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On parsimonious edge-colouring of graphs with maximum degree three

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Abstract

In a graph G of maximum degree Δ let γ denote the largest fraction of edges that can be Δ edge-coloured. Albertson and Haas showed that $\gamma \geq \frac{13}{15}$ when G is cubic. We show here that this result can be extended to graphs with maximum degree 3 with the exception of a graph on 5 vertices. Moreover, there are exactly two graphs with maximum degree 3 (one being obviously the Petersen graph) for which $\gamma = \frac{13}{15}$. This extends a result given by Steffen. These results are obtained by using structural properties of the so called δ -minimum edge colourings for graphs with maximum degree 3.

Keywords : Cubic graph; Edge-colouring;

Mathematics Subject Classification (2010) : 05C15.

1 Introduction

Throughout this paper, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem [15] that these graphs can be edge-coloured with 4 colours. Let $\phi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of G . It is often of interest to try to use one colour (say δ) as few as possible. When it is optimal following this constraint, we shall say that such a parsimonious edge-colouring ϕ is δ -*minimum*. In [3] we gave without proof (in French, see [6] for a translation) results on δ -*minimum* edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [13, 14]. Some other results which were not stated formally in [4] are needed here. The purpose of Section 2 is to give those results as structural properties of δ -minimum edge-colourings as well as others which will be useful in Section 3.

An edge colouring of G using colours $\alpha, \beta, \gamma, \delta$ is said to be δ -*improper* provided that adjacent edges having the same colours (if any) are coloured with δ . It is clear that a proper edge colouring (and hence a δ -minimum edge-colouring) of G is a particular δ -improper edge colouring. For a proper or δ -improper edge colouring ϕ of G , it will be convenient to denote $E_\phi(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$) the set of edges coloured with x by ϕ . For $x, y \in \{\alpha, \beta, \gamma, \delta\}, x \neq y$, $\phi(x, y)$ is the partial subgraph of G spanned by these two colours, that is $E_\phi(x) \cup E_\phi(y)$ (this subgraph being a union of paths and even cycles where the colours x and y alternate). Since any two δ -minimum edge-colourings of G have the same number

of edges coloured δ we shall denote by $s(G)$ this number (the *colour number* as defined by Steffen in [13]).

As usual, for any undirected graph G , we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges and we suppose that $|V(G)| = n$ and $|E(G)| = m$. Moreover, $V_i(G)$ denotes the set of vertices of G of degree i , and when no confusion is possible we shall write V_i instead of $V_i(G)$. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C , or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2 On δ -minimum edge-colouring

The graph G considered in this section will have maximum degree 3.

Lemma 1 *Let ϕ be a δ -improper colouring of G then there exists a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$*

Proof Let ϕ be a δ -improper edge colouring of G . If ϕ is a proper colouring, we are done. Hence, assume that uv and uw are coloured δ . If $d(u) = 2$ we can change the colour of uv to α, β or γ since v is incident to at most two colours in this set.

If $d(u) = 3$ assume that the third edge uz incident to u is also coloured δ , then we can change the colour of uv for the same reason as above.

If uz is coloured with α, β or γ , then v and w are incident to the two remaining colours of the set $\{\alpha, \beta, \gamma\}$ otherwise one of the edges uv, uw can be recoloured with the missing colour. W.l.o.g., consider that uz is coloured α then v and w are incident to β and γ . Since u has degree 1 in $\phi(\alpha, \beta)$ let P be the path of $\phi(\alpha, \beta)$ which ends on u . We can assume that v or w (say v) is not the other end vertex of P . Exchanging α and β along P does not change the colours incident to v . But now uz is coloured α and we can change the colour of uv with β .

In each case, we get hence a new δ -improper edge colouring ϕ_1 with $E_{\phi_1}(\delta) \subsetneq E_{\phi}(\delta)$. Repeating this process leads us to construct a proper edge colouring of G with $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$ as claimed. \square

Proposition 2 *Let $v_1, v_2, \dots, v_k \in V(G)$ be such that $G - \{v_1, v_2, \dots, v_k\}$ is 3-edge colourable. Then $s(G) \leq k$.*

Proof Let us consider a 3-edge colouring of $G - \{v_1, v_2, \dots, v_k\}$ with α, β and γ and let us colour the edges incident to v_1, v_2, \dots, v_k with δ . We get a δ -improper edge colouring ϕ of G . Lemma 1 gives a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$. Hence ϕ' has at most k edges coloured with δ and $s(G) \leq k$. \square

Proposition 2 above has been obtained by Steffen [13] for cubic graphs.

