.*
4. *S well-intersects the cycle Γ_j.*

Proof By construction the walk $R(x_j, y_j)$ well-intersects Γ_j. If the subwalk $R(y_i, \sigma_R(\Gamma_j))$ shares an edge, say e, with Γ_j, this edge is coloured with β. When e belongs to $\Gamma(x_j, y_j)$ the intersection of R with Γ_j will not be changed by e. This is not the case when e is an edge of a β-chain.

Let P be the subpath of Γ_j whose endpoints are x_j and y_j which is distinct from $\Gamma_j(x_j, y_j)$. Observe that P is a β-chain.

If the subwalk $R(y_i, \sigma_R(\Gamma_j))$ does not intersect P we set $S = R(y_i, \sigma_R(\Gamma_j))$ and $S' = R'(x'_j, x'_{\sigma_{R'}(\Gamma_j)})$ and we are done.

If on the contrary $R(y_i, \sigma_R(\Gamma_j))$ shares an edge with P, let ab ($a \prec_R b$) be a such edge where $\Gamma(a, y_j)$ is a subpath of P with maximum length. It must be pointed out that in this case $R(y_i, \sigma_R(\Gamma_j))$ does not intersect with Γ_j.

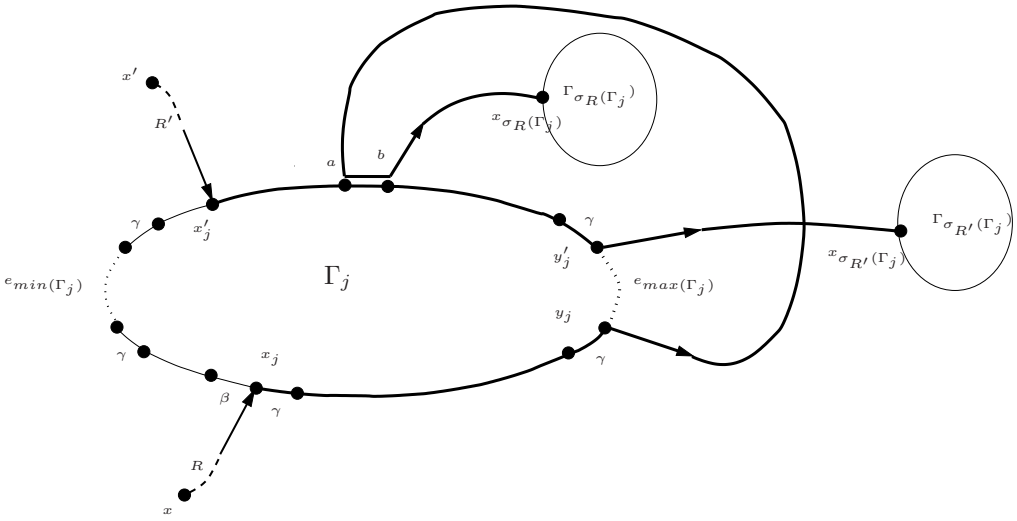

Figure 4: The walks R and R' in Case 1

Case 1. If b is a vertex of $\Gamma(a, y_j)$ (see Figure 4) we write $S = R(x_j, y_j) + \Gamma(y_j, b) + R(b, \sigma_R(\Gamma_j))$ and $S' = R'(x'_j, \sigma'_R(\Gamma_j))$.

Case 2. When b does not belong to $\Gamma(a, y_j)$ the a is a vertex of $\Gamma(b, y_j)$(see Figure 5). We write $S = R(x_j, y_j) + R(y_j, a) + \Gamma(a, y'_j) + R'(y'_j, \sigma_{R'}(\Gamma_j))$ and $S' = R'(x'_j, b) + R(b, \sigma_R(\Gamma_j))$, where $R'(x'_j, b)$ denotes the subpath of Γ with endpoints x'_j and b which does not contain a.

□

4.3 The main results

We use in the sequel the same notations than above.