Lemma 3 *Let ϕ be a δ -improper colouring of G then $|E_{\phi}(\delta)| \geq s(G)$.*

Proof Applying Lemma 1, let ϕ' be a proper edge colouring of G such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$. We clearly have $|E_\phi(\delta)| \geq |E_{\phi'}(\delta)| \geq s(G)$. \square

Theorem 4 [6] *Let G be a graph of maximum degree 3 and ϕ be a δ -minimum colouring of G . Then the following hold.*

1. $E_\phi(\delta) = A_\phi \cup B_\phi \cup C_\phi$ where an edge e in A_ϕ (B_ϕ, C_ϕ respectively) belongs to a uniquely determined cycle $C_{A_\phi}(e)$ ($C_{B_\phi}(e), C_{C_\phi}(e)$ respectively) with precisely one edge coloured δ and the other edges being alternately coloured α and β (β and γ, α and γ respectively).
2. Each edge having exactly one vertex in common with some edge in A_ϕ (B_ϕ, C_ϕ respectively) is coloured γ (α, β , respectively).
3. The multiset of colours of edges of $C_{A_\phi}(e)$ ($C_{B_\phi}(e), C_{C_\phi}(e)$ respectively) can be permuted to obtain a (proper) δ -minimum edge-colouring of G in which the colour δ is moved from e to an arbitrarily prescribed edge.
4. No two consecutive vertices of $C_{A_\phi}(e)$ ($C_{B_\phi}(e), C_{C_\phi}(e)$ respectively) have degree 2.
5. The cycles from 1 that correspond to distinct edges of $E_\phi(\delta)$ are vertex-disjoint.
6. If the edges $e_1, e_2, e_3 \in E_\phi(\delta)$ all belong to A_ϕ (B_ϕ, C_ϕ respectively), then the set $\{e_1, e_2, e_3\}$ induces in G a subgraph with at most 4 edges.

Lemma 5 [4] *Let ϕ be a δ -minimum edge-colouring of G . For any edge $e = uv \in E_\phi(\delta)$ with $d(v) \leq d(u)$ there is a colour $x \in \{\alpha, \beta, \gamma\}$ present at v and a colour $y \in \{\alpha, \beta, \gamma\} - \{x\}$ present at u such that one of connected components of $\phi(x, y)$ is a path of even length joining the two ends of e . Moreover, if $d(v) = 2$, then both colours of $\{\alpha, \beta, \gamma\} - \{x\}$ satisfy the above assertion.*

An edge of $E_\phi(\delta)$ is in A_ϕ when its ends can be connected by a path of $\phi(\alpha, \beta)$, B_ϕ by a path of $\phi(\beta, \gamma)$ and C_ϕ by a path of $\phi(\alpha, \gamma)$. From Lemma 5 it is clear that if $d(u) = 3$ and $d(v) = 2$ for an edge $e = uv \in E_\phi(\delta)$, the A_ϕ , B_ϕ and C_ϕ are not pairwise disjoint; indeed, if the colour γ is present at the vertex v , then $e \in A_\phi \cap B_\phi$.

When $e \in A_\phi$ we can associate to e the odd cycle $C_{A_\phi}(e)$ obtained by considering the path of $\phi(\alpha, \beta)$ together with e . We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when e is in B_ϕ or C_ϕ .

For each edge $e \in E_\phi(\delta)$ (where ϕ is a δ -minimum edge-colouring of G) we can associate one or two odd cycles following the fact that e is in one or two sets among A_ϕ, B_ϕ or C_ϕ . Let \mathcal{C} be the set of odd cycles associated to edges in $E_\phi(\delta)$.

By Theorem 4 any two cycles in \mathcal{C} corresponding to edges in distinct sets A_ϕ, B_ϕ or C_ϕ are at distance at least 2. Assume that $C_1 = C_{A_\phi}(e_1)$ and $C_2 = C_{A_\phi}(e_2)$ for some edges e_1 and e_2 in A_ϕ . Can we say something about the subgraph of G whose vertex set is $V(C_1) \cup V(C_2)$? In general, we have no answer to this problem. However, when G is cubic and any vertex of G

lies on some cycle of \mathcal{C} (we shall say that \mathcal{C} is *spanning*), we have a property which will be useful later. Let us remark first that whenever \mathcal{C} is spanning, we can consider that G is edge-coloured in such a way that the edges of the cycles of \mathcal{C} are alternatively coloured with α and β (except one edge coloured δ) and the remaining perfect matching is coloured with γ . For this δ -minimum edge-colouring of G we have $B_\phi = \emptyset$ as well as $C_\phi = \emptyset$.

Lemma 6 *Assume that G is cubic and \mathcal{C} is spanning. Let $e_1, e_2 \in A_\phi$ and let $C_1, C_2 \in \mathcal{C}$ such that $C_1 = C_{A_\phi}(e_1)$ and $C_2 = C_{A_\phi}(e_2)$. Then at least one of the following is true:*

- (i) C_1 and C_2 are at distance at least 2.
- (ii) C_1 and C_2 are joined by at least 3 edges.
- (iii) C_1 and C_2 have at least two chords each.

Proof Since $e_1, e_2 \in A_\phi$ and \mathcal{C} is spanning we have $B_\phi = C_\phi = \emptyset$. Let $C_1 = v_0v_1 \dots v_{2k_1}$ and $C_2 = w_0w_1 \dots w_{2k_2}$. Assume that C_1 and C_2 are joined by the edge v_0w_0 . By Theorem 4, up to a re-colouring of the edges in C_1 and C_2 , we can consider a δ -minimum edge-colouring ϕ such that $\phi(v_0v_1) = \phi(w_0w_1) = \delta$, $\phi(v_1v_2) = \phi(w_1w_2) = \beta$ and $\phi(v_0v_{2k_1}) = \phi(w_0w_{2k_2}) = \alpha$. Moreover each edge of G (in particular v_0w_0) incident with these cycles is coloured γ . We can change the colour of v_0w_0 in β . We obtain thus a new δ -minimum edge-colouring ϕ' . Performing that exchange of colours on v_0w_0 transforms the edges coloured δ v_0v_1 and w_0w_1 in two edges of $C_{\phi'}$ lying on odd cycles C'_1 and C'_2 respectively. We get hence a new set $\mathcal{C}' = \mathcal{C} - \{C_1, C_2\} \cup \{C'_1, C'_2\}$ of odd cycles associated to δ -coloured edges in ϕ' .

From Theorem 4 C'_1 (C'_2 respectively) is at distance at least 2 from any cycle in $\mathcal{C} - \{C_1, C_2\}$. Hence $V(C'_1) \cup V(C'_2) \subseteq V(C_1) \cup V(C_2)$. It is an easy task to check now that (ii) or (iii) above must be verified. \square

Lemma 7 [4] *Let $e_1 = uv_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G . Then v_1 is the only vertex in $N(u)$ of degree 2 and for any other edge $e_2 \in E_\phi(\delta)$, $\{e_1, e_2\}$ induces a $2K_2$.*

3 Applications

3.1 On a result by Payan

In [10] Payan showed that it is always possible to edge-colour a graph of maximum degree 3 with three maximal matchings (with respect to the inclusion) and introduced henceforth a notion of *strong-edge colouring* where a strong edge-colouring means that one colour is a strong matching while the remaining colours are usual matchings. Payan conjectured that any d -regular graph has d pairwise disjoint maximal matchings and showed that this conjecture holds true for graphs with maximum degree 3.

The following result has been obtained first by Payan [10], but his technique does not exhibit explicitly the odd cycles associated to the edges of the strong matching and their properties.

Theorem 8 *Let G be a graph with maximum degree at most 3. Then G has a δ -minimum edge-colouring ϕ where $E_\phi(\delta)$ is a strong matching and, moreover, any edge in $E_\phi(\delta)$ has its two ends of degree 3 in G .*

Proof Let ϕ be a δ -minimum edge-colouring of G . From Theorem 4, any two edges of $E_\phi(\delta)$ belonging to distinct sets from among A_ϕ, B_ϕ and C_ϕ are at least at distance 2 and thus induce a strong matching. Hence, we have to find a δ -minimum edge-colouring where each set A_ϕ, B_ϕ or C_ϕ induces a strong matching (with the supplementary property that the end vertices of these edges have degree 3). That means that we can work on each set A_ϕ, B_ϕ and C_ϕ independently. Without loss of generality, we only consider A_ϕ here.

Assume that $A_\phi = \{e_1, e_2, \dots, e_k\}$ and $A'_\phi = \{e_1, \dots, e_i\}$ ($1 \leq i \leq k-1$) is a strong matching and each edge of A'_ϕ has its two ends with degree 3 in G . Consider the edge e_{i+1} and let $C = C_{e_{i+1}}(\phi) = u_0, u_1 \dots u_{2p}$ be the odd cycle associated to this edge (Theorem 4).

Let us mark any vertex v of degree 3 on C with a $+$ whenever the edge of colour γ incident to this vertex has its other end which is a vertex incident to an edge of A'_ϕ and let us mark v with $-$ otherwise. By Theorem 4, no consecutive vertices on C have degree 2, that means that a vertex of degree 2 on C has its two neighbours of degree 3 and by Lemma 7 these two vertices are marked with a $-$. By Theorem 4 we cannot have two consecutive vertices marked with a $+$, otherwise we would have three edges of $E_\phi(\delta)$ inducing a subgraph with more than 4 edges, a contradiction. Hence, C must have two consecutive vertices of degree 3 marked with $-$ whatever is the number of vertices of degree 2 on C .