In Propositions 19, 20 and 21 we consider particular cases of cubic graph for which Conjecture 3 holds true.

Proposition 19 *Let G be a cubic bridgeless traceable graph such that $\chi'(G) = 4$. If the sequence $(\Gamma_j)_{j=1...h}$ has only two cycles then there exists four perfect matchings M_α, M_1, M_2 and M_3 such that $M_\alpha \cap M_i \cap M_j = \emptyset$ for $i, j \in \{1, 2, 3\}$, $i \neq j$.*

Proof Since $h = 2$ we have $\Gamma_1 = C_1$ and $\Gamma_2 = C_k$. Moreover, the walks described in Lemma 14 are reduced to paths whose edges are alternately coloured with α and β. Thus, for $i \in \{1, 2, 3\}$, by Lemma 8, the set of α-edges of W_i is a balanced $M_\alpha-$matching. Hence we are done since the walks W_1, W_2 and W_3

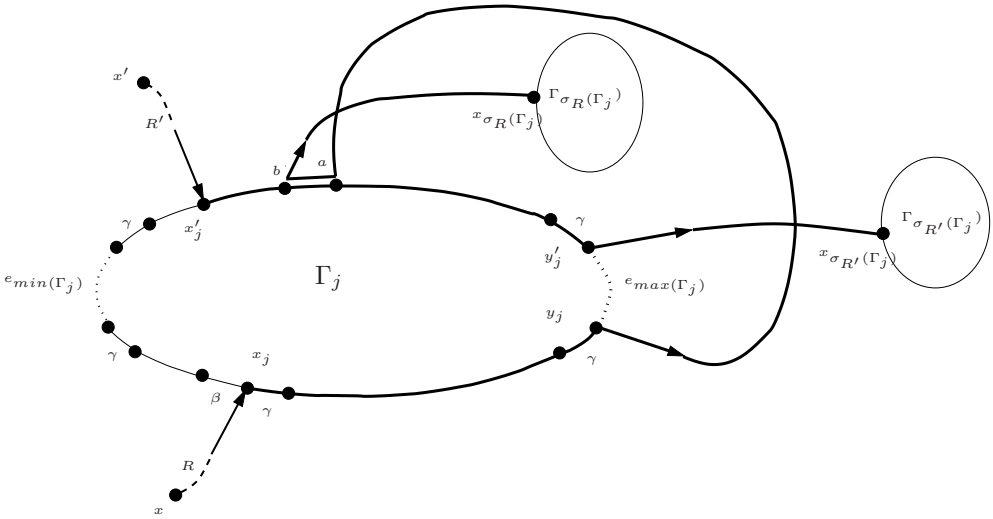

Figure 5: The walks R and R' in Case 2

are α-disjoint. □

Proposition 20 *Let G be a cubic bridgeless traceable graph such that $\chi'(G) = 4$. If the sequence $(\Gamma_j)_{j=1...h}$ has only three cycles then there exists three perfect matchings M_α, M_1, M_2 such that $M_\alpha \cap M_1 \cap M_2 = \emptyset$*

Proof As a matter of fact, when $h = 3$, the sequence $(\Gamma_j)_{j=1...3}$ is reduced to $(\Gamma_1 = C_1, \Gamma_2, \Gamma_3 = C_k)$. Let W_1, W_2 and W_3 be the three walks obtained by Lemma 14. By Lemma 15 those walks well-intersect all cycles which are not in the sequence $(\Gamma_j)_{j=1...h}$. By Lemma 16 we can consider that W_2 well-intersects Γ_2 and by Lemma 18 we can transform W_1 in a walk S_1 well-intersecting this cycle. We get hence two walks α disjoint S_1 and W_2 well-intersecting every cycle of G_{M_α}. Hence the set of edges of W coloured with α, as well as the same set for W', are balanced $M_\alpha-$matchings. By Lemma 8, we get thus two perfect matchings M_1 and M_2 such that $M_\alpha \cap M_1 \cap M_2 = \emptyset$ as claimed. □