Let u_j and u_{j+1} be two vertices of C of degree 3 marked with $-$ (j being taken modulo $2p+1$). We can transform the edge colouring ϕ by exchanging colours on C uniquely, in such a way that the edge of colour δ of this cycle is $u_j u_{j+1}$. In the resulting edge colouring ϕ_1 we have $A_{\phi_1} = A_\phi - \{e_{i+1}\} \cup \{u_j u_{j+1}\}$ and $A'_{\phi_1} = A'_\phi \cup \{u_j u_{j+1}\}$ is a strong matching where each edge has its two ends of degree 3. Repeating this process we are left with a new δ -minimum colouring ϕ' where $A_{\phi'}$ is a strong matching. \square

Corollary 9 *Let G be a graph with maximum degree 3 then there are $s(G)$ vertices of degree 3 pairwise non-adjacent $v_1 \dots v_{s(G)}$ such that $G - \{v_1 \dots v_{s(G)}\}$ is 3-colourable.*

Proof Pick a vertex on each edge coloured δ in a δ -minimum colouring ϕ of G where $E_\phi(\delta)$ is a strong matching (Theorem 8). We get a subset S of vertices satisfying our corollary. \square

Steffen [13] obtained Corollary 9 for bridgeless cubic graphs.

3.2 Parsimonious edge colouring

Let $\chi'(G)$ be the classical chromatic index of G . For convenience let

$$c(G) = \max\{|E(H)| : H \subseteq G, \chi'(H) = 3\}$$

$$\gamma(G) = \frac{c(G)}{|E(G)|}$$

Staton [12] (and independently Locke [9]) showed that whenever G is a cubic graph distinct from K_4 then G contains a bipartite subgraph (and hence a 3-edge colourable graph, by König's theorem [8]) with at least $\frac{7}{9}$ of the edges of G . Bondy and Locke [2] obtained $\frac{4}{5}$ when considering graphs with maximum degree at most 3.

In [1] Albertson and Haas showed that whenever G is a cubic graph, we have $\gamma(G) \geq \frac{13}{15}$ while for graphs with maximum degree 3 they obtained $\gamma(G) \geq \frac{26}{31}$. Our purpose here is to show that $\frac{13}{15}$ is a lower bound for $\gamma(G)$ when G has maximum degree 3, with the exception of the graph G_5 depicted in Figure 1 below.

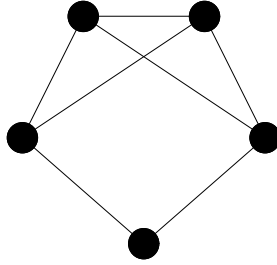


Figure 1: G_5

Lemma 10 *Let G be a graph with maximum degree 3 then $\gamma(G) = 1 - \frac{s(G)}{m}$.*

Proof Let ϕ be a δ -minimum edge-colouring of G . The restriction of ϕ to $E(G) - E_\phi(\delta)$ is a proper 3-edge-colouring, hence $c(G) \geq m - s(G)$ and $\gamma(G) \geq 1 - \frac{s(G)}{m}$.

If H is a subgraph of G with $\chi(H) = 3$, consider a proper 3-edge-colouring $\phi : E(H) \rightarrow \{\alpha, \beta, \gamma\}$ and let $\psi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be the continuation of ϕ with $\psi(e) = \delta$ for $e \in E(G) - E(H)$. By Lemma 1 there is a proper edge-colouring ψ' of G with $E_{\psi'}(\delta) \subseteq E_\psi(\delta)$ so that $|E(H)| = |E(G) - E_{\psi'}(\delta)| \leq |E(G) - E_\psi(\delta)| \leq m - s(G)$, $c(G) \leq m - s(G)$ and $\gamma(G) \leq 1 - \frac{s(G)}{m}$. \square

In [11], Rizzi shows that for triangle-free graphs of maximum degree 3, $\gamma(G) \geq 1 - \frac{2}{3g_o(G)}$ (where the *odd girth* of a graph G , denoted by $g_o(G)$, is the minimum length of an odd cycle in G).

Theorem 11 *Let G be a graph with maximum degree 3 then $\gamma(G) \geq 1 - \frac{2}{3g_o(G)}$.*

Proof Let ϕ be a δ -minimum edge-colouring of G and $E_\phi(\delta) = \{e_1, e_2, \dots, e_{s(G)}\}$. \mathcal{C} being the set of odd cycles associated to edges in $E_\phi(\delta)$, we write $\mathcal{C} = \{C_1, C_2, \dots, C_{s(G)}\}$ and suppose that for $i = 1, 2, \dots, s(G)$, e_i is an edge of C_i . We know by Theorem 4 that the cycles of \mathcal{C} are vertex-disjoint.

Let us write $\mathcal{C} = \mathcal{C}_2 \cup \mathcal{C}_3$, where \mathcal{C}_2 denotes the set of odd cycles of \mathcal{C} which have a vertex of degree 2, while \mathcal{C}_3 is for the set of cycles in \mathcal{C} whose all vertices have degree 3. Let $k = |\mathcal{C}_2|$, obviously we have $0 \leq k \leq s(G)$ and $\mathcal{C}_2 \cap \mathcal{C}_3 = \emptyset$.