Let G be a cubic bridgeless traceable graph such that $\chi'(G) = 4$. Recall that given an even cycle of the sequence $(\Gamma_j)_{j=1...h}$ say Γ_j we have defined two odd paths P_j and P'_j with one end in $\{min(\Gamma_j), min(\Gamma_j) + 1\}$ and the other end in $\{max(\Gamma_j), max(\Gamma_j) + 1\}$. When one of those paths have endpoints in $\{min(\Gamma_j), max(\Gamma_j)\}$ the paths P_j and P'_j are said to be *crossing* , *non crossing* otherwise.

Proposition 21 *Let G be a cubic bridgeless traceable graph such that $\chi'(G) = 4$. If for each even cycle of the sequence $(\Gamma_j)_{j=1\dots h}$, say Γ_j, the paths P_j and P'_j are non crossing then there exists three perfect matchings M_α, M_1, M_2 such that $M_\alpha \cap M_1 \cap M_2 = \emptyset$*

Proof By Propositions 19 and 20, we can consider that $h \geq 4$. Let W_1, W_2 and W_3 be the 3 walks obtained by Lemma 14. It is an easy task to see that, up to the names of the walks, W_1 is obtained by replacing the additional edges with the paths that they represent for the even cycles Γ_j with j even. In the same way, W_2 is obtained by replacing the additional edges with the paths that they represent for the even cycles Γ_j with j odd. At last, W_3 is obtained by replacing the additional edges with the paths that they represent for each even cycles Γ_j.

Starting with the *green* colour for the subpath containing the vertex 1, the maximal subpaths of the Hamiltonian path, say P, not containing the edges $e_{min(\Gamma_j)}$ and $e_{max(\Gamma_j)}$ $(j = 2 \dots h-1)$, as well as the edges $e_{max(\Gamma_1)}$ and $e_{min(\Gamma_h)}$, are coloured alternately with *green* and *red*. One can see that W_3 uses all the red subpaths while W_1 and W_2 use the green subpaths only.

CLAIM *W_i $(i = 1, 2)$ well intersects each cycle of the sequence $(\Gamma_j)_{j=1\dots h}$*

Proof Assume without loss of generality that $i = 1$. From Lemmas 15 and 16 we have just to prove that W_1 well intersects the even cycles Γ_j with $j \geq 2$ even.

We can check that W_1 contains the vertex $min(\Gamma_j)$, moreover since the subpaths P_j and P'_j of Γ_j are not crossing the vertex $max(\Gamma_j) + 1$ belongs to W_1 too.

By construction of Γ_j, the green subpath of P ending with the vertex $min(\Gamma_j)$ has no edge in common with Γ_j, as well as the green subpath starting with $max(\Gamma_j) + 1$. Hence W_1 contains exactly two α-edges one ending on $min(\Gamma_j)$ and the other on $max(\Gamma_j) + 1$ on Γ_j. The two subpaths of Γ_j so determined are odd, which proves that W_1 well intersect Γ_j. □

By Lemma 8, the set of α-edges of W_1 (W_2 respectively) is a balanced M_α-matching. We have thus two perfect matchings M_1 and M_2 with no α-edge in common. Hence M_α, M_1 and M_2 have an empty intersection, as claimed. □

Conjecture 5 is known to be verified for bridgeless cubic graphs of oddness 2 (see [6]), Theorem 22 gives a stronger result for cubic bridgeless traceable graphs of chromatic index 4.

Theorem 22 *Let G be a cubic bridgeless traceable graph of chromatic index 4. Then there exists four perfect matchings M_α, M_1, M_2 and M_3 such that $M_\alpha \cap M_i$ does not contain any odd cut set, for $i \in \{1, 2, 3\}$. Moreover, for $i \in \{1, 2, 3\}$ one can associate to M_i two joins J_i and J'_i such that $M_\alpha \cap M_i \cap J_i = M_\alpha \cap M_i \cap J'_i = \emptyset$.*

Proof We consider the walks W_1, W_2 and W_3 described in Lemma 14. Without loss of generality we choose $i = 1$ and we derive from W_1 a walk S_1 as follows.