If $C_i \in \mathcal{C}_2$ we suppose without loss of generality that $C_i \in A_\phi$ and we have $|C_i| \geq g_o(G)$. Moreover, since any edge in C_i can be coloured δ (Theorem 4),

we may assume that e_i has a vertex of degree 2. We can associate to e_i another odd cycle say $C'_i \in B_\phi$ (Lemma 5) whose edges distinct from e_i form an even path of $\phi(\alpha, \gamma)$ using at least $\frac{g_o(G)-1}{2}$ edges, coloured γ , which are not edges of C_i .

When $|C_i| > g_o(G)$ or $|C'_i| > g_o(G)$ there are either at least $g_o(G) + 2$ edges in C_i or at least $\frac{g_o(G)-1}{2} + 1$ edges coloured γ in C'_i . If $|C_i| = |C'_i| = g_o(G)$ there is at least one edge coloured α in C'_i that is not an edge of C_i , otherwise all the edges coloured γ of C'_i would be chords of C_i , a contradiction since a such chord would form with vertices of C_i an odd cycle of length smaller than $g_o(G)$.

Hence, $C_i \cup C'_i$ contains at least $g_o(G) + \frac{g_o(G)-1}{2} + 1 > \frac{3}{2}g_o(G)$ edges.

Consequently there are at least $\frac{3}{2} \times k \times g_o(G)$ edges in $\bigcup_{C_i \in \mathcal{C}_2} (C_i \cup C'_i)$.

When $C_i \in \mathcal{C}_3$, C_i contains at least $g_o(G)$ edges, moreover, each vertex of C_i being of degree 3, there are at least $\frac{s(G)-k}{2} \times g_o(G)$ additional edges which are incident to a vertex of $\bigcup_{C_i \in \mathcal{C}_3} C_i$.

Since $C_i \cap C_j = \emptyset$ and $C'_i \cap C'_j = \emptyset$ ($1 \leq i, j \leq s(G)$, $i \neq j$), we have

$$m \geq \frac{3}{2}g_o(G) \times k + (s(G) - k) \times g_o(G) + \frac{s(G) - k}{2} \times g_o(G) = \frac{3}{2} \times s(G) \times g_o(G).$$

Consequently $\gamma(G) = 1 - \frac{s(G)}{m} \geq 1 - \frac{2}{3g_o(G)}$. \square

As a matter of fact, $\gamma(G) > 1 - \frac{2}{3g_o(G)}$ when the graph G contains vertices of degree 2. In a further work (see [5]) we refine the bound and prove that $\gamma(G) \geq 1 - \frac{2}{3g_o(G)+2}$ when G is a graph of maximal degree 3 distinct from the Petersen graph.

Lemma 12 [1] *Let G be a graph with maximum degree 3. Assume that $v \in V(G)$ is such that $d(v) = 1$ then $\gamma(G) > \gamma(G - v)$.*

A triangle $T = \{a, b, c\}$ is said to be *reducible* whenever its neighbours are distinct. When T is a reducible triangle in G (G having maximum degree 3) we can obtain a new graph G' with maximum degree 3 by shrinking this triangle into a single vertex and joining this new vertex to the neighbours of T in G .

Lemma 13 [1] *Let G be a graph with maximum degree 3. Assume that $T = \{a, b, c\}$ is a reducible triangle and let G' be the graph obtained by reduction of this triangle. Then $\gamma(G) > \gamma(G')$.*

Theorem 14 *Let G be a graph with maximum degree 3. If $G \neq G_5$ then $\gamma(G) \geq 1 - \frac{\frac{2}{15}}{1 + \frac{2}{3} \frac{|V_2|}{|V_3|}}$.*

Proof From Lemma 12 and Lemma 13 we can consider that G has only vertices of degree 2 or 3 and that G contains no reducible triangle.

Assume that we can associate a set P_e of at least 5 distinct vertices of V_3 for each edge $e \in E_\phi(\delta)$ in a δ -minimum edge-colouring ϕ of G . Assume moreover that

$$\forall e, e' \in E_\phi(\delta) \quad P_e \cap P_{e'} = \emptyset \quad (1)$$

Then

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \geq 1 - \frac{\frac{|V_3|}{5}}{\frac{3}{2}|V_3| + |V_2|}$$

Hence

$$\gamma(G) \geq 1 - \frac{\frac{2}{15}}{1 + \frac{2}{3} \frac{|V_2|}{|V_3|}}$$

It remains to see how to construct the sets P_e satisfying Property (1). Let \mathcal{C} be the set of odd cycles associated to edges in $E_\phi(\delta)$. Let $e \in E_\phi(\delta)$, assume that e is contained in a cycle $C \in \mathcal{C}$ of length 3. By Theorem 4, the edges incident to that triangle have the same colour in $\{\alpha, \beta, \gamma\}$. This triangle is hence reducible, impossible. We can thus consider that each cycle of \mathcal{C} has length at least 5. By Lemma 7, we know that whenever such a cycle contains vertices of V_2 , their distance on this cycle is at least 3. Which means that every cycle $C \in \mathcal{C}$ contains at least 5 vertices in V_3 as soon as C has length at least 7 or C has length 5 but does not contain a vertex of V_2 . For each edge $e \in E_\phi(\delta)$ contained in such a cycle we associate P_e as any set of 5 vertices of V_3 contained in the cycle.

There may exist edges in $E_\phi(\delta)$ contained in a 5-cycle of \mathcal{C} having exactly one vertex in V_2 . Let $C = a_1a_2a_3a_4a_5$ be such a cycle and assume that $a_1 \in V_2$. By Lemma 7, a_1 is the only vertex of degree 2 and by exchanging colours along this cycle, we can suppose that $a_1a_2 \in E_\phi(\delta)$. Since $a_1 \in V_2$, $e = a_1a_2$ is contained in a second cycle C' of \mathcal{C} (see Remark ??). If C' contains a vertex $x \in V_3$ distinct from a_2, a_3, a_4 and a_5 then we set $P_e = \{a_2, a_3, a_4, a_5, x\}$. Otherwise $C' = a_1a_2a_4a_3a_5$ and G is isomorphic to G_5 , impossible.

The sets $\{P_e \mid e \in E_\phi(\delta)\}$ are pairwise disjoint since any two cycles of \mathcal{C} associated to distinct edges in $E_\phi(\delta)$ are disjoint. Hence Property 1 holds and the proof is complete. \square

Albertson and Haas [1] proved that $\gamma(G) \geq \frac{26}{31}$ when G is a graph with maximum degree 3 and Rizzi [11] obtained $\gamma(G) \geq \frac{6}{7}$. From Theorem 14 we get immediately for all graphs $G \neq G_5$ $\gamma(G) \geq \frac{13}{15}$, a better bound. Let us remark that we get also $\gamma(G) \geq \frac{13}{15}$ by Theorem 11 as soon as $g_o(G) \geq 5$.

Lemma 15 *Let G be a cubic graph which can be factored into $s(G)$ cycles of length 5 and has no reducible triangle. Then every 2-factor of G contains $s(G)$ cycles of length 5.*

Proof Since G has no reducible triangle, all cycles in a 2-factor have length at least 4. Let \mathcal{C} be any 2-factor of G . Let us denote n_4 the number of cycles of length 4, n_5 the number of cycles of length 5 and n_{6+} the number of cycles on at least 6 vertices in \mathcal{C} . We have $5n_5 + 6n_{6+} \leq 5s(G) - 4n_4$.

If $n_4 + n_{6+} = 0$, then $n_5 = s(G)$. If $n_4 + n_{6+} > 0$, then the number of odd cycles in \mathcal{C} is at most $n_5 + n_{6+} \leq \frac{5s(G) - 4n_4 - n_{6+}}{5} = \frac{5s(G) - (n_4 + n_{6+}) - 3n_4}{5} < s(G)$. A contradiction since a 2-factor of G contains at least $s(G)$ odd cycles. \square

Corollary 16 *Let G be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. Then G is a cubic graph which can be factored into $s(G)$ cycles of length 5. Moreover every 2-factor of G has this property.*

Proof The optimum for $\gamma(G)$ in Theorem 14 is obtained whenever $s(G) = \frac{|V_3|}{5}$ and $|V_2| = 0$. That is, G is a cubic graph admitting a 2-factor of $s(G)$ cycles of length 5. Moreover by Lemma 13 G has no reducible triangle, the result comes from Lemma 15. \square

As pointed out by Albertson and Haas [1], the Petersen graph with $\gamma(G) = \frac{13}{15}$ supplies an extremal example for cubic graphs. Steffen [14] proved that the only cubic bridgeless graph with $\gamma(G) = \frac{13}{15}$ is the Petersen graph. In fact, we can extend this result to graphs with maximum degree 3 where bridges are allowed (excluding the graph G_5). Let P' be the cubic graph on 10 vertices obtained from two copies of G_5 (Figure 1) by joining by an edge the two vertices of degree 2.

Theorem 17 *Let G be a connected graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. Then G is isomorphic to the Petersen graph or to P' .*

Proof Let G be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$.

From Corollary 16, we can consider that G is cubic and G has a 2-factor of cycles of length 5. Let $\mathcal{C} = \{C_1 \dots C_{s(G)}\}$ be such a 2-factor (\mathcal{C} is spanning). Let ϕ be a δ -minimum edge-colouring of G induced by this 2-factor.

Without loss of generality consider two cycles in \mathcal{C} , namely C_1 and C_2 , and let us denote $C_1 = v_1v_2v_3v_4v_5$ while $C_2 = u_1u_2u_3u_4u_5$ and assume that $v_1u_1 \in G$. From Lemma 6, C_1 and C_2 are joined by at least 3 edges or each of them has two chords. If $s(G) > 2$ there is a cycle $C_3 \in \mathcal{C}$. Without loss of generality, G being connected, we can suppose that C_3 is joined to C_1 by an edge. Applying one more time Lemma 6, C_1 and C_3 have two chords or are joined by at least 3 edges, contradiction with the constraints imposed by C_1 and C_2 . Hence $s(G) = 2$ and G has 10 vertices and no 4-cycle, which leads to a graph isomorphic to P' or the Petersen graph as claimed. \square

We can construct cubic graphs with chromatic index 4 (*snarks* in the literature) which are cyclically 4-edge connected and having a 2-factor of C_5 's.