First we set $S_1 = W_1$. Following the natural order on the vertices of S_1 given by $\prec_{S_1}$ when there is on S_1 a vertex of some edge $e_{max(\Gamma_j)}$ for some cycle Γ_j of the sequence $(\Gamma_j)_{j=2,\dots h-1}$ we set $R = S_1$ and thus either $R' = W_2$ or $R' = W_3$. We apply Lemma 18 on R and R' and we get thus two walks S and S', we know that $R(x, x'_j) + S$ well-intersects Γ_j. The walks S and S' have endpoints in $\{\sigma_R(\Gamma_j), \sigma_{R'}(\Gamma_j)\}$. Hence when $\sigma_R(\Gamma_j)$ is an endpoint of S we write $S_1 = S_1(x, x_j)+S+R(\sigma_R(\Gamma_j), y)$ and $R' = R'(x', x'_j)+S'+R'(\sigma_{R'}(\Gamma_j), y')$. Otherwise $\sigma_{R'}(\Gamma_j)$ is an endpoint of S and we write $S_1 = S_1(x, x'_j) + S + R'(\sigma_{R'}(\Gamma_j), y')$ and $R' = R'(x', x'_j) + S' + R(\sigma_R(\Gamma_j), y)$(recall that either $R' = W_2$ or $R' = W_3$). Finally S_1 well-intersects all concerned cycle of the sequence $(\Gamma_j)_{j=2,\dots h-1}$. Moreover, the walks S_1, W_2 and W_3 are α-disjoint all of them have one endpoint in $\{q_1, q_2, q_3\}$ and the other one in $\{q'_1, q'_2, q'_3\}$.

Due to Lemma 8, the set of edges $A = M_\alpha \cap S_1$ is a balanced M_α-matching, that is there is a perfect matching M_1 such that $M_\alpha \cap M_1 = A$.
But now, if $M_\alpha \cap M_1$ contains an odd cut set, say X, there must be a partition (V_1, V_2) of $V(G)$ into two odd sets such that the edges of X have one end in V_1 and the other end in V_2. Moreover $X \subset M_\alpha$, V_1 and V_2 being odd, each of those sets precisely contains exactly one odd cycle of G_{M_α}. Since W_2 and W_3 are both connecting a vertex of C_1 to a vertex of C_k, there must be an edge of S_1 and an edge of W_3 in X, a contradiction since S_1, W_2 and W_3 are α-disjoint.

Moreover, the set of vertices $S_1 \cup W_2 - S_1 \cap W_2$ together with a sub-path of C_1 whose endpoints are q_1 and q_2 and a sub-path of C_k whose endpoints are q'_1 and q'_2 form a set X of vertices that induce cycles of G. Thus the edge set J_1 of the subgraph induced with $V(G) - X$ is a join that avoids the edges of S_1, in other words $M_\alpha \cap M_1 \cap J_1 = \emptyset$. Similarly we can derive from $S_1 \cup W_3 - S_1 \cap W_3$ another join J'_1 with the same property. □

A direct consequence of Theorem 22 is that Conjecture 4 holds true for cubic bridgeless traceable graphs.

5 Conclusion

As far as we know the techniques developed in Lemmas 14 to 18 as well as in Theorem 22 do not lead to a proof of Conjecture 3 for cubic bridgeless traceable graphs in general. However Conjecture 3 holds true in some particular cases, see for example Propositions 19, 20 or 21 or when in applying Lemma 18 on all the concerned cycles of the sequence $(\Gamma_j)_{j=2,\dots h-1}$, we get two α-disjoint walks that well-intersect the cycles.

In a forthcoming paper ([4]), we prove that a minimal counter-example to Conjecture 3 must have at least 36 vertices (40 vertices when the cyclic edge connectivity of the graph is at least 4).

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