Indeed, let G be a cubic cyclically 4-edge connected graph of order n and M be a perfect matching of G , $M = \{x_iy_i | i = 1 \dots \frac{n}{2}\}$. Let $P_1 \dots P_{\frac{n}{2}}$ be $\frac{n}{2}$ copies of the Petersen graph. For each P_i ($i = 1 \dots \frac{n}{2}$) we consider two edges at distance 1 apart e_i^1 and e_i^2 . Let us observe that $P_i - \{e_i^1, e_i^2\}$ contains a 2-factor of two C_5 's (C_i^1 and C_i^2).

We construct then a new cyclically 4-edge connected cubic graph H with chromatic index 4 by applying the well known operation dot-product (see Figure 2, see Isaacs [7] for a description and for a formal definition) on $\{e_i^1, e_i^2\}$ and the edge x_iy_i ($i = 1 \dots \frac{n}{2}$). We remark that the vertices of G vanish in the operation and the resulting graph H has a 2 factor of C_5 , namely $\{C_1^1, C_1^2, \dots, C_i^1, C_i^2, \dots, C_{\frac{n}{2}}^1, C_{\frac{n}{2}}^2\}$.

We do not know an example of a cyclically 5-edge connected snark (except the Petersen graph) with a 2-factor of induced cycles of length 5.

Problem 18 Is there any cyclically 5-edge connected snark distinct from the Petersen graph with a 2-factor of C_5 's ?

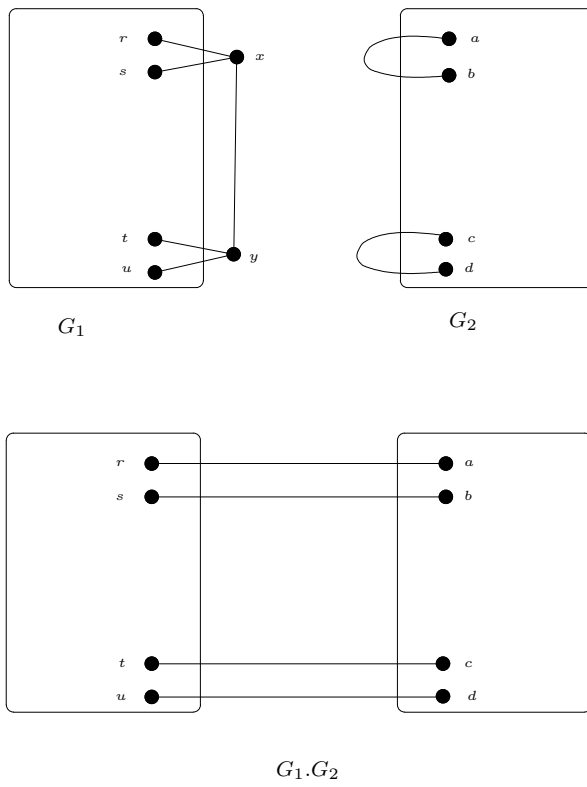


Figure 2: The dot product operation on graphs G_1, G_2 .

As a first step towards the resolution of this Problem we propose the following Theorem. Recall that a *permutation graph* is a cubic graph having some 2-factor with precisely 2 odd cycles.

Theorem 19 *Let G be a cubic graph which can be factored into $s(G)$ induced odd cycles of length at least 5, then G is a permutation graph. Moreover, if G has girth 5 then G is the Petersen graph.*

Proof Let \mathcal{F} be a 2-factor of $s(G)$ cycles of length at least 5 in G , every cycle of \mathcal{F} being an induced odd cycle of G . We consider the δ -minimum edge-colouring ϕ such that the edges of all cycles of \mathcal{F} are alternatively coloured α and β except for exactly one edge per cycle which is coloured with δ , all the remaining edges of G being coloured γ . By construction we have $B_\phi = C_\phi = \emptyset$ and $A_\phi = \mathcal{F}$.

Let xy be an edge connecting two distinct cycles of \mathcal{F} , say C_1 and C_2 ($x \in C_1$, $y \in C_2$). By Theorem 4, since any edge in C_1 or C_2 can be coloured δ , we may assume that there is an edge in C_1 , say e_1 , adjacent to x and coloured with δ , similarly there is on C_2 an edge e_2 adjacent to y and coloured with δ . Let z be the neighbour of y on C_2 such that $e_2 = yz$ and let t be the neighbour of z such that zt is coloured with γ . If $t \notin C_1$, there must be $C_3 \neq C_1$ such that $t \in C_3$, by Theorem 4 again there is an edge e_3 of C_3 , adjacent to t and coloured with δ . But now $\{e_1, e_2, e_3\}$ induces a subgraph with at least 5 edges, a contradiction with Theorem 4.

It follows that \mathcal{C} contains exactly two induced odd cycles and they are of equal length. Consequently G is a permutation graph. When these cycles have length 5, since G has girth 5 G is obviously the Petersen graph. \square

When G is a cubic bridgeless planar graph, we know from the Four Colour Theorem that G is 3-edge colourable and hence $\gamma(G) = 1$. Albertson and Haas [1] gave $\gamma(G) \geq \frac{6}{7} - \frac{2}{35m}$ when G is a planar bridgeless graph with maximum degree 3. Our Theorem 14 improves this lower bound (allowing moreover bridges). On the other hand, they exhibit a family of planar graphs with maximum degree 3 (bridges are allowed) for which $\gamma(G) = \frac{8}{9} - \frac{2}{9n}$.

As Steffen in [14] we denote $g(\mathcal{F})$ the minimum length of an odd cycle in a 2-factor \mathcal{F} and $g^+(G)$ the maximum of these numbers over all 2-factors. We suppose that $g^+(G)$ is defined, that is G has at least one 2-factor (when G is a cubic bridgeless graphs this condition is obviously fulfilled).

When G is cubic bridgeless, Steffen [14] showed that we have :

$$\gamma(G) \geq \max\left\{1 - \frac{2}{3g^+(G)}, \frac{11}{12}\right\}$$

The difficult part being to show that $\gamma(G) \geq \frac{11}{12}$.

Theorem 20 *Let G be a graph with maximum degree 3. Then $\gamma(G) \geq 1 - \frac{2n}{(3n - |V_2|)g^+(G)}$.*

Proof By Lemma 12, we may assume $V_1 = \emptyset$. Hence, $m = \frac{1}{2}(2|V_2| + 3|V_3|)$, moreover $n = |V_2| + |V_3|$, henceforth $m = \frac{3n - |V_2|}{2}$. We have $\gamma(G) = 1 - \frac{s(G)}{m}$, obviously, $s(G) \leq \frac{n}{g^+(G)}$. The result follows. \square

Theorem 21 *Let G be a graph with maximum degree 3 having at least one 2-factor. Assume that $|V_2| \leq \frac{n}{3}$ and $g^+(G) \geq 11$ then $\gamma(G) \geq \max\{1 - \frac{3}{4g^+(G)}, \frac{11}{12}\}$.*

Proof By assumption we have $V_1 = \emptyset$. From Theorem 20 we have just to prove that $\gamma(G) \geq \frac{11}{12}$. Following the proof of Theorem 14, we try to associate a set P_e of at least 8 distinct vertices of V_3 for each edge $e \in E_\phi(\delta)$ in a δ -minimum edge-colouring ϕ of G such that

$$\forall e, e' \in E_\phi(\delta) \quad P_e \cap P_{e'} = \emptyset \quad (2)$$

Indeed, let \mathcal{F} be a 2-factor of G where each odd cycle has length at least 11 and let $C_1, C_2 \dots C_{2k}$ be its set of odd cycles. We have, obviously $s(G) \leq 2k$. Let V'_3 and V'_2 be the sets of vertices of degree 3 and 2 respectively contained in these odd cycles. As soon as $|V'_3| \geq 8s(G)$ we have

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \geq 1 - \frac{\frac{|V'_3|}{8}}{\frac{3}{2}|V_3| + |V_2|} \quad (3)$$

which leads to

$$\gamma(G) \geq 1 - \frac{\frac{2|V'_3|}{24|V_3|}}{1 + \frac{2|V_2|}{3|V_3|}}$$

Since $|V_3| \geq |V'_3|$, we have

$$\gamma(G) \geq 1 - \frac{\frac{2}{24}}{1 + \frac{2|V_2|}{3|V_3|}}$$

and

$$\gamma(G) \geq \frac{11}{12}$$

as claimed.

It remains the case where $|V'_3| < 8s(G)$. Since each odd cycle has at least 11 vertices we have $|V'_2| > 11 \times 2k - |V'_3| > 3s(G)$.

$$\gamma(G) = \frac{m - s(G)}{m} \geq \frac{m - \frac{|V'_2|}{3}}{m}$$

We have

$$\frac{m - \frac{|V'_2|}{3}}{m} \geq \frac{11}{12}$$

when

$$m \geq 4|V'_2| \quad (4)$$

Since $|V_2| \leq \frac{n}{3}$ we have $|V_3| \geq \frac{2n}{3}$ and

$$m = 3\frac{|V_3|}{2} + |V_2| = 3\frac{n - |V_2|}{2} + |V_2| = 3\frac{n}{2} - \frac{|V_2|}{2} \geq 4\frac{n}{3} \geq 4|V'_2| \quad (5)$$

and the result holds. \square

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